Gauge/gravity duality in heterotic string theory

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Gravity duals for little string theories—which give rise to four-dimensional theories that undergo permanent confinement in the infrared—have not been studied in great detail. We address this question in the framework of heterotic $SO(32)$ and $E_8 \times E_8$ string theory, constructing these backgrounds by wrapping heterotic 5-branes on calibrated two-cycles of non-Kähler resolved conifolds. Related to deformations of the underlying little string theories, we find numerous analytic solutions preserving $\mathcal{N} = 1$ supersymmetry in four dimensions. These theories all have non-Abelian global symmetries that generally arise from both the heterotic vector bundle and from certain orbifold states. In the decoupling limit, we argue that the gravity duals are given by non-Kähler manifolds that have both blown-up two-cycles and three-cycles at the origin. We argue this following certain duality sequences that include M-theory torsional manifolds at an intermediate step, which help us to construct new type I′ gauge/gravity duality pairs. In the M-theory duality frame, we also elucidate new sequences of flips and flops.

I. INTRODUCTION

Geometric transitions have long been exploited to explore strong-coupling regimes of field theories, beginning with [1] in the context of type II string theories where powerful topological string methods were available. The basic setup is to wrap branes on compact two- or three-cycles of Calabi–Yau threefolds; in the limit where the number of branes is very large, we can shrink the wrapped cycle until it vanishes so that the manifold develops a singularity. That singularity is then resolved (or deformed) via an extremal transition, where a different vanishing cycle grows and where the branes are replaced by fluxes that thread a dual cycle. In terms of the gauge theory living on the original branes, the size of the new cycle is determined by strong-coupling effects, such as gluino condensation.

A natural question to ask is how this generalizes to heterotic string theory, where, in principle, one could have world sheet descriptions of both sides of the duality. However, there are a few immediate hurdles that must be overcome. The first is that the gauge theory side, described by little string theory (LST), is quite mysterious. This is in contrast to the type II case, where the gauge theories are simply given by super Yang-Mills theories in various dimensions with well-understood dynamics. The second hurdle is on the gravity side, where we generally expect the manifolds to have non-Kähler metrics. For these kinds of

manifolds, it is much more difficult to solve the supersymmetry conditions and Bianchi identity as well as to single out the relevant cycles. Finally, while in the type II cases we could gain more information by using topological methods, the heterotic theory generally only admits a "half-twist" which is not nearly as powerful.

Nevertheless, some recent progress has been made in a series of papers (Refs. [2,3] as well as Ref. [4]), starting with the program initiated in Ref. [5]. In fact, the first hurdle regarding the mysterious gauge theories can actually be turned to our advantage: having a gravity dual to a more mysterious LST could be a fruitful way to extract information about these theories. In earlier works of some of the present authors [2,3], we adopted this viewpoint and argued for the gravity duals using various consistency arguments stemming from the duality cycle advertised in Refs. [5–7]. These consistency arguments also helped us overcome the second hurdle, finding non-Kähler backgrounds, without the need to appeal to half-twisted topological theories. However, no *direct* proof of the gauge/gravity duality was provided in Refs. [2,3], and, furthermore, most of the analysis focussed on the $SO(32)$ heterotic string.

In the current work, we will begin to address these shortcomings in a more unified approach, setting up a geometric transition route in the $E_8 \times E_8$ heterotic theory. Our construction will exploit the F theory and the $E_8 \times E_8$ heterotic duality by using D3-branes to probe a system of two orthogonal sets of D7/O7 branes/orientifolds that wrap a $\mathbf{P}^1$. This additional $\mathbf{P}^1$ makes the issue of supersymmetry much more subtle since we need to switch on additional fluxes to preserve supersymmetry. In Sec. IV, we show that these fluxes force the heterotic geometries to be non-Kähler resolved conifolds, with the NS5-branes

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wrapping calibrated two-cycles. In Sec. IV B, we determine metric, three-form, and dilaton, presenting three distinct cases that correspond to different configurations of wrapped NS5-branes. In Sec. IV C, we present simple non-Abelian vector bundles that, together with the other fields, preserve supersymmetry and satisfy the Bianchi identity. We also discuss the asymptotic behavior and range of applicability of these solutions. To our knowledge, this is the first time most of these backgrounds have been determined for the $E_8 \times E_8$ theory.

Through the course of Sec. V, we argue for the gravity duals by following a duality chain that incorporates type I, type I′, and M theory. Compared to some of our earlier works [2,3], this is a more direct way to integrate these theories into a single duality chain. Using this chain, we obtain two distinct limits: a non-Kähler resolved conifold and a non-Kähler deformed conifold. For the present work, we focus on the resolved conifold side. The LSTs have global degrees of freedom, and we argue that they reside *both* in the heterotic vector bundle as well as in certain orbifold states. Of course, vector bundle degrees of freedom can also be viewed as small instantons in certain limits, which are intimately connected to singularities in the torsion and, hence, NS5-branes [8]. In the three classes of solutions that we study in Sec. IV B, the torsion asymptotes to the constant value required to maintain the non-Kählerity, as well as zero potential (4.13), of the resolved conifold backgrounds—this is shown in Sec. V A. In the dual gravity description, we do not expect these singularities to be present, which we know can happen if the small instantons dissociate into smooth gauge flux. This raises the intriguing possibility that the gauge/gravity duality in the heterotic theory is directly related to a small instanton transition. We discuss this further in Secs. V A and V F, where we also show how to look for the confining strings and new exotic states in the dual LST.

II. HETEROTIC AND LITTLE STRING THEORY BACKGROUND

As shown in Ref. [9], the supersymmetric variations in heterotic string theory allow for a complex but non-Kähler internal manifold, satisfying

$$d^\dagger J = i(\partial - \bar{\partial}) \ln \| \Omega \|$$
$$J^{a\bar{b}} F_{a\bar{b}} = F_{ab} = F_{\bar{a}\bar{b}} = 0 \qquad (2.1)$$
$$i\partial\bar{\partial}J = \alpha'\left[\mathrm{tr}(R_+ \wedge R_+) - \frac{1}{30}\,\mathrm{tr}(F \wedge F) \right],$$

where J is the fundamental (1, 1) Hermitian form on the internal manifold, Ω is the holomorphic (3, 0)-form, F is the $SO(32)$ or $E_8 \times E_8$ gauge field strength, and R_+ is the Ricci two-form constructed from the ω_+ connection: $\omega_{+\mu}^{ij} \equiv \Gamma_\mu^{ij} + \frac{1}{2}\mathcal{H}_\mu^{ij}$, where $\mathcal{H}_\mu^{ij} \equiv \mathcal{H}_{\mu\nu\rho} e^{\nu i} e^{\rho j}$ and $e^{\nu i}$ is the vielbein. The Neveu-Schwarz (NS) three-form flux $\mathcal{H}$ and dilaton ϕ are related via

$$\mathcal{H} = -e^{2\phi} * d(e^{-2\phi} J) = i(\bar{\partial} - \partial)J, \qquad (2.2)$$

so the final equation in Eq. (2.1) is the Bianchi identity for $\mathcal{H}$. Heterotic NS5-branes appear on the right-hand side of the Bianchi identity in Eq. (2.1), as delta-function contributions to $d\mathcal{H}$.

In Ref. [8], six-dimensional theories with $\mathcal{N} = 1$ supersymmetry, arising from the $SO(32)$ heterotic theory compactified on K3, were studied. There it was argued that when k small gauge theory instantons on K3 shrink to zero size, they become a stack of k NS5-branes, emerging from the vector bundle mist.[1] In the Bianchi identity (2.1), this corresponds to tuning vector bundle moduli so that $\mathrm{tr}(F \wedge F)$ develops delta-function contributions, arising from "small instantons" that shrink to zero size, that can be reinterpreted as source terms for NS5-branes. It was also argued there that there is an $Sp(2k)$ gauge group that lives on the k NS5-branes. This was generalized to the $E_8 \times E_8$ heterotic theory in Ref. [11], where it was argued that similar zero-size instantons give rise to LSTs. This was further analyzed in Ref. [12], where the author studied a broad class of six-dimensional theories with $\mathcal{N} = 1$ superconformal symmetry—which could arise, for example, from compactification of the $E_8 \times E_8$ heterotic theory on K3—and found singularities arising from tensionless strings at boundaries between phases with different instanton numbers.[2]

When compactified on tori, LSTs enjoy T-duality symmetry, arising from the fact that they are nonlocal theories [14].[3] They also have no gravitational degrees of freedom. Most familiar examples of LSTs are obtained by studying the dynamics of multiple 5-branes in various limits of string theory. A LST is labeled by N, the number of 5-branes, but it does not become weakly coupled at large N, i.e., there is no $1/N$ expansion [15]. LSTs can also arise from the dynamics of string theory at a singularity; for example, they can arise in type IIA and IIB theories on an orbifold singularity $\mathbb{C}^2/\Gamma_G$, as reviewed in Refs. [13,16].

Holographic duals for LSTs from multiple coincident NS5-branes in type IIA were studied in Ref. [17]. Along these lines, in Ref. [18] the authors found the holographic dual of a single heterotic NS5-brane. In Ref. [19], another LST was studied using a technique called deconstruction, obtaining a six-dimensional LST on a torus as a special limit of four-dimensional gauge theory. There they also

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[1]This can also be seen from a sigma model analysis of the background; see Ref. [10].

[2]For an early history and progress report on LSTs, see the excellent review of Ref. [13].

[3]This is different from D-branes, which have local quantum field theories (QFTs) because the LSTs, being given by the NS5-branes, do not change dimension under longitudinal T dualities. Since the world volume theories just shift from one description to another, they do not have a well-defined energy-momentum tensor [12].

GAUGE **GRAVITY**

Het $E_8 \times E_8$
non-Kähler res. cone,
torsion, k 5-branes

GT ···

Het $E_8 \times E_8$
non-Kähler def. cone,
torsion,
vector bundles

↑ U-dualities ↑ U-dualities

(A) **F-theory** CY4, G-fluxes, k D3s

(C) **Type IIB** fluxes, k D3s, D7/O7 ⊥ D7/O7

(D) **Type I** non-Kähler res. cone, torsion, k D5s

GT ···

Type I non-Kähler def. cone, H_{RR}-fluxes, vector bundles

2T

↓ mirror ↑ mirror

Type IA non-Kähler def. cone, fluxes, k D6s, D6/O6 ⊥ D6/O6

GT ···

Type IA non-Kähler res. cone, fluxes, only one D6/O6

↓ M-theory uplift ↑ dimensional reduction

M-theory G_2-structure manifold, G-fluxes (torsion)

flop

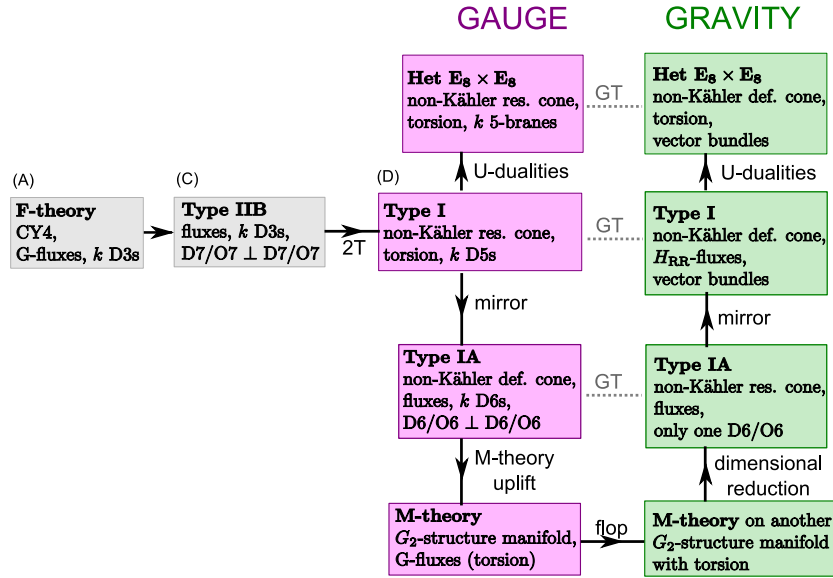**M-theory** on another G_2-structure manifold with torsion

FIG. 1 (color online). The duality sequence that will be used in this paper to argue for the gravity duals in heterotic, type I, and M-theories. The boxes in red in the left-hand column are related to the theories on the branes, i.e., the strongly coupled gauge theories in the IR, except for the M-theory case. The boxes in green in the right-hand column are the possible gravity duals. F-theory origins of these theories are depicted by the boxes in white. The alphabetical ordering of the leftmost boxes are related to a refinement of the parameter regimes of the theories, described in Fig. 2 below. The boxes A, C, and D appearing here will be defined more chronologically in Fig. 2 below.

briefly discussed confinement, but a more detailed study of confinement is still lacking.[4]

In the rest of our paper, we consider similar setups, but without studying the LSTs directly. Instead, we aim to use geometric transitions in heterotic string theory to shed light on LSTs. The usual problem with LSTs is that they cannot be treated on the same footing as many of the systems studied using AdS/CFT techniques. On the other hand, geometric transitions work predominantly in nonconformal cases, so they can be used to study strong-coupling effects in setups that are not covered by AdS/CFT. Our goal, then, is twofold: one is to obtain a large set of supergravity solutions corresponding to wrapped NS5-branes in $E_8 \times E_8$ heterotic string theory, and the other is to conjecture possible gravity duals of these theories, obtained via geometric transitions.

III. TYPE II DUALITY FRAME

Our aim is to extend 5-brane gauge/gravity duality to the $E_8 \times E_8$ heterotic theory. In Sec. IV, we will return to the heterotic setting, but for now we begin in type II, where we have a better handle on gauge/gravity duality. We begin with a large number of 5-branes wrapped on a calibrated two-cycle of a non-Kähler resolved conifold. The duality chain that we will follow is depicted in

Fig. 1. The solutions that we obtain using the duality chain will have ranges of validity that we need to keep track of, so some of these details are illustrated in Fig. 2.

The resolved and the deformed conifold that we will consider support non-Kähler metrics that will be relevant to our work. The main feature we want to highlight is the presentation of them as $\mathbf{P}^1$ fibrations over an asymptotically locally Euclidean (ALE) space, so we describe the geometry next.

F/Het duality

(A) Generic **F-theory** varying τ k D3s

(I) Generic **Type I** varying axion, H_{RR} k 5-branes

S

(H) Generic **Het** vector bundle, torsion varying dilaton, k 5-branes

(B) vary ↑ coupling
Away from (C) O7s split/join

(E) vary ↑ coupling
Strong **Type I** $\mathcal{O}(1)$ coupling

S

(G) vary ↑ coupling
Intermediate **Het** $\mathcal{O}(1)$ coupling torsion, k 5-branes

(C) vary ↑ coupling
Type IIB D7/O7 ⊥ D7/O7, constant τ, k D3s

2T

(D) vary ↑ coupling
Weak **Type I** D9/O9, D5/O5 k 5-branes

S

(F) vary ↑ coupling
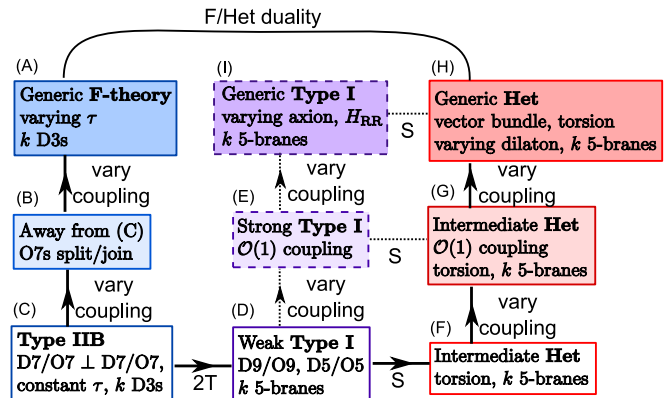Intermediate **Het** torsion, k 5-branes

FIG. 2 (color online). This flow diagram depicts the various refinements and regimes of interest of the theories that we study here. The duality that we study in the early sections of the paper are related to the boxes marked A, C, and D. These are the regions where the orientifold descriptions are most useful. The heterotic analyses that we perform later in the text are related to the boxes marked H and G. Other interconnections between the theories are depicted by arrows.

[4]The theories that we are most interested in are the LSTs compactified on a two-sphere, giving rise to four-dimensional theories below a certain energy scale. In this paper, we will be particularly interested in the confining behavior in the deep infrared, so the UV completion will be beyond the scope of our discussions.

A. Resolved and deformed conifolds, revisited

The standard deformed conifold, as embedded in $\mathbb{C}^4$, has equation

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = \mu^2 \tag{3.1}$$

for any $\mu \in \mathbb{C}^*$. For fixed μ, this can be viewed as a fibration over the z_4 plane, with fiber

$$z_1^2 + z_2^2 + z_3^2 = \mu^2 - z_4^2. \tag{3.2}$$

If z_4 is not $\pm\mu$, this fiber is a noncompact smooth complex surface, a K3 surface in the sense that its canonical bundle is trivial. If z_4 equals $\pm\mu$, the fiber is an ALE space, a singular K3. In other words, this is essentially the same fibration as for the conifold case, with the parameter values altered. This realizes the conifold as an ALE fibration over $\mathbf{P}^1$.[5] In our case, as we will momentarily see, it will be more useful to instead view the warped deformed and warped resolved conifolds as $\mathbf{P}^1$ fibrations over a warped ALE space.

First let us show this for the warped deformed conifold. The metric of the warped deformed conifold can be written in the following way:

$$ds^2 = ds_{\text{ALE}}^2 + ds_{\text{fiber}}^2, \tag{3.3}$$

where the metrics of the ALE base and two-sphere fiber are given by

$$
\begin{aligned}
ds_{\text{ALE}}^2 &= f_1 dr^2 + f_2(d\psi + \cos\theta_1 d\phi_1)^2 \\
&\quad + \alpha_3(d\psi + \cos\theta_1 d\phi_1) + f_3(d\theta_1 + \epsilon_1)^2 \\
&\quad + f_3(\sin\theta_1 d\phi_1 + \epsilon_2)^2 \\
ds_{\text{fiber}}^2 &= \left(f_3 - \frac{f_4^2}{4f_3}\right)(d\theta_2^2 + \alpha_4 \sin^2\theta_2 d\phi_2^2),
\end{aligned} \tag{3.4}
$$

where we have defined various variables, f_i, ϵ_i, and α_i, to be

$$
f_1 = \frac{r^2\hat{\gamma}' - \hat{\gamma}}{r^2} + \frac{r^2\hat{\gamma}}{r^4 - \mu^4}, \qquad f_2 = \frac{r^4 - \mu^4}{4r^2}f_1, \qquad f_3 = \frac{\hat{\gamma}}{4}, \qquad f_4 = \frac{2\mu^2}{r^2}f_3,
$$

$$
\epsilon_1 = \frac{f_4}{2f_3}(\cos\psi\, d\theta_2 + \sin\psi\sin\theta_2 d\phi_2), \qquad \epsilon_2 = \frac{f_4}{2f_3}(\sin\psi\, d\theta_2 - \cos\psi\sin\theta_2 d\phi_2), \tag{3.5}
$$

$$
\alpha_3 = 2\alpha(\theta_1, \phi_1, \psi)f_2\cos\theta_2 d\phi_2, \qquad \alpha_4 = 1 + \frac{4f_2 f_3}{4f_3^2 - f_4^2}\cot^2\theta_2
$$

and where $\hat{\gamma}$ is given by

$$
\hat{\gamma} \equiv r^{\frac{4}{3}}\frac{\left(\sqrt{1 - \frac{\mu^4}{r^4}} - \frac{\mu^4}{r^4}\cosh^{-1}\left(\frac{r^2}{\mu^2}\right)\right)^{\frac{1}{3}}}{\sqrt{1 - \frac{\mu^4}{r^4}}}, \qquad \hat{\gamma}' \equiv \frac{\partial\hat{\gamma}}{\partial r^2}. \tag{3.6}
$$

Finally, the function $\alpha(\theta_1, \phi_1, \psi)$ appearing in Eq. (3.12) is typically an even or odd function of the variables (θ_1, ϕ_1, ψ) under a $\mathbb{Z}_2$ reflection, i.e., $\alpha(-\theta_1, -\phi_1, -\psi) = \pm\alpha(\theta_1, \phi_1, \psi)$. For the standard deformed conifold with Kähler metric, we have $\alpha = 1$, but for a warped deformed conifold, α will be nontrivial.

To see the fibrational structure, fix a point on the $\mathbf{P}^1$ with coordinates (θ_2, ϕ_2); then

$$\epsilon_1 = \epsilon_2 = \alpha_3 = 0, \tag{3.7}$$

and so the metric (3.3) becomes

$$
\begin{aligned}
ds_{\text{ALE}}^2 = f_1 dr^2 &+ f_3(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) \\
&+ f_2(d\psi + \cos\theta_1 d\phi_1)^2.
\end{aligned} \tag{3.8}
$$

Defining a radial variable $\rho \equiv \sqrt{\frac{3}{2}}r^{2/3}$, we see that the metric has the asymptotic form

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$$
\begin{aligned}
ds_{\text{ALE}}^2 \rightarrow d\rho^2 &+ \frac{1}{6}\rho^2(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) \\
&+ \frac{1}{9}\rho^2(d\psi + \cos\theta_1 d\phi_1)^2.
\end{aligned} \tag{3.9}
$$

This is not quite an ALE space because the constant factors $\frac{1}{6}$ and $\frac{1}{9}$ would have to both be $\frac{1}{4}$. Instead, we call this a *warped* ALE space, though we may drop the "warped" adjective at times. Next, fixing a point on the warped ALE space, we see that the fiber metric is that of a squashed $\mathbf{P}^1$, with squashing α_4 depending on where we are in the warped ALE space.

In a similar vein, the warped resolved conifold can be presented as a $\mathbf{P}^1$ fibration over a warped ALE. The metric of the resolved conifold can be written as

$$
\begin{aligned}
ds^2 = \gamma' dr^2 &+ \frac{\gamma + a^2}{4}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) \\
&+ \frac{\gamma' r^2}{4}(d\psi + \cos\theta_1 d\phi_1)^2 + \alpha_1(d\psi + \cos\theta_1 d\phi_1) \\
&+ \frac{\gamma}{4}(d\theta_2^2 + \alpha_2\sin^2\theta_2 d\phi_2^2) \\
&\equiv ds_{\text{ALE}}^2 + ds_{\text{fiber}}^2,
\end{aligned} \tag{3.10}
$$

where a is the resolution parameter; ds_{ALE}^2 denotes the metric of the warped ALE space plus mixed terms with

the fiber; ds^2_{fiber} denotes the metric of a squashed two-sphere with coordinates (θ_2, ϕ_2); γ is given by [20]

$$\gamma = -2a^2 + 4a^4 N(r)^{-\frac{1}{3}} + N(r)^{\frac{1}{3}}, \qquad (3.11)$$

$$N(r) = \frac{1}{2}\Big(r^4 - 16a^6 + \sqrt{r^8 - 32a^6 r^4}\Big);$$

and α_1 and α_2 describe the ALE[6] warping and the squashing of the two-sphere, respectively, given by

$$\alpha_1 \equiv \frac{\gamma' r^2 \alpha(\theta_1, \phi_1, \psi)}{2} \cos\theta_2 d\phi_2,$$

$$\alpha_2 \equiv 1 + \frac{r^2 \gamma'}{\gamma} \cot^2\theta_2, \qquad \gamma' \equiv \frac{\partial\gamma}{\partial r^2}, \qquad (3.12)$$

away from the origin of (θ_2, ϕ_2). At any other point on sphere, we have an ALE space, as seen from the first line of Eq. (3.10).

Just as for the warped deformed conifold, for the warped resolved conifold, we can fix a point on the $\mathbf{P}^1$ fiber by (θ_2, ϕ_2) to constant values. This implies $\alpha_1 = 0$ so the metric (3.10) becomes

$$ds^2_{\text{ALE}} = \gamma' dr^2 + \frac{\gamma + a^2}{4}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2)$$

$$+ \frac{\gamma' r^2}{4}(d\psi + \cos\theta_1 d\phi_1)^2, \qquad (3.13)$$

which is the metric for a warped resolved ALE space.

B. F-theory picture

In our previous papers [2,3], we have argued for the existence of gauge/gravity duality in the $SO(32)$ heterotic theory by duality chasing the original type IIB geometric transition. In particular, the technique for going from a local heterotic description to a global description with wrapped 5-branes gives us a way to generate the gravity solution *before* the geometric transition.

Another way to study the type IIB transition is through compactification of F-theory on fourfolds. For example, consider F theory on an elliptically fibered Calabi–Yau fourfold $\mathbf{X} \xrightarrow{\pi} \mathbf{B}$, where $\mathbf{B}$ is the threefold base. We will assume that $\mathbf{B}$ contains a smooth curve $\mathbf{E} \simeq \mathbf{P}^1$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, which implies that locally, near $\mathbf{E}$, the base looks like a resolved conifold. We can then perform an extremal transition from $\mathbf{B}$ to $\mathbf{B}'$, obtained by contracting the $\mathbf{P}^1$ to a point and then smoothing. Under this transition, we obtain another elliptically fibered Calabi–Yau fourfold

$$\mathbf{X}' \xrightarrow{\pi} \mathbf{B}', \qquad (3.14)$$

which is the manifold that, in the presence of both RR and NS fluxes in type IIB, can be deformed to yield a metrically

non-Kähler manifold (which still has the same topology as $\mathbf{B}'$) [2,7]. This construction yields the same result as the one discussed earlier, namely, that the branes disappear in the process so that the final result just contains the gravitational dual with fluxes and no extra branes.[7]

The duality map between F theory and heterotic theory could now, in principle, help us understand the transition on the heterotic side. Unfortunately, this is easier said than done since the dualities between the heterotic theories and F theory on fourfolds are involved. One has to tread carefully to find the appropriate duality.

As discussed above, our aim is to connect a specfic F-theory compactification with a compactification of $E_8 \times E_8$ heterotic theory. One way to do this would be to exploit the duality between the Gimon–Polchinski model [23] and F theory on a particular Calabi–Yau threefold with Hodge numbers (3, 243), which admits an elliptic fibration over $\mathbf{P}^1 \times \mathbf{P}^1$ [24,25]. This duality is useful because of its connection to the $E_8 \times E_8$ heterotic theory on a K3 manifold, where the 24 instantons are divided equally between the two E_8's. Our next step then would be to express the resolved conifold as a K3 fibration over a $\mathbf{P}^1$. We could then extend the duality to heterotic on a resolved conifold by dualizing along the K3 fiber.[8]

Our starting point is then to study the theory generated by k D3-branes probing type IIB backgrounds with intersecting 7-branes and orientifold planes. We expect the gauge theory on the k D3-branes to be $Sp(2k) \times Sp(2k)$ in the presence of one set of intersecting branes and planes. Local charge cancellation then imposes a global $U(4) \times U(4)$ symmetry, where a set of four D7-branes is placed perpendicular to another set of four D7-branes. We will discuss soon how other global symmetries may appear.

C. Type II picture

Our starting point, as we mentioned above, would be to take k D3-branes in the type IIB theory to probe the

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[6]The warping, as mentioned above, takes us away from the standard ALE metric [21] (see also the recent paper, Ref. [22], for more details on ALE and asymptotically locally flat spaces).

[7]Although, in more general cases, the gravity duals can be *nongeometric*. This feature was discussed in detail in Refs. [2,3].

[8]There are other variants of this story. For example, the F-theory fourfold compactification with a $\mathbf{P}^1 \times \mathbf{P}^1$ is also connected to another Calabi–Yau with Hodge numbers (51, 3) [and probably also to (3, 51)], as pointed out by Refs. [26–28]. The difference between the two compactifications is related to the number of tensor and charged hypermultiplets. In this paper, we will concentrate only on the (3, 243) case where the heterotic dual has only one tensor multiplet. To get more tensors in six dimensions, we have to redefine the orientifold operation in the dual type I side. A way to achieve this would be to define the orientifold operation in such a way that, in addition to reversing the world sheet coordinate σ to $\pi - \sigma$, it also flips the sign of the twist fields at all fixed points (see, for example, Refs. [27,29]). This way the closed string sectors of both the theories match, but in the twisted sector, we get 17 tensor multiplets instead of hypermultiplets. Therefore, even though the orientifolding action looks similar in both cases, the massless multiplets are quite different.

