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Emergent Friedmann equation from the evolution of cosmic space revisited

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Following the recent study on the emergent Friedmann equation from the expansion of cosmic space for a flat universe, we apply this method to a nonflat universe, and modify the evolution equation to lead to the Friedmann equation. In order to maintain the same form with the original evolution equation, we have to define the time-dependent Planck length, which shows that the spatial curvature of $k = 0$ and $k = 1$ is preferable to $k = -1$ since the Planck length of the nonflat open universe is divergent. Finally, we discuss its physical consequences.

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The thermodynamic quantities of black holes such as its temperature and entropy are related to geometrical quantities such as surface gravity and horizon area [1]. It has been shown that the first law of thermodynamics leads to Einstein's field equations for an accelerating observer [2], and then it has been proposed that gravity can be interpreted as an entropic force caused by changes of entropy associated with the information on a holographic screen [3]. By applying the holographic principle and the equipartition rule of energy, the Einstein field equations could be derived. These make us know that gravity is an emergent phenomenon.

Recently, the Friedmann equation governing the Friedmann-Robertson-Walker (FRW) universe has been remarkably derived by Padmanabhan [4] from the expansion of cosmic space due to the difference between the degrees of freedom on a bulk and in its boundary. It has been proposed that in an infinitesimal interval dt of cosmic time, the increase dV of the cosmic volume in the flat universe is given by

$$\frac{dV}{dt} = \ell_{\rm P}^2 (N_{\rm sur} - N_{\rm bulk}), \qquad (1)$$

where $\ell_{\rm P}$ is the Planck length and $N_{\rm sur}$ and $N_{\rm bulk}$ are the degrees of freedom on the surface and in the bulk, respectively, and the boundary of the bulk is characterized by the Hubble radius. The relation (1) yields the standard dynamical equation in the Friedmann model only for the accelerating phase of the universe given by $\rho + 3p < 0$, where ρ and p are the energy density and the pressure of a perfect fluid, respectively. In order to obtain the dynamical equation for any phases, Eq. (1) was extended to $dV/dt = \ell_{\rm P}^2(N_{\rm sur} - \epsilon N_{\rm bulk}) = \ell_{\rm P}^2(N_{\rm sur} + N_{\rm m} - N_{\rm de})$, where $N_{\rm m}$ and $N_{\rm de}$ are the numbers of degrees of freedom of matter with $\rho + 3p > 0$ and dark energy with $\rho + 3p < 0$ in the bulk, respectively, and $\epsilon = +1$ if $\rho + 3p < 0$ and $\epsilon = -1$ if $\rho + 3p > 0$. Now, applying the continuity equation to the dynamical equation derived from Eq. (1), the $(n + 1)$-dimensional Friedmann equation can be nicely obtained for the nonflat universe by taking into account the integration constant which plays a role of the spatial curvature [5]. There is another generalization of the emergence of cosmic space for a $(n + 1)$-dimensional FRW universe corresponding to general relativity, Gauss-Bonnet gravity, and Lovelock gravity [6]. They proposed that the dynamical equation (1) was generalized as $dV/dt = \ell_{\rm P}^2 f(\Delta N, N_{\rm sur})$ with $\Delta N = N_{\rm sur} - N_{\rm bulk}$.

On the other hand, there have been extensive studies for thermodynamics in cosmology [7–10]. In the nonflat universe, the thermodynamical quantities are related to the apparent horizon instead of the Hubble radius; that is, the corresponding temperature is given by the surface gravity on the apparent horizon and the entropy is proportional to the area of the apparent horizon. The thermodynamical quantities associated with the apparent horizon satisfy the first law of thermodynamics. In this regard, the apparent horizon is carefully considered to obtain the Friedmann equation so that $\ell_{\rm P}^2$ in Eq. (1) should be replaced by $\ell_{\rm P}^2 \tilde{r}_A / H^{-1}$ in Ref. [11], where $\tilde{r}_A$ and H are the apparent horizon and the Hubble parameter, respectively. However, the bulk is still regarded as a sphere in the Euclidean space even for the nonflat space, so that the volume is calculated as the volume of the sphere with the radius of the apparent horizon.

In this brief report, we would like to extend the evolution equation by taking into account the appropriate invariant volume corresponding to the nonflat space instead of the volume of a sphere for the flat space. We will maintain the form of Padmanabhan's evolution equation that the expansion of the universe is due to the difference from the degrees of freedom in the holographic surface between those in the emerged bulk. Then, the Planck length in the original evolution equation can be simply redefined by replacing the time-dependent Planck length. Finally, we will discuss its physical consequence.

