

Geodesics at sudden singularitiesJohn D. Barrow¹ and S. Cotsakis²¹*DAMTP, Centre for Mathematical Sciences, Cambridge University, Cambridge CB3 0WA, United Kingdom*²*Research Group of Geometry, Dynamical Systems and Cosmology, University of the Aegean, Karlovassi, Samos 83200, Greece*

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We show that a general solution of the Einstein equations that describes the approach to an inhomogeneous and anisotropic sudden spacetime singularity does not experience geodesic incompleteness. This generalizes the result established for isotropic and homogeneous universes. Further discussion of the weakness of the singularity is also included.

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I. INTRODUCTION

There has been strong interest in the structure and ubiquity of finite-time singularities in general-relativistic cosmological models since they were first introduced by Barrow *et al.* [1] as a counterexample to the belief [2] that closed Friedmann universes obeying the strong energy condition must collapse to a future singularity. They were characterized in detail as sudden singularities in Refs. [3–5] and as “weak” singularities in the senses defined by Tipler [6] and Krolak [7]. A sudden future singularity at t_s is defined informally in terms of the metric expansion scale factor, $a(t)$ with $t_s > 0$, by $0 < a(t_s) < \infty$, $0 < \dot{a}(t_s) < \infty$, $\ddot{a}(t \rightarrow t_s) \rightarrow -\infty$. These archetypal examples have finite values of the metric scale factor, its first time derivative, and the density at a finite time, but they possess infinities in the second time derivative of the scale factor and in the pressure. Higher-order examples exist with infinities in the $(2 + n)$ th derivatives of the scale factor and the n th derivative of the matter pressure [4,5]. Other varieties of finite-time singularities have been found in which a different permutation of physical quantities takes on finite and infinite values.¹

The general isotropic and homogeneous approach to a sudden finite time singularity introduced in [3] for the Friedmann universe has been used [9] to construct a quasi-isotropic, inhomogeneous series expansion around the finite-time singularity which contains nine independently arbitrary spatial functions, as required as a part of the general cosmological solution when the pressure and density are not related by an equation of state. The stability properties of a wide range of possible finite-time singularities were also studied in Ref. [10].

¹There is an interesting example in Newtonian mechanics of motion which formally begins at rest with infinite acceleration. It is motion at a constant power. This means $v\dot{v}$ is constant, where $v = \dot{x}$ is the velocity in the x direction, and so $v \propto t^{1/2}$ and $x \propto t^{3/2}$ if initially $v(0) = x(0) = 0$. Thus, we see that the acceleration formally has $\dot{v} \propto t^{-1/2}$ and diverges as $t \rightarrow 0$. This motion at a constant power is an excellent model of drag-car racing. The singularity in the acceleration as $t \rightarrow 0$ is ameliorated in practice by the inclusion of frictional effects on the initial motion [8].

It has also been shown by Fernández-Jambrina and Lazkoz [11–13] that, in the context of the Friedmann universe, the sudden singularity introduced in [3] has the property that geodesics do not feel the sudden singularity and pass through it. In this paper we will examine the evolution of geodesics in the general nine-function solution in the vicinity of an inhomogeneous and anisotropic sudden singularity to see if this result continues to hold. We will also formulate these earlier results more precisely.

We will use Latin indices for spacetime components and Greek indices for space components, and set $G = c = 1$.

II. GEOMETRIC SETUP

Let Σ_0 be the 3-space defined by the equations $x^i = \phi^i(\xi)$, $\xi = (\xi_1, \xi_2, \xi_3)$, located at $t = 0$. We suppose that the sudden singularity is located at the time t_s in the future, and we denote by Σ_s the 3-space $t = t_s$. We may attach geodesic normal (synchronous) coordinates at any point $B \in \Sigma_s$ as follows. Let $u^i(\xi)$ be a C^0 vector field over Σ_0 , and through any point on Σ_0 we draw causal geodesics tangent to $u^i(\xi)$ in both future and past directions parametrized by t . These geodesics have $dx^i/dt = u^i$ (and $t = 0$ on Σ_0). Then the geodesic $x^i(t)$ that passes through B cuts Σ_0 at the point A with coordinates (ξ_1, ξ_2, ξ_3) where $t = 0$ and $dx^i/dt = u^i$. The coordinates of B are then (t_s, ξ) , where t_s is t evaluated at B and ξ at A .

III. C^1 QUASI-ISOTROPIC METRIC

In [9] we found that near a sudden singularity the general form of the metric in geodesic normal coordinates is

$$ds^2 = dt^2 - \gamma_{\alpha\beta} dx^\alpha dx^\beta, \quad (1)$$

$$\gamma_{\alpha\beta} = a_{\alpha\beta} + b_{\alpha\beta}t + c_{\alpha\beta}t^n + \dots, \quad n \in (1, 2),$$

and the leading orders of the energy-momentum tensor components, defined by

$$T_j^i = (\rho + p)u^i u_j - p\delta_j^i, \quad u_a u^a = 1,$$

are

$$\begin{aligned}
u_\alpha &= -\frac{3(b_{\alpha;\beta}^\beta - b_{;\alpha})}{2n(n-1)c} t^{2-n} \sim t^{2-n}, & u^\alpha &= \gamma^{\alpha\beta} u_\beta \sim t^2, \\
16\pi\rho &= \left(P + \frac{b^2 - b^{\mu\nu}b_{\mu\nu}}{4}\right) - \frac{n}{2}(b^{\mu\nu}c_{\mu\nu} - bc)t^{n-1} + \dots, \\
16\pi p &= -\frac{2n(n-1)c}{3} t^{n-2} - \frac{3b^{\mu\nu}b_{\mu\nu} + b^2 + 4P}{12} - \frac{n}{2}\left(b^{\mu\nu}c_{\mu\nu} + \frac{bc}{3}\right)t^{n-1} + \dots.
\end{aligned} \tag{2}$$

The Ricci scalar is

$$\begin{aligned}
R &= R_i^i \\
&= -n(n-1)ct^{n-2} - \frac{b_{\mu\nu}b^{\mu\nu} + b^2 + 4P}{4} \\
&\quad - \frac{n}{2}(b^{\mu\nu}c_{\mu\nu} + bc)t^{n-1} + \dots,
\end{aligned}$$

where P is the trace of $P_{\alpha\beta}$, the spatial Ricci tensor associated with $a_{\alpha\beta}$.