TABLE I.   The orientations of various branes in the type IIB, intersecting 7-brane duality frame. The check marks denote the directions along which we have either the branes or the manifold.

Direction	0	1	2	3	4	5	6	7	8	9
D7/O7	✓	✓	✓	✓	·	·	✓	✓	✓	✓
D7$'$/O7$'$	✓	✓	✓	✓	✓	✓	✓	✓	·	·
k D3s	✓	✓	✓	✓	·	·	·	·	·	·
ALE	·	·	·	·	✓	✓	·	·	✓	✓
$\mathbf{P}^1$	·	·	·	·	·	·	✓	✓	·	·

intersecting D7/O7 system. This is almost like the Gimon–Polchinski setup [23] but with one crucial difference: the two intersecting D7/O7 systems wrap a $\mathbf{P}^1$. In the absence of the $\mathbf{P}^1$, we would expect the infrared theory to be an $\mathcal{N} = 2$, $Sp(2k) \times Sp(2k)$ gauge theory that would, under some special conditions, flow to a conformal fixed point. In the presence of the $\mathbf{P}^1$, the infrared theory should instead be an $\mathcal{N} = 1$ gauge theory. The moduli space of the gauge theory—ignoring the global symmetry, for now—will be k copies of the following manifold:

$$\mathcal{M} \equiv \frac{\text{ALE} \otimes \mathbf{P}^1}{(-1)^{F_L} \cdot \Omega \cdot \sigma}, \qquad (3.15)$$

where σ is the Nikulin involution [30] if the ALE space is replaced by a K3 manifold; otherwise, it is the usual orbifolding for the ALE case, and $\otimes$ denotes a *local* product, i.e., there is some nontrivial fibrational structure. Note that the moduli space is a noncompact manifold. The global symmetry of the system will be typically $G \times G$, where we will soon discuss the form of G.

The story now is simple. Under two T dualities along the two-cycle of the ALE space, followed by an S duality, will convert this background to the heterotic theory background (a table of the brane setup appears in Table I).

1. Gauss law constraints

Before moving ahead, let us see how Gauss's law is satisfied. Let the k D3-branes in type IIB be oriented along $x^{0,1,2,3}$, the ALE space along $x^{4,5,8,9}$, and the $\mathbf{P}^1$ along $x^{6,7}$ (see Table I). T dualizing along a two-cycle of the ALE space that we take to have coordinates $x^{4,5}$, we see that the heterotic NS5-branes will be along the $x^{0,1,2,3,4,5}$ directions, so the $x^{6,7,8,9}$ directions should be noncompact.

Back in the type IIB picture, there are two sets of (p, q) 7-branes, one set wrapping $x^{0,1,2,3,4,5,6,7}$, and one set wrapping $x^{0,1,2,3,6,7,8,9}$. If we did not have the k D3-branes, this could allow for the $x^{4,5}$ and $x^{8,9}$ directions to be compact, e.g., $\mathbf{P}^1 \times \mathbf{P}^1$, over which the F-theory torus would be nontrivially fibered, giving rise to F theory on a Calabi-Yau threefold with $\mathbf{F}_n$ base [24,25]. In our case, the D3-brane charge is a problem unless we make the $x^{6,7,8,9}$ directions noncompact, while the $x^{4,5}$ directions need to be compact since we will be T dualizing along them. Thus, we

see that the ALE space in Eq. (3.15) really has to be noncompact, as does the $\mathbf{P}^1$:

(i) The set of (p, q) 7-branes along $x^{0,1,2,3,4,5,6,7}$ has fewer than 24 branes. This can be implemented by taking only one O7 plane and set of charge canceling D7-branes.

(ii) The $\mathbf{P}^1$ oriented along $x^{6,7}$ is topologically noncompact, with one or two antipodal points removed. This can be implemented via making ϕ_2 nonperiodic or via changing the definition of α_2 and α_4 in Eqs. (3.10) and (3.3), respectively.

2. Algebrogeometric picture

To better understand the IIB/F-theory setup, consider the standard Weierstrass equation governing the F-theory axiodilaton as it varies over the ALE space (the $\mathbf{P}^1$ will be suppressed for this subsection, as will the D3-brane probes):

$$y^2 = x^3 + f(u, v)x + g(u, v), \qquad (3.16)$$

where the coordinate $u = x^4 + ix^5$ corresponds to the ALE two-cycle along which we will perform T duality (and is the usual u plane of Seiberg–Witten theory [31]), and $v \equiv x^8 + ix^9$ corresponds to the other ALE directions. $f(u, v)$ is a polynomial of bidegree $(8, 8)$, and $g(u, v)$ is a polynomial of bidegree $(12, 12)$. These polynomials give us the physics not only at the orientifold point but also away from it. In fact, at the orientifold point, the description can be made a little simpler by choosing the functional forms for $f(u, v)$ and $g(u, v)$ to be

$$f(u, v) = \prod_{i=1}^{8} \mathcal{A}_1^i W_i + \prod_{i=1}^{4} \mathcal{A}_2^i Z_i^2,$$

$$g(u, v) = \prod_{i=1}^{8} \prod_{j=1}^{4} \mathcal{A}_3^{ij} W_i Z_j + \prod_{i=1}^{4} \mathcal{A}_4^i Z_i^3 + \prod_{i=1}^{12} \mathcal{A}_5^i U_i, \qquad (3.17)$$

where $\mathcal{A}_k^i$ are coefficients that are constrained by using consistency conditions for orientifolds. The other variables Z_i, W_i, and U_i, are defined as

$$
\begin{aligned}
Z_i &\equiv (u - \hat{u}_i)(v - \hat{v}_i), \\
W_i &\equiv (u - u_i)(v - v_i), \\
U_i &\equiv (u - \tilde{u}_i)(v - \tilde{v}_i).
\end{aligned}
\qquad (3.18)
$$

More details about these polynomials (3.17) can be found in Ref. [32], where the coefficients $\mathcal{A}_k^i$'s were derived.[9] The discriminant locus is then given by the curve

$$\sum_{n,p} \mathcal{C}_{np} \prod_{i,j,k} Z_i^{6-2n-3p} W_j^n U_k^p = 0, \qquad (3.19)$$

which can be decomposed as

$$(\mathcal{C}_{10}\mathcal{F}_1 + \mathcal{C}_{11}\mathcal{F}_2 + \mathcal{C}_{02}\mathcal{F}_3 + \mathcal{C}_{00}\mathcal{F}_4 + \mathcal{C}_{30}\mathcal{F}_5)$$
$$\times (\mathcal{C}_{20}\mathcal{F} + \mathcal{C}_{01}\mathcal{G}) = 0, \qquad (3.20)$$

where the polynomials $\mathcal{F}$ and $\mathcal{G}$ are degree 16 in (u, v), and where $\mathcal{C}_{mn} \neq \mathcal{C}_{nm}$. Thus, we have two curves spanning the discriminant locus given by

$$\mathcal{C}_{20}\mathcal{F} + \mathcal{C}_{01}\mathcal{G} = 0, \qquad (3.21)$$
$$\mathcal{C}_{10}\mathcal{F}_1 + \mathcal{C}_{11}\mathcal{F}_2 + \mathcal{C}_{02}\mathcal{F}_3 + \mathcal{C}_{00}\mathcal{F}_4 + \mathcal{C}_{30}\mathcal{F}_5 = 0.$$

The first curve specifies the orientifold condition under which the type IIB theory has a heterotic dual, given by a heterotic compactification on a K3 manifold. The second curve should be interpreted as (p, q) 7-branes that form a nondynamical orientifold plane. Since our concern is mostly the orientifold background specified by the first curve in Eq. (3.21), we will ignore the physics behind the second curve in this paper.

A more general elliptic fibration can be specified, for example, by

$$f_k(u) = \prod_{i=1}^{8} A_{ik}(u - a_{ik}) + \prod_{i=1}^{4} B_{ik}(u - b_{ik})^2$$
$$+ \prod_{i=1}^{2} C_{ik}(u - c_{ik})^4 + D_k(u - d_k)^8,$$

$$g_k(u) = \prod_{i=1}^{12} M_{ik}(u - m_{ik}) + \prod_{i=1}^{6} N_{ik}(u - n_{ik})^2 \qquad (3.22)$$
$$+ \prod_{i=1}^{4} S_{ik}(u - s_{ik})^3 + \prod_{i=1}^{3} P_{ik}(u - p_{ik})^4$$
$$+ \prod_{i=1}^{2} Q_{ik}(u - q_{ik})^6 + R_k(u - r_k)^{12},$$

where, as before, the coefficients are allowed to take values determined by the underlying dynamics of F theory. The polynomials $f_k(u)$ and $g_k(u)$ are now used to determine $f(u, v)$ and $g(u, v)$ by

$$f(u, v) \equiv f_1(u)f_2(v), \qquad g(u, v) \equiv g_1(u)g_2(v). \qquad (3.23)$$

As we mentioned above, these polynomials give us the physics not only at the orientifold point but also away from it, for example, as in Eq. (4.74). We will discuss some of these curves later when we study vector bundles.

_____

[9] We have also corrected a typo in Ref. [32].

3. Orientifold limit and duality chain

After having provided the algebrogeometric details of the type IIB background, we now ask: what happens when we allow k D3-branes to probe the orientifold point? The orientifold action is similar to the one discussed earlier, and the space involutions are done via the involution σ. The type IIB manifold will be

$$\mathcal{M} \equiv \frac{\mathcal{C}_{\rm res}}{(-1)^{F_L} \cdot \Omega \cdot \sigma}, \qquad (3.24)$$

where $\mathcal{C}_{\rm res} \equiv {\rm ALE} \otimes \mathbf{P}^1$. For the K3 case, σ describes the Nikulin involution of the form $(r, a, \delta) = (2, 2, 0)$ on the K3 subspace of $\mathcal{C}_{\rm res}$ (e.g., see Ref. [30] for more details). Making two T dualities along the $x^{4,5}$ directions [which is the resolved $\mathbf{P}^1$ of the ALE fiber in Eq. (3.10)] will take us to type I on the resolved cone $\mathcal{C}_{\rm res}$, which is then S dual to heterotic on $\mathcal{C}_{\rm res}$.

The k D3-branes at the intersecting orientifold point dualize to k small instantons wrapping the $x^{4,5}$ two-cycle of the ALE space on the heterotic side, hence, wrapping the two-cycle of the resolved conifold. The resolved conifold will support a non-Kähler metric with $SU(3)$ structure. Interestingly, as we will soon see, our construction then brings us to the intrinsic torsion and G structures of Ref. [33]. Recall that for a manifold with $SU(3)$ structure, we have the two conditions

$$0 = d(e^{-2\phi}\Omega) = d(e^{-2\phi} * J), \qquad (3.25)$$

where J is the fundamental two-form, Ω is the holomorphic $(3, 0)$ form, and ϕ is the dilaton. The torsion $\mathcal{H}$ will lie in

$$\mathcal{H} \in W_3 \oplus W_4 \oplus W_5, \qquad (3.26)$$

where W_i are torsion classes that describe what type of manifold we have [34]. In the *absence* of the probe D3-branes, the heterotic dual $\mathcal{C}_{\rm res}$ contains a topologically noncompact K3 surface. Of course, this does not imply that the metric $\mathcal{C}_{\rm res}$ is conformally Kähler, and in the presence of the D3-brane probes, the situation is even more different. Before we describe the non-Kähler heterotic geometry, we should ask what happens to the $Sp(2k) \times Sp(2k)$ gauge symmetry on the heterotic side.

To do this, we will have to study the orientations of various branes on the type IIB side. The two $\mathbb{Z}_2$ orientifold transformations are generated by $\{1, g\}$ and $\{1, h\}$, where [35,36]:

$$g = (-1)^{F_L} \cdot \Omega \cdot \mathcal{I}_{45}, \qquad h = (-1)^{F_L} \cdot \Omega \cdot \mathcal{I}_{89}, \qquad (3.27)$$

where $\mathcal{I}_{ab}$ denotes orbifold action along $x^{a,b}$. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ action can be suggestively rewritten as

TABLE II. Configuration from Table I after T dualizing along $x^{4,5}$. $\mathcal{C}_{\text{res}}$ is the resolved conifold.

Direction	0	1	2	3	4	5	6	7	8	9
D9/O9	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
D5′/O5′	✓	✓	✓	✓	.	.	✓	✓	.	.
k D5s	✓	✓	✓	✓	✓	✓	.	.	.	.
$\mathcal{C}_{\text{res}}$	.	.	.	.	✓	✓	✓	✓	✓	✓

$$\{1, g\} \times \{1, h\} = \{1, g, h, gh\}$$
$$= \{1, (-1)^{F_L} \cdot \Omega \cdot \mathcal{I}_{45}, (-1)^{F_L} \cdot \Omega$$
$$\cdot \mathcal{I}_{89}, \mathcal{I}_{4589}\}$$
$$= \{1, (-1)^{F_L} \cdot \Omega \cdot \mathcal{I}_{45}\} \times \{1, \mathcal{I}_{4589}\}. \quad (3.28)$$

This implies the following relation:

$$\frac{\mathbb{R}^4}{\{1, g\} \times \{1, h\}} = \frac{\mathbb{R}^4}{\{1, \mathcal{I}_{4589}\} \times \{1, (-1)^{F_L} \cdot \Omega \cdot \mathcal{I}_{45}\}}$$
$$\equiv \frac{\text{ALE}}{\{1, (-1)^{F_L} \cdot \Omega \cdot \mathcal{I}_{45}\}}. \quad (3.29)$$

Now recall that the warped ALE space is obtained precisely at a fixed point (θ_2, ϕ_2) of the $\mathbf{P}^1$, i.e., we are taking α in Eq. (3.12) to be an even function of (θ_1, ϕ_1, ψ). We can now extend this orientifold action over the full six-dimensional internal space. Thus, allowing a global structure of the form ALE $\otimes$ $\mathbf{P}^1$ leads to the space (3.24), i.e.,

$$\frac{\mathcal{C}_{\text{res}}}{\{1, (-1)^{F_L} \cdot \Omega \cdot \mathcal{I}_{45}\}}. \quad (3.30)$$

In a more general setting with a slightly different $\mathbb{Z}_2$ action [26–28], the type IIB manifold is an orientifold of a compact K3 manifold, related to Eq. (3.22).

T dualizing along $x^{4,5}$ to type IIB, the orientifold actions (3.27) transform into

$$(-1)^{F_L} \cdot \Omega \cdot \mathcal{I}_{45} \longrightarrow \Omega,$$
$$(-1)^{F_L} \cdot \Omega \cdot \mathcal{I}_{89} \longrightarrow \Omega \cdot \mathcal{I}_{4589}, \quad (3.31)$$

where the former would lead to O9 planes and the latter would lead to O5 planes. The two sets of D7-branes become D9- and D5-branes, while the k probe D3-branes become k D5-branes. Thus, we actually arrive at the type I theory. This is the well-known Gimon–Polchinski system [23], except there is an additional $\mathbf{P}^1$. The orientations of various branes are given in Table II.

This can now be S dualized to the heterotic theory. An S-duality transformation will convert the D9/O9 system into equivalent vector bundles, and the D5/O9 generically transforms into NS5-branes stuck to an orbifold plane. In the case that the D5/O5 system has zero net charge, this will S dualize to only an orbifold plane [37]. Geometrically, using the fiber-wise duality, we expect to obtain heterotic on $\mathcal{C}_{\text{res}}$, as discussed in Ref. [38]. The final heterotic configuration is now expressed in Table III and depicted in Fig. 3.

What about the heterotic gauge symmetry? There are two kinds of gauge groups involved here: the type IIB $Sp(2k) \times Sp(2k)$ pulled to the heterotic side by U dualities (including the transformations in Ref. [38]) and the remnant of the $E_8 \times E_8$ gauge symmetry from the type IIB global symmetry. For example, if we focus on a single probe D3-brane in this background, the generic gauge group is $U(1)$ and *not* $U(1) \times U(1)$. At the orientifold intersection point, one might expect the gauge symmetry to enhance to $Sp(2) \times Sp(2)$, but this is broken to $U(1)$ by quantum corrections. The full theory is then given by an $\mathcal{N} = 1$ SCFT with a $U(1)$ vector multiplet and a massless charged hypermultiplet [35,36]. If we move a D3-brane along $u = x^4 + ix^5$ (the ALE two-cycle), the massless charged hypermultiplet can be interpreted as the monopole/dyon point for one of the $Sp(2)$ groups. Similarly, moving the D3-brane along $v = x^8 + ix^9$ (the other ALE directions), the same massless charged hypermultiplet may now be interpreted as the monopole/dyon point of the other $Sp(2)$ gauge group. Thus, the nonperturbative effects in this model convert the monopole/dyon point of one $Sp(2)$ gauge group to the monopole/dyon point of the other $Sp(2)$ gauge group.

One may expect similar behavior for the heterotic NS5-branes obtained through our U-duality chain. The discussion of the gauge symmetry in heterotic will appear in Sec. IV C. Note, however, that the physics of type IIB probe D3-branes must be a bit different from that of the heterotic NS5-branes, although they share similar confining properties, since we expect LST to play a role in the decoupling limit of the heterotic side [14].

TABLE III. Configuration from Table II after a U-duality to heterotic. The orbifold 5-plane is denoted by O$_r$5. The vector bundles are not shown.

Direction	0 (x^0)	1 (x^1)	2 (x^2)	3 (x^3)	4 (θ_1)	5 (ϕ_1)	6 (θ_2)	7 (ϕ_2)	8 (r)	9 (ψ)
O$_r$5/NS5	✓	✓	✓	✓	.	.	✓	✓	.	.
k NS5s	✓	✓	✓	✓	✓	✓	.	.	.	.
$\mathcal{C}_{\text{res}}$	.	.	.	.	✓	✓	✓	✓	✓	✓

IV. HETEROTIC DUALITY FRAME

Our next set of questions is related to each other. The first question is how to find the supergravity solution on the heterotic side with a large number k of NS5-branes and small string coupling $g_s \to 0$, with $g_s^2 k \to \infty$. The second question is the issue of supersymmetry. Since we have NS5-brane sources on the resolved conifold, we need to switch on torsion to preserve supersymmetry, which is where the torsion classes W_i from Eq. (3.26) enter the picture.

Let us start with finding the geometric part of the supergravity solution; we will turn to the gauge bundle in Sec. IV C. The setup was described in Table III and depicted in Fig. 3. We have k small instantons/NS5-branes along $x^{0,1,2,3,4,5}$—wrapping the two-cycle of the ALE given by $x^{4,5}$—and another (small) set of NS5-branes oriented along $x^{0,1,2,3,6,7}$—i.e., wrapping the $\mathbf{P}^1$ given by $x^{6,7}$—on top an orbifold five-plane. See Table III for the mapping of these coordinates to those in Eq. (3.10).

This is somewhat similar to the scenario studied in Ref. [32], where the metric ansatz for this case was called the "conformal K3 ansatz." For the present case, we want to use the resolved conifold metric (3.10), so the simplified ansatz of Ref. [32] becomes

$$ds^2 = ds_{0123}^2 + \Delta^m ds_{\text{ALE}}^2 + ds_{\text{fiber}}^2, \qquad e^\phi = \Delta, \qquad (4.1)$$

where Δ is the warp factor and m is an integer. In our case, $m = 1$, which appears from the consistency conditions in Ref. [33]. Clearly, because of the warp factor Δ, the resolved conifold must have a non-Kähler metric on it in this case.

Let us now see if we can derive the heterotic metric directly from type IIB using duality chasing. In type IIB,

X^{67}

Heterotic five–branes at a fixed orbifold–plane

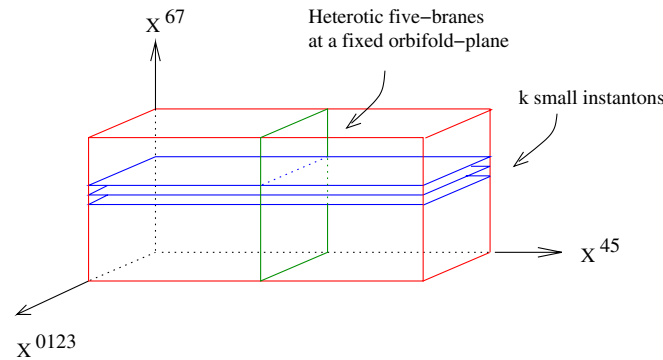k small instantons

X^{45}

X^{0123}

FIG. 3 (color online). The orientations of the NS5-branes on the heterotic side. The k D3-branes become heterotic NS5-branes which we refer them as k small instantons here. One set of type IIB D7/O7 branes/planes becomes the heterotic gauge bundle, while the other set generically becomes another set of NS5-branes on an orbifold plane. In the charge canceled scenario for the second set of D7/O7 branes/planes, which we will concentrate on in this paper, the heterotic dual is just the orbifold five-plane with no NS5-branes on top.

the moduli space is given by Eq. (3.15). Under two T dualities followed by an S duality, the orbifolded ALE space is replaced by just an ALE space. The metric ansatz of the orbifolded space $\text{ALE}/\langle(-1)^{F_L} \cdot \Omega \cdot \sigma\rangle$ can be written as

$$ds^2 = F_0 dr^2 + F_1 d\psi^2 + F_2 d\theta_1^2 + F_3 d\phi_1^2. \quad (4.2)$$

This will solve the equations of motion if we also switch on $B_{\psi\phi_1} \equiv b$, along with a dilaton. Note that this B field is not projected out by the orientifold action. The heterotic dual is then given by

$$ds^2 = dr^2 + \frac{F_1}{F_0}\left(d\psi + \frac{b}{F_1}d\phi_1\right)^2 \\ + \frac{1}{F_0}\left(\frac{1}{F_3} - \frac{b^2}{F_1}\right)d\phi_1^2 + \frac{1}{F_2 F_0}d\theta_1^2, \quad (4.3)$$

with a dilaton that will be assumed to take the form $\phi \equiv -\log F_0$. Note that the metric (4.3) has a fibration structure somewhat similar to the fibration structure in Eq. (3.10). The fibration can be made exactly as in Eq. (3.10) if we make the warp factors functions of (θ_1, r). The Busher rules do not actually allow this, but there are more refined T duality rules given in Ref. [39].

The ALE space that we have here appears as cross sections at a fixed point of the $\mathbf{P}^1$ that is parameterized by $x^{6,7} = (\theta_2, \phi_2)$, just as in the metric (3.13). The $\mathbf{P}^1$ fibration should then convert Eq. (3.13) to Eq. (3.10). In the same vein, we will then take the following as building blocks for the background metric:

$$ds_{45}^2 \equiv \frac{1}{F_0}\left(\frac{1}{F_3} - \frac{b^2}{F_1}\right)d\phi_1^2 + \frac{1}{F_2 F_0}d\theta_1^2,$$

$$ds_{67}^2 \equiv ds_{\text{fiber}}^2,$$

$$ds_{89}^2 \equiv dr^2 + \frac{F_1}{F_0}\left(d\psi + \frac{b}{F_1}d\phi_1\right)^2 \\ + F_5\left(d\psi + \frac{b}{F_1}d\phi_1\right)d\phi_2, \quad (4.4)$$

where F_i are functions of (θ_1, r). The k small instantons along $x^{0,1,2,3,4,5}$ and the other NS5-/O$_r$5-branes/planes make the background more complicated. Let us assume that the two sets of 5-branes come with warp factors $h_i = h_i(x^8, x^9)$, with $i = 1, 2$, such that h_1 denotes the warp factor for k small instantons and h_2 denotes the warp factor for the NS5/O$_r$5-branes/planes.[10] This gives the geometric background

---

[10]This can be seen from the type I picture more clearly. If we have N_2 D5-branes and N_1 O5-planes in type I, then h_2 will be proportional to the difference, i.e., $h_2 \propto 1 + \mathcal{O}(N_2 - N_1)$, since the net D5 charge $(N_2 - N_1)$ gives the number of NS5-branes in the heterotic picture.

$$ds^2 = ds^2_{0123} + h_2(ds^2_{45} + h_1 ds^2_{89}) + h_1 ds^2_{67},$$
$$\mathcal{T}^{(l)}_{abi} = c_l \epsilon_{abik} \partial_k h_l, \tag{4.5}$$

where the $\mathcal{T}^{(l)}_{abi}$ are torsion polynomials, c_l are constants, and (a, b) runs over $(4, 5)$ and $(6, 7)$ with $i = (8, 9)$. The ϵ tensor is related to the volume form on ALE. Note that if

$$
\begin{aligned}
ds^2 &= ds^2_{0123} + h_2(ds^2_{45} + h_1 ds^2_{89}) + h_1 ds^2_{67} \\
&= ds^2_{0123} + h_2[a_1 d\phi_1^2 + a_2 d\theta_1^2 + h_1 dr^2 + h_1 a_3(d\psi + a_4 d\phi_1)^2] + h_1(a_5 d\theta_2^2 + a_6 d\phi_2^2) + h_1 h_2 a_7(d\psi + a_4 d\phi_1)d\phi_2 \\
&= ds^2_{0123} + h_1 h_2 dr^2 + h_1 h_2 a_3\Big(d\psi + a_4 d\phi_1 + \frac{a_7}{2a_3}d\phi_2\Big)^2 + h_2 a_2\Big(d\theta_1^2 + \frac{a_1}{a_2}d\phi_1^2\Big) \\
&\quad + h_1 a_5\Big[d\theta_2^2 + \frac{1}{a_5}\Big(a_6 - \frac{a_7^2 h_2}{4a_3}\Big)d\phi_2^2\Big],
\end{aligned}
\tag{4.6}
$$

where a_1 through a_4 depend on (θ_1, r), a_5 through a_7 depend on (θ_i, r), and a_1 through a_4 are given by

$$a_1 = \frac{1}{F_0}\Big(\frac{1}{F_3} - \frac{b^2}{F_1}\Big), \qquad a_2 = \frac{1}{F_2 F_0}, \qquad a_3 = \frac{F_1}{F_0}, \qquad a_4 = \frac{b}{F_1}. \tag{4.7}$$

a_5 through a_7 will be related to the warp factors h_1 and h_2 via the torsional equations below, where we will use $\mathcal{N} = 1$ supersymmetry to impose additional relations between the metric factors and the torsion.

As one would expect, this metric is somewhat similar to the ones that we considered in Ref. [2] and in Ref. [3]. The difference here is that we also have a set of NS5/O$_r$5-branes/planes. The ultimate question would be what happens if one performs a geometric transition; but first we need to fully understand the solution before we can perform a geometric transition.

A. Supersymmetry and torsion classes

We will start by rewriting Eq. (4.6) as

$$
\begin{aligned}
ds^2 &= ds^2_{0123} + H_2 dr^2 + H_1(d\psi + c_4 d\phi_1 + c_7 d\phi_2)^2 \\
&\quad + H_3(d\theta_1^2 + c_1^2 d\phi_1^2) + H_4(d\theta_2^2 + c_5^2 d\phi_2^2)
\end{aligned}
\tag{4.8}
$$

but now with one crucial difference from Eq. (4.6): we will be away from the orientifold point. This aspect has already been addressed in Fig. 2, where the orientifold regimes were depicted by the boxes marked A, C, and D. The metric (4.6) is derived in this regime. However, we can go to a more generic setup, marked by boxes G and H in Fig. 2, where we are no longer restricted by the type I orientifold constraints.

In this regime, we can allow H_i to depend only r while the c_i depend on θ_1 and θ_2. Note that none of the parameters depend on (ϕ_1, ϕ_2, ψ), which is important since we will perform three T dualities along these directions to study the mirror later in this paper. Therefore, to proceed, we choose the following vielbein [40]:

we make $h_1 = 1$ (i.e., remove all the small instantons) and set $\Delta = h_2$, then we reproduce Eq. (4.1), so our starting ansatz (4.2) is consistent with this limit. For generic choices of h_1, the metric in Eq. (4.5) is non-Kähler, as we will see below.