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Now, let us consider a spatially homogeneous and isotropic spacetime given by the line element of

$$ds^2 = h_{ab}dx^a dx^b + \tilde{r}^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2)$$

where $\tilde{r} = a(t)r$, $x^0 = t$, $x^1 = r$, and $h_{ab} = \mathrm{diag}(-1, a^2/(1-kr^2))$. Here, k denotes the curvature of space with $k = -1$, 0, and 1 corresponding to open, flat, and closed universes, respectively. The apparent horizon can be calculated from the relation $h^{ab}\partial_a\tilde{r}\partial_b\tilde{r}|_{\tilde{r}=\tilde{r}_A} = 0$ and is obtained as

$$\tilde{r}_A = \frac{1}{\sqrt{H^2 + k/a^2}}, \quad (3)$$

where the Hubble parameter is given by $H = \dot{a}/a$ and the overdot denotes the derivative with respect to comoving time t. Note that the apparent horizon becomes $\tilde{r}_A = H^{-1}$ for flat space with $k = 0$. We assume that the number of degrees of freedom on the surface at the apparent horizon is proportional to its area $4\pi\tilde{r}_A^2$ and is given by [4,5,11]

$$N_{\mathrm{sur}} = 4S = \frac{4\pi\tilde{r}_A^2}{\ell_{\mathrm{P}}^2}, \quad (4)$$

where S is the entropy. For simplicity, we set $k_{\mathrm{B}} = c = \hbar = 1$. The Hawking temperature associated with the apparent horizon is given by [8,11]

$$T_{\mathrm{H}} = \frac{1}{2\pi\tilde{r}_A}. \quad (5)$$

At a given time, an invariant volume of space surrounded by the apparent horizon can be written as

$$V_k = 4\pi a^3 \int_0^{\tilde{r}_A/a} dr \frac{r^2}{\sqrt{1-kr^2}}, \quad (6)$$

which reduces to $V_0 = 4\pi\tilde{r}_A^3/3$ for $k = 0$ and $V_k = 2\pi a^2[\sqrt{k}a \sin^{-1}(\sqrt{k}\tilde{r}_A/a) - k\tilde{r}_A^2 H]$ for $k = \pm 1$. Next, the Komar energy within the bulk [4,5,11] is modified by

$$E_k = (\rho + 3p)V_k, \quad (7)$$

where it depends on the spatial curvature k. Using the equipartition rule of energy, the degrees of freedom in the bulk can be written as

$$N_{\mathrm{bulk}} = \frac{2|E_k|}{T_{\mathrm{H}}}. \quad (8)$$

We assume that in an infinitesimal interval dt, an increase dV_k of the invariant volume even in the nonflat FRW universe including the flat universe is still proportional to the difference between N_{sur} and N_{bulk} as [4]

$$\frac{dV_k}{dt} = \ell_{\mathrm{P}}^2 f_k(t)(N_{\mathrm{sur}} - \epsilon N_{\mathrm{bulk}}), \quad (9)$$

where $\epsilon N_{\mathrm{bulk}}$ is given by $\epsilon N_{\mathrm{bulk}} = N_{\mathrm{de}} - N_{\mathrm{m}} = -(2V_k/T_{\mathrm{H}})(\rho + 3p)$ and $f_k(t)$ is a proportional function which can be chosen appropriately to give the Friedmann equation such as

$$f_k(t) = \frac{\bar{V}_k[\dot{\tilde{r}}_A H^{-1}/\tilde{r}_A + (\tilde{r}_A/H^{-1})(H^{-1}/\tilde{r}_A - V_k/\bar{V}_k)]}{V_k(\dot{\tilde{r}}_A H^{-1}/\tilde{r}_A + \bar{V}_k/V_k - 1)}, \quad (10)$$

with $\bar{V}_k \equiv 4\pi\tilde{r}_A^3/3$. We see that $f_0(t) = 1$ for the flat universe with $k = 0$, and so it is compatible with the previous results [8,11].

