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Wilson lines, de Rham theorem, and proton spin decomposition controversy

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The geometric approach based on parallel transport and Wilson lines to gain insight into the decomposition of the gluon gauge 4-vector potential is being developed in the literature to address the proton spin puzzle. Unfortunately, conflicting claims on the uniqueness of the decomposition have been made. We reexamine this controversy and argue that Hodge decomposition and the de Rham theorem resolve this issue in a natural way such that the topological defects acquire a fundamental significance. It is suggested that qualitatively angular momentum holonomy could be an alternative new mechanism to account for the deficit proton spin.

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I. INTRODUCTION

The spin structure of nucleons has been probed by polarized deep inelastic scattering (pDIS) experiments over the past 25 years: the present accepted estimate of the fraction of the spin of the proton carried by its constituent quarks is about one-third [1]. Why is it so small? Wherefrom originates the proton spin? Assuming the proton spin equals the total angular momentum (TAM) of quarks and gluons, it becomes crucial to know the separate contributions of the spin and orbital angular momentum (SAM and OAM) of quarks and gluons. On the other hand, it is well known that perturbative quantum chromodynamics (pQCD) [2] applicable for short-distance processes (due to asymptotic freedom) is only half of the story of QCD; phenomena in a strong coupling regime, for example, the hadrons as bound states, are poorly understood. Therefore, it becomes necessary to make reasonable approximations to interpret the measurements [1,3]. Nevertheless the role of established principles like relativistic and gauge invariance cannot be ignored: the current controversy on the proton spin decomposition [4] underlines this fact.

The main question is that of splitting TAM into SAM and OAM of quarks and gluons in a gauge invariant way. In an extensive elaborative study Wakamatsu has sought to establish that there are only two gauge invariant decompositions of proton spin in which OAM of quarks and gluons have different definitions [5]. At a basic level the idea is to split the gauge 4-vector potential into a pure gauge part and a physical part

$$A^a_\mu = A^{a,\mathrm{pure}}_\mu + A^{a,\mathrm{phys}}_\mu. \tag{1}$$

Here a is the gauge group index that we may suppress sometimes. Lorce develops a geometric approach based on Wilson lines and the notion of parallel transport in [6]. One of the important consequences following from this work is the nonuniqueness of the decomposition (1) *à la* Stueckelberg symmetry; the role of this symmetry was first pointed out by Stoilov in this connection [7]. There are contradictory claims on the question of the uniqueness of the separation (1) in [5,6]. Could one achieve reconciliation based on mathematical or physical argument? The aim of the present paper is to propose a resolution of the controversy invoking the de Rham theorem and topological concepts. First, we present a critical appraisal of the controversy in Sec. II. In Sec. III it is argued that Wilson lines naturally lead to the formalism of differential forms and the de Rham theorem. Gauge transformation in the context of the de Rham theorem is discussed: the uniqueness of the decomposition (1) for simply connected space follows immediately while topological defects lead to infinity of possible decompositions. In the Sec. IV the significance of angular momentum holonomy is discussed along with the concluding remarks.

II. GAUGE 4-VECTOR POTENTIAL DECOMPOSITION

Recall the assertion [8] that simultaneously satisfying the requirements of manifest Lorentz covariance and gauge invariance in the separation of TAM into SAM and OAM is a formidable, if not impossible, task in gauge theories. Note that the Lorentz covariant decomposition (1) articulated by Wakamatsu breaks the covariance once longitudinal and transverse interpretation is introduced since one has to specify a frame of reference [5]. Whether this amounts to the loss of relativistic invariance is a subtle issue remembering that even the Maxwell equations written in terms of electric and magnetic field vectors **E** and **B** are relativistically invariant but not manifestly Lorentz covariant. As regards to the gauge invariance one has

$$A^{\mathrm{pure}}_\mu \to U(x)\left(A^{\mathrm{pure}}_\mu - \frac{i}{g}\partial_\mu\right)U^\dagger(x), \tag{2}$$

and the corresponding field tensor is zero,

$$F^{\mathrm{pure}}_{\mu\nu} = \partial_\mu A^{\mathrm{pure}}_\nu - \partial_\nu A^{\mathrm{pure}}_\mu - ig[A^{\mathrm{pure}}_\mu, A^{\mathrm{pure}}_\nu] = 0. \tag{3}$$

