Nonlinear superposition law and Skyrme crystals

Fabrizio Canfora[1,2,*]

[1]*Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia, Chile*

[2]*Universidad Andres Bello, Avenida Republica 440, Santiago, Chile*

(Received 8 July 2013; published 25 September 2013)

Exact configurations of the four-dimensional Skyrme model are presented. The static configurations have the profile. Which behaves as a kink and, consequently, the corresponding energy-momentum tensor describes a domain wall. Furthermore, a class of exact time-periodic Skyrmions is discovered. Within such a class, it is possible to disclose a remarkable phenomenon that is a genuine effect of the Skyrme term. For a special value of the frequency the Skyrmions admit a nonlinear superposition principle. One can combine two or more exact ‘‘elementary’’ Skyrmions (which may depend in a nontrivial way on all the spacelike coordinates) into a new exact composite Skyrmion. Because of such a superposition law, despite the explicit presence of nonlinear effects in the energy-momentum tensor, the interaction energy between the elementary Skyrmions can be computed exactly. The relations with the appearance of Skyrme crystals are discussed.

DOI: 10.1103/PhysRevD.88.065028 PACS numbers: 12.39.Dc, 05.45.Yv, 11.10.Lm, 11.27.+d

I. INTRODUCTION

The Skyrme theory is one of the most important models of theoretical physics due to its wide range of applications. Skyrme [1] introduced his famous term into the action of the nonlinear sigma model to allow the existence of static soliton solutions with finite energy, called *Skyrmions* (see [2–4] for detailed pedagogical reviews). A further important property of the Skyrme theory is that excitations around Skyrme solitons may represent fermionic degrees of freedom (see the detailed analysis in [5] and references therein) suitable to describe nucleons (nice examples are [6–9]). Besides the great importance of Skyrmions in particles, nuclear physics, and astrophysics (see, for instance, [10] and references therein), many intriguing observations such as [11–19] show that Skyrmions play a very important role also in Bose-Einstein condensates, nematic liquids, multiferric materials, chiral magnets, and condensed matter physics in general. However, in condensed matter systems (such as the ones considered in [17]), Skyrmions are stabilized by specific interactions imposed by handedness of the underlying structure (for details, see [18,19]). This mechanism is absent in the original Skyrme model and, consequently, as it is well known it is extremely difficult to construct analytical configurations in that case. Another issue that makes the task to construct nontrivial analytical configurations of the original Skyrme model particularly difficult is the fact that, unlike what happens, for instance, in the case of monopoles and instantons in Yang-Mills-Higgs theory (see, for instance, [3,4,20]), the Skyrme-Bogomol'nyi-Prasad-Smmerfield (BPS) bound on the energy (which was derived by Skyrme himself well before the modern formulation of soliton theory became available) cannot be saturated for nontrivial spherically symmetric Skyrmions. Nevertheless, in [21] it has been shown that the Skyrme model supports knotted excitations (but they are not known analytically) as it also happens in Yang-Mills theory (see, in particular, [22]).

Among the many open problems of the Skyrme theory (in which exact configurations are very rare[1]) a fundamental theoretical challenge is to construct exact configurations in which the nonlinear effects are explicitly present. This would shed considerable light on the peculiar interactions of the Skyrme model. A further theoretical problem is to explain how, in a nonlinear and nonintegrable theory such as the Skyrme model in four dimensions, nice crystal-like structures (see [24] and references therein) are able to appear. These beautiful configurations, which have been constructed numerically, look almost like nontrivial superpositions of ‘‘elementary solutions’’ but, at first glance, it appears to be impossible to accommodate both nonlinear effects and (any sort of) superposition law.

One often adopts suitable ansatz to make the field equations more tractable. Until recently, the only ansatz able to reduce the complexity of Skyrme field equations is the hedgehog ansatz for spherically symmetric systems and its *rational map* generalization[2] in [25]. However, the field equations are still unsolvable in these cases, since the corresponding Skyrme-BPS bound cannot be saturated.

[1]One of the very few examples (which is not of the type analyzed in the present paper) of exact solutions of the Skyrme field equation is in [23]. Because of the nonlinearities of the theory, explicit solutions are available only under severe simplifying assumptions.

[2]In both cases (the usual spherical hedgehog and the rational map ansatz), it is necessary that the profile of the hedgehog depends on the radius, and, consequently, it is not easy to depart from spherical symmetry.

*canfora@cecs.cl

Many modifications of the Skyrme theory have been proposed (see, for instance, [26–31] and references therein), which change the theory in such a way that suitable bounds on the energy that can be saturated become available. All these models have many interesting mathematical features, and their importance goes beyond the simplification of the original Skyrme theory. However, all such modifications spoil the original motivations of the Skyrme theory whose form is compelling from both phenomenological and theoretical points of view.

In the present paper, it will be shown that using the generalized hedgehog ansatz introduced in [32,33] (which allows one to study systems without spherical symmetry) one can construct exact configurations of the four-dimensional Skyrme theory in its original form that may be static or periodic in time (but other topologies can be considered as well). Within the time-periodic configurations, a remarkable resonance phenomenon emerges: for a special value of the frequency, a nonlinear superposition law appears that allows one to combine two or more "elementary" Skyrmions[3] (whose profile depends in a nontrivial way on all the spacelike coordinates) into a new exact composite Skyrmion. In the present case, despite the explicit presence of nonlinear effects, the interaction energy between the moduli of N elementary Skyrmions *can be computed exactly for any* N. The relations with the appearance of Skyrme crystals is an intriguing property of the present construction.