To show the better motivation for Eq. (10) and discuss the difference from the previous results, we have to mention the work done in Ref. [11]. The extension to a nonflat space in this cosmic model has been performed by carefully treating the general expression of the apparent horizon depending on the spatial curvature. For this reason, the original equation (1) should have been modified so that the author was able to derive the FRW universe with any spatial curvatures. The key ingredient is to assume the nontrivial proportional function instead of the proportional constant in order to get the correct FRW equation even for the nonflat universe. However, the volume of the universe was still defined in the flat space rather than the general expression of V_k defined for the nonflat universe. All physical quantities reflected the nontrivial dependence of k except for the volume in Ref. [11]. It shows that the time evolution of the flat universe generates the nonflat FRW equation. To avoid this interpretation, we have started with the nonflat universe from the beginning. That is the reason why we have considered the volume increase of the nonflat universe such as Eq. (9) along with Eq. (10).

By taking the time derivative of the invariant volume (6), we obtain

$$\frac{dV_k}{dt} = 4\pi\tilde{r}_A(\dot{\tilde{r}}_A H^{-1} - \tilde{r}_A) + 3HV_k$$
$$= 4\pi\tilde{r}_A^2\left[\frac{\dot{\tilde{r}}_A H^{-1}}{\tilde{r}_A} + \frac{\tilde{r}_A}{H^{-1}}\left(\frac{H^{-1}}{\tilde{r}_A} - \frac{V_k}{\bar{V}_k}\right)\right]. \quad (11)$$

The relation between the degrees of freedom is calculated as

$$N_{\mathrm{sur}} - \epsilon N_{\mathrm{bulk}} = \frac{4\pi\tilde{r}_A^2}{\ell_{\mathrm{P}}^2} + 4\pi\tilde{r}_A(\rho + 3p)V_k. \quad (12)$$

Eliminating p in Eq. (12) by the use of the continuity equation $\dot{\rho} + 3H(\rho + p) = 0$, we can obtain

$$N_{\mathrm{sur}} - \epsilon N_{\mathrm{bulk}} = \frac{4\pi\tilde{r}_A^2}{\ell_{\mathrm{P}}^2}\left[1 - \frac{\ell_{\mathrm{P}}^2 V_k}{H\tilde{r}_A}(\dot{\rho} + 2\rho H)\right]$$
$$= \frac{4\pi\tilde{r}_A^2}{\ell_{\mathrm{P}}^2}\left[1 - \frac{\ell_{\mathrm{P}}^2 V_k}{H\tilde{r}_A a^2}\frac{d}{dt}(\rho a^2)\right]. \quad (13)$$

Substituting Eqs. (10), (11), and (13) into Eq. (9), we get

$$\frac{d}{dt}\left(\frac{a^2}{\tilde{r}_A^2}\right) = \frac{d}{dt}\left[a^2\left(H^2 + \frac{k}{a^2}\right)\right] = \frac{8\pi\ell_{\mathrm{P}}^2}{3}\frac{d}{dt}(\rho a^2). \quad (14)$$

Integrating Eq. (14), one can obtain

$$H^2 + \frac{k}{a^2} = \frac{8\pi\ell_{\mathrm{P}}^2}{3}\rho. \tag{15}$$

Note that one can regard the integration constant as the curvature of space $\tilde{r}_A = 1/H$ [5], while one can set the integration constant to zero for $\tilde{r}_A = 1/\sqrt{H^2 + k/a^2}$ [11]. In the present case, we can take the vanishing integration constant for simplicity.