The physical gauge potential under gauge transformation becomes

$$A_\mu^{\rm phys} \rightarrow U(x) A_\mu^{\rm phys} U^\dagger(x). \tag{4}$$

The QCD AM tensor has been shown to have two physically inequivalent decompositions [5] in which SAM parts of quarks and gluons are identical but OAM parts are defined differently; the difference between them is a covariant generalization of the so-called potential angular momentum

$$L_{\rm pot} = \int \rho^a (\mathbf{r} \times \mathbf{A}^a) d^3 r. \tag{5}$$

Note that the gauge group index a is explicitly shown in (5). The corresponding expression in quantum electrodynamics (QED) for the potential angular momentum is

$$L_{\rm pot}^{\rm QED} = \int \rho (\mathbf{r} \times \mathbf{A}_{\rm tr}) d^3 r. \tag{6}$$

Here ρ is charge density and the electromagnetic potential $\mathbf{A}$ is divided into transverse and longitudinal components,

$$\mathbf{A} = \mathbf{A}_l + \mathbf{A}_{\rm tr}, \tag{7}$$

such that

$$\nabla . \mathbf{A}_{\rm tr} = 0, \tag{8}$$

$$\nabla \times \mathbf{A}_l = 0. \tag{9}$$

In QED these components have the following gauge transformation:

$$\mathbf{A}_{\rm tr} \rightarrow \mathbf{A}_{\rm tr}, \tag{10}$$

$$\mathbf{A}_l \rightarrow \mathbf{A}_l + \nabla \chi. \tag{11}$$

Lorce points out that the decomposition (1), on which the construction of the AM tensor and its decomposition are based, is not unique due to Stueckelberg symmetry; see transformation equations (6) and (7) in [6]. In QED the Stueckelberg transformation would be

$$A_\mu^{\rm pure} \rightarrow A_\mu^{\rm pure} + \partial_\mu B, \tag{12}$$

$$A_\mu^{\rm phys} \rightarrow A_\mu^{\rm phys} - \partial_\mu B. \tag{13}$$

Here $B(x)$ is an arbitrary scalar function of space and time. Apparently the decomposition is not unique. However, Wakamatsu asserts that the imposition of the transversality condition in addition to Eqs. (2)–(4) uniquely fixes the decomposition (1).

In the geometric approach [6] the idea of parallel transport is used involving Wilson lines. A gauge link or Wilson line is defined as [2] the path-ordered exponential of the integral of A_μ^a along a path C parametrized by a function $x^\mu(s)$ where the parameter s varies from 0 to 1,

$$W(C) = P\left[\exp\left(-ig \int_0^1 \frac{dx^\mu(s)}{ds} A_\mu^a(x(s)) t_a ds\right)\right]. \tag{14}$$

Here t_a are the generator matrices of the gauge group. Interestingly, the solution of the parallel transport equation can be written in terms of the Wilson line, and the transformations (2) and (4) could be established using the gauge transformation of the Wilson line in this approach [6]. The important point in the present context is that the change in the path C would change the pure and the physical parts of A_μ essentially due to the Stueckelberg transformation: the expressions for the pure and the physical 4-vector gauge potentials are in general path dependent. Note that a Wilson line is invariant under reparametrization but depends on the path even while the end points are fixed. The path dependence of Wilson lines has some important consequences. One implication is that it is imperative to define the measurable quantities carefully. In a practical situation, for example, calculating parton densities using lightlike separation in the minus direction the Wilson line becomes path independent. However, the difference between path ordering and time ordering, the role of the covariant gauge, and the path along the spacelike separation are delicate issues [2]. The contentious nature of the uniqueness of the decomposition (1) and the path dependence of Wilson lines has been reemphasized in recent papers [9,10] and references cited therein.

III. UNIQUENESS: HELMHOLTZ, HODGE, AND DE RHAM THEOREMS

The most intriguing dimension of the controversy concerns the sharp disagreements on what constitutes an observable [4,5,9,10]. A nice review on an observable operator in QED and QCD [4] offers somewhat limited perspective on the measurable quantities; philosophical and conceptual issues are, of course, unavoidable. Let us confine our attention to the formalism aspect.