This paper is organized as follows: in Sec. II, the generalized hedgehog will be described and an exact kink of the Skyrme system will be constructed. In Sec. III, exact time-periodic Skyrmions will be analyzed, and it will be shown how a nonlinear superposition law arises. In Sec. IV will be discussed a mapping from periodic Skyrmions in which the profile is static and the internal orientation is time-periodic to periodic Skyrmions in which the profile depends on time while the internal orientation is periodic in space. In Sec. V, some conclusions will be drawn.

II. GENERALIZED HEDGEHOG AND AN EXACT KINK

The action of the $SU(2)$ Skyrme system in four-dimensional flat (but topologically nontrivial) space-times is

$$S_{\rm Skyrme} = \frac{K}{2}\int d^4x\sqrt{-g}\,{\rm Tr}\left(\frac{1}{2}R^\mu R_\mu + \frac{\lambda}{16}F_{\mu\nu}F^{\mu\nu}\right),$$
$$K>0, \qquad \lambda>0,$$
$$R_\mu := U^{-1}\nabla_\mu U = R^i_\mu t_i,$$
$$F_{\mu\nu} := [R_\mu, R_\nu], \qquad \hbar = 1, \qquad c=1, \tag{1}$$

[3]It is worthwhile to emphasize here that, often, in the literature the term "Skyrmion" is used only to describe configurations with nonvanishing winding numbers. In the following, this term will be used in a broader sense to denote exact solutions of the original four-dimensional Skyrme theory in which nonlinear effects are explicitly present.

where the Planck constant and the speed of light have been set to 1, the coupling constants K and λ are fixed by comparison with experimental data, and the t^i are the basis of the $SU(2)$ generators t^i (where the Latin index i corresponds to the group index).

A mass term can also be included: however, it is well known (see [34] and references therein) that when the mass term is nonvanishing, the theory is considered in $1+1$ dimensions and the group is nontrivial solution corresponding to the Abelian $U(1)$ group saturating suitable BPS bounds appear. Here it will be shown that this is true even *without a mass term* in $3+1$ dimensions and with the $SU(2)$ internal symmetry group. Furthermore, the original action[4] in Eq. (1) without the mass term (besides being a quite good approximation for pion physics when the pion mass can be assumed to be small) is extremely useful in applications in condensed matter physics (see, for instance, [12,14]).

The Skyrme field equations are

$$\nabla^\mu R_\mu + \frac{\lambda}{4}\nabla^\mu[R^\nu, F_{\mu\nu}] = 0. \tag{2}$$

The following standard parametrization of the $SU(2)$-valued scalar $U(x^\mu)$ will be adopted:

$$U^{\pm 1}(x^\mu) = Y^0(x^\mu)\mathbf{1} \pm Y^i(x^\mu)t_i, \quad (Y^0)^2 + Y^iY_i = 1. \tag{3}$$

The generalized hedgehog ansatz [32,33] reads

$$Y^0 = \cos\alpha, \qquad Y^i = \hat{n}^i\sin\alpha, \tag{4}$$

$$\hat{n}^1 = \cos\Theta, \qquad \hat{n}^2 = \sin\Theta,$$
$$\hat{n}^3 = 0, \qquad (\nabla_\mu\alpha)(\nabla^\mu\Theta) = 0, \tag{5}$$

where α is the profile of the hedgehog while the function Θ, through the internal vector $\hat{n}^i$, describes its orientational degrees of freedom in the internal $SU(2)$ space.

As it has been discussed in [32,33], a very convenient choice is to take Θ as a linear function of the coordinates in such a way that, when the metric is flat, one gets

$$(\nabla_\mu\Theta)(\nabla^\mu\Theta) = (\nabla\Theta)^2 = \text{const} \neq 0,$$

since the system becomes rather trivial when $(\nabla\Theta)^2$ vanishes (in which case the field equation for α linearizes and, mainly, the energy-momentum tensor becomes quadratic in α so that the nonlinear effects disappear). Thus, the full Skyrme field equations, Eq. (2), reduce to the following single scalar nonlinear partial differential equation for the Skyrmion profile α:

[4]It is worth mentioning that Skyrme's original idea was precisely that the nucleon's mass is of mesonic origin and that the meson mass originates from the nucleon coupling.

$$0 = (1 + \lambda(\nabla\Theta)^2 \sin^2\alpha)\Box\alpha + \frac{\lambda(\nabla\Theta)^2(\nabla\alpha)^2}{2}\sin(2\alpha) - \frac{(\nabla\Theta)^2}{2}\sin(2\alpha) = 0,$$
$$\Box = \nabla_\mu \nabla^\mu, \tag{6}$$

while the $t-t$ component of the energy momentum (which represents the energy density) reads

$$T_{tt} = K\Big\{(\nabla_t\alpha)(\nabla_t\alpha) + \sin^2\alpha(\nabla_t\Theta)(\nabla_t\Theta) - \frac{g_{tt}}{2}[(\nabla\alpha)^2 + \sin^2\alpha(\nabla\Theta)^2] + \lambda\sin^2\alpha((\nabla\Theta)^2(\nabla_t\alpha)(\nabla_t\alpha) + (\nabla\alpha)^2(\nabla_t\Theta)(\nabla_t\Theta)) - \frac{\lambda g_{tt}}{2}\sin^2\alpha((\nabla\Theta)^2(\nabla\alpha)^2)\Big\}. \tag{7}$$

It is worth emphasizing the following point. When one replaces a suitable ansatz into the equations of motion, the information about the correct boundary conditions are lost. In such cases, the correct boundary conditions for the Skyrmion profile α can be derived analyzing, for instance, the energy density. In the present case, the global minima of the energy density in Eq. (7) are located at $\alpha = n\pi$. Hence, the two most natural possibilities are to require periodic boundary conditions for α (in which cases one is effectively considering the theory defined on a torus with finite volume[5]) or to require that α approaches $n\pi$ at the boundaries of the region one is analyzing. Both cases will be considered in the following.

