

Unitary representations of N -conformal Galilei group

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All unitary irreducible representations of the centrally extended (N -odd) N -conformal Galilei group are constructed. The “on-shell” action of the group is derived and shown to coincide, in a special but most important case, with that obtained by Gomis and Kamimura [Phys. Rev. D **85**, 045023 (2012)].

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I. INTRODUCTION

The N -conformal Galilean algebras/groups provide the generalization of the celebrated Schrödinger algebra/group ($N = 1$) discovered in the nineteenth century in the context of classical mechanics [1] and the heat equation [2] and subsequently rediscovered in the twentieth century as the maximal symmetry group (consisting of point transformations) of free motion in quantum mechanics [3]; the mathematical structure and geometrical as well as physical status of the Schrödinger group has been studied quite extensively [4].

The higher ($N > 1$) N -conformal Galilean groups are also interesting, both from mathematical and physical points of view. Their detailed description was presented in the paper by Negro *et al.* [5]. They were further studied in numerous papers [6].

The N -Galilean conformal algebras split naturally into two classes: for N odd they admit a one-parameter central extension [7–9] while no central extension is admitted for N even (except the case of two-dimensional space when the central extension exists for all N).

One can pose the question concerning the general form of dynamics (both classical and quantum) which is invariant under N -conformal Galilei transformations. For the N -odd and nontrivial central extension, the answer was provided in Ref. [8]. It appeared that the centrally extended odd N -conformal algebra was the symmetry algebra of free dynamics described by the Lagrangian containing $\frac{N+1}{2}$ -th order time derivatives. This conclusion has been confirmed in Ref. [10], where the orbit method [11] was applied to the problem of classification of all invariant Hamiltonian structures. It appears that the most general canonical system invariant under the N -conformal Galilei group consists of the set of “external” canonical variables together with spin and pseudospin ones, corresponding to the $SU(2)$ and $SL(2, R)$ subgroups, respectively; the dynamics of the external variables is described by the Ostrogradski Hamiltonian [12]. The authors of Ref. [8] computed also the action of the N -conformal Galilei group on wave functions obeying the relevant Schrödinger equation (on-shell action) and demonstrated its invariance.

In the present paper we complete the picture by finding all irreducible unitary representations of the centrally extended N -conformal Galilei group. Then we show that, when restricted “on shell,” they yield the generalization of transformation rules derived by Gomis and Kamimura [8]. Our results extend (to any odd N) those obtained by Perroud [13] for the case $N = 1$ (Schrödinger group).

II. THE N -CONFORMAL GALILEI ALGEBRA AND GROUP

The N -conformal centrally extended Galilei algebra is described by the following nontrivial commutation rules:

$$\begin{aligned} [D, H] &= iH, & [D, K] &= -iK, & [K, H] &= 2iD, \\ [J^a, J^b] &= i\epsilon_{abc}J^c, & [J^a, C_j^b] &= i\epsilon_{abc}C_j^c, \\ [H, C_j^a] &= -ijC_{j-1}^a, & [K, C_j^a] &= i(N-j)C_{j+1}^a, & (2.1) \\ [D, C_j^a] &= i\left(\frac{N}{2} - j\right)C_j^a, \\ [C_j^a, C_k^b] &= i\delta_{ab}\delta_{Nj+k}(-1)^{\frac{k-j+1}{2}}k!j!M; \end{aligned}$$

here $a, b, c, \dots = 1, 2, 3$, $j, k, \dots = 0, 1, \dots, N$ and N is odd. For N even the commutation rules look the same except the last one, where one should put $M = 0$.

The structure of the algebra (2.1) is quite simple. We have three subalgebras: $su(2)$, spanned by J 's; $sl(2, \mathbb{R})$, spanned by H, D , and K ; and the Abelian one, c_N , which, for N odd, can be centrally extended to the solvable algebra, \tilde{c}_N . Denoting by g_N the algebra defined by the commutation rules (2.1), we have

$$g_N = (su(2) \oplus sl(2, \mathbb{R})) \ltimes (\tilde{c}_N). \quad (2.2)$$

The semidirect sum is defined by demanding that $c_N \oplus \mathbb{R}$ span the representation $D^{(1, \frac{N}{2})} \oplus D^{(0, 0)}$ of $su(2) \oplus sl(2, \mathbb{R})$.

The conformal Lie algebra can be easily integrated to yield the corresponding group G_n . We present below the form valid for any N (i.e., without central extension; it is not difficult to write out the extended version). It reads [14]

$$\begin{aligned} (g, \tilde{g}, X_{ia}) * (g', \tilde{g}', X'_{ia}) \\ = (gg', \tilde{g}\tilde{g}', R_{ab}(g)X'_{ib} + X_{ja}(D^{\frac{N}{2}}(\tilde{g}'))_i^j); \end{aligned} \quad (2.3)$$

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here $g \in SU(2)$, $\tilde{g} \in SL(2, \mathbb{R})$, $R(g) \in SO(3)$ is the rotation corresponding to g , and $D^{(N)}(\tilde{g})$ is an element of the $(2N + 1)$ -dimensional irreducible representation of $SL(2, \mathbb{R})$. To obtain the universal covering of G_N , one has only to replace $SL(2, \mathbb{R})$ with its universal covering.

III. IRREDUCIBLE UNITARY REPRESENTATIONS OF CENTRALLY EXTENDED N -CONFORMAL GALILEAN SYMMETRY

We are going to construct unitary irreducible representations of the N -conformal Galilean group with N odd under the assumption that the central charge is nontrivial,

$$M = m\mathbf{1}, \quad m \neq 0. \quad (3.1)$$

Let us introduce new operators \hat{q}_k^a , \hat{p}_k^a , $a = 1, 2, 3$, $k = 0, \dots, \frac{N-1}{2}$, by the formulas

$$C_k^a = (-1)^{k-\frac{N-1}{2}} k! p_k^a, \quad C_{N-k}^a = m(N-k)! q_k^a. \quad (3.2)$$

The new operators obey the canonical commutation rules,

$$[q_k^a, p_l^b] = i\delta_{ab}\delta_{kl}, \quad (3.3)$$

and generate the Weyl group. The latter possess unique irreducible unitary representation. Once it is written out, one easily constructs the unitary representation of a solvable subgroup of the (centrally extended) N -conformal Galilei group, generated by C_k^a and M .