The key argument to establish the uniqueness of the decomposition (1) runs as follows [5]. It is admitted that conditions (2) to (4) by themselves are insufficient to uniquely determine the decomposition (1). Next, QED case is considered, and the noncovariant form of the decomposition is identified with the standard transverse and longitudinal separation (7); i.e., $\mathbf{A}^{\rm phys}$ and $\mathbf{A}^{\rm pure}$ satisfy the conditions (8) and (9), respectively. The Stueckelberg transformations (12) and (13) assume the form

$$\mathbf{A}_{\rm tr} \rightarrow \mathbf{A}_{\rm tr} + \nabla B, \tag{15}$$

$$\mathbf{A}_l \rightarrow \mathbf{A}_l - \nabla B. \tag{16}$$

The transformed longitudinal vector potential remains irrotational obeying Eq. (9); however, the transformed transverse vector potential is not divergenceless unless

$$\nabla^2 B = 0. \tag{17}$$

Referring to the Helmholtz theorem it is claimed that "the transverse-longitudinal decomposition is unique once the Lorentz frame of reference is fixed" [5]. Caution needs to be exercised regarding the Helmholtz theorem: the theorem states that a vector field $\mathbf{V}$ can be decomposed into a gradient of a scalar field Φ and a curl of a vector field $\mathbf{W}$ that is unique up to the addition of a gradient of arbitrary scalar S,

$$\mathbf{V} = [\boldsymbol{\nabla}\Phi + \boldsymbol{\nabla}S] + [\boldsymbol{\nabla} \times \mathbf{W} - \boldsymbol{\nabla}S], \tag{18}$$

such that S is a harmonic function

$$\nabla^2 S = 0. \tag{19}$$

We refer to Sec. (2) of [11] for a precise discussion on the Helmholtz theorem. It is clear that an additional assumption on B, namely that it vanishes at the spatial infinity, is required to make it zero, and it is not correct to conclude that Stueckelberg symmetry does not preserve the transversality condition since the Laplace equation for B [S], Eq. (17) [Eq. (19)], ensures that transversality is maintained.

Regarding the specification of a Lorentz frame and breaking Lorentz covariance in the process, we refer to our argument [8] that a formalism to maintain rigorous manifest Lorentz covariance and gauge invariance is nontrivial. That gauge fixing and the transverse-longitudinal decomposition are interrelated [5] seems to be a valid physical argument since gauge fields are massless.

The path dependence of the Wilson line [6] also indicates the nonuniqueness of the separation (1); however, it has been argued in [5] citing previous works that there are instances where path independence and gauge invariance could coexist, and that the decomposition is unique [5]. Lorce, in an attempt to reconcile contradictory claims, introduces two types of gauge transformations: active and passive [10]. It is not clear whether terminology could resolve the problem.

Recall that in the standard interpretation the gauge invariance implies the nonobservability of A_μ. However, the Aharonov-Bohm effect, experimentally demonstrated many times, ascribes physical reality to the electromagnetic vector potential if one considers the phase integral over a closed path,

$$W_{AB} = \exp\left(-\frac{ie}{\hbar c}\oint \mathbf{A}.\mathbf{dl}\right), \tag{20}$$

the covariant form being

$$W_{AB} = \exp\left(-\frac{ie}{\hbar c}\oint A_\mu dx^\mu\right). \tag{21}$$

Using Stoke's theorem the phase integral in (20) could be converted in terms of the flux of the enclosed magnetic field: it may be said that it is a nonlocal field effect; i.e., the local vector potential is not observable. It is straightforward to recognize the significance of the 1-form in Eq. (21),

$$A = A_\mu dx^\mu. \tag{22}$$

The language of exterior differential forms *à la* Cartan seems natural and provides new insights as we discuss it in the following. The definition (14) of the Wilson line is transcripted to

$$W = P\left[\exp -ig \oint A\right], \tag{23}$$

where the 1-form is defined as

$$A = A^a_\mu t_a dx^\mu. \tag{24}$$

The Helmholtz theorem (18) is generally applicable to a continuous differentiable vector field, though it has been extended to the case when singularities are present [11]. What is the equivalent decomposition for differential forms? Note that the Aharonov-Bohm phase integral and the Wilson line can be interpreted as integrations of differential forms, and over closed loops or cycles these are called periods. The homology and cohomology classes are established by the de Rham theorem.