As an interesting static example, one can consider the following ansatz for the Skyrme field:

$$\alpha = \alpha(x),$$
$$\Theta = \Theta(y, z) = \frac{1}{l}(n_2 y + n_3 z), \quad n_2, \quad n_3 \in \mathbb{Z}, \tag{8}$$

$$(\nabla\alpha)^2 = (\alpha')^2, \qquad \alpha' = \frac{d\alpha}{dx},$$
$$(\nabla\Theta)^2 = L^{-2} = ((n_2)^2 + (n_3)^2)l^{-2}, \tag{9}$$

where L and l are constants with dimension of length (so that both Θ and the product λL^{-2} are adimensional) while x, y, and z are the spatial coordinates. The choice in Eq. (8) corresponds to the internal vector $\hat{n}^i$ periodic in the y and z spatial directions while the profile α only depends on the spatial coordinate x. The full Skyrme field equations, Eq. (6), and the energy density in Eq. (7) reduce to

[5]This situation is especially relevant for the applications of Skyrmions in condensed matter physics in which the relevant dynamics happen in finite domains.

$$0 = \alpha'' - \frac{L^{-2}}{2}\sin(2\alpha) + \lambda L^{-2}\Big[\alpha''\sin^2\alpha + \frac{(\alpha')^2}{2}\sin(2\alpha)\Big], \tag{10}$$

$$T_{tt} = \frac{K}{2}[(\alpha')^2 + L^{-2}\sin^2\alpha + \lambda L^{-2}(\alpha')^2\sin^2\alpha]. \tag{11}$$

In this case, the correct boundary conditions correspond to require that α at $x = \pm\infty$ approaches the absolute minima of the energy density located at $n\pi$. It is worth mentioning that such a boundary condition makes sense not only from the energetic point of view. The reason is that, when α is equal to $n\pi$, the Skyrmion configurations defined in Eqs. (3)–(5) reduce to $\pm\mathbf{1}$, and it is customary to require that the Skyrme configuration reduces to (plus or minus) the identity at the boundaries of the domains in which one is interested.

At a first glance, it is a very difficult task to find exact solutions of Eq. (10). However, one can observe that any solution of the following first order differential equation for α:

$$\alpha' = \pm\frac{L^{-1}\sin\alpha}{(F(\alpha))^{1/2}}, \tag{12}$$

$$F(\alpha) = 1 + \lambda L^{-2}\sin^2\alpha, \tag{13}$$

is automatically a solution of Eq. (10) as well. Indeed, if α satisfies Eq. (12), then one can deduce [deriving both sides of Eq. (12)] the following expression for the second derivative of α:

$$\alpha'' = \frac{L^{-2}}{2}\Big\{\frac{\sin(2\alpha)}{1 + \lambda L^{-2}\sin^2\alpha} - \frac{\lambda L^{-2}\sin(2\alpha)\sin^2\alpha}{(1 + \lambda L^{-2}\sin^2\alpha)^2}\Big\}. \tag{14}$$

Then, replacing the expressions in Eqs. (12) and (14) for α' and α'' in Eq. (10), one can easily see that Eq. (10) is satisfied identically. By writing Eq. (12) in the form of a one-dimensional Newton equation, one can show there exist kinklike solutions that interpolate between neighboring minima of the energy density in Eq. (11) located at $n\pi$. Because of the fact that the profile α interpolates between neighboring minima, this type of solutions cannot be deformed continuously to the trivial solutions $\alpha = n\pi$. The obvious reason is that, to perform such deformation, one would need an infinite amount of energy per unit of area in the y-z plane. In particular, the above configuration actually represents a domain wall (for a detailed pedagogical review see [35]). Indeed, the full energy momentum is

$$T_{\mu\nu} = K\{(\nabla_\mu \alpha)(\nabla_\nu \alpha) + \sin^2\alpha(\nabla_\mu \Theta)(\nabla_\nu \Theta) - \frac{\eta_{\mu\nu}}{2}[(\nabla\alpha)^2 + \sin^2\alpha(\nabla\Theta)^2] + \lambda \sin^2\alpha((\nabla\Theta)^2 \times (\nabla_\mu\alpha)(\nabla_\nu\alpha) + (\nabla\alpha)^2(\nabla_\mu\Theta)(\nabla_\nu\Theta)) - \frac{\lambda\eta_{\mu\nu}}{2}\sin^2\alpha((\nabla\Theta)^2(\nabla\alpha)^2)\}, \quad (15)$$