The remaining nontrivial commutation relations of N -conformal algebra, written in terms of q 's and p 's, read

$$\begin{aligned} [J^a, q_k^b] &= i\epsilon_{abc} q_k^c, & [J^a, p_k^b] &= i\epsilon_{abc} p_k^c, \\ [D, q_k^a] &= i\left(k - \frac{N}{2}\right) q_k^a, \\ [D, p_k^a] &= i\left(\frac{N}{2} - k\right) p_k^a, \quad k = 0, \dots, \frac{N-1}{2}, \\ [K, q_k^a] &= ik(N-k+1)q_{k-1}^a, \quad k = 0, \dots, \frac{N-1}{2}, \\ [K, p_k^a] &= -i(N-k)(k+1)p_{k+1}^a, \quad k = 0, \dots, \frac{N-3}{2}, \\ [K, p_{\frac{N-1}{2}}^a] &= im\left(\frac{N+1}{2}\right) q_{\frac{N-1}{2}}^a, \\ [H, q_k^a] &= -iq_{k+1}^a, \quad k = 0, \dots, \frac{N-3}{2}, \\ [H, q_{\frac{N-1}{2}}^a] &= \frac{-i}{m} p_{\frac{N-1}{2}}^a, \\ [H, p_k^a] &= ip_{k-1}^a, \quad k = 0, \dots, \frac{N-1}{2}. \end{aligned} \quad (3.4)$$

The important point is that one can construct the operators belonging to the universal enveloping algebra of Heisenberg algebra, which [8,10]

- (i) obey the commutation rules of $su(2) \oplus sl(2, \mathbb{R})$;
- (ii) obey the same commutation rules with q 's and p 's as \hat{J} , H , D , and K .

They are defined as follows:

$$\begin{aligned} A &= m \sum_{k=0}^{\frac{N-3}{2}} p_k^a q_{k+1}^a + \frac{1}{2} (p_{\frac{N-1}{2}})^2, \\ B &= -m \sum_{k=0}^{\frac{N-3}{2}} (k+1)(N-k) p_{k+1}^a q_k^a + \frac{m^2}{2} \left(\frac{N+1}{2}\right)^2 (q_{\frac{N-1}{2}}^a)^2, \\ C &= \frac{m}{2} \sum_{k=0}^{\frac{N-1}{2}} \left(\frac{N}{2} - k\right) (q_k^a p_k^a + p_k^a q_k^a), \\ L^a &= \epsilon_{abc} \sum_{k=0}^{\frac{N-1}{2}} q_k^b p_k^c. \end{aligned} \quad (3.5)$$

It is easy to check that

$$\begin{aligned} [B, C] &= imB, & [C, A] &= imA, \\ [A, B] &= -2miC, & [L^a, L^b] &= i\epsilon_{abc} L^c, \end{aligned} \quad (3.6)$$

and that the commutation rules (3.4) are obeyed with J^a , D , K , H replaced by L^a , $\frac{1}{m}C$, $\frac{1}{m}B$, and $\frac{1}{m}A$, respectively. Therefore the operators

$$\begin{aligned} s^a &= J^a - L^a, & d &= D - \frac{1}{m}C, \\ h &= H - \frac{1}{m}A, & k &= K - \frac{1}{m}B \end{aligned} \quad (3.7)$$

obey the $su(2) \oplus sl(2, \mathbb{R})$ algebra commutation rules and commute with q 's and p 's. Consequently, Eqs. (3.7) define the decomposition of $su(2) \oplus sl(2, \mathbb{R})$ generators into ‘‘external’’ and ‘‘internal’’ parts, in full analogy with the well-known decomposition of total angular momentum into the orbital one and spin. This allows us to conclude that the space carrying a unitary representation of the N -conformal Galilei group is the tensor product of $L^2(\mathbb{R}^{3\frac{N+1}{2}})$, which carries the representation of the Weyl group generated by q_k^a and p_k^a , $k = 0, 1, \dots, \frac{N-1}{2}$, $a = 1, 2, 3$ and the carrier space of the representation of the $SU(2) \times SL(2, \mathbb{R})$ group generated by \vec{s} , d , h and k .

The irreducible representations under consideration are classified by the eigenvalues m of the central charge M and the choice of the irreducible representations of $SU(2)$ and $SL(2, \mathbb{R})$ groups mentioned above. The representations of $SU(2)$ are uniquely determined by the values of the Casimir operator

$$C_1 = \vec{s}^2, \quad (3.8)$$

while in the $SL(2, \mathbb{R})$ case apart from the eigenvalue of the second Casimir operator

$$C_2 = \hat{h} \hat{k} + \hat{k} \hat{h} - 2\hat{d}^2, \quad (3.9)$$

we need some additional information concerning the spectrum of one generator [15].

Let us start with the case $\mathcal{C}_1 = \mathcal{C}_2 = 0$. Then the representation of internal $SU(2) \times SL(2, \mathbb{R})$ group is trivial and the total representation space is $L^2(\mathbb{R}^{3\frac{(N+1)}{2}})$ consisting of functions $\psi(\vec{q}_0, \dots, \vec{q}_{\frac{N-1}{2}})$ square integrable with respect to the standard Lebesgue measure. Due to the identification (3.2) the action of a centrally extended Abelian subgroup generated by the C_N subalgebra is easily expressible in terms of standard action of the Weyl group. It reads

$$(e^i \sum_{k=0}^N \vec{x}_k \vec{C}_k \psi)(\vec{q}_0, \dots, \vec{q}_{\frac{N-1}{2}}) = e^{\frac{im}{2} \sum_{k=0}^{\frac{N-1}{2}} (-1)^{k-\frac{N-1}{2}} k!(N-k)! \vec{x}_k \vec{x}_{N-k}} e^{im \sum_{k=0}^{\frac{N-1}{2}} (N-k)! \vec{x}_{N-k} \vec{q}_k} \times \psi\left(\vec{q}_0 + (-1)^{\frac{1-N}{2}} \vec{x}_0, \dots, \vec{q}_j + (-1)^{j+\frac{1-N}{2}} j! \vec{x}_j, \dots, \vec{q}_{\frac{N-1}{2}} + \left(\frac{N-1}{2}\right)! \vec{x}_{\frac{N-1}{2}}\right). \quad (3.10)$$

The action of the $SU(2)$ subgroup is standard

$$(U(g)\psi)(\vec{q}_0, \dots, \vec{q}_{\frac{N-1}{2}}) = \overrightarrow{\psi(R^{-1}(g)q_0, \dots, R^{-1}(g)q_{\frac{N-1}{2}})}, \quad (3.11)$$

where $g \in SU(2)$ and $R(g) \in SO(3)$ is the corresponding rotation.