A physically intuitive picture on the rudiments of the differential forms is presented; we refer to [12] for technical details. One defines a chain in analogy to a line integral, and a cycle corresponds to a loop integral. The set of chains is dual to the set of differential forms. The exterior differential operator d on a p-form gives a $(p+1)$-form, for example, a 0-form f is a scalar function and df is a gradient of f. The adjoint of d is δ, and the Laplacian is $d\delta + \delta d$. A differential form γ is harmonic if

$$(d\delta + \delta d)\gamma = 0. \tag{25}$$

The Hodge theorem states that any p-form ω on a compact manifold can be decomposed uniquely into a closed form, a coclosed form and a harmonic form

$$\omega = d\alpha + \delta\beta + \gamma. \tag{26}$$

It is the harmonic part γ that determines the cohomology class and contains the topological information.

Note that in the Euclidean topology the Hodge decomposition (26) is equivalent to the Helmholtz decomposition (18). A distinctly new significance of the harmonic form becomes manifest in the de Rham theorem. Recall the duality of chains and forms mentioned earlier. Using the boundary operator ∂ of a chain, one defines $C = \partial B$ as a boundary and $\partial C = 0$ as a cycle. The integral of ω over a cycle C defines a period of the form: it depends on the cohomology class of ω and the homology class of C. The first de Rham theorem says that for a given set of periods $[\nu_i]$ there exists a closed p-form such that

$$\nu_i = \oint_{C_i} \omega. \tag{27}$$

Here $[C_i]$ is a set of independent cycles. The second theorem states that if all the periods for a p-form α vanish,

$$0 = \oint_{C_i} \alpha, \tag{28}$$

then α is exact. Thus the closed form is determined up to the addition of an exact form: this corresponds to the gauge invariance. The period of the harmonic form characterizes the number of "holes" in the manifold

$$\nu_{\mathrm{Rham}} = \oint \gamma. \tag{29}$$

A typical widely discussed example in the literature, also mentioned in [12], is that of a vector field having

$$\nabla \times \mathbf{A} = 0, \tag{30}$$

given in Cartesian coordinates by

$$\mathbf{A} = \frac{y\hat{i} - x\hat{j}}{x^2 + y^2}. \tag{31}$$

It can be rewritten as a closed form $\omega = df$ where $f = \tan^{-1}\frac{y}{x}$, but it is not exact due to the presence of the hole at the origin. The nonvanishing period is the loop integral encircling the origin given by 2π.

In the light of the preceding discussion we make the following proposition.

Proposition: The natural decomposition of the 4-vector potential in QED and QCD has to be based on the Hodge decomposition (26), and the de Rham theorem paves the way for a unified description incorporating the Wilson line approach: the geometric approach generalizes to a topological one.

The above proposition is essentially a conceptual abstraction of what we have learned from the Aharonov-Bohm effect: the enclosed magnetic flux changes the Euclidean topology, creating a hole. Unless the hole is encircled, the uniqueness of the decomposition (7) is ensured. A simply connected space by virtue of the presence of the magnetic flux acquires a multiply connected structure, and the de Rham theorem shows that the separation of the vector potential is not unique. QCD, unlike QED, is a non-Abelian gauge theory, and there arise new complications. However, so-called gauge invariant extensions, as repeatedly emphasized by Wakamatsu [5,9], for an open straight line path in spacetime have both path independence and gauge invariance. This argument fails for closed paths where the harmonic form *à la* the de Rham theorem may allow an infinity of decompositions if the topology is nontrivial. The assertions made by Lorce [10] are justified only in such cases.

What is the role of our proposition in the context of gauge fixing in the geometric approach? A careful analysis is required to define the path in spacetime that serves as the base space of a principal fiber bundle, the fiber being the gauge group. The 1-form defined in (24) has a nontrivial structure. At least in the so-called contour gauges the gauge fixing can be formulated explicitly [5] and the physical interpretation on the question of unique gauge invariant decomposition can be understood.