$\eta_{\mu\nu}$ being the flat Minkowski metric. Since Eq. (12) implies that the profile α is a monotone function interpolating between neighboring minima of the energy density, one can see that the full energy-momentum tensor is different from zero only in a narrow region around $x = 0$ (for any y and z) and it approaches exponentially fast to zero far from the $x = 0$ plane. Consequently, as domain walls are defined by energy-momentum tensors that are localized around a plane and that approach zero exponentially fast far from the plane (see [35]), the above energy-momentum tensor corresponding to the Skyrmion profile α satisfying Eq. (12) represents a domain wall. To the best of the author's knowledge, this is the first exact domain wall in the four-dimensional Skyrme theory: because of the great importance of domain walls in many applications (see [35]), this result is quite interesting. It is also worth noting that, obviously, the total energy of a domain wall is divergent because of the integration along the transverse y and z directions. In fact, the relevant quantity in these cases is the total energy per unit of area in the y and z directions (which will be denoted as E_A), which reads

$$E_A = \frac{K}{2}\int_{-\infty}^{+\infty}[(\alpha')^2 + L^{-2}\sin^2\alpha + \lambda L^{-2}(\alpha')^2\sin^2\alpha]dx \rightarrow$$
$$E_A = \frac{K}{2}\int_{-\infty}^{+\infty}(1 + \lambda L^{-2}\sin^2\alpha) \times \left[(\alpha')^2 + \frac{L^{-2}}{(1 + \lambda L^{-2}\sin^2\alpha)}\sin^2\alpha\right]dx.$$

One can rewrite the above equation as

$$E_A = \frac{K}{2}\int_{-\infty}^{+\infty}(1 + \lambda L^{-2}\sin^2\alpha)\left[(\alpha')^2 + \frac{L^{-2}}{(1 + \lambda L^{-2}\sin^2\alpha)} \times \sin^2\alpha + 2\alpha'\left[\frac{L^{-2}}{(1 + \lambda L^{-2}\sin^2\alpha)}\sin^2\alpha\right]^{1/2} - 2\alpha'\left[\frac{L^{-2}}{(1 + \lambda L^{-2}\sin^2\alpha)}\sin^2\alpha\right]^{1/2}\right]dx \rightarrow$$
$$E_A = \frac{K}{2}\int_{-\infty}^{+\infty}(1 + \lambda L^{-2}\sin^2\alpha) \times \left[\alpha' \pm \left(\frac{1}{(1 + \lambda L^{-2}\sin^2\alpha)}\right)^{1/2} L^{-1}\sin\alpha\right]^2 dx \mp Q \quad (16)$$

$$Q = K\int_{-\infty}^{+\infty}[L^{-1}\sin\alpha(1 + \lambda L^{-2}\sin^2\alpha)^{1/2}]\alpha' dx. \quad (17)$$

Since the expression for Q in Eq. (17) is a total derivative and that, because of the boundary conditions, α approaches at $x = \pm\infty$ the absolute minima of the energy density itself, and the expression for Q is convergent. Moreover, when the profile α satisfies Eq. (12), the expression for E_A in Eq. (16) reduces to Q, and so it is convergent as well. Therefore, the above configuration has the expected physical behavior of a domain wall.

It is worth emphasizing here that the above example shows that, within Skyrme theory, there are many more nontrivial topological effects than the ones encoded in the winding number.[6]

III. TIME-PERIODIC SKYRMIONS AND NONLINEAR SUPERPOSITION LAW

Now, a class of exact time-periodic solutions (denoted as "periodic Skyrmions") will be presented. Such a class corresponds to the following alternative choices of the profile of the hedgehog α and of the function Θ:

$$\alpha = \alpha(x, y, z), \quad (\nabla\alpha)^2 = (\partial_x\alpha)^2 + (\partial_y\alpha)^2 + (\partial_z\alpha)^2, \quad (18)$$

$$\Theta = \omega t, \qquad \omega \in \mathbb{R}, \qquad (\nabla\Theta)^2 = -\omega^2, \quad (19)$$

where the metric is the flat Minkowski metric in Cartesian coordinates, x, y, and z are the spatial coordinates, while boundary conditions along the spatial direction will be specified below. The ansatz in Eq. (19) describes Skyrmions with a profile α, which depends on all the spacelike coordinates.[7] On the other hand, the internal vector $\hat{n}^i$ that describes the orientation of the Skyrmion in the internal $SU(2)$ space oscillates in time with frequency ω between the first and the second generators of the $SU(2)$ algebra: the description of this dynamical situation would be impossible with the usual spherical hedgehog ansatz.

With the above choice of α and Θ, the Skyrme field equations, Eq. (2), reduce to (see [32,33]) the following scalar elliptic nonlinear partial differential equation for the Skyrmion profile:

$$0 = (1 - \lambda\omega^2\sin^2\alpha) \triangle \alpha + \frac{\omega^2}{2}\sin(2\alpha) - \frac{\lambda\omega^2\sin(2\alpha)}{2}(\nabla\alpha)^2, \quad (20)$$

where $\triangle$ is the flat three-dimensional Laplacian.

In the following, periodic boundary conditions for the profile α will be considered. This corresponds effectively to compactify the space to a three-dimensional torus of

[6] The fact that the winding number by itself is not enough to describe all the subtle topological properties of the Skyrme model can be understood, for instance, following the analysis of [21].

[7] In the cases in which α depends on only one spacelike variable, one can find exact solutions following the same construction described in the previous section.

finite volume. Skyrme theory in finite domains can be quite relevant in applications in condensed matter physics as well (as the references cited in the Introduction show). How the analysis changes with different boundary conditions will be mentioned at the end of this section.