It remains to find the action of the $SL(2, \mathbb{R})$ subgroup. To this end we use the Iwasawa decomposition [13,16] of $SL(2, \mathbb{R})$

$$\tilde{g} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}, \quad u > 0. \quad (3.12)$$

By noting that the $sl(2, \mathbb{R})$ generators in defining representation can be written as

$$D = -\frac{i}{2}\sigma_3, \quad K = -i\sigma_+, \quad H = i\sigma_-, \quad (3.13)$$

we rewrite Eq. (3.12) in the form

$$\tilde{g} = e^{i\theta(H+K)} e^{i\lambda D} e^{ivK}, \quad u = e^{\frac{\lambda}{2}}. \quad (3.14)$$

It is, therefore, sufficient to determine the action of one-parameter subgroups generated by D , K , and $\mathcal{H} = H + K$. The relevant generators in the representation under consideration are $\frac{1}{m}C$, $\frac{1}{m}B$, and $\frac{1}{m}(A + B)$, respectively.

Let us start with the dilatation generator. Taking into account the explicit form of the operator C , we find $[U(\lambda) = e^{i\lambda D}]$

$$(U(\lambda)\psi)(\vec{q}_0, \dots, \vec{q}_{\frac{N-1}{2}}) = e^{\frac{3\lambda(N+1)^2}{16}} \psi(e^{\frac{3\lambda}{2}} \vec{q}_0, \dots, e^{(\frac{N}{2}-j)\lambda} \vec{q}_j, \dots, e^{\frac{\lambda}{2}} \vec{q}_{\frac{N-1}{2}}). \quad (3.15)$$

The action of $U(v) = e^{ivK}$ is slightly more complicated. Detailed computations are given in Appendix A; we quote here only the final result

$$(U(v)\psi)(\vec{q}_0, \dots, \vec{q}_{\frac{N-1}{2}}) = e^{iF(v)} \psi(\vec{q}_0(v), \dots, \vec{q}_{\frac{N-1}{2}}(v)), \quad (3.16)$$

where

$$\vec{q}_j(v) = \sum_{p=0}^j \binom{j}{p} \frac{(N-j+p)!}{(N-j)!} (-v)^p \vec{q}_{j-p} \quad (3.17)$$

$$F(v) = \frac{m}{2} \left(\frac{N+1}{2}\right)^2 \int_0^v du \vec{q}_{\frac{N-1}{2}}^2(u).$$

Finally, consider the action of a compact subgroup generated by $\mathcal{H} = H + K = \frac{1}{m}(A + B)$. In this case the generator is a second order differential operator; therefore, the action of the compact subgroup is nonlocal. Denoting by $\mathcal{H}(\vec{q}; \vec{q}'; \theta)$ the kernel of $\exp(i\theta\mathcal{H})$, one can write

$$(e^{i\theta\mathcal{H}}\psi)(\vec{q}_0, \dots, \vec{q}_{\frac{N-1}{2}}) = \int d^3\vec{q}'_0 \cdots d^3\vec{q}'_{\frac{N-1}{2}} \mathcal{H}(\vec{q}_0, \dots, \vec{q}_{\frac{N-1}{2}}; \vec{q}'_0, \dots, \vec{q}'_{\frac{N-1}{2}}; \theta) \times \psi(\vec{q}'_0, \dots, \vec{q}'_{\frac{N-1}{2}}). \quad (3.18)$$

The algorithm of constructing the kernel $\mathcal{H}(\vec{q}; \vec{q}'; \theta)$ is presented in Appendix B. Due to the fact that \mathcal{H} is defined by a quadratic form in canonical variables, the problem is obviously exactly solvable; the final form is, however, quite involved.

Let us now consider the general case $\mathcal{C}_1 \neq 0$ and/or $\mathcal{C}_2 \neq 0$. Due to our decomposition of generators into external and internal parts, we conclude that the carrier space consists of square integrable functions taking their values in the representation space of the representation of $SU(2) \times SL(2, \mathbb{R})$ [or, more generally, the universal covering of $SL(2, \mathbb{R})$ generated by \vec{s} , d , k , and h ; note that due to noncompactness of $SL(2, \mathbb{R})$ the relevant unitary representation is infinite dimensional (except the case $\mathcal{C}_2 = 0$)]. The complete classification of unitary irreducible representation of $SL(2, \mathbb{R})$ and its universal covering is well known [15,17]. For convenience we use the matrix notation both for representations of $SU(2)$ and $SL(2, \mathbb{R})$ (in the latter case, the matrices are infinite dimensional so in practice it is better to work with functional realizations). Let D and Δ denote the relevant matrix representations of $SU(2)$ and $SL(2, \mathbb{R})$, respectively. In order to write out the action of the group, we first note that our relations (3.10) remain unchanged. The $SU(2)$ subgroup acts as follows:

$$(U(g)\psi)_{\alpha p}(\vec{q}_0, \dots, \vec{q}_{\frac{N-1}{2}}) = D_{pr}(g) \psi_{\alpha r}(\overrightarrow{R^{-1}(g)q_0}, \dots, \overrightarrow{R^{-1}(g)q_{\frac{N-1}{2}}}). \quad (3.19)$$

As far as $SL(2, \mathbb{R})$ is concerned we obtain

$$(U(e^{i\lambda D})\psi)_{\alpha p}(\vec{q}_0, \dots, \vec{q}_{\frac{N-1}{2}}) = e^{\frac{3\lambda}{16}(N+1)^2} \Delta_{\alpha\beta}(e^{i\lambda D})\psi_{\beta p}(e^{\frac{N}{2}}\vec{q}_0, \dots, e^{\frac{\lambda}{2}}\vec{q}_{\frac{N-1}{2}}), \quad (3.20)$$

$$(U(v)\psi)_{\alpha p}(\vec{q}_0, \dots, \vec{q}_{\frac{N-1}{2}}) = e^{iF(v)} \Delta_{\alpha\beta}(e^{ivK})\psi_{\beta p}(\vec{q}_0(v), \dots, \vec{q}_{\frac{N-1}{2}}(v)), \quad (3.21)$$

and

$$(e^{i\theta(H+K)}\psi)_{\alpha p}(\vec{q}_0, \dots, \vec{q}_{\frac{N-1}{2}}) = \int d^3\vec{q}'_0 \cdots d^3\vec{q}'_{\frac{N-1}{2}} \mathcal{H}(\vec{q}_0, \dots, \vec{q}_{\frac{N-1}{2}}; \vec{q}'_0, \dots, \vec{q}'_{\frac{N-1}{2}}; \theta) \times \Delta_{\alpha\beta}(e^{i\theta(H+K)})\psi_{\beta p}(\vec{q}'_0, \dots, \vec{q}'_{\frac{N-1}{2}}), \quad (3.22)$$

for dilations, conformal, and compact transformations, respectively.