We find it useful to parameterize the background as

$$
\begin{aligned}
e_1 &= \sqrt{H_3}(\cos\psi_1 d\theta_1 + \sin\psi_1 c_1 d\phi_1), \\
e_2 &= \sqrt{H_3}(-\sin\psi_1 d\theta_1 + \cos\psi_1 c_1 d\phi_1), \\
e_3 &= \sqrt{H_4}(\cos\psi_2 d\theta_2 + \sin\psi_2 c_5 d\phi_2), \\
e_4 &= \sqrt{H_4}(-\sin\psi_2 d\theta_2 + \cos\psi_2 c_5 d\phi_2), \\
e_5 &= \sqrt{H_1}(d\psi + c_4 d\phi_1 + c_7 d\phi_2), \quad e_6 = \sqrt{H_2} dr.
\end{aligned}
\tag{4.9}
$$

This choice suggests a fundamental two-form:

$$
\begin{aligned}
J &= e_1 \wedge e_2 + e_3 \wedge e_4 + e_5 \wedge e_6 \\
&= H_3 c_1 d\theta_1 \wedge d\phi_1 + H_4 c_5 d\theta_2 \wedge d\phi_2 \\
&\quad + \sqrt{H_1 H_2}(d\psi + c_4 d\phi_1 + c_7 d\phi_2) \wedge dr.
\end{aligned}
\tag{4.10}
$$

The torsion $\mathcal{H}$ then follows from supersymmetry:[11]

$$\mathcal{H} = -e^{2\phi} * d(e^{-2\phi} J) \equiv 2 * d\phi \wedge J - *dJ, \tag{4.11}$$

along with a dilaton ϕ that generically may be a function of the internal coordinates (r, θ_1, θ_2).

The torsion is not closed and must additionally satisfy the Bianchi identity:

$$
\begin{aligned}
d\mathcal{H} &= \text{sources} + \alpha'\Big[\text{tr}(R_+ \wedge R_+) - \frac{1}{30}\text{tr}(F \wedge F)\Big] \\
&\equiv \alpha'\Big[\text{tr}(R_+ \wedge R_+) - \frac{1}{30}\text{tr}(\tilde{F} \wedge \tilde{F})\Big],
\end{aligned}
\tag{4.12}
$$

---

[11]Note that the sign for $\mathcal{H}$ follows the convention of Ref. [41] which, in turn, differs from the sign choice of Ref. [9]. Additionally, our choice of the dilaton ϕ is -4 times the choice of the dilaton in Ref. [41].

where by *sources* we mean contributions from all the NS5-branes. The second equality in Eq. (4.12) is an alternative way to view the background where the sources are directly absorbed into the definition of the vector bundle (i.e., viewing the NS5-branes as small instantons of the heterotic gauge theory). We will use the latter interpretation for the background throughout the paper.

Interestingly, both from our choice of absorbing the sources into the definition of the vector bundle (4.12) and from the torsion (4.11), it is easy to argue that the total scalar potential of the effective four-dimensional theory, obtained by compactification, is given by the following expression [41]:

$$V = -\frac{1}{2} \int e^{-2\phi} \Big(\mathcal{H} + \frac{1}{2} e^{2\phi} * d(e^{-2\phi} J) \Big) \wedge * \Big(\mathcal{H} + \frac{1}{2} e^{2\phi} * d(e^{-2\phi} J) \Big) \tag{4.13}$$

plus a D term. The fact that extra 5-brane sources *do not* break any supersymmetry is related to the concept of *generalized calibration* from Ref. [33]: unbroken supersymmetry is restored by 5-branes wrapping the two-cycle that is calibrated by the same invariant form as the one calibrating the solution involving the full backreactions. This way, the potential V would be exactly zero for the background satisfying Eq. (4.11). Note that although the vanishing of the potential (4.13) is necessary to have supersymmetry preserved, it is not sufficient, so we will show the vanishing of the F and the D terms directly.[12]

Now, using the torsional equation (4.11), we find

$$\begin{aligned}
\mathcal{H} = {} & (2B_6 - A_1)H_4\sqrt{H_1}c_5 d\theta_2 \wedge d\phi_2 \wedge d\psi - ((A_1 - 2B_6)H_4\sqrt{H_1}c_5c_4 - 2(B_1\cos\psi_1 + B_2\sin\psi_1)H_4\sqrt{H_3}c_1c_5)d\theta_2 \\
& \wedge d\phi_2 \wedge d\phi_1 - (A_2 - 2B_6)H_3\sqrt{H_1}c_1 d\theta_1 \wedge d\phi_1 \wedge d\psi - ((A_2 - 2B_6)H_3\sqrt{H_1}c_1c_7 \\
& - (A_4 + 2B_3\cos\psi_2 - 2B_4\sin\psi_2)H_3\sqrt{H_4}c_1c_5)d\theta_1 \wedge d\phi_1 \wedge d\phi_2 - 2\sqrt{H_1H_2H_4}(B_4\cos\psi_2 + B_3\sin\psi_2)d\theta_2 \\
& \wedge (d\psi + c_4 d\phi_1 + c_7 d\phi_2) \wedge dr - 2\sqrt{H_1H_2H_4}(B_4\sin\psi_2 + B_3\cos\psi_2)c_5 d\theta_2 \wedge (d\psi + c_4 d\phi_1) \wedge dr \\
& - 2\sqrt{H_1H_2H_3}(B_1\sin\psi_1 - B_2\cos\psi_1)d\theta_1 \wedge (d\psi + c_4 d\phi_1 + c_7 d\phi_2) \wedge dr \\
& + 2\sqrt{H_1H_2H_3}(B_1\cos\psi_1 - B_2\sin\psi_1)c_1 d\phi_1 \wedge (d\psi + c_7 d\phi_2) \wedge dr - 2H_3\sqrt{H_4}(B_3\cos\psi_2 - B_3\sin\psi_2)c_1 d\theta_1 \\
& \wedge d\theta_2 \wedge d\phi_1 - 2H_4\sqrt{H_3}(B_1\sin\psi_1 - B_2\cos\psi_1)c_1 d\theta_1 \wedge d\theta_2 \wedge d\phi_2 + B_5\sqrt{H_2}dr \wedge (c_1 d\theta_1 \wedge d\phi_1 - c_5 d\theta_2 \wedge d\phi_2) \\
& - \frac{H_1 c_5 c_{4\theta_2}}{c_1} d\theta_1 \wedge d\phi_2 \wedge (d\psi + c_4 d\phi_1 + c_7 d\phi_2),
\end{aligned} \tag{4.14}$$

where to preserve the isometry along ψ direction, we will take $\psi_1 = \psi_2 = \frac{\langle\psi\rangle}{2}$ with $\langle\psi\rangle$ being a constant. Furthermore, we will demand all the coefficients above to be completely independent of (ϕ_1, ϕ_2, ψ), although they could be functions of (θ_1, θ_2) in addition to being functions of r. The coefficients A_i are defined as

$$A_1 = \frac{(H_3 c_1)_r + \sqrt{H_1 H_2}c_{4\theta_1}}{H_3\sqrt{H_2}c_1}, \qquad A_2 = \frac{(H_4 c_5)_r + \sqrt{H_1 H_2}c_{7\theta_2}}{H_4\sqrt{H_2}c_5}, \qquad A_3 = \frac{(H_3 c_1)_{\theta_2}}{\sqrt{H_4}H_3 c_1}, \qquad A_4 = \frac{(H_1 H_2)_{\theta_2}}{2H_1 H_2\sqrt{H_4}}, \tag{4.15}$$

and the B_i coefficients are defined as

$$\begin{aligned}
B_1 &= \left(\frac{\phi_{\phi_1}\sin\psi_1}{\sqrt{H_3}c_1} + \frac{\phi_{\theta_1}\cos\psi_1}{\sqrt{H_3}} - \frac{\phi_\psi\sin\psi_1 c_4}{\sqrt{H_1 H_3}c_1} \right), \qquad & B_2 &= \left(\frac{\phi_{\phi_1}\cos\psi_1}{\sqrt{H_3}c_1} - \frac{\phi_{\theta_1}\sin\psi_1}{\sqrt{H_3}} - \frac{\phi_\psi\cos\psi_1 c_4}{\sqrt{H_1 H_3}c_1} \right), \\
B_3 &= \left(\frac{\phi_{\phi_2}\sin\psi_2}{\sqrt{H_4}c_5} + \frac{\phi_{\theta_2}\cos\psi_2}{\sqrt{H_4}} - \frac{\phi_\psi\sin\psi_2 c_7}{\sqrt{H_1 H_4}c_5} \right), \qquad & B_4 &= \left(\frac{\phi_{\phi_2}\cos\psi_2}{\sqrt{H_4}c_5} - \frac{\phi_{\theta_2}\sin\psi_2}{\sqrt{H_4}} - \frac{\phi_\psi\cos\psi_2 c_7}{\sqrt{H_1 H_4}c_5} \right), \\
B_5 &= \frac{\phi_\psi}{\sqrt{H_1}}, \qquad B_6 = \frac{\phi_r}{\sqrt{H_2}}. &&
\end{aligned} \tag{4.16}$$

---

[12]Indeed, a contradiction was shown in Ref. [42], where imaginary self-dual fluxes were switched on to break supersymmetry without generating a potential.

The subscript on f_α, where α is a coordinate, means derivative $\partial_\alpha f$. The torsion contains all the information of the heterotic 5-branes as well as information about the vector bundle via the relation (4.12). From above, we also see that the torsion has the following nonzero components[13]:

$$\mathcal{H}_{\theta_1\theta_2\phi_1}, \quad \mathcal{H}_{\theta_1\theta_2\phi_2}, \quad \mathcal{H}_{\theta_1\phi_1\phi_2}, \quad \mathcal{H}_{\theta_1\phi_1 r}, \quad \mathcal{H}_{\theta_1\phi_2 r}, \quad \mathcal{H}_{\theta_1\phi_2\psi}, \quad \mathcal{H}_{\theta_1\psi r}, \quad \mathcal{H}_{\theta_2\phi_1\phi_2},$$

$$\mathcal{H}_{\theta_2\phi_1 r}, \quad \mathcal{H}_{\theta_2\phi_2\psi}, \quad \mathcal{H}_{\theta_2\phi_2 r}, \quad \mathcal{H}_{\theta_2\psi r}, \quad \mathcal{H}_{\phi_1\phi_2 r}, \quad \mathcal{H}_{\phi_1\psi r}, \quad \mathcal{H}_{\phi_2\psi r}. \tag{4.17}$$

All the 5-brane components of the torsion will receive $\mathcal{O}(\alpha'^0)$ contributions from the sources in addition to the $\mathcal{O}(\alpha')$ anomaly term. All other components will only have the $\mathcal{O}(\alpha')$ anomaly term. These can be worked out for the generic case. In Sec. IV C we will study a scenario—with the torsional components as functions of the radial coordinate r only—where the Bianchi identity to $\mathcal{O}(\alpha')$ will be satisfied by switching on vector bundles on the internal manifold.

Next, we note the relation to the torsion classes in Eq. (3.26). They are given by

$$W_1 = \frac{c_{4\phi_2} - c_{7\phi_1} - (c_{4\psi}c_7 - c_{7\psi}c_4) + ic_{4\theta_2}}{\sqrt{H_2 H_3 H_4}}$$

$$W_4 = \frac{1}{4}\Bigg[\frac{1}{c_1 H_3}((c_1 H_3)_r dr + (c_1 H_3)_\psi d\psi + (c_1 H_3)_{\phi_2} d\phi_2 + (c_1 H_3)_{\theta_2} d\theta_2) + \frac{1}{c_5 H_4}((c_5 H_4)_r dr + (c_5 H_4)_\psi d\psi)$$

$$+ \frac{1}{\sqrt{H_1 H_2}}\Bigg((\sqrt{H_1 H_2})_\psi(c_4 d\phi_1 + c_7 d\phi_2 + 2d\psi) + (\sqrt{H_1 H_2})_{\phi_2}\Big(\frac{d\psi + c_4 d\phi_1}{c_7} + 2d\phi_2\Big) + (\sqrt{H_1 H_2})_{\theta_2} d\theta_2\Bigg)$$

$$+ \sqrt{H_1 H_2}\, dr\Big(\frac{c_{7\theta_2}}{H_4 c_5} + \frac{c_{4\theta_1}}{H_3 c_1}\Big)\Bigg]$$

$$\text{Re } W_5 = \frac{1}{12}\Bigg\{\frac{4}{\sqrt{H_3 H_4}}\Big[\frac{1}{2}(\sqrt{H_3 H_4})_r dr + (\sqrt{H_3 H_4})_\psi d\psi + (\sqrt{H_3 H_4})_{\phi_2} d\phi_2 + (\sqrt{H_3 H_4})_{\theta_2} d\theta_2\Big]$$

$$+ \frac{2(\sqrt{H_{1r}} dr + \sqrt{H_{1\phi_2}} d\phi_2 + \sqrt{H_{1\theta_2}} d\theta_2 + \sqrt{H_{1\psi}} d\psi)}{\sqrt{H_1}} + \frac{2(\sqrt{H_{2\phi_2}} d\phi_2 + \sqrt{H_{2\theta_2}} d\theta_2 + \sqrt{H_{2\psi}} d\psi)}{\sqrt{H_2}} - 2\sqrt{\frac{H_2}{H_1}} dr\Bigg\}, \tag{4.18}$$

and we can similarly determine the W_2 torsion class from above.[14] In the language of torsion classes, the supersymmetry conditions can be written as[15]

$$W_1 = W_2 = 0, \qquad 2W_4 = \text{Re } W_5. \tag{4.19}$$

Next, we will find solutions to these conditions.

B. Infinite class of solutions

Some recent studies (for example, Ref. [4]) have found similar types of backgrounds of heterotic 5-branes wrapped on a resolved conifold. What we will find here is that there is a huge class of solutions related to various possible LSTs [14] on the heterotic 5-branes. The story then is similar to what we encountered in the $SO(32)$ case [3]: there is an infinite class of LSTs. A small subset of these theories are dual to geometric backgrounds of the type studied in Refs. [2,3,6]. Most of these theories will be dual to nongeometric backgrounds [2,3].

---

[13]Note that the heterotic torsion comes from two different sources in type IIB. The first one is from the B_{RR} fields that have one leg along the orientifolding direction (so that they are not projected out by the orientifold operation). The second one is from the axion and four-form fields that survive the orientifold operation. Some part of the torsion, along with the size of the two-sphere on which we have the wrapped heterotic 5-branes, will eventually be responsible for generating the renormalization group flow in the theory. This will be clearer from the gravity dual. Furthermore, the powerful machinery of the torsion classes that we are going to use to justify many of the subsequent results can be compared with the interesting work of Ref. [43]. It would certainly be interesting to make a precise comparison with the results of Ref. [43], but we think that this comparison would be more appropriately addressed by a separate work.

[14]The expression for W_2 is very long, so we will not write it explicitly.

[15]In our conventions, the torsion classes $\mathcal{W}_4$ and $\mathcal{W}_5$ of Ref. [41] are related to W_4 and W_5 of Eq. (4.18) as $\mathcal{W}_4 = -2W_4$ and $\mathcal{W}_5 = 2W_5$. This means that the supersymmetry condition $2\mathcal{W}_4 + \mathcal{W}_5 = 0$ will become $2W_4 - W_5 = 0$, which is Eq. (4.19) above. Furthermore, $\mathcal{W}_4 \equiv 4d\Phi$, where Φ is the dilaton. In our conventions, the dilaton ϕ is $\phi = -4\Phi$ (see Footnote 11); this implies that $d\phi = 2W_4$. For more details, see Appendix E.

Keeping this in mind, let us now fix the starting coefficients $F_0, \ldots, F_3$ in Eq. (4.2) assuming we are away from the orientifold point as discussed earlier and independent of (ϕ_1, ϕ_2, ψ) coordinates. A hint may come from the heterotic metric (4.3) because we expect this to be a warped ALE space of the form

$$ds^2 = dr^2 + a_3(d\psi + \cos\theta_1 d\phi_1)^2$$
$$+ \frac{r^2 f^2}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2), \qquad (4.20)$$

where a_3 and f are generic functions for which the values will be determined later.[16] This suggests that we choose

$$F_0 = \frac{b}{a_3 \cos\theta_1}, \qquad F_1 = \frac{b}{\cos\theta_1},$$
$$F_2 = \frac{6a_3 \cos\theta_1}{br^2 f^2}, \qquad F_3 = \frac{6a_3 \cos\theta_1}{6ba_3\cos^2\theta_1 + br^2 f^2\sin^2\theta_1}, \qquad (4.21)$$

which implies

$$a_1 = \frac{r^2 f^2\sin^2\theta_1}{6}, \quad a_2 = \frac{r^2 f^2}{6}, \quad a_4 = \cos\theta_1, \qquad (4.22)$$

with a_3 thus far unfixed. Note that the values for F_i (or, equivalently, for a_i) cannot be determined from T duality since that would require they be independent of (θ_1, ϕ_1).

Now looking at the resolved conifold metric of Eq. (4.6), we can argue from the fibrational structure for $d\psi$ that

$$a_4 = \cos\theta_1, \qquad a_7 = 2a_3 \cos\theta_2, \qquad (4.23)$$

where the a_4 is consistent with the value quoted in Eq. (4.22). Similarly, the dependence of the background (4.6) on the resolution parameter a^2 implies we should set

$$a_2 h_2 - a_5 h_1 = a^2. \qquad (4.24)$$

Since a is a constant, it is easy to see that

$$a_5 = \frac{h_2 a_2 - a^2}{h_1} \equiv \frac{\Delta}{h_1}, \qquad a_6 - \frac{a_7^2 h_2}{4a_3} = \frac{\Delta\sin^2\theta_2}{h_1},$$
$$a_1 = \frac{(\Delta + a^2)\sin^2\theta_1}{h_2}. \qquad (4.25)$$

Comparing a_1 from above with a_1 from Eq. (4.22) and using $h_2 = 1$ when there are no NS5-branes on top of the orbifold five plane (see Footnote 10), we see that

$$h_2 = \frac{6(\Delta + a^2)}{r^2 f^2} = 1, \qquad (4.26)$$

which implies that

$$f = \frac{\sqrt{6(\Delta + a^2)}}{r}, \qquad (4.27)$$

where Δ and a_3 are still undetermined functions of r. Defining

$$\mathcal{G}_1 \equiv e^{-\phi}H_2, \qquad \mathcal{G}_2 \equiv e^{-\phi}H_1,$$
$$\mathcal{G}_3 \equiv e^{-\phi}H_3, \qquad \mathcal{G}_4 \equiv e^{-\phi}H_4, \qquad (4.28)$$

the supersymmetry condition (4.19) becomes

$$\frac{5}{6}\left(\frac{\mathcal{G}_{3r}}{\mathcal{G}_3} + \frac{\mathcal{G}_{4r}}{\mathcal{G}_4}\right) - \frac{1}{6}\frac{\mathcal{G}_{2r}}{\mathcal{G}_2} + \frac{3}{2}\phi_r$$
$$= \left(\frac{1}{\mathcal{G}_3} + \frac{1}{\mathcal{G}_4}\right)\sqrt{\mathcal{G}_1\mathcal{G}_2} - \frac{1}{3}\sqrt{\frac{\mathcal{G}_1}{\mathcal{G}_2}}. \qquad (4.29)$$

We already know that $H_3 = \Delta + a^2$ and $H_4 = \Delta$, and now we find that

$$e^\phi = \frac{4\Delta}{r^2}, \qquad h_1 = \frac{2\Delta}{r\sqrt{a_3}}. \qquad (4.30)$$

We will also find it useful to define

$$f_1(r) \equiv e^{-\phi}\frac{h_1 a_3}{r^2}, \qquad f_2(r) \equiv e^{-\phi}h_1, \qquad (4.31)$$

which, using Eq. (4.30), we see satisfy[17]

$$a_3 = 4r^2 f_1^2, \qquad f_1 = \frac{1}{4f_2}. \qquad (4.32)$$

Then the supersymmetry equation (4.29) becomes

$$\frac{\partial\Delta}{\partial r} - \left[\frac{\frac{1}{2}\frac{\partial}{\partial r}(\log a_3) - \frac{2}{\sqrt{a_3}} + \frac{1}{r}(11 + \frac{12\Delta}{\Delta + a^2})}{4 + \frac{5\Delta}{\Delta + a^2}}\right]\Delta = 0. \qquad (4.33)$$

Note that in the limit that a is much smaller than any other scale in the theory, the differential equation (4.33) simplifies to

$$r\frac{\partial\Delta}{\partial r} - \left[\frac{r}{18}\frac{\partial(\log a_3)}{\partial r} - \frac{2r}{9\sqrt{a_3}} + \frac{23}{9}\right]\Delta + \mathcal{O}(a^2) = 0, \qquad (4.34)$$

where the $\mathcal{O}(a^2)$ terms involve powers of Δ and its first derivative.[18] The solution for Δ from Eq. (4.34) then is

$$\Delta(r) = \Delta_0 \exp\int dr\left[\frac{1}{18}\frac{\partial(\log a_3)}{\partial r} - \frac{2}{9\sqrt{a_3}} + \frac{23}{9r}\right],$$
$$r > 0, \qquad (4.35)$$

where Δ_0 is a constant. Then the background metric is

---

[16]The radial coordinate chosen here is not quite the same as in Eq. (3.13). Abusing notation, if we call the radial coordinate in Eq. (3.13) $\tilde{r}$, then $r = \int\sqrt{\gamma'}d\tilde{r}$. Similarly, f^2 in Eq. (4.20) can be related to γ in Eq. (3.13).

[17]See Appendix D for a proof for Eqs. (4.30) and (4.32).
[18]In fact, the $\mathcal{O}(a^2)$ term is given by $-\frac{a^2}{9}[\frac{a_{3r}}{2a_3} - \frac{2}{\sqrt{a_3}} + \frac{11}{r} - 4\frac{\partial}{\partial r}(\log\Delta)]$. This is followed by $\mathcal{O}(a^4)$ terms as can be derived from Eq. (4.33). It is now easy to see that the nth term can be derived from $(1 + \frac{a^2}{\Delta})^{-1}[(1 + \frac{4a^2}{9\Delta})\frac{\partial}{\partial r}(\log\Delta) - \frac{12\Delta}{r}]$; therefore, no higher powers of Δ_r appear in the series.

$$ds^2 = ds_{0123}^2 + \frac{2\Delta}{r\sqrt{a_3}}dr^2 + \frac{2\Delta\sqrt{a_3}}{r}(d\psi + \cos\theta_1 d\phi_1$$
$$+ \cos\theta_2 d\phi_2)^2 + (\Delta + a^2)(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2)$$
$$+ \Delta(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2). \tag{4.36}$$

Calculating the torsion classes W_1 and W_2 from Eq. (4.18), we see that they vanish and, therefore, that the manifold admits a complex structure.

The heterotic torsion can now be read off from Eq. (4.14). This simplifies quite a bit in the limit where all the H_i in Eq. (4.8) are just functions of the radial coordinate r because all $B_i = 0$ except B_6 in Eq. (4.15). The result is

$$\mathcal{H} = -\sqrt{\frac{H_1}{H_2}}(G_1 \sin\theta_2 d\theta_2 \wedge d\phi_2 + G_2 \sin\theta_1 d\theta_1 \wedge d\phi_1) \wedge e_\psi$$
$$= -\sqrt{a_3}\left(\frac{\partial\Delta}{\partial r} - \frac{2\Delta}{r}\right)(\sin\theta_2 d\theta_2 \wedge d\phi_2$$
$$+ \sin\theta_1 d\theta_1 \wedge d\phi_1) \wedge e_\psi, \tag{4.37}$$

where $e_\psi \equiv (d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)$ and we see that the torsion is asymmetric over the two-spheres because of the G_1 and G_2 factors. The precise form shows an amazing simplification,[19] and to $\mathcal{O}(a^2)$ $G_1 = G_2$ with the simplified form being

$$G_1 = \frac{H_4}{H_3}(\sqrt{H_1 H_2} + 2\phi_r H_3 - H_{3r}) = \frac{\partial\Delta}{\partial r} - \frac{2\Delta}{r}$$
$$G_2 = \frac{H_3}{H_4}(\sqrt{H_1 H_2} + 2\phi_r H_4 - H_{4r}) = \frac{\partial\Delta}{\partial r} - \frac{2\Delta}{r}. \tag{4.38}$$

A simple way to relate Δ and a_3 is to relate h_1 to the 5-brane harmonic function—though when r and a are small, we will instead use the 3-brane harmonic function since the k NS5-branes will wrap a collapsed cycle, appearing as $(3+1)$-dimensional sources—i.e.

$$h_1 = \frac{2\Delta}{r\sqrt{a_3}} \equiv f_3(r)\left[1 + \frac{\alpha'k}{r^2}\right], \tag{4.39}$$

with f_3 being a dimensionless function. Plugging this into the supersymmetry condition (4.29) leads to the following differential equation for a_3:

$$\frac{da_3}{dr}\left[1 + \frac{3a^2 r}{4f_3\sqrt{a_3}(e_0 + r^2)}\right] + \frac{\sqrt{a_3}}{2} - \frac{a_3}{2r}\left(\frac{7r^2 + 16e_0}{r^2 + e_0}\right)$$
$$+ \frac{9}{4}\frac{f_{3r}a_3}{f_3} + a^2\left[1 + \frac{2f_{3r}\sqrt{a_3}}{f_3} - \frac{\sqrt{a_3}}{2r}\left(\frac{7r^2 + 15e_0}{r^2 + e_0}\right)\right]$$
$$\times \frac{r}{f_3(r^2 + e_0)} = 0, \tag{4.40}$$

---

[19]Consistent with the fact that for both the conifold as well as the resolved conifold, where $\Delta = \frac{r^2}{6}$ and $a_3 = \frac{r^2}{9}$, the torsion (4.37) vanishes, as expected for a Calabi–Yau geometry.

where we have defined

$$e_0 \equiv \alpha'k. \tag{4.41}$$

In the limit that $a^2 \ll \alpha'$, then Eq. (4.40) simplifies to

$$\frac{da_3}{dr} + \frac{\sqrt{a_3}}{2} - \frac{a_3}{2r}\left(\frac{7r^2 + 16e_0}{r^2 + e_0}\right) + \frac{9}{4}\frac{f_{3r}a_3}{f_3} + \mathcal{O}(a^2) = 0, \tag{4.42}$$

for which the solution can be easily determined if the functional form for f_3 is known. In general, however, to solve Eq. (4.40), we will analyze different choices for f_3.