One can observe that in terms of the following function $H(\alpha)$ of the profile α:

$$H(\alpha) = \int^{\alpha} \sqrt{1 - \lambda\omega^2 \sin^2 s} ds \Rightarrow \frac{\triangle H}{\sqrt{1 - \lambda\omega^2 \sin^2 \alpha}} = \left[\triangle\alpha - \frac{\lambda\omega^2}{2} \frac{\sin(2\alpha)(\nabla\alpha)^2}{1 - \lambda\omega^2 \sin^2\alpha}\right], \tag{21}$$

Eq. (20) can be written as

$$\triangle H + \frac{\omega^2 \sin(2\alpha)}{2\sqrt{1 - \lambda\omega^2 \sin^2\alpha}} = 0, \tag{22}$$

where one should express α in terms of H inverting the elliptic integral in Eq. (21). A very surprising phenomenon is now apparent: if

$$\omega = \omega^* = \frac{1}{\sqrt{\lambda}}, \tag{23}$$

then Eq. (20) with the change of variable in Eq. (21) reduces to the following (*linear*) Helmholtz equation

$$\triangle H + \frac{1}{\lambda} H = 0, \qquad \alpha = \arcsin H, \tag{24}$$

which, as it will be discussed below, allows one to define an exact *nonlinear superposition law*. The energy density [defined in Eq. (7)] in terms of H becomes

$$T_{tt} = K\left\{\frac{1}{2\lambda} H^2 + \frac{1}{2}\left(\frac{1 + H^2}{1 - H^2}\right)(\nabla H)^2\right\}. \tag{25}$$

The explicit presence of nonlinear effects in the energy-momentum tensor in Eq. (25) despite the fact that H satisfies the linear Helmholtz equation is related to the breaking of the homogeneous scaling symmetry. Unlike what happens in free field theories, the energy density does not scale homogeneously under the rescaling

$$H \to \rho H, \qquad \rho \in \mathbb{R}.$$

In particular, by definition [see Eqs. (24) and (4)], $|H|$ cannot be larger than 1 and, moreover, from the energetic point of view, it may be very "expensive" for $|H|$ to get close to 1 as it is clear from Eq. (25). Therefore, one can multiply a given solution $H_{(0)}$ for a constant ρ whenever the (absolute value of the) new solution $\rho H_{(0)}$ of the Helmholtz equation does not exceed 1.

The nonanalytic dependence of the energy density in Eq. (25) on the Skyrme coupling λ clearly shows both the nonperturbative nature of the present effect and the fact that it is closely related to the Skyrme term.[8] Here only the cases in which Eq. (24) has a unique solution up to integration constants (which play the role of moduli of the Skyrmions) will be considered since they allow a more transparent physical interpretation (but more general situations can be analyzed as well). The periodic Skyrmions corresponding with such a unique solution will be denoted as *elementary Skyrmions*.

[8] Namely, it disappears when $\lambda \to 0$ as it is clear from Eq. (21).

Let us consider the case in which the soliton profiles depend on three spacelike coordinates x, y, and z. Periodic boundary conditions (with periods $2\pi L_i$) in the spatial directions will be considered. Let

$$H_i = H(\vec{x} - \vec{x}_i) = A_i(\sin\mu_1(x - x_i)\sin\mu_2(y - y_i)\sin\mu_3(z - z_i)), \tag{26}$$

$$\frac{1}{\lambda} = \sum_{i=1}^{3} \mu_i^2, \qquad \mu_i = \frac{1}{L_i}, \qquad \vec{x}_i = (x_i, y_i, z_i), \qquad \alpha_i = \arcsin H_i, \tag{27}$$

where $H(\vec{x} - \vec{x}_i)$ is the solution of Eq. (24). As one can check in Eq. (25), the positions of the peaks in the energy density in Eq. (25) corresponding to the elementary Skyrmions $\alpha_i = \arcsin H_i$ are determined by $\vec{x}_i$, which therefore plays the role of the moduli of α_i since $\vec{x}_i$ identifies the "position" of the elementary Skyrmion. On the other hand, the overall constant A_i strictly speaking does not represent moduli of the elementary Skyrmion since, when one replaces the expression in Eq. (26) into Eq. (25), one can see that the total energy depends on A_i while it does not depend on $\vec{x}_i$.

The most natural way to define a continuous composition of N elementary Skyrmions $\alpha_i = \arcsin H_i$ with moduli $\vec{x}_i$ [defined in Eqs. (26) and (27)] with the property that, when all the H_i are small, the profile of the sum reduces to the sum of the profiles is

$$\alpha_{1+2+\cdots+N} = \arcsin(H_1 + H_2 + \cdots + H_N), \quad \text{if } |H_1 + H_2 \cdots + H_N| \leq 1, \tag{28}$$

$$\alpha_{1+2+\cdots+N} = \frac{\pi}{2}, \quad \text{if } H_1 + H_2 \cdots + H_N > 1, \tag{29}$$

$$\alpha_{1+2+\cdots+N} = -\frac{\pi}{2}, \quad \text{if } H_1 + H_2 \cdots + H_N < -1, \tag{30}$$

where one must take into account that $\arcsin x$ is only defined when $|x| \leq 1$. At a first glance, in the cases in which $|H_1 + H_2 + \cdots| > 1$, discontinuities in the first derivatives of the composite Skyrmion can appear. In fact, it is unlikely that such nonsmooth solutions can survive since they are very expensive energetically (since the corresponding gradient would be unbounded). Hence, the nonlinear superposition of elementary Skyrmions is allowed only when they satisfy $|H_1 + H_2 + \cdots| < 1$; otherwise it is not energetically convenient to combine the elementary Skyrmions into the composite Skyrmion.