IV. THE ON-SHELL REALIZATION

In this section we consider the action of N -conformal Galilei transformations on the solutions of the Schrödinger equation. For simplicity we assume $\mathcal{C}_1 = \mathcal{C}_2 = 0$ (the results are easily extendible to the general case), so our Hamiltonian takes the Ostrogradski form [8,10,12].

The action of Galilean conformal transformations is defined as follows. Given wave function at the time t , we

translate it back to $t = 0$, then act with an element of our group, and finally translate the result again to time t in formulas

$$\tilde{U}(g, t) = e^{-itH}U(g)e^{itH}. \quad (4.1)$$

Note the identities

$$e^{-itH}e^{ix_k^a C_k^a}e^{itH} = e^{ix_k^a(t)C_k^a}, \quad x_k^a(t) = \sum_{j=k}^N \binom{j}{k} (-t)^{j-k} x_j^a, \quad (4.2a)$$

$$e^{-itH}U(g)e^{itH} = U(g), \quad g \in SU(2), \quad (4.2b)$$

$$e^{-itH}e^{i\lambda D}e^{itH} = e^{i\lambda D}e^{i(1-e^\lambda)tH}, \quad (4.2c)$$

$$e^{-itH}e^{ivK}e^{itH} = e^{-2t \ln(1+vt)D}e^{iv(1+vt)K}e^{i(t-\frac{t}{1+vt})H}, \quad (4.2d)$$

$$e^{-itH}e^{i\tau H}e^{itH} = e^{i\tau H}. \quad (4.2e)$$

According to the first formula (4.2a) the on-shell action of the subgroup generated by C_N is given by Eq. (3.10), with x_k^a replaced by $x_k^a(t)$. The action of the $SU(2)$ subgroup remains unchanged. On the other hand, Eq. (4.2c) yields

$$(\tilde{U}(e^{i\lambda D}, t)\psi)(\vec{q}_0, \dots, \vec{q}_{\frac{N-1}{2}}; t) = e^{\frac{3\lambda}{16}(N+1)^2} \psi(e^{\frac{N}{2}}\vec{q}_0, \dots, e^{\frac{\lambda}{2}}\vec{q}_{\frac{N-1}{2}}; e^\lambda t). \quad (4.3)$$

The action of conformal transformation is slightly more complicated. By virtue of Eqs. (3.15), (3.16), (3.17), and (4.2d), one obtains

$$(\tilde{U}(e^{ivK}, t)\psi)(\vec{q}_0, \dots, \vec{q}_{\frac{N-1}{2}}; t) = (1+vt)^{-\frac{3}{8}(N+1)^2} e^{iF(\frac{v}{1+vt})} \psi\left((1+vt)^{-N}\vec{q}_0\left(\frac{v}{1+vt}\right), \dots, (1+vt)^{-1}\vec{q}_{\frac{N-1}{2}}\left(\frac{v}{1+vt}\right); \frac{t}{1+vt}\right). \quad (4.4)$$

Finally,

$$(\tilde{U}(e^{i\tau H}, t)\psi)(\vec{q}_0, \dots, \vec{q}_{\frac{N-1}{2}}) = \psi(\vec{q}_0, \dots, \vec{q}_{\frac{N-1}{2}}; t - \tau). \quad (4.5)$$

The results obtained may be compared with those contained in Ref. [8]. We find that they are in full agreement.

V. CONCLUDING REMARKS

We constructed all unitary representations of centrally extended N -conformal Galilei groups. They provide the framework for constructing the “elementary” quantum mechanical systems with the N -conformal Galilei group as a symmetry group.

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APPENDIX A

We find here the action of conformal transformations. To this end, let us write it in the form

$$(e^{ivK}\psi)(\vec{q}_0, \dots, \vec{q}_{\frac{N-1}{2}}) = \psi(e^{ivK}\vec{q}_0 e^{-ivK}, \dots, e^{ivK}\vec{q}_{\frac{N-1}{2}} e^{-ivK})e^{ivK} \cdot 1. \quad (A1)$$

Denoting

$$\vec{q}_j(v) = e^{ivK}\vec{q}_j e^{-ivK}, \quad (A2)$$

we find

$$\frac{d\vec{q}_j(v)}{dv} = ie^{ivK}[K, \vec{q}_j]e^{ivK} = -j(N-j+1)\vec{q}_{j-1}(v). \quad (A3)$$

Together with the initial conditions $\vec{q}_j(0) = \vec{q}_j$, Eqs. (A3) yield

$$\tilde{q}_j(v) = \sum_{p=0}^j \binom{j}{p} \frac{(N-j+p)!}{(N-j)!} (-v)^p \tilde{q}_{j-p}. \quad (\text{A4})$$

To compute $e^{ivK} \cdot 1$, we decompose $K = \frac{1}{m}B$ as follows:

$$K = \left(i \sum_{k=0}^{\frac{N-3}{2}} (k+1)(N-k) \tilde{q}_k \frac{\partial}{\partial \tilde{q}_{k+1}} \right) + \left(\frac{m}{2} \left(\frac{N+1}{2} \right)^2 \tilde{q}_{\frac{N-1}{2}}^2 \right) \\ = X + Y, \quad (\text{A5})$$

and put

$$e^{iv(X+Y)} = Z(v)e^{ivX}. \quad (\text{A6})$$

Then $Z(v)$ obeys

$$\frac{dZ(v)}{dv} = ie^{iv(X+Y)} Y e^{-iv(X+Y)} Z(v) \\ = \frac{im}{2} \left(\frac{N+1}{2} \right)^2 \tilde{q}_{\frac{N-1}{2}}^2(v) Z(v), \quad (\text{A7})$$

which yields [$Z(0) = 1$]

$$Z(v) = \exp \left(\frac{im}{2} \left(\frac{N+1}{2} \right)^2 \int_0^v du \tilde{q}_{\frac{N-1}{2}}^2(u) \right). \quad (\text{A8})$$

By virtue of (A6) and (A8) one obtains

$$e^{ivK} \cdot 1 = \exp(iF(v)), \quad (\text{A9}) \\ F(v) = \frac{m}{2} \left(\frac{N+1}{2} \right)^2 \int_0^v du \tilde{q}_{\frac{N-1}{2}}^2(u).$$

APPENDIX B

Our aim is to describe the dynamics (both classical and quantum) generated by the ‘‘Hamiltonian’’ resulting from Iwasawa decomposition (3.14). To this end, we define the following Hamiltonian:

$$\mathcal{H} = \left(\sum_{k=0}^{n-1} p_k q_{k+1} + \frac{1}{2m} p_n^2 \right) \\ + \left(- \sum_{k=0}^{n-1} (k+1)(2n+1-k) p_{k+1} q_k \right. \\ \left. + \frac{m(n+1)^2}{2} q_n^2 \right), \quad (\text{B1})$$

where for simplicity we define $N = 2n + 1$ and skip vector indices.