Case I: $f_3 = 1$

This is the simplest case where h_1 in Eq. (4.39) is exactly the 5-brane harmonic function for k coincident 5-branes. In this case, a_3 will satisfy

$$\frac{da_3}{dr}\left[1 + \frac{3a^2 r}{4\sqrt{a_3}(r^2 + e_0)}\right] + \frac{\sqrt{a_3}}{2} - \frac{a_3}{2r}\left(\frac{7 + \frac{16e_0}{r^2}}{1 + \frac{e_0}{r^2}}\right)$$
$$+ \frac{a^2 r}{r^2 + e_0}\left[1 - \frac{\sqrt{a_3}}{2r}\left(\frac{7r^2 + 15e_0}{r^2 + e_0}\right)\right] = 0, \tag{4.43}$$

for which the solution will determine the full metric of the system. In the limit $a^2 \ll \alpha'$, this reduces to Eq. (4.42), with $f_{3r} \equiv \partial_r f_3 = 0$, of course. In fact, $a_3(r)$ can be solved for exactly, and, in the limit of small a^2, the result is

$$a_3(r) = \frac{r^2}{(192)^2}\left[\frac{9r^4 \mathcal{F}_{1,2}}{e_0^{7/8}(r^2 + e_0)^{9/8}} - \left(\frac{9\sqrt{\pi}\Gamma(3/8)r^3}{e_0^{3/8}(r^2 + e_0)^{9/8}}\right.\right.$$
$$\left.\left. + \frac{8\Gamma(7/8)(4e_0 + 13r^2)}{e_0 + r^2}\right)\right]^2 + \mathcal{O}(a^2), \tag{4.44}$$

where

$$\mathcal{F}_{1,2} \equiv {}_2F_1\left(\frac{1}{2}, \frac{7}{8}, \frac{3}{2}, -\frac{r^2}{e_0}\right). \tag{4.45}$$

We can also now solve for Δ, which gives

$$\Delta(r) = \frac{r^2}{384}\left(1 + \frac{e_0}{r^2}\right)\left[\frac{9r^4 \mathcal{F}_{1,2}}{e_0^{7/8}(r^2 + e_0)^{9/8}}\right.$$
$$\left. - \left(\frac{9\sqrt{\pi}\Gamma(3/8)r^3}{e_0^{3/8}(r^2 + e_0)^{9/8}} + \frac{8\Gamma(7/8)(4e_0 + 13r^2)}{e_0 + r^2}\right)\right]. \tag{4.46}$$

The small r and large r behaviors of Δ and a_3 are given by

$$\frac{e_0}{24} \overset{0 \leftarrow r}{\longleftarrow} \Delta \overset{r \rightarrow \infty}{\longrightarrow} \frac{r^2}{6} \tag{4.47}$$

$$\frac{r^2}{144} \overset{0 \leftarrow r}{\longleftarrow} a_3 \overset{r \rightarrow \infty}{\longrightarrow} \frac{r^2}{9}, \tag{4.48}$$

which would imply that at $r = 0$, the two two-spheres have radii $\frac{e_0}{24}$ and $\frac{e_0}{24} + a^2$.[20] So, the gravity dual should be given by a warped resolved deformed conifold with torsion.[21]

However, as mentioned earlier, very close to the origin $r \to 0$ and in the limit that a^2 is small, the k 5-branes are wrapped on an almost vanishing cycle and therefore appear as 3-brane sources. While the ansatz (4.39) suffices in a delocalized limit, in the localized limit, a better ansatz would be to use a 3-brane harmonic function rather than 5-brane harmonic function—i.e., to replace the $\frac{e_0}{r^2}$ term in Eq. (4.39) with $\alpha' \frac{e_0}{r^4}$. Indeed, we can also replace our above ansatz (4.39) with a more generic one of the form

$$h_1 \equiv 1 + \frac{e_0}{r^2} + \frac{\alpha' e_0}{r^4}, \qquad (4.49)$$

where we see that for $r \gg \sqrt{\alpha'}$, we recover the warp factor (4.39), while for small r, $r \ll \sqrt{\alpha'}$, this will convert to the localized 3-brane ansatz. Taking this into account converts Eq. (4.43) to the following differential equation for a_3 (with $\alpha' = 1$ for convenience) near $r \to 0$:

$$\frac{da_3}{dr} + \frac{\sqrt{a_3}}{2} - \frac{a_3}{2r}\left(\frac{7r^4 + 25e_0}{r^4 + e_0}\right) + \mathcal{O}(a^2) = 0. \quad (4.50)$$

As expected, the large r behavior is unaffected, but the small r behavior does change. Now,

$$a_3 \xrightarrow{r \to 0} \frac{r^2}{441}, \qquad \Delta \xrightarrow{r \to 0} \frac{e_0}{42r^2}, \qquad (4.51)$$

which means that the dilaton diverges at the origin and that the metric is affected by the branes near the origin. Naturally, this change also affects the torsion and the vector bundle, as we will see later.

To complete the story, of course, we still must find a vector bundle that satisfies the Donaldson–Uhlenbeck–Yau equations and the Bianchi identity, which we postpone until Sec. IV C.

Case II: $f_3 = f_2 \equiv \frac{r}{2\sqrt{a_3}}$

In this case, the dilaton and Δ are simply

$$e^\phi = 1 + \frac{\alpha' k}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right), \qquad \Delta(r) = \frac{r^2 + e_0}{4}. \quad (4.52)$$

As in case I, the two two-cycles at $r = 0$ will have sizes $\frac{e_0}{4}$ and $\frac{e_0}{4} + a^2$, until we replace the 5-brane harmonic function with the 3-brane harmonic function. Thus, the gravity dual should also be a resolved warped deformed conifold with torsion.

[20]In fact, to state the latter radius, we really need to know Δ at order a^2, not just a^0. This can easily be done by solving (4.43) to the next order.

[21]Interestingly, this case somewhat resembles the type IIB case studied in [44], where the authors studied the wrapped five-brane scenario with closed three-form fluxes. Additionally one may refer to [45,46] where the $d = 4$ example studied therein fits into one of our large class of models.

The differential equation for a_3, Eq. (4.40), becomes

$$\frac{da_3}{dr}\left(1 + \frac{a^2}{\Delta}\right) - 4\sqrt{a_3} + \frac{2a_3}{r}\left(\frac{5r^2 + 23e_0}{r^2 + e_0}\right) - \frac{16a^2}{r^2 + e_0}$$

$$\times \left[\sqrt{a_3} - \frac{a_3}{2r}\left(\frac{3r^2 + 11e_0}{r^2 + e_0}\right)\right] = 0. \qquad (4.53)$$

Again, this equation can be solved exactly in terms of Appell hypergeometric functions, but for simplicity we will focus on the small a^2 limit. In this case, the a_3 equation (4.53) is

$$\frac{da_3}{dr} - 4\sqrt{a_3} + \frac{2a_3}{r}\left(\frac{5r^2 + 23e_0}{r^2 + e_0}\right) + \mathcal{O}(a^2) = 0, \quad (4.54)$$

which appears quite different from the $a^2 \ll \alpha'$ case for the first scenario. The value for $a_3(r)$ at zeroth order in a^2 can now be written as

$$a_3(r) = \frac{r^2}{(168)^2}\left(1 + \frac{e_0}{r^2}\right)^2\left[1 + 495\left(\frac{e_0}{r^2}\right)^{10} + \frac{7425}{2}\left(\frac{e_0}{r^2}\right)^9 \right.$$

$$+ 12045\left(\frac{e_0}{r^2}\right)^8 + \frac{87945}{4}\left(\frac{e_0}{r^2}\right)^7 + 24519\left(\frac{e_0}{r^2}\right)^6$$

$$+ \frac{33789}{2}\left(\frac{e_0}{r^2}\right)^5 + \frac{47619}{7}\left(\frac{e_0}{r^2}\right)^4 + \frac{75339}{56}\left(\frac{e_0}{r^2}\right)^3$$

$$+ 55\left(\frac{e_0}{r^2}\right)^2 - \frac{11}{2}\left(\frac{e_0}{r^2}\right) - 495\left(\frac{e_0}{r^2}\right)^3\left(1 + \frac{e_0}{r^2}\right)^8$$

$$\left. \times \log\left(1 + \frac{r^2}{e_0}\right)\right]^2 + \mathcal{O}(a^2).$$

Surprisingly, the leading behavior at both large and small r is precisely the same as in case I!

To get a better feel for how the metric behaves, let us define a function $X(r)$ in the following way:

$$X(r) = \frac{11}{2}\left(\frac{e_0}{r^2}\right) - 495\left(\frac{e_0}{r^2}\right)^{10} - \frac{7425}{2}\left(\frac{e_0}{r^2}\right)^9$$

$$- 12045\left(\frac{e_0}{r^2}\right)^8 - \frac{87945}{4}\left(\frac{e_0}{r^2}\right)^7 - 24519\left(\frac{e_0}{r^2}\right)^6$$

$$- \frac{33789}{2}\left(\frac{e_0}{r^2}\right)^5 - \frac{47619}{7}\left(\frac{e_0}{r^2}\right)^4 - \frac{75339}{56}\left(\frac{e_0}{r^2}\right)^3$$

$$- 55\left(\frac{e_0}{r^2}\right)^2 + 495\left(\frac{e_0}{r^2}\right)^3\left(1 + \frac{e_0}{r^2}\right)^8 \log\left(1 + \frac{r^2}{e_0}\right),$$

$$(4.55)$$

using which we can define another function $F_5(r)$ as

$$F_5(r) \equiv \frac{1}{168} - \frac{1}{168}\left[X(r) + \frac{e_0 X(r)}{r^2} - \frac{e_0}{r^2}\right]. \quad (4.56)$$

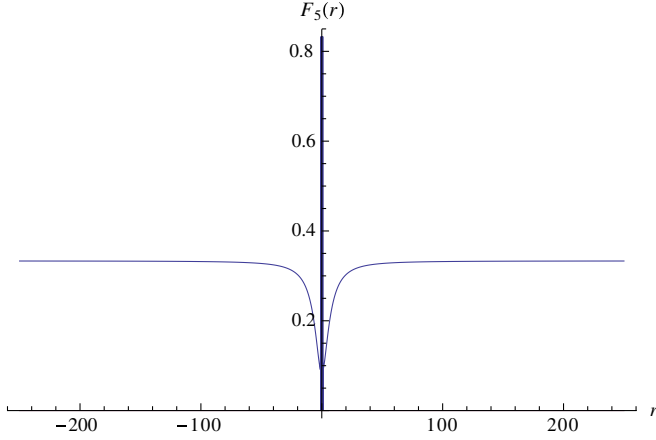With these definitions, we can write the metric as

FIG. 4 (color online).   The behavior of the function $F_5(r)$ plotted with respect to the radial coordinate. Note that the F_5 function asymptotes to $\frac{1}{3}$ at large r, while there is an expected singularity at the origin $r = 0$. For the plot, we have chosen $e_0 = 10$ in units of α'.

$$ds^2 = ds^2_{0123} + \left(1 + \frac{e_0}{r^2}\right)\left[\frac{dr^2}{2F_5} + \frac{r^2}{2}F_5(d\psi + \cos\theta_1 d\phi_1\right.$$
$$\left. + \cos\theta_2 d\phi_2)^2 + \left(\frac{r^2}{4} + \tilde{a}^2\right)(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2)$$
$$\left. + \frac{r^2}{4}(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2)\right], \tag{4.57}$$

where $\tilde{a} \equiv \frac{2a}{\sqrt{1+\frac{e_0}{r^2}}}$ and the behavior of $F_5(r)$ is plotted in Fig. 4. It is interesting that even for very large r, the manifold does not quite become a Calabi–Yau resolved conifold as the coefficients differ (see Ref. [47] for details on the Calabi–Yau resolved conifold).

As in case I, to be careful about the limit that $r \to 0$, we should employ the 3-brane harmonic function instead of the 5-brane harmonic function behavior using similar point of view as employed for case I. This also applies to the string coupling and to Δ, which will now be (with $\alpha' = 1$)

$$e^\phi = 1 + \frac{e_0}{r^4}, \qquad \Delta = \frac{r^2}{4} + \frac{e_0}{4r^2}. \tag{4.58}$$

Then the equation for a_3 (4.54) becomes

$$\frac{da_3}{dr} - 4\sqrt{a_3} + \frac{2a_3}{r}\left(\frac{5r^4 + 41e_0}{r^4 + e_0}\right) + \mathcal{O}(a^2) = 0. \tag{4.59}$$

The components of the torsion will blow up near the origin because of the sources but will asymptote to a constant value for large r.

We will solve for the vector bundle in Sec. IV C.

Case III: $f_3 \neq f_2$

Now the equation for a_3 is simply Eq. (4.40). In the limit with small resolution parameter, Eq. (4.42) tells us that

$$|f_{3r}| < |f_3|, \tag{4.60}$$

where the inequality should be viewed order by order in a $\frac{1}{r}$ expansion; the result will be similar to case I studied above.

For a class of examples in this case, we consider a form for h_1 that has the following piecewise behavior:

$$h_1 = \begin{cases} 1 + \frac{e_0}{r^2}\left(1 + \frac{b}{r^8}\right)^{-1}, & r^8 \gg \tilde{b} \\ \frac{e_0}{r^2}\left(1 + \frac{\tilde{b}}{r^8}\right), & r^8 \ll \tilde{b}, \end{cases} \tag{4.61}$$

where $\tilde{b} \ll b$ are two parameters defining the class.[22]

The ansatz (4.61) is similar to the one considered in Ref. [4], where the $r \to \infty$ behavior was like case I studied above. The difference is in the intermediate r behavior. The equation for a_3 in the intermediate region $\tilde{b}^{\frac{1}{8}} \ll r \ll b^{\frac{1}{8}}$ becomes

$$\frac{da_3}{dr} + \frac{\sqrt{a_3}}{2} - \frac{2a_3}{r}\left(\frac{4r^8 - 5b}{r^8 + b}\right) + \mathcal{O}(a^2) = 0. \tag{4.62}$$

Note that e_0 does not appear in the a_3 equation but does appear in the definition of Δ because of the relation (4.61). In the limit that a^2 much smaller than any other scale in the theory, the solution for a_3 becomes

$$a_3(r) = \frac{(b + r^8)^2\left[24c\sqrt[8]{b + r^8} - \frac{6r^6}{b}\left(1 - \frac{5}{6}\sqrt[8]{1 + \frac{r^8}{b}}\mathcal{D}_{1,2}\right)\right]^2}{576r^{10}}, \tag{4.63}$$

where we have defined another hypergeometric function $\mathcal{D}_{1,2}$ and a constant c in the following way:

$$\mathcal{D}_{1,2} \equiv {}_2F_1\left(\frac{1}{8}, \frac{3}{4}, \frac{7}{4}, -\frac{r^8}{b}\right), \quad c = \frac{5\Gamma(5/8)\cdot\Gamma(7/4)}{24b^{3/8}\Gamma(5/8)}. \tag{4.64}$$

The large r behavior of a_3 is consistent with our earlier ansatz in Eq. (4.32), with leading behavior proportional to r^2. On the other hand, for $\tilde{b}^{\frac{1}{8}} \ll r \ll b^{\frac{1}{8}}$, the leading behavior of $a_3(r)$ is proportional to $\frac{b^{3/2}}{r^{10}}$, as one might have expected from our ansatz (4.61).

Once we know a_3, we can readily get Δ from the ansatz (4.61) and find

$$\Delta(r) = \frac{e_0 r^2}{24}\left[24c\sqrt[8]{b + r^8} - \frac{6r^6}{b}\left(1 - \frac{5}{6}\sqrt[8]{1 + \frac{r^8}{b}}\mathcal{D}_{1,2}\right)\right]. \tag{4.65}$$

Δ has the following asymptotics:

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[22]Alternatively, we could have followed a consistent set of conventions for h_1 across the cases and instead defined f_3 in this of piecewise fashion. See Fig. 6 for details.

$$\Delta(r)|_{\tilde{b}^{1/8} \ll r \ll b^{1/8}} = \frac{5e_0}{48}\left\{\frac{\Gamma(5/8)\Gamma(7/4)}{\Gamma(1/4)}\left(\frac{r^8}{b}\right)^{1/4}\right.$$

$$\times\left[1 + \frac{1}{8}\left(\frac{r^8}{b}\right)\right] - \frac{1}{5}\left(\frac{r^8}{b}\right)\right\} + \mathcal{O}(r^{11})$$

$$\Delta(r)|_{r \gg b^{1/8}} = \frac{\Gamma(7/4)}{\Gamma(3/4)}\left[\frac{e_0}{18} - \frac{e_0}{99}\left(\frac{b}{r^8}\right) + \frac{10e_0}{1881}\left(\frac{b}{r^8}\right)^2\right.$$

$$\left. - \frac{20e_0}{5643}\left(\frac{b}{r^8}\right)^3\right] + \mathcal{O}\left(\frac{1}{r^{32}}\right). \quad (4.66)$$

Since Δ is vanishing at small r, the background behaves like a resolved conifold in this regime, but at large r, $r \gg b^{\frac{1}{8}}$, both cycles attain finite sizes.

To simply encapsulate the metric behavior, we define another dimensionless function $f_6(r)$ as (see Fig. 5):

$$f_6(r) \equiv r^2\left[24c\sqrt[8]{b + r^8} - \frac{6r^6}{b}\left(1 - \frac{5}{6}\sqrt[8]{1 + \frac{r^8}{b}}\mathcal{D}_{1,2}\right)\right],$$

$$(4.67)$$

which asymptotes to $\frac{8\Gamma(7/4)}{3\Gamma(3/4)} + \mathcal{O}(\frac{1}{r^8})$ at large r. Then the background metric for this case can be expressed as

$$ds^2 = ds_{0123}^2 + \frac{e_0}{r^2}\left[\frac{dr^2}{1 + \frac{b}{r^8}} + \frac{r^2}{576}\left(1 + \frac{b}{r^8}\right)f_6^2(d\psi + \cos\theta_1 d\phi_1\right.$$

$$+ \cos\theta_2 d\phi_2)^2 + r^2\left(\frac{f_6}{48} + a^2\right)(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2)$$

$$\left. + r^2\left(\frac{f_6}{48}\right)(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2)\right], \quad (4.68)$$

where for $r \gg b^{\frac{1}{8}}$ the metric resembles the one of Ref. [4] at the leading order, albeit with a positive value for b. The metric (4.68) does not cover the patches $r \to 0$ and $r \to \infty$,
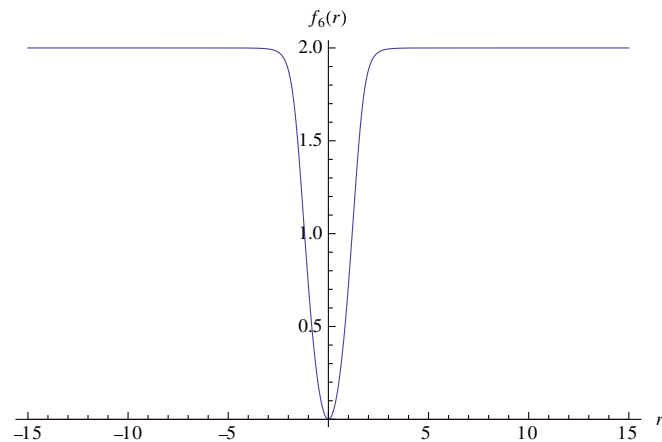

FIG. 5 (color online). The behavior of the function $f_6(r)$ for case III that determines the metric (4.68). Note that the f_6 function asymptotes to $\frac{8\Gamma(7/4)}{3\Gamma(3/4)}$, while at the origin $r = 0$, it vanishes. Of course the behavior at the origin, as well as at large r, will be different if we include Eq. (4.61). For simplicity we have chosen $b = 100$ in units of α'.

as the two asymptotes are given by the warp factor (4.61) and the one studied for case I, respectively.

If we now consider the small r behavior, $r \ll \tilde{b}^{\frac{1}{8}}$, the equation for a_3 becomes

$$\frac{da_3}{dr} + \frac{\sqrt{a_3}}{2} - \frac{2a_3}{r}\left(\frac{4r^8 + 13\tilde{b}}{r^8 + \tilde{b}}\right) + \mathcal{O}(a^2) = 0. \quad (4.69)$$

The behavior of (a_3, Δ, e^ϕ) near the origin $r \to 0$ is then given by

$$a_3 \to \frac{r^2}{2304} + \frac{r^2}{512}\left(\frac{r^8}{\tilde{b}}\right) + \frac{c_1 r^2}{4096}\left(\frac{r^8}{\tilde{b}}\right)^{3/2} + \mathcal{O}(r^{16})$$

$$e^\phi \to \frac{13e_0}{96r^2} + \frac{e_0}{24r^2}\left(\frac{\tilde{b}}{r^8}\right) + \frac{3c_1 e_0}{256r^2}\left(\frac{r^8}{\tilde{b}}\right)^{1/2} + \mathcal{O}(r^6) \quad (4.70)$$

$$\Delta \to \frac{13e_0}{384} + \frac{e_0}{96}\left(\frac{\tilde{b}}{r^8}\right) + \frac{3c_1 e_0}{1024}\left(\frac{r^8}{\tilde{b}}\right)^{1/2} + \mathcal{O}(r^6),$$

where $c_1 = \frac{\sqrt{\pi}\Gamma(3/8)}{\Gamma(7/8)}$. We see that the dilaton diverges in an expected fashion.

C. Analysis of the vector bundles: Global and local symmetries

Now that we have determined the background geometry, we must find a suitable vector bundle. In heterotic theories there are two sources of vector bundles related to the global and the local symmetries. From the original F-theory perspective, the two symmetries are easy to see: the global symmetries come from the intersecting system of D7/O7s, and the local symmetries come from the probe D3-branes. In the following we will determine these symmetries explicitly.

1. Global symmetries and the torsional backgrounds

F-theory models have many enhanced global symmetry points. In the absence of the second set of D7/O7s in the type IIB setup, there would be multiple points with constant couplings [48,49]. One of the simplest ones is the D_4 point, leading to a $SO(8)$ global symmetry [48]. One would then think that, in the presence of the second set of D7/O7s, the enhanced symmetry group would be given by the $D_4 \times D_4$ point from Tate's algorithm [50]. One might worry that there would be tensionless strings at such a colliding D_4 singularity [51], but this does not happen in our case because the orbifold singularity associated with the generator gh in Eq. (3.27) hides half a unit of a B_{NS} flux wrapping the collapsed two-cycle (for example, see Ref. [52]).

Beyond this, the colliding D_4 singularities actually do not even survive the orientifolding operation that we performed in earlier sections. What we get instead of colliding D_4 singularities is colliding A_3 singularities, leading to a $U(4) \times U(4)$ global symmetry. This is because the orientifold projection is coupled with a gauge transformation, so the surviving symmetry group is the subgroup of $SO(8)$ that commutes with this gauge transformation:

$$D_4 \begin{pmatrix} \mathcal{M} & 0 \\ 0 & -\mathcal{M} \end{pmatrix} A_3, \qquad (4.71)$$

where the 4×4 matrix $\mathcal{M}$ is given by

$$\mathcal{M} \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \qquad (4.72)$$

The adjoint hypermultiplet of $SO(8)$ picks up a minus sign when conjugated with the matrix $\mathcal{M}$, leading to two hyper-multiplets in the **6** of $SU(4)$. Thus, putting four D7 branes on top of the O7 plane, and including the necessary $U(1)$ factors, we get a $U(4)$ global symmetry. The full global symmetry group at the constant coupling point is then given by $U(4)_u^4 \times U(4)_v^4$.

Of course, if we arrange the branes slightly differently, we can have other global symmetry groups. For example, we could go to break to $SU(2)_u^8 \times SU(2)_v^8$ by allowing the 7-branes to move in pairs [35,36], though this will no longer correspond to constant coupling. This is exemplified by the curve (3.22) with the choice given by Eq. (3.23). An example of this would be the following choices:

$$f(u, v) = (u - u_1)F(u, v) \qquad g(u, v) = (u - u_1)G(u, v), \qquad (4.73)$$

where F and G are chosen not to have additional zeros at $u = u_1$. In terms of the coefficients in Eq. (3.22), this is equivalent to the following choices:

$$a_{11} = b_{11} = c_{11} = d_1 = m_{11} = n_{11} = s_{11} = p_{11}$$
$$= q_{11} = r_1 \equiv u_1, \qquad (A_{11}, M_{11}) \neq 0, \qquad (4.74)$$

with all other coefficients not equal to u_1. The discriminant locus will then be given by (the discriminant Δ should not be confused with the parameter in the metric of the same name)

$$\Delta = (u - u_1)^2[4(u - u_1)F^3 + 27G^2] \equiv (u - u_1)^2\tilde{\Delta}(u, v) \qquad (4.75)$$

so that we have only a pair of 7-branes together, resulting in a classical $U(2)$ global symmetry [23]. This is the type of singularity we will study in this paper.[23]

Of course, we expect that the variety of global symme-tries that arise in the F-theory construction can be

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[23]Of course, we could have moved in the opposite direction, enhancing the global symmetry instead. For example, if we modify Eq. (4.75) to

$$\Delta = (u - u_1)^{10}(v - v_1)^{10}\tilde{\Delta}(u, v),$$

such that the vanishing of $\tilde{\Delta}(u, v)$ leads to no new enhanced symmetry points [other than the $U(1)$'s], then we could achieve the full $E_8 \times E_8$ global symmetry in our setup, though it is believed that such a point does not occur along the constant coupling branches of F theory (see, for example, Ref. [53]).

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reproduced by varying moduli on the heterotic side. Actually verifying this, of course, is quite involved since we have to satisfy both the Donaldson–Uhlenbeck–Yau equations as well as the Bianchi identity:

$$F_{ab} = F_{\bar{a}\bar{b}} = g^{a\bar{b}}F_{a\bar{b}} = 0,$$
$$\frac{1}{30}\text{tr}F \wedge F = \text{tr}R_+ \wedge R_+ - \frac{1}{\alpha'}d\mathcal{H}, \qquad (4.76)$$

where, again, R_+ is the Ricci two-form constructed from a metric-compatible connection with torsion—the "plus" connection. When we have torsion, the dilaton is not constant, and so on the type II side, we will not be at the constant coupling point of the Gimon–Polchinski model. The functional form for the axiodilaton would determine the resulting positions of the branes and planes in this scenario and therefore the gauge bundle.

Another reason this is challenging on the heterotic side is because the global symmetry generically will also arise from the orbifold 5-planes. This is different from the $SO(32)$ case studied earlier [2], where the global symme-tries appeared only from the usual heterotic vector bundles with appropriate Wilson lines. In our case, we expect part of the non-Abelian global symmetry to appear from the twisted sectors states of the orbifold.[24]

In this paper, we will study only $U(2)$ global symmetries arising from a nontrivial vector bundle and none from the orbifold. This corresponds to an F-theory curve of the form (4.73) with discriminant locus given by Eq. (4.75), leading to a $U(2) \times U(1)^n$ symmetry. We then decouple the $U(1)^n$ so that the heterotic global symmetry is just $U(2)$, all coming from the vector bundle and none from the orbifold planes.[25]

We now work out an explicit example. We take the metric ansatz (4.8) with $c_4 = \cos\theta_1$, $c_7 = \cos\theta_2$, $c_1 = \sin\theta_1$, and $c_5 = \sin\theta_2$, and as before H_i only de-pends on the radial coordinate r [we could just as well have chosen the metric ansatz (4.36)]. The torsion polynomial (4.37) then gives us

$$d\mathcal{H} = \frac{1}{\sqrt{H_2}}(A_d e_3 \wedge e_4 + B_d e_1 \wedge e_2) \wedge e_5 \wedge e_6, \qquad (4.77)$$

where e_i are the vielbeins (4.9), and A_d and B_d are defined as

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[24]Unfortunately, this non-Abelian enhancement is not visible from string perturbation theory. An alternative way to see this would be to dualize to a singular type IIA geometry, where the symmetry enhancement could be computed using the techniques of Refs. [54,55]. We thank Ashoke Sen for clarifying this point.