As far as the appearance of crystals is concerned, from Eqs. (7) and (25) one can see that the energy density of the composition $\alpha_{1+2+\cdots+N}$ of N elementary Skyrmions (whose integral represents the interaction energy between the N elementary Skyrmions and can be computed in principle for any N) depends in a complicated way on the moduli $\vec{x}_i$. However, one can observe in Eqs. (7) and (25) that, in the expression of the energy density of the composition $\alpha_{1+2+\cdots+N}$, in order to find configurations of the $\vec{x}_i$ that are favorable energetically, one should minimize with respect to the moduli $\vec{x}_i$ quadratic sums of the following type:

$$\Psi = \left(\sum_{i=1}^{N} H(\vec{x} - \vec{x}_i)\right)^2.$$

In all the cases in which $H(\vec{x})$ involves trigonometric functions, the theory of interference in optics[9] can be applied to minimize sums of the type appearing in the above equation since one can interpret Ψ as the interference of many elementary waves. Hence, placements in which the $\vec{x}_i$ follow patterns of negative interference are always favorable energetically (although other local minima of the total energy appear as well). In the cases in which $H(\vec{x})$ involves a different basis of functions (such as the Bessel functions that naturally appear when analyzing the Helmholtz equation in unbounded domains) the known results in optics cannot be applied directly, but it is reasonable to expect that also in those cases the $\vec{x}_i$ follow patterns associated with "negative interference of Bessel functions."

It is worth emphasizing here that the present composite Skyrmions do not correspond to the Skyrme crystals already known numerically (see, for instance, [3,24]) since, in the latter case, the internal $SU(2)$ orientation depends nontrivially on spacelike coordinates while, in the present case, it depends nontrivially on time. However, the above result is quite remarkable since, to the best of the author's knowledge, it is the first example constructed in a nonlinear nonintegrable four-dimensional theory in which it is possible to accommodate both explicit nonlinear effects in the energy-momentum tensor and a superposition law of elementary solutions. Hence, the present results shed new light on the nonlinear interactions that are responsible for the appearance of composite structures such as the ones in [3,24].

It is natural to wonder whether the intriguing picture described above is a "coincidence" that can appear only when the frequency of the periodic Skyrmion satisfies Eq. (23). In fact, standard results in perturbation theory (see the detailed analysis in [36]) ensure that there exists a nontrivial left neighborhood of $\omega = 1/\sqrt{\lambda}$ in which the above construction provides a good approximation of the multi-Skyrmions solutions. To see this, let us consider $\lambda\omega^2 = 1 - \eta$ where $0 < \eta \ll 1$. In this case, one can treat the term proportional to η in the Skyrme field equation, Eq. (20), as a perturbation and η as the perturbation parameter. Well-known techniques in the theory of nonlinear partial differential equations can be applied to the present case (see, in particular, p. 206, Chaps. 4.5B and 4.5C of [36]). The kind of results that one gets in this way is that, given a solution $\alpha_{(0)}$ of Eq. (20) with $\eta = 0$, there exists a unique solution $\alpha_{(\eta)}$ of Eq. (20) with $\eta > 0$ and small such that $\alpha_{(\eta)} \to_{\eta\to 0} \alpha_{(0)}$ uniformly apart from a narrow region close to the boundary. The uniqueness result implies that, for small enough η, the composite periodic Skyrmions described above when $\eta = 0$ will exist also when η is positive and small enough. Of course, they will be slightly distorted and their energies will acquire corrections of order η, but the broad theoretical picture will remain the same.

Besides periodic boundary conditions, there are two more interesting cases that can be analyzed easily within the present framework. The first one corresponds to a finite domain with a nontrivial boundary. In these situations, it is reasonable to require that the profile α approaches a global minimum of the energy density at the boundary. In terms of the variable H defined in Eq. (21) this implies that H must vanish on the boundary of the region one is considering. Because of the fact that, in the sector in which the nonlinear superposition law is available, H satisfies the Helmholtz equation, one arrives at the well-defined mathematical problem to solve the Helmholtz equation with vanishing Dirichlet boundary conditions. This problem is well analyzed in mathematical textbooks, and explicit solutions exist when the domain has enough symmetries. As it has been already discussed, the total energy is finite whenever the absolute value of the solution is strictly less than one.[10] The second case corresponds to infinite domains, and so the mathematical problem is to solve the Helmholtz equation with vanishing Dirichlet boundary conditions in infinite domains. In these cases, however, to require that the absolute value of the solution is strictly less than 1 is not enough to get a finite total energy: in the second case, the total energy diverges as for plane waves. Moreover, the time dependence of the present periodic Skyrmions is similar to the usual behavior of plane waves. As it has been discussed in [37] in the case of exact plane waves in Yang-Mills theory, in order to have well-behaved configurations it is enough to require that the energy

[9]Namely, if one considers sums such as $\sum_j^{N-1} A \exp(i\xi_j)$, one can show that it vanishes when $\xi_j - \xi_{j-1}$ is equal to $2\pi/N$ for any j. In the present case, the difference between the arguments in the summands is related to the distance between peaks of neighboring Skyrmions.