The relevant canonical (or Heisenberg) equations of motion read

$$\dot{q}_k = q_{k+1} - (2(n+1) - k)kq_k, \\ \dot{p}_k = (k+1)(2n+1-k)p_{k+1} - p_{k-1}, \\ \dot{q}_n = \frac{1}{m} p_n - (n+2)nq_{n-1}, \\ \dot{p}_n = -m(n+1)^2 q_n - p_{n-1}. \quad (\text{B2})$$

Equations (B2) can be solved as follows. Define

$$q_k = k! \xi_k, \quad p_k = (-1)^{n-k} (2n+1-k)! m \xi_{2n+1-k}, \quad (\text{B3})$$

for $k = 0, \dots, n$. In terms of new variables, Eqs. (B3) read (cf. Ref. [18])

$$\dot{\xi}_k = (k+1)\xi_{k+1} - (2(n+1) - k)\xi_{k-1}, \quad k = 0, \dots, 2n+1. \quad (\text{B4})$$

The boundary conditions $q_{-1} = 0, p_{-1} = 0$ yield $\xi_{-1} = 0, \xi_{2(n+1)} = 0$, which, together with (B4), implies $\xi_k = 0$ for $k \leq -1$ or $k \geq 2(n+1)$. Let x^k be a real variable and

$$\xi(x, t) = \sum_{k=0}^{2n+1} \xi_k(t) x^k. \quad (\text{B5})$$

Equations (B4) can be summarized as follows:

$$\frac{\partial \xi(x, t)}{\partial t} = (1 + x^2) \frac{\partial \xi(x, t)}{\partial x} - Nx \xi(x, t). \quad (\text{B6})$$

By virtue of the boundary conditions for ξ_k we are looking for the solutions of Eq. (B6) which are polynomials of degree $2n + 1$ in x . They read

$$\xi(x, t) = \sum_{\substack{l=-(2n+1) \\ l\text{-odd}}}^{2n+1} s_l e^{ilt} P_l(x), \quad (\text{B7})$$

$$P_l(x) = (1 + ix)^{\frac{2n+1+l}{2}} (1 - ix)^{\frac{2n+1-l}{2}}. \quad (\text{B8})$$

Let us note the following properties of the polynomials $P_l(x)$:

$$\overline{P_l(x)} = P_{-l}(x), \quad (\text{B9})$$

and

$$\int_{-\infty}^{\infty} d\mu(x) \overline{P_l(x)} P_{l'}(x) = \delta_{ll'}, \quad d\mu(x) = \frac{1}{\pi} \frac{dx}{(1+x^2)^{2(n+1)}}. \quad (\text{B10})$$

The polynomials $P_l(x)$ form an orthonormal basis in the space of polynomials of degree $2n + 1$.

Note that $\xi(x, t)$ is real, so Eq. (B9) implies

$$\bar{s}_l = s_{-l}. \quad (\text{B11})$$

By comparing Eqs. (B5) and (B7), one finds

$$\xi_k(t) = \sum_{l=-(2n+1)}^{2n+1} i^k \beta_{kl} e^{ilt} s_l, \quad (\text{B12})$$

where the coefficients β_{kl} are defined through

$$\sum_{k=0}^{2n+1} \beta_{kl} x^k = (1+x)^{\frac{2n+1+l}{2}} (1-x)^{\frac{2n+1-l}{2}}. \quad (\text{B13})$$

Note the following properties of β_{kl} :

$$\beta_{kl} = \sum_{m=0}^k (-1)^m \binom{\frac{2n+1-l}{2}}{m} \binom{\frac{2n+1+l}{2}}{k-m}, \quad (\text{B14})$$

$$\beta_{k-l} = (-1)^k \beta_{kl}, \quad \beta_{2n+1-k,l} = (-1)^{\frac{2n+1-l}{2}} \beta_{kl}.$$

We will see later that the coefficient β_{kl} has a nice interpretation in the language of finite-dimensional representations of $SL(2, \mathbb{R})$ [see Eq. (B41)].

Equations (B12) define new dynamical variables

$$s_l(t) = e^{ilt} s_l, \quad (\text{B15})$$

$$l = -(2n+1), -(2n-1), \dots, (2n-1), 2n+1;$$

in order to compute their Poisson brackets, we use (B3) and (B5) to write

$$\xi(x, t) = \sum_{k=0}^n \frac{q_k(t)}{k!} x^k + \sum_{k=n+1}^{2n+1} \frac{(-1)^{n+1-k} p_{2n+1-k}(t) x^k}{mk!}. \quad (\text{B16})$$

Equation (3.13), together with the canonical Poisson brackets, yields

$$\{\xi(x, t), \xi(y, t)\} = \frac{(-1)^{n+1}}{m(2n+1)!} (x-y)^{2n+1}. \quad (\text{B17})$$

On the other hand,

$$\{\xi(x, t), \xi(y, t)\} = \sum_{\substack{l, l' = -(2n+1) \\ l, l' \text{ odd}}}^{2n+1} \{s_l, s_{l'}\} e^{i(l+l')t} P_l(x) P_{l'}(y). \quad (\text{B18})$$

The left-hand side does not depend on t . Therefore,

$$\mathcal{H} = \frac{m}{2} \sum_{\substack{l, l' = -(2n+1) \\ l, l' \text{ odd}}}^{2n+1} s_l s_{l'} e^{i(l+l')t} \left(\sum_{k=0}^{2n} (-1)^{2n+1-k} (k+1)! (2n+1-k)! (\beta_{2n+1-k,l} \beta_{k+1,l'} - \beta_{2n-k,l} \beta_{k,l'}) \right). \quad (\text{B25})$$