[25]Actually, there could still be a single localized $U(1)$ that would be difficult to decouple. We will ignore this subtlety and only consider a $U(2)$ bundle for simplicity.

$$A_d \equiv A \frac{\partial}{\partial r} \log\left(H_4\sqrt{H_1}\right) - A_r, \qquad B_d \equiv B \frac{\partial}{\partial r} \log\left(H_3\sqrt{H_1}\right) - B_r, \tag{4.78}$$

with the subscript r being the derivative with respect to r, as before. The two functions A are B are now defined from the torsion polynomial as

$$A = -\frac{2\phi_r}{\sqrt{H_2}} + \frac{H_{3r}}{H_3\sqrt{H_2}} - \frac{\sqrt{H_1}}{H_3}, \qquad B = -\frac{2\phi_r}{\sqrt{H_2}} + \frac{H_{4r}}{H_4\sqrt{H_2}} - \frac{\sqrt{H_1}}{H_4}. \tag{4.79}$$

The Bianchi identity in Eq. (4.12) then implies that the rhs of Eq. (4.77) should have three contributions: one from the k 5-brane sources, one from $\mathrm{tr}R_+ \wedge R_+$, and one from $\mathrm{tr}F \wedge F$. The torsional connections have been worked out in Appendix C. Using these, we obtain

$$\mathrm{Tr}R_+ \wedge R_+ = (R_1 e_1 \wedge e_2 + R_2 e_3 \wedge e_4) \wedge e_5 \wedge e_6, \tag{4.80}$$

where R_1 and R_2 are given by

$$R_1 = \frac{3}{2}\left[\left(8 + 6B\sqrt{H_2} - \frac{3H_2}{H_3} + B^2 H_3^2 + \frac{H_3'}{H_1 H_3}\right)\left(\frac{2H_2'}{H_3} - \frac{BH_2'}{\sqrt{H_2}} - \frac{2H_2 H_3'}{H_3^2} - 2\sqrt{H_2}B'\right)\right.$$
$$+ \frac{4(\sqrt{H_2} - AH_4)}{H_4^3}(2H_2 H_4^2 A' - 2\sqrt{H_2}H_2' H_4 + AH_4^2 H_2' + 2H_2^{3/2}H_4')$$
$$\left. + \frac{2}{H_1^2 H_2^2 H_3}(H_2 H_3' + H_3(2H_2' + B\sqrt{H_2}H_3'))(H_1 H_2'^2 + H_2(H_1' H_2' - 2H_1 H_2'')) \right] \tag{4.81}$$

$$R_2 = \frac{3}{2}\left[\left(8 + 6A\sqrt{H_2} - \frac{3H_2}{H_4} + A^2 H_4^2 + \frac{H_4'}{H_1 H_4}\right)\left(\frac{2H_2'}{H_4} - \frac{AH_2'}{\sqrt{H_2}} - \frac{2H_2 H_4'}{H_4^2} - 2\sqrt{H_2}A'\right)\right.$$
$$+ \frac{4(\sqrt{H_2} - BH_3)}{H_3^3}(2H_2 H_3^2 B' - 2\sqrt{H_2}H_2' H_3 + BH_3^2 H_2' + 2H_2^{3/2}H_3') + \frac{2}{H_1^2 H_2^2 H_4}(H_2 H_4' + H_4(2H_2' + A\sqrt{H_2}H_4'))$$
$$\left. \times (H_1 H_2'^2 + H_2(H_1' H_2' - 2H_1 H_2'')) \right]. \tag{4.82}$$

Away from the origin of r, $d\mathcal{H}$ receives no contributions from delta-function sources in Eq. (4.12), so we next work on the vector bundle, which must satisfy

$$\frac{1}{30}\mathrm{tr}\,F \wedge F = \left[\left(R_1 - \frac{A_d}{\alpha'\sqrt{H_2}}\right)e_1 \wedge e_2 + \left(R_2 - \frac{B_d}{\alpha'\sqrt{H_2}}\right)e_3 \wedge e_4\right] \wedge e_5 \wedge e_6. \tag{4.83}$$

Our aim now is to determine the $U(2)$ bundle, which we will assume comes from the remnants of the 7-branes and 7-planes in the $u \equiv x^4 + ix^5$ plane, ignoring $U(1)$s from the orbifold states. The $U(2)$ bundle can then be expressed in terms of the Pauli matrices, and one simple ansatz will be

$$F = (\mathfrak{f}_1 e_1 \wedge e_2 + \mathfrak{f}_2 e_3 \wedge e_4 + \mathfrak{f}_3 e_5 \wedge e_6)I + (\mathfrak{f}_4 e_1 \wedge e_2 + \mathfrak{f}_5 e_3 \wedge e_4)\sigma^1, \tag{4.84}$$

where σ^1 is the first Pauli matrices and $\mathfrak{f}_i$ are real functions of r that will be determined shortly. The choice (4.84) then immediately implies the following value for $\mathrm{tr}F \wedge F$:

$$\frac{1}{30}\mathrm{tr}\,F \wedge F = \frac{2}{15}[(\mathfrak{f}_1\mathfrak{f}_2 - \mathfrak{f}_4\mathfrak{f}_5)e_1 \wedge e_2 \wedge e_3 \wedge e_4 + \mathfrak{f}_2\mathfrak{f}_3 e_3 \wedge e_4 \wedge e_5 \wedge e_6 + \mathfrak{f}_1\mathfrak{f}_3 e_1 \wedge e_2 \wedge e_5 \wedge e_6]. \tag{4.85}$$

Comparing Eq. (4.83) with Eq. (4.85), we see the following conditions on the $\mathfrak{f}_i$:

$$\mathfrak{f}_1\mathfrak{f}_2 = \mathfrak{f}_4\mathfrak{f}_5, \qquad \mathfrak{f}_1\mathfrak{f}_3 = R_1 - \frac{A_d}{\alpha'\sqrt{H_2}}, \qquad \mathfrak{f}_2\mathfrak{f}_3 = R_2 - \frac{B_d}{\alpha'\sqrt{H_2}}. \tag{4.86}$$

We still have to satisfy the Donaldson–Uhlenbeck–Yau conditions, which are equivalent to $F_{mn}J^{mn} = 0$. Imposing this on our ansatz immediately implies the following two additional constraints on $\mathfrak{f}_i$:

$$\mathfrak{f}_1 + \mathfrak{f}_2 + \mathfrak{f}_3 = 0, \qquad \mathfrak{f}_4 + \mathfrak{f}_5 = 0. \tag{4.87}$$

These turn out to be enough to determine the functional forms for $\mathfrak{f}_i$ uniquely. Combining Eqs. (4.87) and (4.86) gives us the following:

$$\check{f}_1 = \pm \frac{R_1 - A_d/\alpha'\sqrt{H_2}}{\sqrt{-R_1 - R_2 + (A_d + B_d)/\alpha'\sqrt{H_2}}}, \qquad \check{f}_2 = \pm \frac{R_2 - B_d/\alpha'\sqrt{H_2}}{\sqrt{-R_1 - R_2 + (A_d + B_d)/\alpha'\sqrt{H_2}}},$$

$$\check{f}_3 = \pm\sqrt{-R_1 - R_2 + (A_d + B_d)/\alpha'\sqrt{H_2}}, \qquad \check{f}_4 = -\check{f}_5 = \pm\sqrt{\frac{(R_1 - A_d/\alpha'\sqrt{H_2})(R_2 - B_d/\alpha'\sqrt{H_2})}{R_1 + R_2 - (A_d + B_d)/\alpha'\sqrt{H_2}}}, \qquad (4.88)$$

with R_1 & R_2, A & B, and the warp factors H_i, given by Eqs. (4.80), (4.79), and (4.8), respectively. Note that $\check{f}_i$ are defined for $r > 0$. At the origin, we have to account for the sources in Eq. (4.12) to determine corrections to $\check{f}_i$. Therefore with Eqs. (4.84) and (4.88), we have the $U(2)$ global symmetries for all the three cases discussed earlier. We now go to the issue of local symmetries.

2. Local symmetries and gauge groups in the strongcoupling limit

To study the local symmetries, or the gauge groups on the wrapped heterotic 5-branes, it will be easier to study them without going to the decoupling limit. As we discussed briefly in the introduction, once we are away from the resolved conifold point, there are two possible theories in six dimensions: theories with six-dimensional vector multiplets and theories with six-dimensional tensor multiplets. These two theories are related to $SO(32)$ and $E_8 \times E_8$ heterotic theories, respectively. However, our heterotic theories appeared in conjunction with other string theories that were related by a series of dualities. In fact, we considered three different theories, which are related by T or S dualities:

  (i) Type IIB theory with k D3-branes probing two sets of D7/O7-branes/orientifolds, as in Table I.
  (ii) Type I theory with k D5-branes and D5′/O5′-branes/orientifolds, as in Table II.
  (iii) Heterotic $E_8 \times E_8$ with a set of k NS5-branes and a set of orbifold five-planes O_r5 (on top of which we could also layer NS5-branes), as in Table III.

In the type IIB model, we can displace the D3-branes along the $u = x^4 + ix^5$ direction and along the $v = x^8 + ix^9$ direction. The gauge group on the k D3-branes is $Sp(2k) \times Sp(2k)$. What happens in the strong coupling regime? We know that when the type IIB model is lifted to an elliptically fibered F-theory model, the orientifold planes split into two sets of nonlocal 7-branes, and so we would expect to have four different local 7-branes.

Nevertheless, the story is different as we know from Ref. [35]. There it was shown that the location of a set of 7-branes for large v is identical to the the location of the other set of 7-branes for large u, which implies that the two 7-branes in which the u plane splits join the two 7-branes in which the v plane splits. In the nonperturbative regime, the breaking of the $Sp(2k) \times Sp(2k)$ group is related to the existence of massless hypermultiples which are identified with the monopoles/dyons for either one of the two $Sp(2k)$ groups. The interpretation of the results in Ref. [35] is that these two monopoles can be deformed into each other.

Our question is how we translate the results of Ref. [35] into type I and then to the heterotic picture. To do this, we are going to invoke the discussion of Ref. [56]. Since u and $\bar{u}$ are the directions that must be T dualized to relate the type IIB frame to the type I frame, the IIB position of the D3-branes in the u plane maps into Wilson lines that are switched on on both the type I D5-branes as well as the background D9-/O9-branes/planes. For generic Wilson lines, one of the $Sp(2k) \subset Sp(2k) \times Sp(2k)$ groups is completely broken while the other $Sp(2k)$ is kept intact; the gauge group on the D9-/O9-branes/planes is broken to $U(1)^{16}$.

After a S duality, one obtains a heterotic string with a generic $U(1)^{16}$ vector bundle along with NS5-branes at the O_r5 plane, implying an $Sp(k)$ local gauge group. In the previous section, we saw that we then preserved only a $U(2)$ subgroup [ignoring other $U(1)$ factors] of the full global group.

What happens now if one displaces the D3-branes in the v and $\bar{v}$ directions? The D3-branes would then be moved away from the D7′ branes, and it would then be expected that the only orientifold projection would be due to the other set of D7/O7-branes/planes. This then implies that the gauge group is just $Sp(2k)$. After T dualizing in the $x^{4,5}$ directions, the k type I D5-branes would again be separated from the D5′-/O5′-brane/plane system, and the only orientifold projection would now be due to the D9-/O9-branes/planes. In this case, we again have a gauge group $Sp(2k)$ on the D5-branes with the gauge group on the D9/O9 system appearing as a global symmetry on the D5-branes [of which again we can only keep the $U(2)$ subgroup as an illustrative example].

The continuous transformation between the monopoles of the two $Sp(2k)$ groups in the $Sp(2k) \times Sp(2k)$ theory on the type IIB D3-branes is mapped in the type I language into a deformation of the group on the D9-/O9-branes/ planes from the unbroken subgroups that are invariant under orientifold operations to the completely broken $U(1)^{16}$. In this scenario the $U(2)$ global symmetry may be assumed to come from the twisted sector states alone.

V. GEOMETRIC TRANSITIONS AND HETEROTIC GAUGE/GRAVITY DUALITY

Our analysis in the previous sections yield a heterotic background with wrapped NS5-branes on a non-Kähler warped resolved conifold. The full background is[26]

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[26]A summary of the various backgrounds before and after geometric transitions is given in Appendix B.

$$ds^2 = ds_{0123}^2 + \frac{2\Delta}{r\sqrt{a_3}}dr^2 + \frac{2\Delta\sqrt{a_3}}{r}(d\psi + \cos\theta_1 d\phi_1$$
$$+ \cos\theta_2 d\phi_2)^2 + (\Delta + a^2)(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2)$$
$$+ \Delta(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2)$$
$$\mathcal{H} = \sqrt{\frac{H_1}{H_2}}(G_1\sin\theta_2 d\theta_2 \wedge d\phi_2 + G_2\sin\theta_1 d\theta_1 \wedge d\phi_1) \wedge e_\psi$$
$$e^\phi = \frac{4\Delta}{r^2}, \tag{5.1}$$

where G_1 and G_2 are given in Eq. (4.38) in terms of Δ and the resolution parameter a^2. All we need to do is substitute the values for a_3 and Δ for the three cases that we studied in Sec. IV B. For case I, a_3 and Δ are given, respectively, in Eqs. (4.44) and (4.46); whereas for case II, a_3 and Δ are given, respectively, in Eqs. (4.55) and (4.52). For both cases, we see that the string coupling e^ϕ has standard behavior near the NS5-branes; namely, it blows up as $\frac{1}{r^4}$. Thus, the core of the 5-brane is described by a strongly coupled theory.

For case III, the situation is slightly different because there are three regimes of interest. For region I, which is close to the origin, the values for a_3 and Δ can be read off from Eq. (4.70). For the intermediate region, a_3 and Δ can be read off from Eqs. (4.63) and (4.65), respectively. Finally for the asymptotic region $r \to \infty$, a_3 and Δ can be read from the asymptotic region for case I studied earlier. If we extrapolate the value of the string coupling in the intermediate region to the origin, then it will appear as though the string coupling e^ϕ does not blow up at the origin but attains the following finite value:
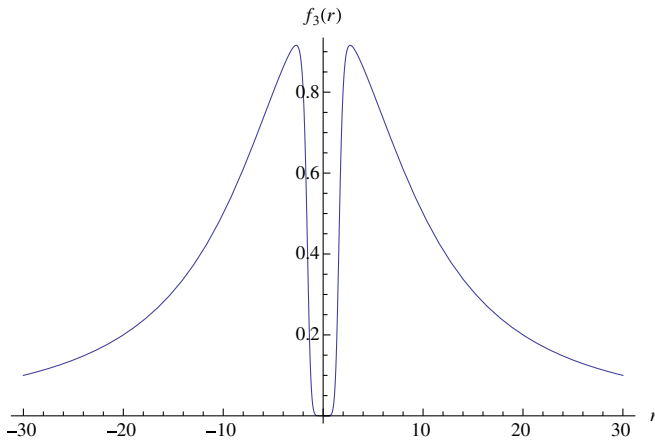

FIG. 6 (color online). The behavior of the function $f_3(r)$ for case III plotted along the direction orthogonal to the wrapped 5-branes without taking into account the source contributions and asymptotic modification as in Eq. (4.61). Note that the f_3 function asymptotes to 0, whereas at the origin $r = 0$, it vanishes. For simplicity we have chosen $e_0 = 100$ and $b = 50$ in units of α'. If we include Eq. (4.61), the behavior at $r \to 0$ and at $r \to \infty$ will change.
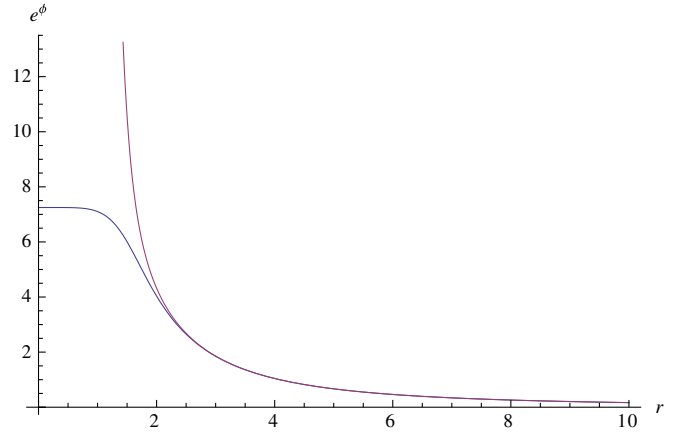

FIG. 7 (color online). The behavior of the string coupling e^ϕ along the direction orthogonal to the wrapped 5-branes for case III. If we only use the warp factor (4.61) then near the origin, with $e_0 = 100$ and $b = 50$ in units of α', the string coupling becomes $e^\phi = \frac{5e_0}{12b^{1/4}}\frac{\Gamma(5/8)\Gamma(7/4)}{\Gamma(1/8)}$, which is large but finite. This is where the regime of validity of the curve terminates. The modification near the origin is shown by the red curve where we take the warp factor (4.61). We see that the dilaton blows up near the core of the 5-branes in an expected fashion.

$$e^\phi = \frac{5e_0}{12b^{1/4}}\frac{\Gamma(5/8)\Gamma(7/4)}{\Gamma(1/8)}, \tag{5.2}$$

whereas it vanishes at infinity. However, from Eq. (4.70) we know that e^ϕ blows up when $r \to 0$, so the two curves for the two regions have to be attached at the value where the string coupling is Eq. (5.2). The plot for the dilaton is depicted in Fig. 7.

In the absence of the 5-brane sources, the behavior of the dilaton is completely governed by the warp factor (4.61). One of the main reasons for such a behavior of the dilaton may stem from the fact that the $f_3(r)$ function for case III vanishes at the origin (as can be seen from Fig. 6). This is reminiscent of the configuration studied in Ref. [57], where the string coupling has somewhat similar behavior. The difference therein is that the 5-branes are distributed over some orthogonal S^3 in the case of Ref. [57], whereas in our case, this effect is captured by the $f_3(r)$ function that is *distributed* over the radial direction. This, however, does not mean that the 5-branes in our case are distributed along r, but it implies only that the effective warp factor has the distribution given by Eq. (4.61).[27]

A. Torsion in the heterotic theory

The torsion for all the three cases can be computed from Eq. (4.37) by including the asymmetry factor (4.38). However, it will be instructive to analyze the functional form of the torsion to gain more information of the

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[27]For example, $f_3 = 1$ for case I does not imply that the 5-branes are distributed equally along the radial direction.
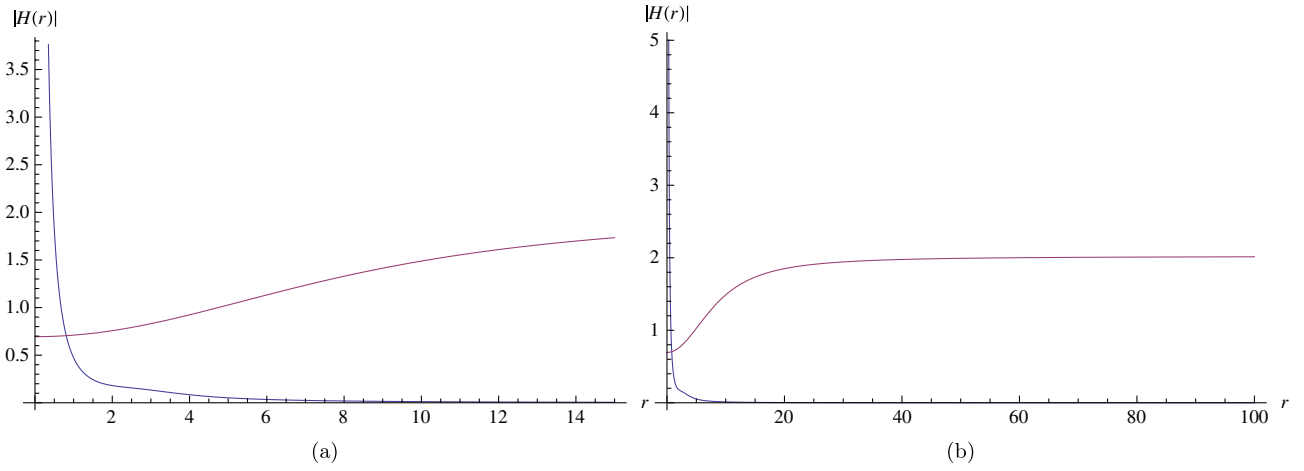
FIG. 8 (color online).   The plot of the torsion coefficient $|H(r)|$ for case I, i.e., $H(r) = -\sqrt{a_3}(\partial_r\Delta - 2\Delta/r)$. The curve in blue represents the localized 3-brane ansatz (4.50), while the curve in red is the standard ansatz without the modification at small r. At large r, the effect of the blue curve is negligible, and the red curve dominates, asymptoting to a constant value, while near the origin, the blue curve captures the source contribution. For simplicity, we choose $\alpha' = 1$ and $e_0 = 100$. (a) Behavior near the origin, (b) Behavior far from the origin.

background. If we ignore the $\mathcal{O}(a^2)$ correction in the expression for the dilaton, ϕ, then the torsion can be written in the following form:

$$\mathcal{H} = -\sqrt{a_3}\left(\Delta_r - \frac{2\Delta}{r}\right)(\Omega_1 + \Omega_2) \wedge e_\psi$$

$$= -\sqrt{a_3}\Delta\phi_r(\Omega_1 + \Omega_2) \wedge e_\psi + \mathcal{O}(a^2), \qquad (5.3)$$

where $\Omega_i = \sin\theta_i d\theta_i \wedge d\phi_i$ and ϕ_r is the derivative of the dilaton without the a^2 contribution, i.e., $\phi_r = \frac{\Delta_r}{\Delta} - \frac{2}{r} + \mathcal{O}(a^2)$. This form of the torsion, with a derivative of the dilaton, is reminiscent of the standard 5-brane background.

   The plots of the three cases are given in Figs. 8–10. Note that in all three cases, the torsion blows up near the origin, signaling the existence of the wrapped 5-brane sources,

while they become constants at large r. The reason for this asymptotic behavior is because of the potential (4.13), which needs to vanish for all three cases. We can also plot the coefficient of $d\mathcal{H}$ and can see the presence of delocalized sources in Fig. 11.

   Note that while we performed this analysis at leading order in small a^2, it can be performed to arbitrary orders in the resolution parameter.

FIG. 10 (color online).   The plot of the torsion coefficient $|H(r)|$ for case III plotted for the $r \rightarrow 0$ and the intermediate region. Near the origin the torsion blows up, signifying the presence of the sources, as depicted by the blue curve using Eq. (4.61). For large r the blue curve coincides with the red curve drawn using Eq. (4.61), confirming our ansatz that the warp factor for small r, which is Eq. (4.61), should smoothly transform to the warp factor for $r > |\tilde{b}|$, which is Eq. (4.61). Notice that for $r \rightarrow \infty$, the torsion becomes finite as in cases I and II. Again, this behavior is necessary for maintaining zero energy in the system (4.13). We have chosen $e_0 = 100$, $\tilde{b} = 50$, and $b = -50$ in units of α'.
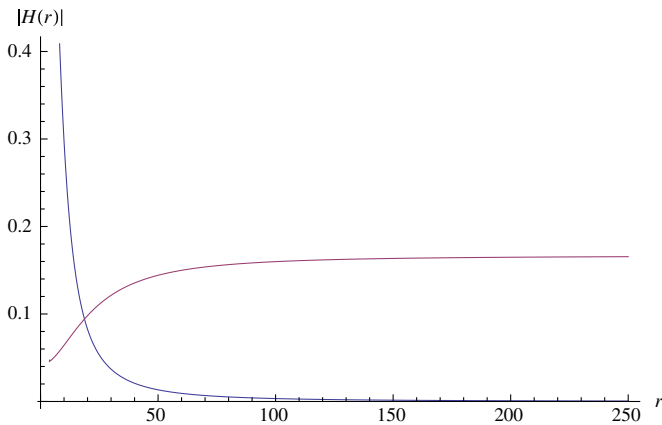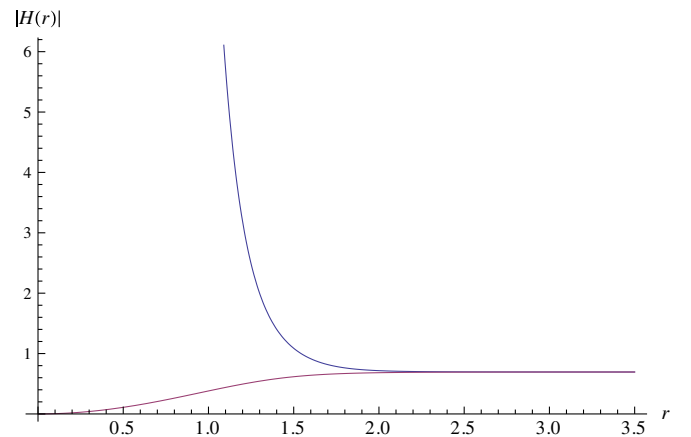

FIG. 9 (color online).   The plot of the torsion coefficient $|H(r)|$ for case II. As before, note that the torsion becomes finite at infinity. We have again chosen $e_0 = 100$ in units of α'. The blue curve is plotted using the $r \rightarrow 0$ modification as Eq. (4.59), while the red curve is for Eq. (4.54).
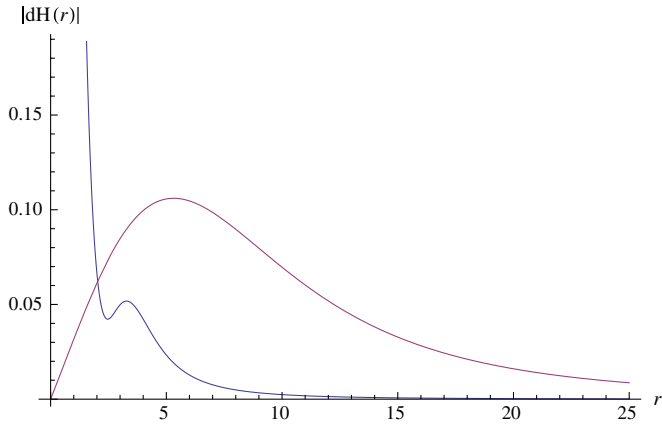
FIG. 11 (color online). A plot for the coefficient of $d\mathcal{H}$ for case I, with $e_0 = 100$ as before. The blue curve clearly depicts the presence of the sources, while the red curve vanishes near the origin. The color codings are same as in the earlier plots.

B. Type I duality frame

Under an S duality, we go back to the type I background that we studied earlier. The metric in type I is the usual S-dual transform of the metric (4.6), or of the metric (4.8),

$$ds^2 = e^{-\phi}[ds^2_{0123} + H_2 dr^2 + H_1(d\psi + c_4 d\phi_1 + c_7 d\phi_2)^2 + H_3(d\theta_1^2 + c_1^2 d\phi_1^2) + H_4(d\theta_2^2 + c_5^2 d\phi_2^2)], \quad (5.4)$$

where the torsion $\mathcal{H}$ becomes the RR three-form F_3. The type I background is useful because it is the closest to the well-known type IIB background with wrapped D5-branes on the two-cycle of a resolved conifold, namely, the one studied in Refs. [2,5,58]. The difference now is that we have an additional set of D5-branes, which we can trace through a geometric transition and obtain the S dual of the gravity dual of the heterotic side configuration. Finding the heterotic gravity dual this way also involves understanding the following:

(i) The type I D9/O9 system will undergo some changes on its world volume after the geometric transition, which corresponds to changing the vector bundle.

(ii) The set of type I D5-branes wrapped on the two-cycle $\mathbf{P}^1$, parameterized by $x^{6,7}$, will also dissolve into geometry and flux. This means that after the geometric transition, we will only see torsion, and all the heterotic 5-branes will have dissolved into geometry and flux.

We can make the duality more precise as follows: the type I background that we are interested in, under a mirror transformation, leads to a type I$'$ background given by

$$\text{Type IIB on } \frac{\mathcal{C}_{\text{res}}}{\{1, \Omega\}} \xrightarrow{\text{mirror}} \text{Type IIA on } \frac{\mathcal{C}_{\text{def}}}{\{1, \Omega \cdot \mathcal{I}_{\phi_1\phi_2\psi} \cdot (-1)^{F_L}\}}, \quad (5.5)$$

where $\mathcal{C}_{\text{def}}$ is the deformed conifold and the mirror symmetry is defined in the usual way by three T dualities along the isometry directions ϕ_1, ϕ_2, and ψ [59]. The fixed points of the orientifold action in the mirror deformed conifold are two sets of O6 planes with bound D6-branes. This D6/O6 system intersects the other set of k wrapped D6-branes to form an intersecting brane/plane system. As we saw in the heterotic case, the system is supersymmetric in the absence of the deformed conifold background. In the presence of the deformed conifold, supersymmetry is achieved by turning on fluxes (which are mirror to the torsion in the heterotic setup). The brane configuration is shown in Table IV.

C. Type I$'$ and type IIA duality frames

Our next step would be to find the mirror type IIA configuration. Naively, this can be obtained by performing three T dualities along ψ, ϕ_1, and ϕ_2, but this would lead to an erroneous result [2,3,5–7]. The subtlety lies in making the base of the manifold, parametrized by θ_1, θ_2, and r, very large. The simplest way to do this would be to make the following replacements in the background:

$$d\psi \mapsto d\psi + f_1 c_4 d\theta_1 + f_2 c_7 d\theta_2$$
$$d\phi_1 \mapsto d\phi_1 - f_1 d\theta_1, \quad (5.6)$$
$$d\phi_2 \mapsto d\phi_2 - f_2 d\theta_2,$$

TABLE IV. Configuration after taking a mirror transformation of Table II. The deformed conifold is denoted as $\mathcal{C}_{\text{def}}$.