To conclude this section, we have shown that the contradictory claims made on the uniqueness of the decomposition (1) in the literature can be satisfactorily resolved based on the de Rham theorem and Hodge decomposition.

IV. DISCUSSION AND CONCLUSION

Though the proton size is of the order of a Fermi, high energy scattering experiments provide useful data on the three-dimensional internal structure of the proton. The hadron tensor is split into four components characterized by four structure functions $F_1(x, Q^2)$, $F_2(x, Q^2)$, $g_1(x, Q^2)$, $g_2(x, Q^2)$. Here Q denotes the invariant momentum transfer in the process and x the Bjorken variable. In pQCD essentially the same description holds [2]; however, the most challenging task is to make realistic calculations and obtain physical quantities from the measurements on deep inelastic scattering (DIS) and pDIS. One of the most important quantities measured in the laboratories the world over and showing reasonable consistency is the spin structure function g_1; see Fig. 4 in [1]. The crucial factor that determines the first moment of g_1, $\int_0^1 g_1(x, Q^2)dx$, is the flavor-singlet axial charge $g_A^{(0)}$. The second spin structure function seems to be

$$\int_0^1 g_2(x, Q^2)dx = 0. \tag{32}$$

The charge $g_A^{(0)}$ is related with the renormalization scale invariant singlet axial charge that corresponds to $g_A^{(0)}(Q^2)$ obtained in the limit $Q^2 \to \infty$. The scale dependence is embodied [1] in a renormalization group factor $E(\alpha_s)$. Next-to-leading order calculations in pQCD show that the separation of g_1 into hard and soft contributions depends on the factorization schemes: modified minimal subtraction and chiral invariant are two important schemes discussed in the literature [1]. However, it has been emphasized in [13] that both factorization schemes are on the same footing.

The value of $g_A^{(0)}$ extracted from pDIS is about 0.3. The estimates of the octet axial charge $g_A^{(8)}$ are model-dependent, and also alter the value of $g_A^{(0)}$. The QCD result may be written as

$$g_A^{(0)} = \mathrm{spin}(\Delta q, \Delta g) + C_\infty. \tag{33}$$

Here the first term corresponds to the quark and polarized gluon contributions [see Eq. (22) in [1]]. The constant term C_∞ is interesting: it is related to a nonlocal gluon topological structure in the proton, vanishes in pQCD, and is believed to reveal the nontrivial QCD vacuum.

Recent experiments give strong indications that going beyond spin sums we may have to include OAM of quarks and gluons. Note that even for gluon polarization the gauge fixing plays a significant role; in the case of the decomposition of the spin of the proton into SAM and OAM of the constituent quarks and gluons, gauge invariance becomes crucial. Assuming specific forms of the 4-vector gauge potential (1), there appear to be mainly four decompositions of TAM ($\mathbf{J}$) in the literature [14–17]. The resolution of the uniqueness controversy of decomposition (1) proposed in the preceding section leads to a new insight: TAM may have a constant topological contribution. Using the simple notation

$$\mathbf{J} = \mathbf{S}_q + \mathbf{S}_g + \mathbf{L}_q + \mathbf{L}_g, \tag{34}$$

where subscripts q and g refer to quarks and gluons, respectively, Lorce notes that $\mathbf{S}_q$ is the same in all four schemes [10],

$$\mathbf{S}_q = \int \left(\Psi^\dagger \frac{1}{2} \Sigma \Psi \right) d^3 r. \tag{35}$$

We suggest (34) is modified by a topological term

$$\mathbf{J}_{\mathrm{mod}} = \mathbf{J} + \mathbf{J}_{\mathrm{top}}. \tag{36}$$

Expression (36) is motivated by angular momentum holonomy conjecture [18]. Geometric phases in optics, analogous to the Aharonov-Bohm effect found extensive renewed activity after the discovery of the Berry phase in 1984. Angular momentum exchange was proposed as a physical mechanism to explain the geometric phases in optics [18]. Nonvanishing de Rham periods could manifest as $\mathbf{J}_{\mathrm{top}}$. Is $\mathbf{J}_{\mathrm{top}}$ associated with gluons? Does it have any relation with C_∞? It may seem reasonable to associate $\mathbf{J}_{\mathrm{top}}$ with gluons but it is not clear whether it is a part of gluon SAM or OAM.