[10]This can always be achieved since once one solves the Dirichlet problem one can multiply the solution by a small enough factor to satisfy the above requirement, satisfying at the same time the same boundary conditions.

density is bounded, and this happens in the present case as well. Actually, the present periodic multi-Skyrmions in infinite domains are more general than the exact multiplane waves considered in the case of the Yang-Mills theory, for instance, in [38]. The reason is that, both in [37] and in [38], in order to construct exact plane waves of the Yang-Mills theory in which the nonlinear effects are present, only configurations with flat wave fronts were considered. On the other hand, in the present case, the spatial dependence of the profile of the periodic multi-Skyrmion (which determines the form of the wave front) can be found by solving the Helmholtz equation, and this allows one to construct wave front with nontrivial spatial geometries while keeping both the nonlinear superposition law and the nonlinear effects alive.

IV. MAPPING BETWEEN DUAL PERIODIC SKYRMIONS

In the previous section, it has been shown how to construct exact Skyrme configurations that exhibit a nonlinear superposition whose internal orientation changes periodically in time while the Skyrmions profile α is static. Hence, it is natural to wonder whether it is possible to realize also the "dual configurations" in which the profile α depends on time while the internal orientation changes periodically along some spacelike direction. The equations of motion in both cases are very similar, but there is an important difference as will now be shown.

Let us consider the following choices of the profile of the hedgehog α and of the function Θ:

$$\alpha = \alpha(t, x, y),\qquad (\nabla\alpha)^2 = -(\partial_t\alpha)^2 + (\partial_x\alpha)^2 + (\partial_y\alpha)^2, \tag{31}$$

$$\Theta = L^{-1}z, \qquad (\nabla\Theta)^2 = L^{-2}, \tag{32}$$

where the flat Minkowski metric in Cartesian coordinates is used, x, y, and z are the corresponding spatial coordinates, and the constant L has the dimension of length. The ansatz in Eqs. (31) and (32) [which is the natural extension of the one in Eqs. (8) and (9)] describes Skyrmions with a time-dependent profile α that depends on two spacelike coordinates as well. The internal vector $\hat{n}^i$ that describes the orientation of the Skyrmion in the internal $SU(2)$ space depends periodically on z.

With the above choice, the full Skyrme field equations reduce to

$$0 = (1 + \lambda L^{-2}\sin^2\alpha)\Box\alpha + \frac{\lambda L^{-2}(\nabla\alpha)^2}{2} \times \sin(2\alpha) - \frac{L^{-2}}{2}\sin(2\alpha) = 0,\qquad \Box = -\partial_t^2 + \partial_x^2 + \partial_y^2. \tag{33}$$

In terms of the new variable u defined as

$$\alpha = u + \frac{\pi}{2},$$

Eq. (33) becomes

$$0 = (1 - \lambda(\omega_{\rm eff})^2\sin^2 u)\Box u + \frac{(\omega_{\rm eff})^2}{2}\sin(2u) - \frac{\lambda(\omega_{\rm eff})^2\sin(2u)}{2}(\nabla u)^2, \tag{34}$$

$$(\omega_{\rm eff})^2 = \frac{L^{-2}}{1 + \lambda L^{-2}}, \tag{35}$$

where trivial trigonometric identities have been used. Hence, in terms of the variable u, the equation is the same as Eq. (20) analyzed in the previous section with an effective frequency $\omega_{\rm eff}$ given in Eq. (35). In fact, it is now apparent that the fundamental difference between this case and the periodic Skyrmions is described in the previous section. One of the key observations of the previous section is that Eq. (20), when the critical condition in Eq. (23) is satisfied, can be mapped into a linear equation. In the present case, the critical condition corresponding to Eq. (23) would be

$$\lambda(\omega_{\rm eff})^2 = 1,$$

so that, as it is evident from Eq. (35), the above condition cannot be fulfilled for the ansatz in Eqs. (31) and (32). This is the crucial difference between the previous case (in which $\hat{n}^i$ depends periodically on time) and the one discussed in this section. Although a sort of duality mapping between the two cases can be constructed, the nonlinear superposition law appears only when the profile α is static and the internal orientation changes periodically in time.

V. CONCLUSIONS

In the present paper, exact configurations of the four-dimensional Skyrme model have been constructed. Such configurations can be static and kinklike or time periodic. The static configurations represent domain walls: to the best of author's knowledge these are the first exact domain walls of the original four-dimensional Skyrme theory. Within the class of the time-periodic configurations (periodic Skyrmions), which can depend in a nontrivial way on all the spacelike coordinates, it is possible to disclose a remarkable phenomenon. For a special value of the frequency, a nonlinear superposition law arises that allows one to compose two (or more) periodic Skyrmions into a new one. The nonlinear superposition leads to the appearance of crystals of periodic Skyrmions in which the peaks in the energy densities of elementary Skyrmions are placed according to patterns of negative interference. Such composite Skyrmions do not disappear if the frequency is close enough to the resonance frequency (although they will suffer small distortions). In a nonlinear theory such as the four-dimensional Skyrme model, these results are very surprising and shed considerable new light on the interactions of Skyrmions.