Direction	0	1	2	3	4	5	6	7	8	9
D6/O6	✓	✓	✓	✓	✓	·	✓	·	✓	·
D6$'$/O6$'$	✓	✓	✓	✓	·	✓	✓	·	·	✓
k D6	✓	✓	✓	✓	✓	·	·	✓	·	✓
$\mathcal{C}_{\text{def}}$	·	·	·	·	✓	✓	✓	✓	✓	✓

assuming that f_1, c_4 are functions of θ_1 and f_2, c_7 are functions of θ_2. Furthermore, as shown in Refs. [2,5], we need to change the $d\psi$ fibration structure in Eq. (4.8) to

$$H_1[(1 - \sqrt{\epsilon})d\psi + c_4 d\phi_1 + c_7 d\phi_2][(1 + \sqrt{\epsilon})d\psi + c_4 d\phi_1 + c_7 d\phi_2], \tag{5.7}$$

where we have to take the limit where (f_1, f_2) is very large and ϵ is very small, which we can do on the mirror metric. Performing the Strominger-Yau-Zaslow (SYZ) mirror transformation, we find the type IIA mirror metric:

$$ds^2 = e^{-\phi}\Bigg[ds_{0123}^2 + H_2 dr^2 + \frac{e^{2\phi}A}{H_1 H_3 H_4 c_1^2 c_5^2}\Big(d\phi - \frac{H_1 H_4 c_4 c_5^2}{A}d\phi_1 - \frac{H_1 H_3 c_1^2 c_7}{A}d\phi_2\Big)^2 + \frac{e^{2\phi}(H_1 c_7^2 + H_4 c_5^2)}{A}d\phi_1^2$$

$$+ \frac{e^{2\phi}(H_1 c_4^2 + H_3 c_1^2)}{A}d\phi_2^2 - \frac{2H_1 c_4 c_7}{A}d\phi_1 d\phi_2 + (H_3 - \epsilon H_1 f_1^2 c_4^2)d\theta_1^2 + (H_4 - \epsilon H_1 f_2^2 c_7^2)d\theta_2^2$$

$$- 2H_1 c_4 c_7 (\epsilon f_1 f_2) d\theta_1 d\theta_2 \Bigg], \tag{5.8}$$

with a dilaton $\phi_{(b)}$ and with A, f_1, and f_2 defined by

$$A = H_3 H_4 c_1^2 c_5^2 + H_1 H_4 c_4^2 c_5^2 + H_1 H_3 c_1^2 c_7^2, \qquad f_i = \frac{\beta_i}{\sqrt{\epsilon}}. \tag{5.9}$$

Similarly, the B_{NS} field is given by

$$B_{\mathrm{NS}} = \alpha f_1 H_3 c_1^2 (H_1 c_7^2 + H_4 c_5^2)d\theta_1 \wedge d\phi_1 + \alpha f_2 H_4 c_5^2 (H_1 c_4^2 + H_3 c_1^2)d\theta_2 \wedge d\phi_2$$

$$+ \Big(1 - \frac{\epsilon}{\alpha H_1 H_4 c_1^2 c_5^2}\Big)(f_1 c_4 d\theta_1 + f_2 c_7 d\theta_2) \wedge d\psi, \tag{5.10}$$

where $\alpha \equiv A^{-1}$. In the limit $\epsilon \to 0$, the second line is a pure gauge, but the other two components are large. This is expected as the B_{NS} field appears because we made the base of the SYZ T^3 fibration large.[28]

In addition to metric and B field, we also have gauge flux F_2 as well as four-form flux F_4. The nonzero components of F_2 are given by

$$F_{\psi\theta_1} = \mathcal{H}_{\phi_1 \phi_2 \theta_1}, \quad F_{\psi\theta_2} = \mathcal{H}_{\phi_1 \phi_2 \theta_2}, \quad F_{\psi r} = \mathcal{H}_{\phi_1 \phi_2 r},$$

$$F_{\phi_1 r} = \mathcal{H}_{r\phi_2\psi} + \frac{2H_1 c_4 c_7}{H_1 c_4^2 + H_3 c_1^2}\mathcal{H}_{\phi_1 r\psi} + \frac{2H_1 c_4}{H_1 c_4^2 + H_3 c_1^2}\mathcal{H}_{r\phi_1\phi_2},$$

$$F_{\phi_2 r} = \mathcal{H}_{\phi_1 r\psi} + 2\alpha[H_1 c_7(H_1 c_4^2 + H_3 c_1^2) - H_1^2 c_4^2 c_7]\mathcal{H}_{r\phi_1\phi_2}, \quad F_{\phi_1\theta_2} = \mathcal{H}_{\psi\theta_2\phi_2} + 2\frac{H_1 c_4}{H_1 c_4^2 + H_3 c_1^2}H_{\phi_1\phi_2\theta_2},$$

$$F_{\phi_1\theta_1} = 2\frac{H_1 c_4 c_7}{H_1 c_4^2 + H_3 c_1^2}\mathcal{H}_{\phi_1\theta_1\psi} + 2\frac{H_1 c_4}{H_1 c_4^2 + H_3 c_1^2}\mathcal{H}_{\phi_1\phi_2\theta_1},$$

$$F_{\phi_2\theta_1} = \mathcal{H}_{\psi\phi_1\theta_1} + 2\alpha[H_1 c_7(H_1 c_4^2 + H_3 c_1^2) - H_1^2 c_4^2 c_7]\mathcal{H}_{\phi_1\phi_2\theta_1}, \quad F_{\phi_2\theta_2} = 2\alpha[H_1 c_7(H_1 c_4^2 + H_3 c_1^2) - H_1^2 c_4^2 c_7]\mathcal{H}_{\phi_1\phi_2\theta_2}, \tag{5.11}$$

where one may read off the components $\mathcal{H}_{mnp}$ from Eq. (4.14). Similarly, the nonzero components of F_4 are

$$F_{\theta_1\theta_2\phi_1\psi}, \quad F_{\theta_1\phi_1\phi_2 r}, \quad F_{\theta_1\phi_1\psi r}, \quad F_{\theta_1\phi_2\psi r}, \quad F_{\theta_1\theta_2\phi_2\psi}, \quad F_{\theta_2\phi_1\phi_2 r},$$

$$F_{\theta_2\phi_2\psi r}, \quad F_{\theta_2\phi_1\psi r}, \quad F_{r\psi\theta_1\theta_2}, \quad F_{\theta_1\theta_2\phi_1\phi_2}, \quad F_{r\phi_1\theta_1\theta_2}, \quad F_{r\phi_2\theta_1\theta_2}. \tag{5.12}$$

The whole configurations preserves $\mathcal{N} = 1$ supersymmetry in four dimensions.

The metric (5.8) can also be written in a more suggestive way:

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[28]We will discuss another case later where we can gauge away such a B_{NS} field.

$$ds^2 = e^{-\phi}\Bigg\{ds_{0123}^2 + H_2 dr^2 + \frac{e^{2\phi}A}{H_1 H_3 H_4 c_1^2 c_5^2}\Bigg(d\psi - \frac{H_1 H_4 c_4 c_5^2}{A}d\phi_1 - \frac{H_1 H_3 c_1^2 c_7}{A}d\phi_2\Bigg)^2$$

$$+ \Bigg[(H_4 - H_1\beta_2^2 c_7^2)d\theta_2^2 + \frac{e^{2\phi}(H_1 c_7^2 + H_4 c_5^2)}{A}d\phi_1^2\Bigg] + \Bigg[(H_3 - H_1\beta_1^2 c_4^2)d\theta_1^2 + \frac{e^{2\phi}(H_1 c_4^2 + H_3 c_1^2)}{A}d\phi_2^2\Bigg]$$

$$- 2H_1 c_4 c_7\Bigg(\beta_1\beta_2 d\theta_1 d\theta_2 + \frac{1}{A}d\phi_1 d\phi_2\Bigg)\Bigg\}. \tag{5.13}$$

Now we identify the two two-spheres with the sets of coordinates (θ_1, ϕ_2) and (θ_2, ϕ_1). The supersymmetry variations again lead to

$$c_4 = \cos\theta_1, \qquad c_7 = \cos\theta_2, \qquad c_5 = \sin\theta_2, \qquad c_1 = \sin\theta_1, \tag{5.14}$$

with (H_1, H_2, H_3, H_4) all being functions of r, the radial coordinate. This means the metric along the (θ_1, ϕ_2) direction will become[29]

$$d\theta_1^2 + \frac{e^{2\phi}}{A}\Bigg(\frac{H_1 c_4^2 + H_3 c_1^2}{H_3 - H_1\beta_1^2 c_4^2}\Bigg)d\phi_2^2 = d\theta_1^2 + \frac{e^{2\phi}}{A}\Bigg(\frac{H_1\cos^2\theta_1 + H_3\sin^2\theta_1}{H_3 - H_1\beta_1^2\cos^2\theta_1}\Bigg)d\phi_2^2$$

$$= d\theta_1^2 + \frac{e^{2\phi}}{A}\Bigg[\frac{H_1 + (H_3 - H_1)\sin^2\theta_1}{H_3 - H_1\beta_1(\theta_1)^2 + H_1\beta_1(\theta_1)^2\sin^2\theta_1}\Bigg]d\phi_2^2, \tag{5.15}$$

meaning that the length of the ϕ_2 cycle will vary between $e^{\phi}\sqrt{\frac{H_1}{H_3 - H_1\beta(0)^2}}$ and e^{ϕ} as θ_1 varies as $0 < \theta_1 \leq \frac{\pi}{2}$. Similarly, the metric along (θ_2, ϕ_1) directions will become

$$d\theta_2^2 + \frac{e^{2\phi}}{A}\Bigg[\frac{H_1 + (H_4 - H_1)\sin^2\theta_2}{H_4 - H_1\beta_2(\theta_2)^2 + H_1\beta_2(\theta_2)^2\sin^2\theta_2}\Bigg]d\phi_1^2. \tag{5.16}$$

Note that we can now absorb A into a redefinition of the dilaton[30] as $e^{\phi} = e^{\hat{\phi}}\sqrt{A}$.

The brane setup is composed of k D6-branes wrapping the three-cycle parametrized by (θ_1, ϕ_2, ψ), one set of D6/O6 oriented along (ϕ_1, θ_2, ψ), and the other set of D6/O6 (coming from type I D9/O9) oriented along (θ_1, θ_2, r). This is summarized in Table IV.

D. Type IIA detour to brane constructions

At this point, let us take a short detour to discuss geometrical interpretations of the cycles where we wrap our three types of D6-branes. In the language of Ref. [60], there are several Lagrangian submanifolds that we can wrap our D6-branes on. Specifying the deformed conifold, as before, by[31]

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = \mu^2, \tag{5.17}$$

then the base S^3 can be identified with the fixed point set of the antiholomorphic involution $z_i \rightarrow \bar{z}_i$, which for $z_i = x_i + iy_i$ and real μ is given by

$$y_i = 0, \qquad x_1^2 + x_2^2 + x_3^2 + x_4^2 = \mu^2. \tag{5.18}$$

The k D6-branes are wrapped on this S^3 with coordinates (θ_1, ϕ_2, ψ); i.e., this is $S_{(1)}^3$ above.

What about the two pairs of D6/O6? One of the D6/O6 systems is along (ϕ_1, θ_2, ψ), and the other is along (θ_1, θ_2, r). In the language of Ref. [60], there are other Lagrangian submanifolds identified as the fixed point sets of involutions like $(z_1, z_2) \rightarrow (\bar{z}_1, \bar{z}_2)$ and $(z_3, z_4) \rightarrow (-\bar{z}_3, -\bar{z}_4)$. This is given by

$$y_1 = y_2 = x_3 = x_4 = 0, \qquad x_1^2 + x_2^2 = \mu^2 + y_3^2 + y_4^2. \tag{5.19}$$

The unconstrained values for y_3, y_4 and the phase of $x_1 + ix_2$ imply that we have a three-cycle of topology $\mathbb{C} \times S^1$. Note that the S^3 and the $\mathbb{C} \times S^1$ intersect along

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[29]Note that we will always stay away from the points $(\theta_1, \theta_2) = (0, 0)$, (π, π) since the metric is singular at those points and the T^3 fibration degenerates.

[30]Not to be confused with the dilaton $\phi_{(b)}$ in that frame.

[31]Since, for the specific purpose of this section we do not need the added complication of non-Kählerity, we will analyze the branes wrapped on cycles using the Kähler deformed conifold. The analysis can be easily extended to include non-Kählerity.

$$x_3 = x_4 = y_i = 0, \qquad x_1^2 + x_2^2 = \mu^2, \qquad (5.20)$$

which represents a cycle S^1.

The next question then is: out of the two distinct D6/O6 systems, which one is wrapped on the geometric cycle? From the discussion of Ref. [60], the branes wrapped on $\mathbb{C} \times S^1$ survive the geometric transition, so they must be the ones wrapped on the (θ_1, θ_2, r) three-cycle. On the other hand, the D6$'$/O6$'$ system wrapped along (ϕ_1, θ_2, ψ) become geometry and flux after the geometric transition, so they cannot be wrapped on $\mathbb{C} \times S^1$. From Eqs. (5.15) and (5.16) above, we see that ψ combines with ϕ_1 and ϕ_2 so that the k D6-branes and the D6$'$/O6$'$ system are each wrapped on a Hopf fibration of ψ over a two-cycle given by (ϕ_1, θ_2) and (θ_1, ϕ_2), respectively. To complete the story, two additional ingredients are required:

(i) Two-form fluxes through the bases of the Hopf fibrations: these are indeed present, as we see from the nonzero (ϕ_1, θ_2) and (θ_1, ϕ_2) components of F_2 in Eq. (5.11).

(ii) Supersymmetry: of course, this is true since we determined the background by demanding it preserve supersymmetry (4.19), in addition to the Bianchi identity.

E. M-theory duality frame and new flips and flops

Our next step is to lift this configuration to M theory. As we know, in M theory, the k D6-branes become a k-centered Taub–NUT space while the two sets of D6-/O6-branes/planes become two sets of Atiyah–Hitchin spaces, as shown in Table V.

The uplifted geometry of k D6-branes looks like a Taub–NUT space along the $S^3_{(2)}$ given by $(\theta_2, \phi_1, x^{11})$ and stretched along the radial r direction $\mathbb{R}^+$. Locally, the geometry would then look like $\mathbb{R}^+ \times S^3_{(1)} \times S^3_{(2)}$, where $S^3_{(1)}$ is along (θ_1, ϕ_2, ψ). Similarly, one of the D6/O6 systems becomes an Atiyah–Hitchin space with the local geometry $\mathbb{R}^+ \times S^3_{(3)} \times S^3_{(4)}$, where $S^3_{(3)}$ is along (θ_2, ϕ_1, ψ), and $S^3_{(4)}$ is along $(\theta_1, \phi_2, x^{11})$. The M-theory metric then takes the following form:

$$ds^2 = e^{-\phi - \frac{2\phi_{(b)}}{3}} \Bigg\{ ds_{0123}^2 + H_2 dr^2 + \frac{e^{2\phi} A}{H_1 H_3 H_4 c_1^2 c_5^2} \Big(d\psi - \frac{H_1 H_4 c_4 c_5^2}{A} d\phi_1 - \frac{H_1 H_3 c_1^2 c_7}{A} d\phi_2 \Big)^2$$

$$+ \Big[(H_4 - H_1 \beta_2^2 c_7^2) d\theta_2^2 + \frac{e^{2\phi}(H_1 c_7^2 + H_4 c_5^2)}{A} d\phi_1^2 \Big] + \Big[(H_3 - H_1 \beta_1^2 c_4^2) d\theta_1^2 + \frac{e^{2\phi}(H_1 c_4^2 + H_3 c_1^2)}{A} d\phi_2^2 \Big]$$

$$- 2H_1 c_4 c_7 \Big(\beta_1 \beta_2 d\theta_1 d\theta_2 + \frac{1}{A} d\phi_1 d\phi_2 \Big) \Bigg\} + e^{\frac{4\phi_{(b)}}{3}} (dx_{11} + A_r dr + A_{\theta_1} d\theta_1 + A_{\theta_2} d\theta_2$$

$$+ A_{\phi_1} d\phi_1 + A_{\phi_2} d\phi_2 + A_\psi d\psi)^2, \qquad (5.21)$$

which is a noncompact G_2-structure manifold with G fluxes.[32]

Next we will perform a flop. This will be similar to the one in Ref. [61], in the sense that we will have to exchange three-cycles, but it will be slightly different. We impose the following flop operation:

$$S^3_{(1)} \leftrightarrow S^3_{(2)} \qquad (5.22)$$

and simultaneously

$$S^3_{(3)} \leftrightarrow S^3_{(4)}, \qquad (5.23)$$

where the $S^3_{(3),(4)}$ are the same as $S^3_{(1),(2)}$ but with Hopf fibers exchanged. In coordinates,

$$\theta_1 \leftrightarrow \theta_2, \qquad \phi_1 \leftrightarrow \phi_2, \qquad \psi \leftrightarrow x^{11}. \qquad (5.24)$$

Under this flip and flop,[33] the Taub–NUT space will be along $S^3_{(1)}$, and the first Atiyah–Hitchin space will be along $S^3_{(3)}$. The second Atiyah–Hitchin space is actually unchanged under the flop, which means that when we dimensionally reduce on x^{11}, it will convert back to the same D6/O6 system it came from. This is depicted in Table VI. The dilaton in this frame is $\phi_{(c)}$, which is different from $\phi_{(b)}$ because of Eq. (5.24).

In the flopped setup, the D6/O6 system arising from the second Atiyah–Hitchin space converts to a D9/O9 system after mirror symmetry, returning to a type I model. All other branes/planes in the original type I configuration have dissolved into geometry to become fluxes, consistent with the predictions in Refs. [3,6,7]. S dualizing to heterotic then yields the gravity dual that is composed only of geometry and fluxes, with no localized sources. Following the procedure in Ref. [2], we deduce that the gravity dual is also a non-Kähler warped resolved conifold.

F. Gravity duals in the heterotic theories

So, after dimensional reduction back to type IIA, we will have a manifold that is topologically a resolved conifold

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[32]One could make further local rotations to the M-theory metric to bring it into a more standard form (see Ref. [2]), but we will not do so here.

[33]By *flip* we mean the exchange $\psi \leftrightarrow x^{11}$, whereas *flop* is the standard flop operation of a three-sphere.

TABLE V. The uplift to M theory of the type IIA configuration in Table IV. These configurations of Taub–NUT and Atiyah–Hitchin spaces give rise to a supersymmetric G_2-structure manifold.

Direction	θ_1	ϕ_2	θ_2	ϕ_1	ψ	r	x^{11}
Taub–NUT (TN)	·	·	✓	✓	·	✓	✓
Atiyah–Hitchin (AH$_1$)	✓	✓	·	·	·	✓	✓
Atiyah–Hitchin (AH$_2$)	·	✓	·	✓	✓	·	✓

TABLE VI. Type IIA and M branes/planes/geometry before and after the flop (5.22) and (5.23).

Type IIA	M theory before flop	M theory after flop	Type IIA reduction
k D6	k centered TN along $S^3_{(2)}$	k centered TN along $S^3_{(1)}$	geometry + fluxes
D6$'$/O6$'$	AH$_1$ along $S^3_{(4)}$	AH$_1$ along $S^3_{(3)}$	geometry + fluxes
D6/O6	AH$_2$ along $(\phi_1, \phi_2, \psi, x^{11})$	AH$_2$ along $(\phi_1, \phi_2, \psi, x^{11})$	D6/O6

TABLE VII. Field content traced through much of the duality chain depicted in Fig. 1, starting from the type I theory and ending with the type I$'$ theory.

Original (I)	$g^{(a)}_{\mu\nu}$	ϕ	$\mathcal{H}^{\mathrm{RR}}_{\theta_1\phi_1\psi}$	$\mathcal{H}^{\mathrm{RR}}_{\theta_2\phi_2\psi}$	$\mathcal{H}^{\mathrm{RR}}_{\theta_1\theta_2\phi_1}$	$\mathcal{H}^{\mathrm{RR}}_{\theta_1\theta_2\phi_2}$	$g^{(b)}_{\mu\nu}$	
Mirror (I$'$)	$g_{\mu\nu}$	$\phi_{(b)}$	$F^{(2)}_{\theta_1\phi_2}$	$F^{(2)}_{\theta_2\phi_1}$	$F^{(4)}_{\theta_1\theta_2\phi_2\psi}$	$F^{(4)}_{\theta_1\theta_2\phi_1\psi}$	$\mathcal{H}^{\mathrm{NS}}_{\theta_i\phi_i r}$	$\mathcal{H}^{\mathrm{NS}}_{\theta_i\phi_i\theta_k}$
11D lift (M)			$G_{\mu\nu}$		$G^{(4)}_{\theta_1\theta_2\phi_2\psi}$	$G^{(4)}_{\theta_1\theta_2\phi_1\psi}$	$G^{(4)}_{\theta_i\phi_i r,11}$	$G^{(4)}_{\theta_i\phi_i\theta_k,11}$
Flop (M)			$G'_{\mu\nu}$		$G^{(4)}_{\theta_1\theta_2\phi_1,11}$	$G^{(4)}_{\theta_1\theta_2\phi_2,11}$	$G'^{(4)}_{\theta_k\phi_k r\psi}$	$G'^{(4)}_{\theta_k\phi_k\theta_i\psi}$
Reduce (I$'$)	$\tilde{G}_{\mu\nu}, \phi_{(c)}$		$\tilde{F}^{(2)}_{\theta_1\phi_1}$	$\tilde{F}^{(2)}_{\theta_2\phi_2}$	$\mathcal{H}^{\mathrm{NS}}_{\theta_1\theta_2\phi_1}$	$\mathcal{H}^{\mathrm{NS}}_{\theta_1\theta_2\phi_2}$	$G'^{(4)}_{\theta_k\phi_k r\psi}$	$G'^{(4)}_{\theta_k\phi_k\theta_i\psi}$

with two key differences from the usual geometric transition of Refs. [58,62]: the first is already known from Ref. [5], namely, that the metric should be non-Kähler, and the second is the appearance of a D6-/O6-brane/plane system (see Table VI). Following the steps and notation in Ref. [2], we find the metric:

$$ds^2_{\mathrm{IIA}} = e^{-\tilde{\phi}}[ds^2_{0123} + H_2 dr^2] + \tilde{H}_1(d\psi + \Delta_1\cos\theta_1 d\phi_1$$
$$+ \Delta_2\cos\theta_2 d\phi_2)^2 + (\tilde{H}_{3a}d\theta_1^2 + \tilde{H}_{3b}d\phi_2^2)$$
$$+ (\tilde{H}_{4a}d\theta_2^2 + \tilde{H}_{4b}d\phi_1^2), \qquad (5.25)$$

where $\tilde{\phi}$ is the remnant of the dilaton factor from M theory after flop and H_2 is the same as in Eq. (4.8). For the other coefficients, one may look up their values in sec. 4.4 of Ref. [2].[34] The various components of the fluxes could be traced from the original type I side or from Eqs. (5.11) and (5.12). If we start off with only two components of torsion in the heterotic side as in Eq. (5.3), then the field contents in the subsequent theories will be simple. The list of all the field contents are depicted in Table VII, where we see that in the final type I$'$ theory, the field contents are the two-form and the NS three-form fields along with the dilaton $\phi_{(c)}$ and the metric $\tilde{G}_{\mu\nu}$ given by Eq. (5.25).

The type I$'$ configuration that we get is on a non-Kähler resolved conifold, and therefore we need to perform coordinate transformations of the form (5.6) before performing SYZ mirror transformation. There are a few subtleties now compared to Eq. (5.6) that we performed earlier. First, the coefficients Δ_i appearing in the metric (5.25) may not necessarily be a function of θ_i only. Second, even if Δ_i are functions of θ_i, the coordinate transformation (5.6) is not feasible because this will generate a B_{NS} field like Eq. (5.10) after a mirror transformation to type I, which should have been projected out by the orientifold action.[35] The solution to this puzzle is rather simple and illuminating. There are two ways to make the base of the T^3 fibration bigger in Eq. (5.25): 1) by making a coordinate transformation of the form (5.6) and 2) by changing the complex structures of the two base spheres. Performing the second operation implies that the metric (5.25) picks up additional terms of the form

$$\tilde{f}_1 d\theta_1 d\phi_1 + \tilde{f}_2 d\theta_2 d\phi_2, \qquad (5.26)$$

where $\tilde{f}_i$, unlike the f_i before in Eq. (5.6), do not have to be functions of θ_i only. This means that we can define the functional forms for the $\tilde{f}_i$ in such a way so as to make the

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[34]Note that one has to set to zero all of the b_{ij} fields that appear in the fibrational structure in Ref. [2].

[35]Note that this subtlety was absent in the case studied in Ref. [2] because the mirror was type IIB theory, where such a B_{NS} field can exist.

TABLE VIII. The field content we obtain in the heterotic theory by following the final stages of the duality chain in Fig. 1, starting from the type I′ theory.

Original (I′)	$\tilde{G}'_{\mu\nu}, \phi_{(c)}$	$\mathcal{H}'^{\mathrm{NS}}_{\theta_1\theta_2\phi_1}$	$\mathcal{H}'^{\mathrm{NS}}_{\theta_1\theta_2\phi_2}$	$\tilde{G}^{(4)}_{\theta_k\phi_k r\psi}$	$\tilde{G}^{(4)}_{\theta_k\phi_k\theta_i\psi}$	$\tilde{F}'^{(2)}_{\theta_1\phi_1}$	$\tilde{F}'^{(2)}_{\theta_2\phi_2}$
Mirror (I)	$G'^{\mathrm{NK}}_{\mu\nu}, -\tilde{\phi}$			$\mathcal{H}^{\mathrm{RR}}_{\theta_k\phi_j r}$	$\mathcal{H}^{\mathrm{RR}}_{\theta_k\phi_i\phi_l}$	$\mathcal{H}^{\mathrm{RR}}_{\theta_1\phi_2\psi}$	$\mathcal{H}^{\mathrm{RR}}_{\theta_2\phi_1\psi}$
U dual (Het)	$G^{\mathrm{NK}}_{\mu\nu}, \tilde{\phi}$			$\mathcal{H}_{\theta_k\phi_j r}$	$\mathcal{H}_{\theta_k\theta_i\phi_l}$	$\mathcal{H}_{\theta_1\phi_2\psi}$	$\mathcal{H}_{\theta_2\phi_1\psi}$

B_{NS} field in the mirror type I theory to be a pure gauge. The additional warpings of the coefficients $d\theta_i$ and $d\phi_i$ of the base spheres in the type I′ frame can be absorbed in the definition of $\tilde{H}_{3a,3b,4a,4b}$. Of course, such a change will also affect the background fields, which in turn are shown in the list of fields depicted in Tables VII and VIII.

In Table VII we represented the NS three-form fields in the mirror type I′ theory, coming from the cross-term $g^{(b)}_{\mu\nu}$ of the original type I metric, by $\mathcal{H}^{\mathrm{NS}}_{\theta_i\phi_i r}$ and $\mathcal{H}^{\mathrm{NS}}_{\theta_i\phi_i\theta_k}$. This is the simplest scenario. In general we could have additional components like

$$\mathcal{H}^{\mathrm{NS}}_{\theta_1\phi_1\psi}, \quad \mathcal{H}^{\mathrm{NS}}_{\theta_2\phi_2\psi}, \quad \mathcal{H}^{\mathrm{NS}}_{\theta_1\phi_1\phi_2}, \quad \mathcal{H}^{\mathrm{NS}}_{\theta_2\phi_2\phi_1}. \tag{5.27}$$

The latter two are projected out by orientifolding operation. For the first two fields[36]—under M-theory lift, flop, and dimensional reduction— we will get back three-form fields with the same components but with different magnitudes, in addition to the ones mentioned in Table VII. Changing the complex structure of the base spheres will further change the values of the fields in type I′ theory similar to the ones depicted in Table VIII. Finally, after the mirror and U duality, we get our required heterotic background with a dilaton and torsion as shown in Table VIII. From this table and Eq. (5.27), we can easily see that there are four possible components of the B_{NS} field in the type I′ reduction:

$$B_{\theta_1\phi_1} \equiv b_1, \quad B_{\theta_2\phi_2} \equiv b_2, \quad B_{\psi\phi_1} \equiv b_3, \quad B_{\psi\phi_2} \equiv b_4, \tag{5.28}$$

where all other components can be gauged away. Note that we have components like (b_3, b_4), which have both legs along the T duality directions. This would imply the possibility of nongeometric background after SYZ mirror transformation. We will, however, only concentrate on the geometrical aspect of the mirror and leave the intriguing possibility of nongeometric backgrounds for future works.