Experiments providing information on generalized parton distributions (GPDs) and transverse momentum dependent (TMD) distributions pose delicate and challenging issues regarding the formalism and physical interpretation [1,2]. Wakamatsu shows that GPDs and polarized PDFs are insensitive to a continuous deformation of the path of Wilson lines [5]. Since the moments of these distributions are related to the dynamical OAM of quarks and gluons, it is interesting to ask how this result will be changed for closed paths, not just deformations. In fact, theoretically Hatta shows that a light-cone path leads to canonical OAM [19], while Ji *et al.* [20] argue that straight paths connecting spacetime points give dynamical OAM. In an interesting paper Burkardt [21] reconsiders the path choice for the gauge links and offers a physical interpretation for the difference between the Jaffe-Manohar definition of OAM [14] and that of Ji [15]. The difference between canonical and dynamical OAMs and the role of potential angular momentum [17] have been sought to be understood in terms of the color Lorentz force and the color magnetic flux; however, as noted by the author no experiment measures the implicit torque. Therefore, in our view the work in this direction is inconclusive on the nature of OAMs *vis-à-vis* GPDs and TMDs.

In contrast to this the alternative possibility suggested in the present work, namely the nontrivial topological defects encircled by closed paths leading to a contribution $\mathbf{J}_{\mathrm{top}}$ to gluon OAM $\mathbf{L}_g$, is worth exploring. In conclusion, the controversy regarding the uniqueness of the gauge invariant decomposition of the 4-vector gauge potential is resolved invoking Hodge decomposition and the de Rham theorem, and plausible arguments are presented to suggest a topological angular momentum contribution to the proton spin arising as a consequence of nonvanishing de Rham periods.

[1] C. A. Aidala, S. D. Bass, D. Hasch, and G. K. Mallot, Rev. Mod. Phys. **85**, 655 (2013).
[2] J. Collins, *Foundations of Perturbative QCD* (Cambridge University Press, Cambridge, England, 2011).
[3] A. W. Thomas, Chinese Phys. C **34**, 1153 (2010).
[4] E. Leader, Phys. Rev. D **83**, 096012 (2011).
[5] M. Wakamatsu, Phys. Rev. D **87**, 094035 (2013).
[6] C. Lorce, Phys. Lett. B **719**, 185 (2013).
[7] M. N. Stoilov, arXiv:1011.5617.
[8] S. C. Tiwari, arXiv:0807.0699.
[9] M. Wakamatsu, arXiv:1304.1589.
[10] C. Lorce, arXiv:1306.0456.
[11] G. Dassios and I. V. Lindell, J. Phys. A **35**, 5139 (2002).
[12] Y. Choquet-Bruhat, C. DeWitt-Morette, and M. Dillard-Bleick, *Analysis, Manifolds and Physics Part I: Basics* (Elsevier, The Netherlands, 1982), revised edition.
[13] H. Y. Cheng, Int. J. Mod. Phys. A **11**, 5109 (1996).
[14] R. L. Jaffe and A. Manohar, Nucl. Phys. **B337**, 509 (1990).
[15] X. D. Ji, Phys. Rev. Lett. **78**, 610 (1997).
[16] X. S. Chen, X. F. Lu, W. M. Sun, F. Wang, and T. Goldman, Phys. Rev. Lett. **100**, 232002 (2008).
[17] M. Wakamatsu, Phys. Rev. D **81**, 114010 (2010).
[18] S. C. Tiwari, Phys. Lett. A **149**, 223 (1990); J. Mod. Opt. **39**, 1097 (1992).
[19] Y. Hatta, Phys. Lett. B **708**, 186 (2012).
[20] X. Ji, X. Xiong, and F. Yuan, Phys. Rev. Lett. **109**, 152005 (2012).
[21] M. Burkardt, Phys. Rev. D **88**, 014014 (2013).