Using (B14) we find

$$\begin{aligned} & \sum_{k=0}^{2n} (-1)^{2n+1-k} (k+1)! (2n+1-k)! (\beta_{2n+1-k,l} \beta_{k+1,l'} - \beta_{2n-k,l} \beta_{k,l'}) \\ &= \sum_{k=0}^{2n} (k+1)! (2n+1-k)! (\beta_{2n+1-k,l} \beta_{k+1,-l'} + \beta_{2n-k,l} \beta_{k,-l'}) \\ &= (-1)^{\frac{2n+1-l}{2}} \sum_{k=0}^{2n} (k+1)! (2n+1-k)! (\beta_{k,l} \beta_{k+1,-l'} + \beta_{k+1,l} \beta_{k,-l'}). \end{aligned} \quad (\text{B26})$$

To proceed further, note that $P_l(x)$ obeys

$$\left((1+x^2) \frac{d}{dx} - (2n+1)x \right) P_l(x) = il P_l(x). \quad (\text{B27})$$

Inserting the expansion

$$P_l(x) = \sum_{k=0}^{2n+1} i^k \beta_{kl} x^k, \quad (\text{B28})$$

$$\{s_l, s_{l'}\} = F(n, l) \delta_{l,-l'}, \quad (\text{B19})$$

and

$$\{\xi(x, t), \xi(y, t)\} = \sum_{\substack{l, l' = -(2n+1) \\ l \text{ odd}}}^{2n+1} F(n, l) P_l(x) P_{-l}(y). \quad (\text{B20})$$

Putting $y = 0$ and using (B17), one arrives at the following relation:

$$\frac{(-1)^{n+1}}{m(2n+1)!} x^{2n+1} = \sum_{\substack{l = -(2n+1) \\ l \text{ odd}}}^{2n+1} F(n, l) P_l(x). \quad (\text{B21})$$

One can find $F(n, l)$ using orthogonality relation (B10). Alternatively, putting $x = \tan \phi$ one gets

$$\frac{(-1)^{n+1}}{m(2n+1)!} \sin^{2n+1}(\phi) = \sum_{\substack{l = -(2n+1) \\ l \text{ odd}}}^{2n+1} F(n, l) e^{il\phi}, \quad (\text{B22})$$

leading to

$$F(n, l) = \frac{-i(-1)^{\frac{2n+1+l}{2}}}{m 2^{2n+1} \left(\frac{2n+1+l}{2}\right)! \left(\frac{2n+1-l}{2}\right)!}, \quad (\text{B23})$$

or, by virtue of (B19),

$$\{s_l, s_{l'}\} = \frac{-i(-1)^{\frac{2n+1+l}{2}}}{m 2^{2n+1} \left(\frac{2n+1+l}{2}\right)! \left(\frac{2n+1-l}{2}\right)!} \delta_{l,-l'}. \quad (\text{B24})$$

It remains to express the Hamiltonian (B1) in terms of new variables. Using (B1), (B3), and (B12), one obtains

one derives the recurrence relation

$$(k+1)\beta_{k+1,l} + (2(n+1)-k)\beta_{k-1,l} - l\beta_{k,l} = 0, \quad k = 1, \dots, 2n+1 \quad \beta_{1l} - l\beta_{0l} = 1. \quad (\text{B29})$$

It easy to check with the help of (B29) that

$$\begin{aligned} & (l-l') \sum_{k=0}^{2n+1} k!(2n+1-k)!\beta_{k,l}\beta_{k,-l'} \\ &= 2 \sum_{k=0}^{2n} (k+1)!(2n+1-k)!(\beta_{k+1,l}\beta_{k,-l'} + \beta_{kl}\beta_{k+1,-l'}). \end{aligned} \quad (\text{B30})$$

We will show that

$$\sum_{k=0}^{2n+1} k!(2n+1-k)!\beta_{k,l}\beta_{k,l'} = G(n,l)\delta_{ll'}, \quad (\text{B31})$$

where

$$G(n,l) = 2^{2n+1} \left(\frac{2n+1+l}{2}\right)! \left(\frac{2n+1-l}{2}\right)!. \quad (\text{B32})$$

We prove (B31) and (B32) in two ways. In the first approach we note that the matrix $i^k\beta_{kl}$ is invertible because it relates two bases. Putting

$$x^k = \sum_{\substack{l=-(2n+1) \\ l\text{-odd}}}^{2n+1} \gamma_{lk} P_l(x), \quad (\text{B33})$$

one gets

$$\sum_{\substack{l=-(2n+1) \\ l\text{-odd}}}^{2n+1} i^k \beta_{kl} \gamma_{lk'} = \delta_{kk'}, \quad \sum_{k=0}^{2n+1} i^k \gamma_{lk} \beta_{kl'} = \delta_{ll'}. \quad (\text{B34})$$

Equation (B31) is equivalent to

$$k!(2n+1-k)!\beta_{kl} = G(n,l)\gamma_{lk}(-i)^k. \quad (\text{B35})$$

We prove (B35) by deriving the recurrence relation for γ_{lk} . To this end, note that (B33) implies

$$\gamma_{lk} = \int_{-\infty}^{\infty} d\mu(x) x^k \overline{P_l(x)}. \quad (\text{B36})$$

The operator $(1+x^2)\frac{d}{dx} - (2n+1)x$ is anti-Hermitian with respect to the product defined by $d\mu(x)$. Therefore,

$$\begin{aligned} (2n+1-k)\gamma_{l,k+1} - k\gamma_{l,k-1} - il\gamma_{lk} &= 0, \\ (2n+1)\gamma_{1l} - il\gamma_{l0} &= 0. \end{aligned} \quad (\text{B37})$$

Now, $\frac{i^k k!(2n+1-k)!\beta_{kl}}{G(n,l)}$ obeys the same recurrence. To find $G(n,l)$, we note that $\beta_{0l} = 1$ and

$$\begin{aligned} \gamma_{l,0} &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi \cos^{2n+1}(\phi) e^{-il\phi} \\ &= \frac{(2n+1)!}{2^{2n+1} \left(\frac{2n+1-l}{2}\right)! \left(\frac{2n+1+l}{2}\right)!}, \end{aligned} \quad (\text{B38})$$

which concludes the proof of (B35).