After a SYZ mirror transformation (converting D6/O6 to D9/O9) and a U duality, we wind up in heterotic with the metric

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[36]These components do not appear from Eq. (5.10) that we got using T dualities. We, however, expect that a more generic choice of the warp factors for the original type I background may lead to these components.

$$ds^2 = ds^2_{0123} + \frac{2\Delta}{r\sqrt{a_3}}dr^2$$

$$+ \frac{1}{\tilde{H}_1 + \mathcal{A}}[d\psi + \tilde{\Delta}_1\cos\theta_1(d\phi_1 + \alpha_1 d\theta_1)$$

$$+ \tilde{\Delta}_2\cos\theta_2(d\phi_2 + \alpha_2 d\theta_2)]^2$$

$$+ [g_{\theta_1\theta_1}d\theta_1^2 + g_{\phi_1\phi_1}(d\phi_1 + \alpha_1 d\theta_1)^2]$$

$$+ [g_{\theta_2\theta_2}d\theta_2^2 + g_{\phi_2\phi_2}(d\phi_2 + \alpha_2 d\theta_2)^2]$$

$$+ g_{\theta_1\theta_2}d\theta_1 d\theta_2 + g_{\phi_1\phi_2}(d\phi_1 + \alpha_1 d\theta_1)$$

$$\times (d\phi_2 + \alpha_2 d\theta_2). \tag{5.29}$$

Note that there is no coefficient in front of the space-time part. The coefficient, which appears in Eq. (5.25), is in fact the dilaton in the type I/heterotic frame and is related to the dilaton $\phi_{(c)}$ in the type I′ frame by mirror/U-duality transformation. Therefore, in the heterotic frame, we see no factor of dilaton in front of the space-time part. Additionally, preserving G_2 structure in M theory and $SU(3)$ structure in heterotic theory gives the following relation between the dilaton values in various frames [2]:

$$\tilde{\phi} = \phi + \frac{2\phi_{(b)}}{3} - \frac{2\phi_{(c)}}{3}, \tag{5.30}$$

which is perfectly consistent with our duality chain, equations of motion, and the metrics in various frames.

The final metric (5.29) resembles somewhat the metric in Eq. (5.13), but the ϕ_i's are nontrivially fibered. The nontrivial fibrations are given in terms of α_i, which we define as

$$\alpha_1 = \frac{\alpha}{2}[b_1 b_4^2 + \tilde{H}_1 b_1(\tilde{H}_{3b} + \tilde{H}_1\tilde{\Delta}_2^2\cos^2\theta_2$$

$$- b_1\tilde{H}_1\tilde{\Delta}_2^2\cos^2\theta_2)]$$

$$\alpha_2 = \frac{\alpha}{2}[b_2 b_3^2 + \tilde{H}_1 b_2(\tilde{H}_{4b} + \tilde{H}_1\tilde{\Delta}_4^2\cos^2\theta_1$$

$$- b_2\tilde{H}_1\tilde{\Delta}_1^2\cos^2\theta_1)], \tag{5.31}$$

where b_i are the components of the B field given in Eq. (5.28) and $\tilde{H}_k$ are the components of the metric (5.25). The coefficient α in Eq. (5.31) is given by

$$\alpha \equiv \frac{1}{\tilde{H}_{3b}\tilde{H}_{4b} + \tilde{H}_1(\tilde{H}_{3b}\tilde{\Delta}_1^2\cos^2\theta_1 + \tilde{H}_{4b}\tilde{\Delta}_2^2\cos^2\theta_2)}, \tag{5.32}$$

with $\tilde{H}_{3b}$, etc., to be viewed as the modified warp factors. Note also that the coefficient of the $d\psi$ term is not just $\tilde{H}_1^{-1}$ but has a correction term given by $\mathcal{A}$ in the following way:

$$\mathcal{A} = \alpha[(\tilde{H}_{4b} + \tilde{H}_1\Delta_1^2\cos^2\theta_1)(b_4^2 - \tilde{H}_1\Delta_2\cos\theta_2)$$
$$+ (\tilde{H}_{3b} + \tilde{H}_1\Delta_2^2\cos^2\theta_2)(b_3^2 - \tilde{H}_1\Delta_1\cos\theta_1)$$
$$\times 2\Delta_1\Delta_2\cos\theta_1\cos\theta_2(\tilde{H}_1^2\Delta_1\Delta_2\cos\theta_1\cos\theta_2 - b_3b_4)],$$
$$(5.33)$$

which is nonzero even in the absence of b_3 and b_4 components of the B_{NS} field. The other variables in the metric (5.29) are defined in the following way:

$$\tilde{\Delta}_1 \equiv -\alpha\Delta_1\tilde{H}_{3b}, \qquad \tilde{\Delta}_2 \equiv -\alpha\Delta_2\tilde{H}_{4b}. \qquad (5.34)$$

The rest of the g_{ij} defined in Eq. (5.29) are related to the corresponding type IIA warped resolved conifold metric (5.25) by the relations given in appendix C of Ref. [2]. Additionally, we also have nonzero $\mathcal{H}$ flux, dilaton, and vector bundle as depicted in Table VIII. Our claim then is that the gravity dual of the wrapped heterotic 5-branes in the heterotic $E_8 \times E_8$ theory can be extracted from Eq. (5.29) by performing a geometric transition on the metric (4.8). In the decoupling limit of Eq. (4.8) and (5.29) will provide the precise gravity dual. Defining $u = \frac{\alpha'}{r}$, we see that the five-dimensional part of Eq. (5.29) takes the following form:

$$ds^2 = ds^2_{0123} + \frac{du^2}{G(u)}, \qquad (5.35)$$

where $G(u)$ can be extracted from $\frac{2\mathcal{A}}{r\sqrt{a_3}}$ in Eq. (5.29) in the decoupling limit. The spacetime part is flat, so we conclude that, in the string frame, a flat Minkowski space is dual to LST compactified to four dimensions on a non-Kähler resolved conifold. Additionally, irrespective of the details of the internal space, the five-dimensional dual metric will always be of the form (5.35).

Note that the metric that we get on the heterotic side *cannot* be derived using any techniques other than the duality chains given in Figs. 1 and 2, as our knowledge of the dual gauge theory, i.e., the dimensionally reduced LST, is very minimal. Thus, we think that the gravity dual (5.29) may be the only way to extract nontrivial information about the gauge theory.

To complete the story, we should perform a few tests on the type IIB background to check that the background describes a dual to the four-dimensional confining theory. One important consistency check is to find confining strings, which we examine in Sec. V F 2, but before that we further discuss the geometry.

1. Geometry and instanton transitions

Looking at the metric (5.29), we see that the two two-spheres are now given by (θ_1, ϕ_1) and (θ_2, ϕ_2). The fact that the two-spheres have asymmetrical coefficients suggests that the metric describes a warped resolved-deformed conifold, consistent with the expectation from the three cases studied earlier in Sec. IV B. We can also compute the
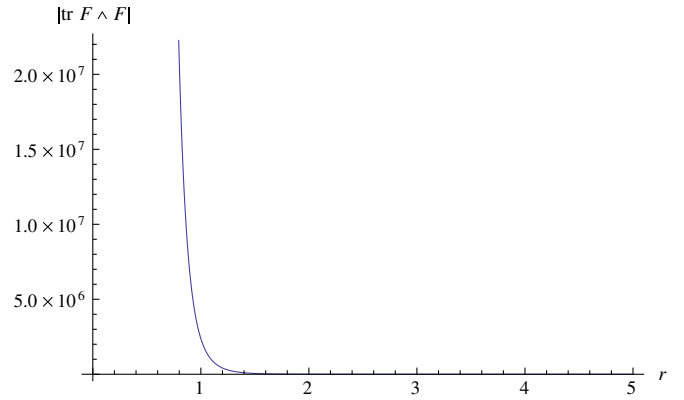

FIG. 12 (color online). The behavior of $|\text{tr}(F \wedge F)|$ plotted near the origin for case I. Note that the singularity at $r = 0$ signifies the presence of zero size instantons. After geometric transition, we expect that the singularity in the vector bundles will be absent. This way the branes will convert to geometry and fluxes. As before we have chosen $e_0 = 100$ in units of α' with $\alpha' = 1$.

torsion $\mathcal{H}$, using the dilaton $\tilde{\phi}$, using the standard formula,

$$\mathcal{H} = -e^{2\tilde{\phi}} * d(e^{-2\tilde{\phi}}J), \qquad (5.36)$$

which we expect this to satisfy the Bianchi identity (4.12) without a source term; therefore, $\text{tr}(F \wedge F)$ should be regular.

In the heterotic frame before the geometric transition, we saw in case I, for example, that $\text{tr}(F \wedge F)$ had a singularity at $r \to 0$ (see Fig. 12), signifying the presence of the k heterotic NS5-branes.[37] Now, after the transition, the branes convert to fluxes (5.36), and the singularity in the vector bundle should vanish. A puzzling aspect, however, is what happens to the small instanton singularities? This can be answered from the M-theory picture that we have. Before the flop the small instanton singularities become multicentered TN space. Thus, after the flop the vector bundle singularities become singularities of the manifold (as there are no moduli for the small instanton singularities to blow up). In other words, before transition we have a vector bundle that has two parts: one singular piece from the small instantons and another nonsingular piece that is supported by the underlying non-Kählerity of the background. After transition, the singular piece becomes part of the manifold (so it is no longer part of the vector bundle), and the nonsingular piece gets further deformed by the flop operation. This is almost like a small instanton transition, and this is why we assume that geometric transitions in heterotic theories are closely related to small instanton transitions.

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[37]This singularity can be easily seen from Eq. (4.85) using the values for $\tilde{f}_i$ in Eq. (4.88) at $r \to 0$.
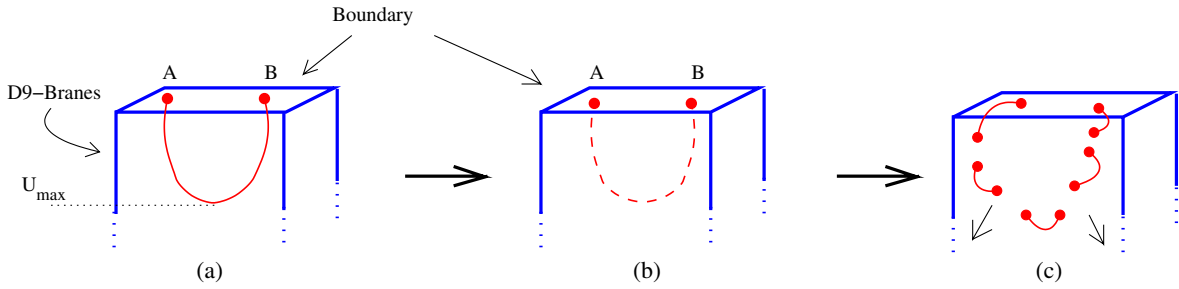
FIG. 13 (color online).   A long fundamental string in type I theory is unstable and is prone to breaking into small strings on the D9-brane. In the dual theory, this captures the phenomena of a confining string breaking into small strings connecting quark-antiquark pairs. (a) An unstable fundamental type I string hanging between two points A and B at the boundary; (b) The type I string breaking up into small open strings on the D9-brane; and (c) The small open strings are dynamical and move on the D9-brane simulating the quark-antiquark jet production.

In this way, a heterotic background with branes is converted into another heterotic background without branes but with fluxes instead. Thus, the little string background (4.36) is converted to a closed string background (5.29), where the singularities of the branes are transformed to the manifold (and also to the torsion to satisfy the Bianchi identity), and the vector bundles become smooth. The whole duality chain depicted in Fig. 1 transforms the brane background of Eq. (4.36) to a closed string background (5.29). In the decoupling limit of the LST, the metric given in Eq. (5.29) will be the gravity dual.

2. In search of the confining strings

Our next check is to look for the confining strings and potentials from the heterotic gravity dual, studying the behavior of a string connecting two points on the boundary. This is easier said than done for two reasons:

(i) Taking a string to the boundary means that we need to know the $r \rightarrow \infty$ behavior of the background—i.e., we need to know the UV completion of the background for which the IR physics is captured by Eq. (5.29).

(ii) Looking at the metric (5.29), we see that the spacetime part of the metric is flat and comes with no warp factor. This suggests that even if we know the UV behavior, a Nambu–Goto action for the string will not produce a nontrivial result.

The second concern is more subtle because the string coupling is also typically strong on the heterotic side. The fact that it is strong on the heterotic side means we could instead ask this question on the S-dual, type I side where it will be weak. So we will consider fundamental strings in the type I frame, which appears to be an appropriate candidate for the confining string. This clearly shows that the confining string from the heterotic frame is a more exotic object.

To proceed, let us define $g_{rr} = e^{2\tilde{\phi}}\tilde{g}_{rr}$ in Eq. (5.29).[38] With this the type I metric becomes

$$ds^2 = e^{-\tilde{\phi}}(-dt^2 + dx_i dx^i) + e^{\tilde{\phi}}\tilde{g}_{uu}du^2 + e^{-\phi}ds^2_{\text{internal}},$$
(5.37)

where $u = 1/r$ and ds^2_{internal} is the internal metric that can be read off from Eq. (5.29). Note that there are no NS fluxes in the type I frame, which will be useful below.

Now we can address the first concern regarding UV completion. We will assume that for large r, i.e., for $u \rightarrow 0$, the four-dimensional gauge theory becomes almost conformal. This means, without loss of generality, that we can take[39]

$$e^{-\tilde{\phi}} = \sum_{n=0}^{\infty} c_n u^{n-2}, \qquad \tilde{g}_{uu} = \sum_{n=0}^{\infty} b_l u^{l-4},$$
(5.38)

where c_n and b_l are independent of u and of the four spacetime directions (for simplicity, we will just take them to be constants with $c_1 = c_2 = 0$). With these choices, the type I metric becomes

$$ds^2 = c_n u^{n-2}(-dt^2 + dx_i dx^i) + \frac{b_l u^l}{c_m u^{m+2}}du^2$$
$$+ c_p u^{p-2}ds^2_{\text{internal}}.$$
(5.39)

Next, we want to study a long fundamental string in this background, but we must first discuss one thing: the type I theory has no stable fundamental strings. Instead, the only stable string is the type I D string. Of course, to describe fluctuations of the D9 background, type I should allow some form of fundamental strings; otherwise, duality to type IIB will not work properly. So this means that if we construct a long open fundamental string, it will break into small open strings and other radiation. This is exactly what we want from the gravity dual of a confining theory: the fundamental string represents the gravity dual of the confining flux tube, which, as we know, is susceptible to breaking up into quark-antiquark pairs. These quark-antiquark pairs are precisely the little open strings that our big

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[38]This is consistent with our convention described in Footnote 43.

[39]Note that, although we are using a specific asymptotic expansion, the confining nature of the string will *only* depend on the IR physics! This will be clarified below. The UV completion is done only to have renormalized results.

fundamental string breaks into as depicted in Fig. 13. Note that this also tells us that the matter sector of our theory contains quarks, which was not easy to decipher directly from the dimensional reduction of the LSTs. Note that this was also discussed in Ref. [4].

Now, consider a fundamental string connecting two boundary points so that it projects into the five-dimensional space $(x^{0,1,2,3}, u)$, and let $X^\mu: (\sigma, \tau) \rightarrow (x^{0,1,2,3}, u, \psi, \theta_i, \phi_i)$ be the embedding of the world sheet into spacetime. If we choose a parametrization $\tau \equiv t$ and $\sigma \equiv x$ and the configuration

$$X^0 = t, \quad X^1 = x, \quad X^2 = X^3 = 0, \quad X^8 = u(x),$$

$$X^9 = \psi = 0, \quad (X^4, X^5) = (\theta_1, \phi_1) = \left(\frac{\pi}{2}, 0\right),$$

$$(X^6, X^7) = (\theta_2, \phi_2) = \left(\frac{\pi}{2}, 0\right), \quad (5.40)$$

then this will traverse a path in the five-dimensional space provided we equip this with an additional boundary condition:

$$u(\pm d/2) = 0, \quad (5.41)$$

where $x = \pm d/2$ is the value of x at the two endpoints of the string. We then have to consider the equation of motion for $u(x)$.

The Nambu–Goto action for the string is given by

$$S_{NG} = \frac{T_0}{2\pi} \int d\sigma d\tau \left[\sqrt{-\det\left[(g_{\mu\nu} + \partial_\mu \phi \partial_\nu \phi)\partial_a X^\mu \partial_b X^\nu\right]} + \epsilon^{ab} \partial_a X^m \partial_b X^n \bar\Theta \Gamma_m \Gamma^{pqs} \Gamma_n \Theta \mathcal{H}_{pqs}\right], \quad (5.42)$$

where $\mathcal{H}_{pqs}$ is the type I RR three-form field strength, T_0 is the fundamental string tension, and $a, b = 1, 2$, $\partial_1 \equiv \frac{\partial}{\partial \tau}$, $\partial_2 \equiv \frac{\partial}{\partial \sigma}$. Note that once we switch off the fermions $\bar\Theta = \Theta = 0$, the RR three-form decouples from the Nambu–Goto action. Furthermore, the absence of type I NS three-form fields simplifies things further. Plugging in Eq. (5.38) in Eq. (5.42), we get

$$S_{NG} = \frac{T_0}{2\pi} \int_{-d/2}^{+d/2} \frac{dx}{u^2} \sqrt{(c_n u^n)^2 + g_l u^l (\partial u/\partial x)^2}, \quad (5.43)$$

with $g_l u^l = b_k u^k +$ dilaton contributions that can be derived from Eqs. (5.42) and (5.38).

Now the analysis is very similar to that performed in Refs. [63–65], so the reader may want to look there for more details. From Refs. [64,65] we know that the string will take a U shape because of symmetry. Letting u_{max} be the maximum value of u, one can then show that for d to be real, we should satisfy Refs. [64,65]:

$$\frac{1}{2}(m+1)c_{m+3}u_{max}^{m+3} \leq 1. \quad (5.44)$$

There will generally be multiple solutions for $u(x)$ that satisfy this bound (and, likely, solutions that saturate it as well). Define $\mathbf{u}_{max}$ to be the largest value of u_{max} attained

by solutions $u(x)$ satisfying the bound (5.44) (which is possibly just the value of u_{max} that saturates the bound), and define $\mathbf{M}$ to be

$$\mathbf{M} \equiv \frac{n^2 c_n \mathbf{u}_{max}^n}{c_m \mathbf{u}_{max}^m} - 4. \quad (5.45)$$

It is then easy to see from Eq. (5.43) that the Nambu–Goto action has the following dominant behavior [64,65]:

$$d = \lim_{\epsilon \to 0} \frac{2\sqrt{g_n \mathbf{u}_{max}^n} \mathbf{u}_{max}}{c_m \mathbf{u}_{max}^m} \frac{\log \epsilon}{\sqrt{\mathbf{M}}},$$

$$S_{NG} = \lim_{\epsilon \to 0} \frac{T_0}{\pi} \frac{\sqrt{g_n \mathbf{u}_{max}^n}}{\mathbf{u}_{max}} \frac{\log \epsilon}{\sqrt{\mathbf{M}}}, \quad (5.46)$$

which means that both d and S_{NG} have identical logarithmic behavior. Thus, the finite quantity is the ratio between the two, i.e., the action per unit length, which is

$$\frac{S_{NG}}{T_0} \equiv V_{Q\bar{Q}} \equiv \left(\frac{c_m \mathbf{u}_{max}^m}{\pi \mathbf{u}_{max}^2}\right) d. \quad (5.47)$$

Then $V_{Q\bar{Q}}$ is the expected confining potential between the quark and the antiquark.

Note that while the above computation tells us where to look for the confining string, it does not tell us how the string breaks into smaller strings. For the latter, we would presumably require more machinery, such as string field theory. Furthermore, the result (5.47) is independent of the UV completion as the details of the UV completion appears on both d and S_{NG} via the coefficient c_m but not via b_l. The details of the full series $\{c_m\}$ appearing in the dilaton (5.38) are not required to study the IR confining behavior. Even if the UV completion is not made, the result (5.47) will continue to hold.

3. New exotic states in LST from the gravity dual

Another interesting direction is to look for exotic states in the dimensionally reduced LST using exotic stable states in the type I and heterotic theories. One candidate is the stable non-BPS D0-brane state in type I $SO(32)$ theory that is constructed by switching on a tachyonic kink solution on a D1–anti-D1 pair in type IIB [66]. The heterotic dual of this construction is the lightest spinor state of the $SO(32)$ theory [66]. Unfortunately, this state was constructed in a flat background, but quantization of open strings in a curved background is not well understood, so it could be difficult to properly analyze the spectrum of a type IIB open string with one endpoint on the D1 and the other on the anti-D1.

What we can try, however, is to go to a regime where the metric is approximately flat.[40] Thus, the configuration of a

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[40] For example, if we define $R \equiv \int_\infty^u \frac{du}{\sqrt{G(u)}}$, where $G(u)$ is as given in Eq. (5.35), then in the local coordinate system $(x^{0,1,2,3}, R)$, the five-dimensional metric is flat. On the other hand, the internal space parametrized by (θ_i, ϕ_i, ψ) in Eq. (5.29), however, is now much more complicated.

D1–anti-D1 pair with one endpoint at the boundary and the other endpoint at $u \to \infty$ in Eq. (5.35) [or in Eq. (5.29)] should appear as a stable non-BPS state in the dual gauge theory. From the heterotic $SO(32)$ point of view, we could refine this even further by taking the nontrivial T^3 fibration of the metric (5.29) and looking for the stability region of this state on a T^3. This has recently been discussed in Ref. [67], where the stability region was shown to satisfy the inequalities

$$16R_a^2 + \frac{1}{R_b^2} + \frac{1}{R_c^2} > 8, \qquad R_a^2 + R_b^2 + R_c^2 > \frac{1}{2}, \quad (5.48)$$

where R_a, R_b, and R_c, are the radii of the three cycles of T^3 measured in units of $\sqrt{\alpha'}$. Thus, in our five-dimensional space, all regimes where the internal tori parametrized by (ϕ_1, ϕ_2, ψ) satisfy the inequalities (5.48) should have corresponding stable non-BPS states that should be visible in the dual gauge theory at all energy scales (including the far IR). The question of what this state corresponds to in the E_8 theory will be discussed elsewhere.

VI. DISCUSSIONS AND CONCLUSIONS

In this paper we have identified gravity duals for little string theories that have been dimensionally reduced to four dimensions—these theories have permanent confinement in the deep infrared. Our constructions focused on heterotic $SO(32)$ and $E_8 \times E_8$ theories, where we wrapped heterotic NS5-branes on a $\mathbf{P}^1$ of a non-Kähler resolved conifold. The form of the metric we considered was

$$ds^2 = ds_{0123}^2 + \frac{2\Delta}{r\sqrt{a_3}} dr^2 + \frac{2\Delta\sqrt{a_3}}{r}(d\psi + \cos\theta_1 d\phi_1$$
$$+ \cos\theta_2 d\phi_2)^2 + (\Delta + a^2)(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2)$$
$$+ \Delta(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2), \quad (6.1)$$

where Δ and a_3 were determined in Sec. IV B, along with the values for the torsion and the dilaton. We also discussed the global and local symmetries in Sec. IV C.

We determined the respective gravity duals through a series of dualities depicted in Figs. 1 and 2. The gravity duals take the following form:

$$ds^2 = ds_{0123}^2 + \frac{2\Delta}{r\sqrt{a_3}} dr^2 + \frac{1}{\tilde{H}_1 + \mathcal{A}}[d\psi + \tilde{\Delta}_1 \cos\theta_1(d\phi_1 + \alpha_1 d\theta_1) + \tilde{\Delta}_2 \cos\theta_2(d\phi_2 + \alpha_2 d\theta_2)]^2$$
$$+ [g_{\theta_1\theta_1} d\theta_1^2 + g_{\phi_1\phi_1}(d\phi_1 + \alpha_1 d\theta_1)^2] + [g_{\theta_2\theta_2} d\theta_2^2 + g_{\phi_2\phi_2}(d\phi_2 + \alpha_2 d\theta_2)^2] + g_{\theta_1\theta_2} d\theta_1 d\theta_2$$
$$+ g_{\phi_1\phi_2}(d\phi_1 + \alpha_1 d\theta_1)(d\phi_2 + \alpha_2 d\theta_2), \quad (6.2)$$

where the components of the metric are given in Sec. V F, along with the other background fields. The gravity duals have noncompact two- and three-cycles and resemble a non-Kähler deformed conifold but with an interesting fibration structure for ϕ_1 and ϕ_2. Also, note that the space-time directions are flat, so the Einstein-frame background does not contain an AdS space (because of a nontrivial dilaton). It will be interesting to see if aspects of the linear dilaton background [68] could be recovered in some regime of parameter space.

It would also be informative to study a proper decoupling limit of our backgrounds and check whether the final result will differ significantly from Eq. (6.2).[41] One possible ansatz for the five-dimensional part of the dual metric, in the decoupling limit, is

$$ds^2 = ds_{0123}^2 + \frac{du^2}{G(u)}, \quad (6.3)$$

where $u = \alpha'/r$ and $G(u)$ can be extracted from $\frac{2\Delta}{r\sqrt{a_3}}$ in Eq. (6.2). The various choices of $G(u)$ will determine the gravity duals for various configurations of LSTs studied here. Note also that the duality sequences given in Figs. 1 and 2 are possibly the only way to determine the gravity duals for the configurations studied in Sec. IV B, in part because our knowledge of the dual gauge theory side is extremely limited. Of course, this means that the gravity

dual is particularly useful as a way to discover interesting information about the gauge theory.

One such interesting piece of information is to look for a candidate confining string in the gauge theory, which we discussed in Sec. V F 2. The most natural candidate seems to be the type I fundamental string, which is prone to breaking into smaller strings, similar to the breaking of a confining string into quark-antiquark pairs. Our simple analysis showed a linear potential between quarks and antiquarks, but a full understanding of the string breaking remains a challenge. Additionally, it is not clear whether the theory that we study using our gravity dual has a simple mass gap, implying that the matter content of the gauge theory is nontrivial, as can also be seen from the possible existence of stable, uncharged, non-BPS states in some region of the moduli space of the gauge theory. A detailed study of this is left for future work.

Another future direction that could improve our understanding of gauge/gravity duality in the context of heterotic

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[41]For example, one of the simplest decoupling limits generically applied is the so-called double scaling limit, where $\alpha' \to 0$, $r \to 0$ such that $u \equiv \alpha'/r$ is kept fixed. Using this we can easily check that the form of Eq. (6.2) does not change. However, we have not verified whether the double scaling limit is effective in decoupling the bulk degrees of freedom from the brane for our case.

string theory stems from the fact that all the background fluxes are from the NS sector. Thus, one might hope to gain a better understanding of the backgrounds, and perhaps even the geometric transition itself, through Ramond-Neveu-Schwarz world sheet techniques, similar to those in Refs. [4,69,70], where a nonperturbative (in α') understanding of a class of non-Kähler backgrounds was given. We hope to return to this in the future.