It is interesting to note that the original form of the evolution equation (1) can be maintained if we replace the proportional function along with the Planck length by the effective Planck length as $\ell_{\mathrm{P}}^2 f_k(t) = (\ell_{\mathrm{P}}^2)_k^{\mathrm{eff}}$ in the modified equation (9). What it means is that the Newton constant may be running depending on the curvature of space as long as one can take the square of the Planck length as a gravitational coupling. In particular, the Newton constant is merely constant for the flat universe of $k = 0$. In order to examine the late time behavior of the effective Planck length, we now consider three cases: the radiation-dominant, the matter-dominant, and the Λ-dominant universes, where Λ denotes the positive cosmological constant. First, the radiation-dominant universe is described by the scale factor $a = a_0 t^{1/2}$, where a_0 is a positive constant. The square of the effective Planck length goes to $(\ell_{\mathrm{P}}^2)_{k=-1}^{\mathrm{eff}} \approx \ell_{\mathrm{P}}^2(1/3)(1/2 - t^{1/2}/a_0)^{-1/2}$ with $t < a_0^2/4$. At $t = a_0^2/4$, it is divergent for the open universe of $k = -1$ while it is asymptotically constant since $(\ell_{\mathrm{P}}^2)_{k=1}^{\mathrm{eff}} = \ell_{\mathrm{P}}^2[2 - (3\pi a_0/8)t^{-1/2} + O(t^{-1})]$ for the closed universe of $k = 1$. Second, in the matter-dominant universe described by $a = a_0 t^{2/3}$, the square of the effective Planck length behaves as $(\ell_{\mathrm{P}}^2)_{k=-1}^{\mathrm{eff}} = \ell_{\mathrm{P}}^2[(4/9)(4/9 - t^{2/3}/a_0^2)^{-1/2} + O((4a_0^2/9 - t^{2/3})^{1/2})]$ with $t < (2a_0/3)^3$, and it is divergent at $t = (2a_0/3)^3$. However, it becomes a constant for the closed universe of $k = 1$ since $(\ell_{\mathrm{P}}^2)_{k=1}^{\mathrm{eff}} = \ell_{\mathrm{P}}^2[2 - (\pi a_0/2)t^{-1/3} + O(t^{-2/3})]$ at the late time. Thus for the radiation- and matter-dominant universes, the effective Planck length is divergent for $k = -1$ while it approaches $(\ell_{\mathrm{P}}^2)_{k=1}^{\mathrm{eff}} = 2\ell_{\mathrm{P}}^2$ for $k = 1$. Finally, the Λ-dominant nonflat universe is governed by the scale factor $a = a_0 e^{\alpha t}$ with $\alpha \equiv \sqrt{\Lambda/3}$. The square of the effective Planck length behaves as $(\ell_{\mathrm{P}}^2)_{k=\pm 1}^{\mathrm{eff}} = \ell_{\mathrm{P}}^2[12/7 + (61/(343\alpha^2 a_0^2))e^{-2\alpha t} + O(e^{-4\alpha t})]$, and both of them approach $(\ell_{\mathrm{P}}^2)_k^{\mathrm{eff}} = (12/7)\ell_{\mathrm{P}}^2$ at the asymptotic infinity. Therefore, for the radiation- and matter-dominant cases, the Newton constant can be divergent for the nonflat open universe while it becomes the constant for the nonflat closed universe. In the vacuum-energy-dominant case, the Newton constant can be finite at $k = \pm 1$ and $k = 0$. It shows that if the nonflat open universe evolves eternally without encountering the vacuum-energy era, then it undergoes the divergent gravitational interaction.

As seen from the above calculations, the gravitational coupling diverges for the nonflat spaces, so that it is natural to ask why the present calculation is not compatible with the observations. Before the big bang, all forces were expected to be unified, and then the gravitational force was decoupled from three forces at 10^{19} GeV where the temperature was about 10^{32} K. It has been claimed that the radiation-dominant era, matter-dominant era, and the recent vacuum-energy-dominant era, all can be described in terms of thermodynamics by assuming a thermal or quasithermal state of our universe. However, we found some deviations from the standard results, and the worst case is that the severely divergent gravitational coupling at $k = -1$ appears for these eras. The essential drawback of our formulation is due to the time-dependent temperature (5) and equipartition law (8). Note that these have something to do with the assumption of thermal equilibrium. Therefore, we think that this incompatibility with the observations should be related to the validity of the temperature (5) and equipartition law (8) when we consider the nonflat universe. Of course, the best fit appears at the Λ-dominant era of the flat universe since the Hawking temperature (5) is no more time dependent, which is definitely constant, so that we can use the equipartition law and the related thermal quantities in equilibrium, which is compatible with the qualitative behavior of the accelerated expansion of the universe with $k = 0$. As a result, we have tried to extend the original idea that the cosmic expansion appears from the difference of the degrees of freedom on the bulk and that of the boundary to the nonflat space; however, it is not easy to obtain meaningful interpretations except for the flat universe since the time-dependent temperature (5) and the equipartition law (8) can be inappropriate to the nonflat universe.

In conclusion, following the proposal that the space evolution is due to the different degrees of freedom on the holographic surface and its bulk, we have extended the evolution equation to give the Friedmann equation even in the nonflat universe corresponding to $k = \pm 1$ by taking into account the invariant volume surrounded by the apparent horizon. Moreover, the limit to the flat universe of $k = 0$ can be easily recovered.

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