The second proof of Eqs. (B31) and (B32) makes use of the properties of finite-dimensional irreducible representations of $SL(2, \mathbb{R})$. In fact, finite-dimensional irreducible representations of $SL(2, \mathbb{R})$ are classified by natural number N and they are matrices of the form (see, e.g., [14])

$$\begin{aligned} \Phi(A)_{mm'} &= \sum_{\max(0, m'-m)}^{\min(N-m, m')} \binom{N-m}{k} \\ &\times \binom{m}{m-k} a^{N-m-k} b^k c^{m-m'+k} d^{m'-k}, \end{aligned} \quad (\text{B39})$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$$

and $m, m' = 0, \dots, N$; in our case $N = 2n+1$. Now let A be of the form

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in SL(2, \mathbb{R}). \quad (\text{B40})$$

It is not difficult to show that

$$\Phi(A)_{mm'} = \frac{1}{\sqrt{2^{2n+1}}} \beta_{m', -(2n+1)+2m}. \quad (\text{B41})$$

Since $A^T A = A A^T = Id$, one finds

$$\Phi(A)\Phi(A^T) = \Phi(A^T A) = Id. \quad (\text{B42})$$

Moreover, let us note that in our case we have

$$\Phi(A)_{mm'} = (-1)^{m'-m} \frac{\binom{2n+1}{m'}}{\binom{2n+1}{m}} (\Phi(A))_{m'm}, \quad (\text{B43})$$

and

$$\Phi(A^T)_{mm'} = (-1)^{m-m'} \Phi(A)_{mm'}. \quad (\text{B44})$$

Inserting (B44) into Eq. (B42) and using Eqs. (B41) and (B43), we obtain the relations (B31) and (B32).

Collecting all formulas, we find

$$\mathcal{H} = \frac{m}{2} \sum_{\substack{l=1 \\ l\text{-odd}}}^{2n+1} (-1)^{\frac{2n+1-l}{2}} l 2^{2n+1} \times \left(\frac{2n+1-l}{2}\right)! \left(\frac{2n+1+l}{2}\right)! (s_l \bar{s}_l + \bar{s}_l s_l); \quad (\text{B45})$$

upon defining ($l = 1, \dots, 2n+1$, l -odd)

$$a_l = \begin{cases} 2^{\frac{2n+1}{2}} \sqrt{m} \sqrt{\left(\frac{2n+1-l}{2}\right)! \left(\frac{2n+1+l}{2}\right)!} s_l, & \frac{2n+1+l}{2} \text{-even} \\ 2^{\frac{2n+1}{2}} \sqrt{m} \sqrt{\left(\frac{2n+1-l}{2}\right)! \left(\frac{2n+1+l}{2}\right)!} \bar{s}_l, & \frac{2n+1+l}{2} \text{-odd,} \end{cases} \quad (\text{B46})$$

one arrives finally at the following result:

$$\mathcal{H} = \frac{1}{2} \sum_{\substack{l=0 \\ l\text{-odd}}}^{2n+1} (-1)^{\frac{2n+1-l}{2}} l (a_l \bar{a}_l + \bar{a}_l a_l), \quad (\text{B47})$$

$$\{a_l, \bar{a}_{l'}\} = -i \delta_{ll'}, \quad l, l' = 1, 3, \dots, 2n+1.$$

Quantization yields

$$\mathcal{H} = \frac{1}{2} \sum_{\substack{l=0 \\ l\text{-odd}}}^{2n+1} (-1)^{\frac{2n+1-l}{2}} l \left(a_l^\dagger a_l + \frac{1}{2} \right), \quad (\text{B48})$$

$$[a_l, a_{l'}^\dagger] = \delta_{ll'}, \quad l, l' = 1, 3, \dots, 2n+1.$$

Define further, for $k = 0, 1, \dots, n$, $l = 1, 3, \dots, 2n+1$,

$$\rho_{kl} = \begin{cases} \frac{i^k k! \beta_{kl}}{2^{\frac{2n+1}{2}} \sqrt{m} \sqrt{\left(\frac{2n+1-l}{2}\right)! \left(\frac{2n+1+l}{2}\right)!}}, & \frac{2n+1+l}{2} \text{-even} \\ \frac{(-i)^k k! \beta_{kl}}{2^{\frac{2n+1}{2}} \sqrt{m} \sqrt{\left(\frac{2n+1-l}{2}\right)! \left(\frac{2n+1+l}{2}\right)!}}, & \frac{2n+1+l}{2} \text{-odd,} \end{cases} \quad (\text{B49})$$

and then

$$q_k = \sum_{\substack{l=1 \\ l\text{-odd}}}^{2n+1} \rho_{kl} a_l + \sum_{\substack{l=1 \\ l\text{-odd}}}^{2n+1} \bar{\rho}_{kl} a_l^\dagger, \quad k = 0, \dots, n. \quad (\text{B50})$$

The coordinate operators commute, which implies

$$\rho \bar{\rho}^T = \bar{\rho} \rho^T, \quad (\text{B51})$$

so $\rho \bar{\rho}^T$ is real symmetric [Eq. (B51) can be checked directly using (B31)] and hence diagonalizable by real orthogonal transformation. Assume $\det \rho = 0$; then $\det(\rho \bar{\rho}^T) = 0$ and there exists real nonzero vector u such that $u^T \rho = 0$. Taking the complex conjugate, one finds $u^T \bar{\rho} = 0$. Therefore, by virtue of Eq. (B50),

$$\sum_{k=0}^n u_k q_k = 0, \quad (\text{B52})$$

which contradicts the canonical commutation rules. So we conclude that ρ is invertible.

Finally, define for $k = 0, \dots, n$, $l = 1, 3, \dots, 2n+1$

$$\tau_{kl} = \begin{cases} \frac{(-1)^{n-k} (2n+1-k)! \sqrt{m} i^{2n+1-k} \beta_{2n+1-k,l}}{2^{\frac{2n+1}{2}} \sqrt{\left(\frac{2n+1-l}{2}\right)! \left(\frac{2n+1+l}{2}\right)!}}, & \frac{2n+1+l}{2} \text{-even} \\ \frac{(-1)^{n-k} (2n+1-k)! \sqrt{m} (-i)^{2n+1-k} \beta_{2n+1-k,l}}{2^{\frac{2n+1}{2}} \sqrt{\left(\frac{2n+1-l}{2}\right)! \left(\frac{2n+1+l}{2}\right)!}}, & \frac{2n+1+l}{2} \text{-odd.} \end{cases} \quad (\text{B53})$$

Then, for $k = 0, \dots, n$,

$$p_k = \sum_{l=1}^{2n+1} \tau_{kl} a_l + \sum_{l=1}^{2n+1} \bar{\tau}_{kl} a_l^\dagger. \quad (\text{B54})$$

Note that $\rho_{kl} = \bar{\rho}_{kl}$ for k even while $\tau_{kl} = \bar{\tau}_{kl}$ for k odd.