APPENDIX A: EXPLICIT FORMS FOR $\mathcal{H}$ AND dJ

In this section, we will work out the explicit forms for $\mathcal{H}$ that we used in earlier sections. One can also see from here that $dJ \neq 0$. The expression for $\mathcal{H}$ is given by

$$\mathcal{H} = -e^{2\phi} * d(e^{-2\phi}J) = 2 * d\phi \wedge J - *dJ. \quad \text{(A1)}$$

The term $*dJ$ has the following form:

$$
\begin{aligned}
*dJ = {} & \frac{(H_3 c_1)_r + \sqrt{H_1 H_2} c_{4\theta_1}}{H_3 \sqrt{H_2} c_1} e_3 \wedge e_4 \wedge e_5 + \frac{(H_4 c_5)_r + \sqrt{H_1 H_2} c_{7\theta_2}}{H_4 \sqrt{H_2} c_5} e_5 \wedge e_1 \wedge e_2 \\
& - \frac{(H_3 c_1)_{\theta_2}}{\sqrt{H_4} H_3 c_1} (\cos \psi_2 e_4 \wedge e_5 \wedge e_6 + \sin \psi_2 e_5 \wedge e_6 \wedge e_3) - \frac{(H_1 H_2)_{\theta_2}}{2 H_1 H_2 \sqrt{H_4}} (\cos \psi_2 e_1 \wedge e_2 \wedge e_4 + \sin \psi_2 e_1 \wedge e_2 \wedge e_3) \\
& + \frac{\sqrt{H_1} c_{4\theta_2}}{\sqrt{H_3 H_4} c_1} (\cos \psi_2 \sin \psi_1 e_2 \wedge e_4 \wedge e_5 - \cos \psi_1 \cos \psi_2 e_1 \wedge e_4 \wedge e_5 + \sin \psi_1 \sin \psi_2 e_2 \wedge e_3 \wedge e_5 \\
& - \sin \psi_2 \cos \psi_1 e_1 \wedge e_3 \wedge e_5),
\end{aligned}
\quad \text{(A2)}
$$

while the first term, involving the dilaton ϕ, has the following form:

$$
\begin{aligned}
*d\phi \wedge J = {} & \left(\frac{\phi_{\phi_1} \sin \psi_1}{\sqrt{H_3} c_1} + \frac{\phi_{\theta_1} \cos \psi_1}{\sqrt{H_3}} - \frac{\phi_\psi \sin \psi_1 c_4}{\sqrt{H_1 H_3} c_1} \right) e_2 \wedge (e_5 \wedge e_6 + e_3 \wedge e_4) \\
& + \left(\frac{\phi_{\phi_1} \cos \psi_1}{\sqrt{H_3} c_1} - \frac{\phi_{\theta_1} \sin \psi_1}{\sqrt{H_3}} - \frac{\phi_\psi \cos \psi_1 c_4}{\sqrt{H_1 H_3} c_1} \right) e_1 \wedge (e_5 \wedge e_6 + e_3 \wedge e_4) \\
& + \left(\frac{\phi_{\phi_2} \sin \psi_2}{\sqrt{H_4} c_5} + \frac{\phi_{\theta_2} \cos \psi_2}{\sqrt{H_4}} - \frac{\phi_\psi \sin \psi_2 c_7}{\sqrt{H_1 H_4} c_5} \right) e_4 \wedge (e_5 \wedge e_6 + e_1 \wedge e_2) \\
& + \left(\frac{\phi_{\phi_2} \cos \psi_2}{\sqrt{H_4} c_5} - \frac{\phi_{\theta_2} \sin \psi_2}{\sqrt{H_4}} - \frac{\phi_\psi \cos \psi_2 c_7}{\sqrt{H_1 H_4} c_5} \right) e_3 \wedge (e_1 \wedge e_2 - e_5 \wedge e_6) + \frac{\phi_\psi}{\sqrt{H_1}} e_6 \wedge (e_1 \wedge e_2 - e_3 \wedge e_4) \\
& + \frac{\phi_r}{\sqrt{H_2}} e_5 \wedge (e_1 \wedge e_2 + e_3 \wedge e_4).
\end{aligned}
\quad \text{(A3)}
$$

These are the components of $\mathcal{H}_{\text{NS}}$ presented in Eq. (4.14).

APPENDIX B: BACKGROUND SOLUTIONS USING GEOMETRIC TRANSITIONS

Our starting solution in the heterotic theory before geometric transition, i.e., with wrapped heterotic 5-branes, takes the following form:

$$ds^2 = ds_{0123}^2 + H_2 dr^2 + H_1(d\psi + c_4 d\phi_1 + c_7 d\phi_2)^2 + H_3(d\theta_1^2 + c_1^2 d\phi_1^2) + H_4(d\theta_2^2 + c_5^2 d\phi_2^2), \quad \text{(B1)}$$

where we have not taken the decoupling limit. For the duality chain that we will follow, taking the decoupling limit at this stage is not necessary. The metric (B1) contains the backreactions of the wrapped 5-branes. The background also has a nontrivial dilaton ϕ; a three-form $\mathcal{H}$ given by

$$
\begin{aligned}
\mathcal{H} = {} & (2B_6 - A_1)H_4\sqrt{H_1}c_5 d\theta_2 \wedge d\phi_2 \wedge d\psi - ((A_1 - 2B_6)H_4\sqrt{H_1}c_5 c_4 - 2(B_1\cos\psi_1 + B_2\sin\psi_1)H_4\sqrt{H_3}c_1 c_5)d\theta_2 \\
& \wedge d\phi_2 \wedge d\phi_1 - (A_2 - 2B_6)H_3\sqrt{H_1}c_1 d\theta_1 \wedge d\phi_1 \wedge d\psi - ((A_2 - 2B_6)H_3\sqrt{H_1}c_1 c_7 \\
& - (A_4 + 2B_3\cos\psi_2 - 2B_4\sin\psi_2)H_3\sqrt{H_4}c_1 c_5)d\theta_1 \wedge d\phi_1 \wedge d\phi_2 - 2\sqrt{H_1 H_2 H_4}(B_4\cos\psi_2 + B_3\sin\psi_2)d\theta_2 \\
& \wedge (d\psi + c_4 d\phi_1 + c_7 d\phi_2) \wedge dr - 2\sqrt{H_1 H_2 H_4}(B_4\sin\psi_2 + B_3\cos\psi_2)c_5 d\phi_2 \wedge (d\psi + c_4 d\phi_1) \wedge dr \\
& - 2\sqrt{H_1 H_2 H_3}(B_1\sin\psi_1 - B_2\cos\psi_1)d\theta_1 \wedge (d\psi + c_4 d\phi_1 + c_7 d\phi_2) \wedge dr \\
& + 2\sqrt{H_1 H_2 H_3}(B_1\cos\psi_1 - B_2\sin\psi_1)c_1 d\phi_1 \wedge (d\psi + c_7 d\phi_2) \wedge dr - 2H_3\sqrt{H_4}(B_3\cos\psi_2 - B_3\sin\psi_2)c_1 d\theta_1 \\
& \wedge d\theta_2 \wedge d\phi_1 - 2H_4\sqrt{H_3}(B_1\sin\psi_1 - B_2\cos\psi_1)c_1 d\theta_1 \wedge d\theta_2 \wedge d\phi_2 + B_5\sqrt{H_2}dr \\
& \wedge (c_1 d\theta_1 \wedge d\phi_1 - c_5 d\theta_2 \wedge d\phi_2) - \frac{H_1 c_5 c_{4\theta_2}}{c_1}d\theta_1 \wedge d\phi_2 \wedge (d\psi + c_4 d\phi_1 + c_7 d\phi_2),
\end{aligned} \tag{B2}
$$

where the coefficients A_i's and B_i's appearing in Eq. (B2) are given in Eqs. (4.15) and (4.16); and a non-Abelian vector bundle F, for which the simple $U(2)$ form is given as

$$
F = (\mathfrak{f}_1 e_1 \wedge e_2 + \mathfrak{f}_2 e_3 \wedge e_4 + \mathfrak{f}_3 e_5 \wedge e_6)I + (\mathfrak{f}_4 e_1 \wedge e_2 + \mathfrak{f}_5 e_3 \wedge e_4)\sigma^1, \tag{B3}
$$

where σ^1 is the first Pauli matrix and $\mathfrak{f}_i$'s are given in Eq. (4.88). One may note that a simplified form of the above background can be presented as Eq. (5.1), where the coefficients of the metric (B1), the three-form flux (B2), and the dilaton ϕ take a specific form. The values of these coefficients can be inserted in Eq. (4.88) to determine the vector bundle for the background (5.1).

For the simple case of Eq. (5.1), one may determine the metric for the gravity dual in the heterotic theory. The full set of transformations and the duality chain required to describe the final result is depicted in details in the text. The result is

$$
\begin{aligned}
ds^2 = {} & ds^2_{0123} + \frac{2\Delta}{r\sqrt{a_3}}dr^2 + \frac{1}{\tilde{H}_1 + \mathcal{A}}[d\psi + \tilde{\Delta}_1\cos\theta_1(d\phi_1 + \alpha_1 d\theta_1) + \tilde{\Delta}_2\cos\theta_2(d\phi_2 + \alpha_2 d\theta_2)]^2 \\
& + [g_{\theta_1\theta_1}d\theta_1^2 + g_{\phi_1\phi_1}(d\phi_1 + \alpha_1 d\theta_1)^2] + [g_{\theta_2\theta_2}d\theta_2^2 + g_{\phi_2\phi_2}(d\phi_2 + \alpha_2 d\theta_2)^2] + g_{\theta_1\theta_2}d\theta_1 d\theta_2 \\
& + g_{\phi_1\phi_2}(d\phi_1 + \alpha_1 d\theta_1)(d\phi_2 + \alpha_2 d\theta_2).
\end{aligned} \tag{B4}
$$

Note that there is no coefficient in front of the space-time part. The coefficient, which appears in Eq. (5.25), is in fact the dilaton in the type I/heterotic frame and is related to the dilaton $\phi_{(c)}$ in the type I′ frame by mirror/U-duality transformation. Therefore, in the heterotic frame we see no factor of dilaton in front of the space-time part.

Our claim then is that the gravity dual of the wrapped heterotic 5-branes in the heterotic $E_8 \times E_8$ theory can be extracted from Eq. (B4) by performing a geometric transition on the metric (B1) or, in the simplified form, Eq. (4.8). In the decoupling limit of Eq. (4.8) and (B4) will provide the precise gravity dual. Defining $u = \frac{\alpha'}{r}$, we see that the five-dimensional part of Eq. (B4) takes the following form:

$$
ds^2 = ds^2_{0123} + \frac{du^2}{G(u)}, \tag{B5}
$$

where $G(u)$ can be extracted from $\frac{2\Delta}{r\sqrt{a_3}}$ in Eq. (B4) in the decoupling limit. The space-time part is flat, so we conclude that, in the string frame, a flat Minkowski space is dual to LST compactified to four dimensions on a two-cycle of the non-Kähler resolved conifold. Additionally, irrespective of the details of the internal space, the five-dimensional dual metric will always be of the form (B5).

APPENDIX C: TORSIONAL CONNECTION Ω^m_{np}

The connection that we use throughout this paper is the so-called +-connection, constructed from the Christoffel symbol and $\mathcal{H}$ as

$$
\Omega^\rho_{+\,\mu\nu} \equiv \Omega^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} + \frac{1}{2}\mathcal{H}^\rho_{\mu\nu}. \tag{C1}
$$

The various components of $\Omega^\rho_{\mu\nu}$ are given as follows:

$$\Omega^r_{\psi\psi} = -\frac{H_1'}{2\sqrt{H_2}\sqrt{H_1}}, \qquad \Omega^\psi_{\psi r} = \frac{H_1'}{2\sqrt{H_2}\sqrt{H_1}}, \qquad \Omega^{\phi_1}_{\psi\theta_1} = \frac{1}{2}B\sqrt{H_1} - \frac{H_1}{2H_3}, \qquad \Omega^{\phi_2}_{\psi\theta_2} = \frac{1}{2}A\sqrt{H_1} - \frac{H_1}{2H_4},$$

$$\Omega^{\theta_1}_{\psi\phi_1} = \frac{H_1 - B\sqrt{H_1}H_3}{2H_3}, \qquad \Omega^{\theta_2}_{\psi\phi_2} = \frac{H_1 - A\sqrt{H_1}H_4}{2H_4}, \qquad \Omega^r_{\phi_1\psi} = -\frac{\cos\theta_1 H_1'}{2\sqrt{H_2}\sqrt{H_1}}, \qquad \Omega^r_{\phi_1\phi_1} = -\frac{\sin\theta_1 H_3'}{2\sqrt{H_2}\sqrt{H_3}},$$

$$\Omega^\psi_{\phi_1 r} = \frac{\cos\theta_1 H_1'}{2\sqrt{H_2}\sqrt{H_1}}, \qquad \Omega^\psi_{\phi_1\theta_1} = -\frac{(\sqrt{H_1}+BH_3)\sin\theta_1}{2\sqrt{H_3}}, \qquad \Omega^\psi_{\phi_1 r} = \frac{\sin\theta_1 H_3'}{2\sqrt{H_2}\sqrt{H_3}}, \qquad \Omega^{\phi_1}_{\phi_1\theta_1} = \frac{1}{2}\cos\theta_1\left(2 + B\sqrt{H_1} - \frac{H_1}{H_3}\right),$$

$$\Omega^{\phi_2}_{\phi_1\theta_2} = \frac{1}{2}A\cos\theta_1\sqrt{H_1} - \frac{\cos\theta_1 H_1}{2H_4}, \qquad \Omega^{\theta_1}_{\phi_1\psi} = \frac{(\sqrt{H_1}+BH_3)\sin\theta_1}{2\sqrt{H_3}}, \qquad \Omega^{\theta_1}_{\phi_1\phi_1} = \frac{\cos\theta_1(H_1 - 2H_3 - B\sqrt{H_1}H_3)}{2H_3},$$

$$\Omega^{\theta_2}_{\phi_1\phi_2} = -\frac{1}{2}A\cos\theta_1\sqrt{H_1} + \frac{\cos\theta_1 H_1}{2H_4}, \qquad \Omega^r_{\phi_2\psi} = -\frac{\cos\theta_2 H_1'}{2\sqrt{H_2}\sqrt{H_1}}, \qquad \Omega^r_{\phi_2\phi_2} = -\frac{\sin\theta_2 H_4'}{2\sqrt{H_2}\sqrt{H_4}},$$

$$\Omega^\psi_{\phi_2 r} = \frac{\cos\theta_2 H_1'}{2\sqrt{H_2}\sqrt{H_1}}, \qquad \Omega^\psi_{\phi_2\theta_2} = -\frac{(\sqrt{H_1}+AH_4)\sin\theta_2}{2\sqrt{H_4}} \qquad \Omega^{\phi_1}_{\phi_2\theta_1} = \frac{1}{2}B\cos\theta_2\sqrt{H_1} - \frac{\cos\theta_2 H_1}{2H_3},$$

$$\Omega^{\phi_2}_{\phi_2 r} = \frac{\sin\theta_2 H_4'}{2\sqrt{H_2}\sqrt{H_4}}, \qquad \Omega^{\phi_2}_{\phi_2\theta_2} = \frac{1}{2}\cos\theta_2\left(2 + A\sqrt{H_1} - \frac{H_1}{H_4}\right), \qquad \Omega^\psi_{\theta_1\phi_1} = \frac{\sqrt{H_1}+BH_3}{2\sqrt{H_3}}$$

$$\Omega^{\theta_1}_{\phi_2\phi_1} = -\frac{1}{2}B\cos\theta_2\sqrt{H_1} + \frac{\cos\theta_2 H_1}{2H_3}, \qquad \Omega^{\theta_2}_{\phi_2\psi} = \frac{(\sqrt{H_1}+AH_4)\sin\theta_2}{2\sqrt{H_4}} \qquad \Omega^{\theta_2}_{\phi_2\phi_2} = \frac{\cos\theta_2(H_1 - 2H_4 - A\sqrt{H_1}H_4)}{2H_4},$$

$$\Omega^r_{\theta_1\theta_1} = -\frac{H_3'}{2\sqrt{H_2}\sqrt{H_3}} \qquad \Omega^{\phi_1}_{\theta_1\psi} = \frac{-\sqrt{H_1}-BH_3}{2\sqrt{H_3}}, \qquad \Omega^{\theta_1}_{\theta_1 r} = \frac{H_3'}{2\sqrt{H_2}\sqrt{H_3}}, \qquad \Omega^r_{\theta_2\theta_2} = -\frac{H_4'}{2\sqrt{H_2}\sqrt{H_4}}$$

$$\Omega^\psi_{\theta_2\phi_2} = \frac{\sqrt{H_1}+AH_4}{2\sqrt{H_4}}, \qquad \Omega^{\phi_2}_{\theta_2\psi} = \frac{-\sqrt{H_1}-AH_4}{2\sqrt{H_4}}, \qquad \Omega^{\theta_2}_{\theta_2 r} = \frac{H_4'}{2\sqrt{H_2}\sqrt{H_4}}. \tag{C2}$$

APPENDIX D: PROOFS FOR EQS. (4.30) AND (4.32)

Let us now clarify the functional forms for the dilaton and h_1 that we used in Eq. (4.30). In the process we will also justify Eq. (4.32).

A simple dimensional analysis will tell us that all three warp factors in Eq. (4.6), namely, $h_1 a_3$, a_2, and $h_1 a_5$, are proportional to the square of the radius r^2.[42] Additionally, the generic form for the heterotic metric (4.6) should be[43]

$$ds^2 = ds^2_{0123} + e^\phi ds^2_{\text{internal}}, \tag{D1}$$

where the 5-branes wrap the space-time $x^{0,1,2,3}$ and the two-cycle (θ_1, ϕ_1), and ds^2_{internal} is the internal non-Kähler resolved conifold metric that we will typically demand takes the following form:

$$ds^2_{\text{internal}} \equiv \frac{dr^2}{\mathcal{F}_3(r)} + \frac{r^2\mathcal{F}_3(r)}{4}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \frac{r^2 + a^2 e^{-\phi}}{4}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2)$$

$$+ \frac{r^2}{4}(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2), \tag{D2}$$

where $\mathcal{F}_3(r)$ will turn out to be $\frac{2\sqrt{a_3}}{r}$, a fact that will be justified below. Thus, if we isolate the metric of the two-sphere on which we will have the wrapped 5-branes from Eq. (D2), we will get the metric of the four-dimensional internal space, which is locally an ALE space. This means, looking at Eqs. (4.6) and (4.25), that we find $h_1 a_5 = \Delta$, so using Eq. (D1) and dimensional analysis, the dilaton becomes

$$e^\phi = \frac{4\Delta}{r^2} + \mathcal{O}(a^2), \tag{D3}$$

where we have inserted a factor of 4 just for convenience to comply with the non-Kähler resolved conifold metric (D2) that we consider here. Interestingly, one may also verify Eq. (D3) using the torsion classes that we derived earlier. Using the normalization of the dilaton factor that we consider here, the relation is $d\phi = 2W_4$, as discussed in Footnote 15. Thus, from the W_4 computed in Eq. (4.18), we get the following equation for the dilaton:

$$\phi(r) = \frac{1}{2} \int dr \left(1 + \frac{\Delta}{\Delta + a^2}\right) \frac{d}{dr} \log\left(\frac{4\Delta}{r^2}\right), \quad r > 0. \tag{D4}$$

In the limit that a^2 is much smaller than other scales, we immediately reproduce Eq. (D3). For finite a^2, Eq. (D4) can give us the correction to the dilaton due to the resolution parameter of the underlying non-Kähler resolved conifold metric.

From the dimensional analysis, since the dilaton (D3) is the simplest possible ansatz, one might ask whether a more generic ansatz, with a dimensionless function $f_0(r)$, could be envisioned for the dilaton, i.e.,

$$e^\phi = \frac{4\Delta}{r^2 f_0(r)}, \qquad f_0(r) \equiv 1 + \sum_{n \geq 1} c_n \left(\frac{r}{\sqrt{\alpha'}}\right)^n + \sum_{m \geq 1} d_m \left(\frac{\sqrt{\alpha'}}{r}\right)^m. \tag{D5}$$

Without doing any computations, we might say that $c_n = 0$ because for large r, where the effect of the wrapped heterotic 5-branes is negligible, we expect the standard metric of a resolved conifold. The rest of f_0 could then be absorbed into the definition of Δ, implying that the form of the dilaton (D3) is generic enough. The above conclusions are *almost* true, but we will consider a case where positive powers of r will sum up asymptotically to yield a result that will behave as *inverse* powers of r as r tends to a large value. Furthermore, the underlying manifold may not exactly become a Calabi–Yau resolved conifold at large r. Despite these subtleties, we can generically absorb $f_0(r)$ into the definition of the dilaton as long as we have good asymptotic behavior. This is also consistent with the torsion class analysis (D4).

The dimensional analysis now tells us that the coefficient a_2 in Eq. (4.6) should be $a_2 \equiv \frac{e^\phi r^2}{4} + a^2$, which is of course consistent with $a_2 = \Delta + a^2$ in Eq. (4.36). The other two coefficients in Eq. (4.6) now take the form

$$h_1 a_3 = r^2 e^\phi f_1(r), \qquad h_1 = e^\phi f_2(r), \tag{D6}$$

where $f_1(r)$ and $f_2(r)$ are dimensionless functions of r. Only a_3 could carry a dimensionful factor of r^2. The question now is: how are f_1 and f_2 related?

To find the relation between f_1 and f_2, let us go back to the metric ansatz (4.20), where a_3 first appears. Dimensionally, a_3 should be r^2, so we can only attach a dimensionless factor to it. Motivated by the form of the other term in Eq. (4.20), let us then make the following choice for a_3:

$$a_3 \equiv 4r^2 f_1^2, \tag{D7}$$

where again we put a factor of 4 for later convenience. Plugging this into Eq. (D6), we immediately get the required value for h_1 in Eq. (4.30) and

$$f_1 = \frac{1}{4f_2} = \frac{\sqrt{a_3}}{2r}. \tag{D8}$$

Thus, everything can be expressed either in terms of Δ or a_3 or both, provided Eq. (D7) holds. Of course, this is not the only ansatz since we can always express a_3 as a function proportional to $r^2 f_1^n$ or to $r^2 g_1^m$ with (n, m) being

any number (integer or fractional). However, in this paper we will stick with the simplest choice (D7).

APPENDIX E: SUPERSYMMETRY CONDITION REVISITED

The supersymmetry condition for the conventions in this paper is Eq. (4.19), i.e., $2W_4 = W_5$, as our choice of the dilaton is minus the choice of the dilaton in Ref. [41].

However, it is interesting to see, just for the sake of curiosity, how the backgrounds change if we take the supersymmetry condition to be $2W_4 = -W_5$, without worrying about the sign of the dilaton. To be concrete, let us just consider the scenario depicted in case I. The equation for a_3 changes from Eq. (4.42) to the following:

$$\frac{da_3}{dr} - \frac{\sqrt{a_3}}{4} + \frac{5a_3}{12r}\left(\frac{17r^2 + 8e_0}{r^2 + e_0}\right) + \mathcal{O}(a^2) = 0. \tag{E1}$$

This can again be exactly solved, and the solution for a_3 will become

$$a_3(r) = \left(\frac{3}{6649}\right)^2$$
$$\times \frac{[45e_0^2 + 106e_0 r^2 + 61r^4 - 45e_0^2 \left(\frac{e_0 + r^2}{e_0}\right)^{1/16} \mathcal{J}_{1,2}]^2}{r^2 (e_0 + r^2)^2}, \tag{E2}$$

which can be compared to Eq. (4.44). The hypergeometric function now, denoted $\mathcal{J}_{1,2}$, is

$$\mathcal{J}_{1,2} \equiv {}_2F_1\left(\frac{1}{16}, \frac{1}{3}, \frac{4}{3}, -\frac{r^2}{e_0}\right). \tag{E3}$$

The behavior of a_3 at the origin and at infinity is

$$a_3|_{r \to 0} = \frac{9r^2}{4096} - \frac{405r^4}{229376e_0} + \mathcal{O}(r^6),$$
$$a_3|_{r \to \infty} = \frac{9r^2}{11881} + \frac{810e_0}{724741} + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{E4}$$

Having obtained a_3, we can now work out Δ from the warp factor. This is easy to write down since a_3 is expressible again as a square [similar to the scenario we encountered in Eq. (4.46)]. The functional form for Δ then is
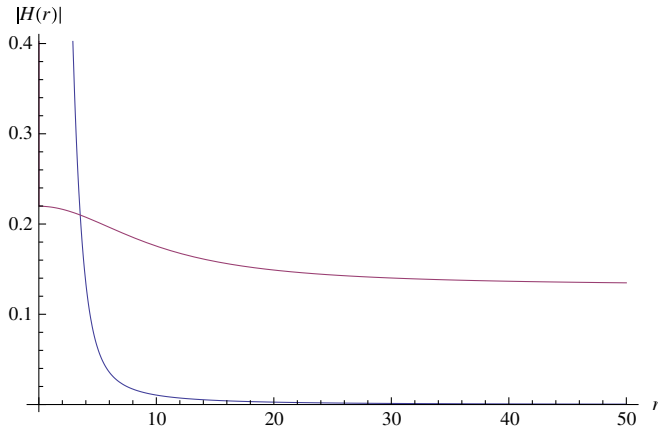
FIG. 14 (color online). The plot of the torsion coefficient $|H(r)|$ for $r \to 0$ and the intermediate region. Near the origin the torsion blows up, signifying the presence of the sources, as depicted by the blue curve using Eq. (E7). For large r the story is captured by the red curve plotted using Eq. (E1). Notice that for $r \to \infty$, the torsion becomes finite to maintain zero energy in the system (4.13) as for the three cases studied earlier. We have chosen $e_0 = 100$ in units of α'.

$$\Delta(r) = \frac{3}{13298r^2}\left[45e_0^2 + 106e_0r^2 + 61r^4 - 45e_0^2\left(\frac{e_0 + r^2}{e_0}\right)^{1/16}\mathcal{J}_{1,2}\right], \quad \text{(E5)}$$

which immediately tell us that the boundary values for this will be

$$\Delta|_{r\to\infty} = \frac{3r^2}{218} + \frac{159e_0}{6649} + \mathcal{O}\left(\frac{1}{r^2}\right),$$
$$\Delta|_{r\to 0} = \frac{3e_0}{128} + \frac{201r^2}{14336} + \mathcal{O}(r^4), \quad \text{(E6)}$$

consistent with the parallel story in Eq. (4.46), although the numerical coefficients are different from Eq. (4.47).

The picture is not complete until we figure out the $r \to 0$ behavior as the small r behavior in Eq. (E6) does not take into account the subtlety that the wrapped 5-branes behave like 3-branes. Therefore, as before, near the origin, we will assume that the warp factor is given by $1 + \alpha'e_0/r^4$ with $\alpha' = 1$. The a_3 equation now changes from Eq. (E1) to

$$\frac{da_3}{dr} - \frac{\sqrt{a_3}}{4} + \frac{5a_3}{12r}\left(\frac{17r^4 - e_0}{r^4 + e_0}\right) + \mathcal{O}(a^2) = 0, \quad \text{(E7)}$$

mirroring the change from Eq. (4.42) to Eq. (4.50). The solution for a_3 now is similar to Eq. (E2):

$$a_3(r) = \left(\frac{3r}{2071}\right)^2\frac{[19(e_0 + r^4) + 90e_0(1 + \frac{r^4}{e_0})^{1/16}\tilde{\mathcal{J}}_{1,2}]^2}{(e_0 + r^4)^2}, \quad \text{(E8)}$$

with $\tilde{\mathcal{J}}_{1,2}$ being another hypergeometric function. On the other hand, Δ becomes

$$\Delta(r) = \left(\frac{3}{4142r}\right)\left[19(e_0 + r^4) + 90e_0\left(1 + \frac{r^4}{e_0}\right)^{1/16}\tilde{\mathcal{J}}_{1,2}\right], \quad \text{(E9)}$$

which is similar to Eq. (E5), as one might have expected. We now only care about the $r \to 0$ behaviors of a_3 and Δ since the large r behavior should be captured by Eqs. (E2) and (E5), respectively. They are given by

$$a_3 \to \frac{9r^2}{361} - \frac{324r^6}{8303e_0} + \frac{2142936r^{10}}{40294459} + \mathcal{O}(r^{14}),$$
$$\Delta \to \frac{3e_0}{38r^2} + \frac{15r^2}{874} - \frac{162r^6}{92207e_0} + \mathcal{O}(r^{10}). \quad \text{(E10)}$$

One final computation needed to complete the story is to determine the torsions $\mathcal{H}$ for both the cases (E1) and (E7). Near $r = 0$, the torsion is given by

$$\mathcal{H} = \frac{9}{34312328e_0r^2}\left(\frac{e_0}{e_0 + r^4}\right)^{31/16}\left[361e_0^2\left(1 + \frac{r^4}{e_0}\right)^{31/16}\right.$$
$$+ 285(121e_0^2 + 230e_0r^4 + 109r^8)\tilde{\mathcal{J}}_{1,2}$$
$$\left.+ 1350e_0\left(1 + \frac{r^4}{e_0}\right)^{1/16}(115e_0 + 109r^4)\tilde{\mathcal{J}}_{1,2}^2\right]$$
$$\times (\Omega_1 + \Omega_2) \wedge e_\psi. \quad \text{(E11)}$$

This is plotted as the blue curve in Fig. 14. The behavior is no different from the three cases studied earlier. This confirms the statement that we made earlier: we can go from one convention to another by redefinitions of the variables in the problem.

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