ACKNOWLEDGMENTS

I warmly thank Hideki Maeda for many comments, and I also thank very much Francisco Correa, Hideki Maeda, and Cedric Troessaert for carefully checking the computations of this work. This work has been funded by the Fondecyt Grant No. 1120352. The Centro de Estudios Científicos (CECs) is funded by the Chilean Government through the Centers of Excellence Base Financing Program of Conicyt.

[1] T. Skyrme, Proc. R. Soc. A **260**, 127 (1961); **262**, 237 (1961); Nucl. Phys.**31**, 556 (1962).
[2] H. Weigel, *Chiral Soliton Models for Baryons*, Lecture Notes in Physics Vol. 743 (Springer-Verlag, Berlin, 2008).
[3] N. Manton and P. Sutcliffe, *Topological Solitons* (Cambridge University Press, Cambridge, 2007).
[4] *Duality and Supersymmetric Theories*, edited by D. I. Olive and P. C. West (Cambridge University Press, Cambridge, 1999).
[5] A. P. Balachandran, H. Gomm, and R. D. Sorkin, Nucl. Phys. **B281**, 573 (1987).
[6] A. P. Balachandran, A. Barducci, F. Lizzi, V. G. J. Rodgers, and A. Stern, Phys. Rev. Lett. **52**, 887 (1984).
[7] A. P. Balachandran, F. Lizzi, V. G. J. Rodgers, and A. Stern, Nucl. Phys. **B256**, 525 (1985).
[8] G. S. Adkins, C. R. Nappi, and E. Witten, Nucl. Phys. **B228**, 552 (1983).
[9] E. Guadagnini, Nucl. Phys. **B236**, 35 (1984).
[10] H. Pais and J. R. Stone, Phys. Rev. Lett. **109**, 151101 (2012).
[11] U. Al Khawaja and H. Stoof, Nature (London) **411**, 918 (2001).
[12] J.-I. Fukuda and S. Zumer, Nat. Commun. **2**, 246 (2011).
[13] C. Pfleiderer and A. Rosch, Nature (London) **465**, 880 (2010).
[14] S. Mühlbauer, B. Binz, F. Jonietz, C. Pfleiderer, A. Rosch, A. Neubauer, R. Georgii, and P. Böni, Science **323**, 915 (2009).
[15] F. Jonietz, S. Mühlbauer, C. Pfleiderer, A. Neubauer, W. Münzer, A. Bauer, T. Adams, R. Georgii, P. Böni, R. A. Duine, K. Everschor, M. Garst, and A. Rosch, Science **330**, 1648 (2010).
[16] S. Seki, X. Z. Yu, S. Ishiwata, and Y. Tokura, Science **336**, 198 (2012).
[17] U. K. Roessler, A. N. Bogdanov, and C. Pfleiderer, Nature (London) **442**, 797 (2006).
[18] A. N. Bogdanov and D. A. Yablonsky, Sov. Phys. JETP **68**, 101 (1989).
[19] A. Bogdanov and A. Hubert, J. Magn. Magn. Mater. **138**, 255 (1994); **195**, 182 (1999).
[20] N. S. Manton, Phys. Lett. **110B**, 54 (1982).
[21] Y. M. Cho, Phys. Rev. Lett. **87**, 252001 (2001).
[22] L. Faddeev and A. Niemi, Nature (London) **387**, 58 (1997).
[23] M. Hirayama and J. Yamashita, Phys. Rev. D **66**, 105019 (2002).
[24] R. A. Battye and P. M. Sutcliffe, Phys. Rev. Lett. **79**, 363 (1997); Phys. Lett. B **391**, 150 (1997).
[25] C. J. Houghton, N. S. Manton, and P. M. Sutcliffe, Nucl. Phys. **B510**, 507 (1998).
[26] B. M. A. G. Piette, B. J. Schoers, and W. J. Zakrzewski, Z. Phys. C **65**, 165 (1995); Nucl. Phys. **B439**, 205 (1995).
[27] R. A. Leese, M. Peyrard, and W. J. Zakrzewski, Nonlinearity **3**, 773 (1990); B. M. A. G. Piette and W. J. Zakrzewski, Chaos Solitons Fractals **5**, 2495 (1995); P. M. Sutcliffe, Nonlinearity **4**, 1109 (1991).
[28] T. Weidig, Nonlinearity **12**, 1489 (1999).
[29] P. Eslami, M. Sarbishaei, and W. J. Zakrzewski, Nonlinearity **13**, 1867 (2000).
[30] A. E. Kudryavtsev, B. Piette, and W. J. Zakrzewski, Eur. Phys. J. C **1**, 333 (1998).
[31] C. Adam, J. Sanchez-Guillen, and A. Wereszczynski, Phys. Lett. B **691**, 105 (2010); Phys. Rev. D **82**, 085015 (2010).
[32] F. Canfora and P. Salgado-Rebolledo, Phys. Rev. D **87**, 045023 (2013).
[33] F. Canfora and H. Maeda, Phys. Rev. D **87**, 084049 (2013).
[34] P. M. Sutcliffe, Nucl. Phys. **B431**, 97 (1994).
[35] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Other Topological Defects* (Cambridge University Press, Cambridge, 2000).
[36] M. Berger, *Non-linearity and Functional Analysis* (Academic Press, New York, 1977).
[37] S. Coleman, Phys. Lett. **70B**, 59 (1977).
[38] H. J. de Vega, Commun. Math. Phys. **116**, 659 (1988).