This solution is only \mathcal{C}^1 , meaning that the metric and its first derivatives, as well as the Christoffel symbols, will be continuous through the 3-slice Σ_s containing the sudden singularity at B , but we expect discontinuities in the second and higher derivatives of the metric, and at least in the first derivatives of the Christoffel symbols.

IV. GEODESIC BEHAVIOR AT t_s

The Christoffel symbols are \mathcal{C}^0 , and so the geodesic equations,

$$\ddot{x}^i + \Gamma_{jk}^i u^j u^k = 0, \tag{3}$$

will have solutions $x^i(t)$ with continuous derivatives up to and including d^2x^i/dt^2 . Therefore, we can Taylor estimate these solutions as follows. For any $\delta > 0$ and $t \in (t_s - \delta, t_s + \delta)$, we have

$$x^i(t) = x^i(t_s) + (t - t_s)u^i(t_s) - \frac{1}{2}(t - t_s)^2(\Gamma_{\alpha\beta}^i u^\alpha u^\beta)(t_*), \tag{4}$$

with t_* between t and t_s . The last term is given in the Lagrange form for the remainder. Since the error term is quadratic in $t - t_s$, it vanishes asymptotically for both past and future sudden singularities. This means that the geodesic equations (3) have complete \mathcal{C}^2 solutions through the sudden singularity at B to the future and the past given by this form. In higher-order Lagrangian theories of gravity it is possible for sudden singularities to arise because there are infinities in the third, or higher, time derivatives of the metric scale factor. In these cases the effect of the singularity on the geodesics is weaker still and avoids a violation of the dominant energy condition [5,14].

A spacetime is Tipler (T) strong [6] iff, as the affine parameter $\tau \rightarrow t_s$, the integral

$$T(u) \equiv \int_0^\tau d\tau' \int_0^{\tau'} R_{ij} u^i u^j d\tau'' \rightarrow \infty. \tag{5}$$

The spacetime is Krolak (K) strong [7] iff, as $\tau \rightarrow t_s$, the integral

$$K(u) \equiv \int_0^\tau R_{ij} u^i u^j d\tau' \rightarrow \infty. \tag{6}$$

If these conditions do not hold, the spacetime is T weak or K weak, respectively. It is possible for a singularity to be K strong but T weak; for example, the so-called [15] type III singularities with $\rho \rightarrow \infty$, $|p| \rightarrow \infty$ as $a \rightarrow a_s$ have this property. In our case, the various components of the Ricci curvature have leading orders of the following forms: $R_{00} \sim t^{n-2}$, $R_{0\alpha} \sim t^0$, $R_{\alpha\gamma} \sim t^{2(n-1)}$, while $u^0 \sim t^0$, $u^\alpha \sim t^2$. Therefore,

$$R_{ij} u^i u^j \sim t^{n-2} + 2t^2 + t^{2n+2}. \tag{7}$$

But since at the sudden singularity, $1 < n < 2$, we find that

$$R_{ij} u^i u^j \sim t^{n-2}, \quad \text{as } t \rightarrow t_s, \tag{8}$$

and so after one integration we have

$$K(u) \sim \tau^{n-1} \rightarrow t_s^{n-1}, \quad \text{as } \tau \rightarrow t_s, \tag{9}$$

and after a second integration,

$$T(u) \sim \tau^n \rightarrow t_s^n, \quad \text{as } \tau \rightarrow t_s, \tag{10}$$

and so the generic sudden singularity (1) is T weak and K weak.² This weakness also suggests that we do not expect these singularity structures to be modified by quantum particle production effects. Some studies of the quantum cosmology of sudden singularities which confirm this have

²If $0 < n < 1$, and the metric contains a power of $(t - t_s)^n$, then it will only have a well-defined meaning as a real function when $t - t_s > 0$. So, at any point t_s (e.g., when $t_s = 0$), it will be defined asymptotically only in the past direction and not in the future. Our argument also requires continuity of the Christoffel symbols, and so it will not be valid when the metric contains a power of $(t - t_s)^n$ with $0 < n < 1$, even if we restrict only to the past direction. For an isotropic solution with a sudden singularity at $t = 0$, see [16].

been made in Ref. [17], but quantum modifications can occur for particular regularization procedures [18]. There are also interesting classical questions about the passage through a sudden singularity in certain examples where the background matter variables, ρ and p , do not continue to be well defined. These problems can be avoided by a distributional redefinition of the cosmological quantities involved [19]. It is also interesting to note that extended objects like fundamental string loops can pass through weak singularities without their invariant sizes becoming infinite [20].

V. CONCLUSION

This result generalizes the studies of Fernández-Jambrina and Lazkoz [11–13] by showing that there is no geodesic incompleteness at a general inhomogeneous and anisotropic sudden singularity. The inclusion of anisotropy and inhomogeneity does not introduce geodesic incompleteness. We expect that these results will also hold for sudden singularities in loop quantum gravity cosmologies of the sort studied in Ref. [21] and in higher-order Lagrangian gravity theories [4].

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