Let us introduce new canonical variables

$$Q_l = \frac{1}{\sqrt{2l}} (a_l + a_l^\dagger), \quad P_l = i \sqrt{\frac{l}{2}} (-a_l + a_l^\dagger). \quad (\text{B55})$$

Then the Hamiltonian takes the form

$$\mathcal{H} = \sum_{\substack{l=0 \\ l\text{-odd}}}^{2n+1} (-1)^{\frac{2n+1-l}{2}} \left(\frac{P_l^2}{2} + \frac{l^2 Q_l^2}{2} \right). \quad (\text{B56})$$

Moreover,

$$\begin{aligned} q_k &= \sum_{\substack{l=0 \\ l\text{-odd}}}^{2n+1} \sqrt{2l} \rho_{kl} Q_l, & k\text{-even;} \\ q_k &= \sum_{\substack{l=0 \\ l\text{-odd}}}^{2n+1} i \sqrt{\frac{2}{l}} \rho_{kl} P_l, & k\text{-odd} \\ p_k &= \sum_{\substack{l=0 \\ l\text{-odd}}}^{2n+1} i \sqrt{\frac{2}{l}} \tau_{kl} P_l, & k\text{-even;} \\ p_k &= \sum_{\substack{l=0 \\ l\text{-odd}}}^{2n+1} \sqrt{2l} \tau_{kl} Q_l, & k\text{-odd.} \end{aligned} \quad (\text{B57})$$

The above formulas allow us to give a simple prescription for computing the kernel $\mathcal{H}(\vec{q}, \vec{q}'; \theta)$ of $\exp(i\theta \mathcal{H})$. Namely, the canonical variables (Q_l, P_l) diagonalize the Hamiltonian. Therefore, in terms of them, the propagator kernel is simply the product of single propagators for harmonic oscillators; to account for the sign on the right-hand side of Eq. (B56) in every second term in the product, the replacement $\theta \rightarrow -\theta$ should be made. Next, note that (B57) can be viewed as the composition of the point transformation

$$\tilde{q}_k = \begin{cases} \sum_{\substack{l=1 \\ l\text{-odd}}}^{2n+1} \sqrt{2l} \rho_{kl} Q_l, & k\text{-even} \\ \sum_{\substack{l=1 \\ l\text{-odd}}}^{2n+1} \sqrt{2l} \tau_{kl} Q_l, & k\text{-odd,} \end{cases} \quad (\text{B58})$$

$$\tilde{p}_k = \begin{cases} \sum_{\substack{l=1 \\ l\text{-odd}}}^{2n+1} i \sqrt{\frac{2}{l}} \tau_{kl} P_l, & k\text{-even} \\ - \sum_{\substack{l=1 \\ l\text{-odd}}}^{2n+1} i \sqrt{\frac{2}{l}} \rho_{kl} P_l, & k\text{-odd,} \end{cases} \quad (\text{B59})$$

with a simple canonical transformation

$$\begin{aligned} q_k &= \tilde{q}_k, & \text{k-even,} \\ q_k &= -\tilde{p}_k, & \text{k-odd,} \\ p_k &= \tilde{p}_k, & \text{k-even,} \\ p_k &= \tilde{q}_k, & \text{k-odd.} \end{aligned} \quad (\text{B60})$$

(B58) and (B59) result in expressing the initial kernel in terms of new variables \tilde{q}_k and multiplying it by an appropriate constant factor according to the formula

$$\langle \tilde{q} | \underline{Q} \rangle = \left| \det \left(\frac{\partial f_k}{\partial Q_m} \right) \right|^{-\frac{1}{2}} \delta(\tilde{q} - \underline{f}(\underline{Q})). \quad (\text{B61})$$

The final step is to perform the canonical transformation (B60). The corresponding kernel for it reads

$$\langle \underline{q} | \tilde{q} \rangle = \prod_{k\text{-even}} \delta(q_k - \tilde{q}_k) \prod_{k\text{-odd}} \frac{1}{\sqrt{2\pi}} e^{iq_k \tilde{q}_k}, \quad (\text{B62})$$

i.e., we perform the Fourier transform with respect to odd variables.

Finally, let us remind the reader that up to now we have skipped the vector indices. However, the dynamics is diagonal with respect to them so one has only to multiply kernels for the propagation for all separate components.

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- [1] C. G. J. Jacobi, *Gesammelte Werke* (Reimer, Berlin, 1884).
 [2] S. Lie, *Arch. Math.* **6**, 328 (1881).
 [3] R. Jackiw, *Phys. Today* **25**, No. 1, 23 (1972); C. R. Hagen, *Phys. Rev. D* **5**, 377 (1972); U. Niederer, *Helv. Phys. Acta* **45**, 802 (1973); C. Duval, G. Burdet, H. P. Künzle, and M. Perrin, *Phys. Rev. D* **31**, 1841 (1985); C. Duval, G. W. Gibbons, and P. A. Horvathy, *Phys. Rev. D* **43**, 3907 (1991); M. Henkel, *J. Stat. Phys.* **75**, 1023 (1994); P. Havas and J. Plebański, *J. Math. Phys. (N.Y.)* **19**, 482 (1978); C. Duval and P. A. Horvathy, *J. Phys. A* **42**, 465206 (2009).
 [4] C. Duval, *Lect. Notes Phys.* **261**, 162 (1986); C. Duval, M. Hassaine, and P. A. Horvathy, *Ann. Phys. (Amsterdam)* **324**, 1158 (2009); C. Duval and S. Lazzarini, *J. Phys. A* **45**, 395203 (2012); M. Blau, J. Hartong, and B. Rollier, *J. High Energy Phys.* **7** (2010) 069; O. Jahn and V. V. Sreedhar, *Am. J. Phys.* **69**, 1039 (2001).
 [5] J. Negro, M. A. del Olmo, and A. Rodriguez-Marco, *J. Math. Phys. (N.Y.)* **38**, 3786 (1997); M. Henkel, *Phys. Rev. Lett.* **78**, 1940 (1997); N. Aizawa and V. K. Dobrev, *Nucl. Phys.* **B828**, 581 (2010); R. Cherniha and M. Henkel, *J. Math. Anal. Appl.* **369**, 120 (2010).
 [6] J. Lukierski, P. C. Stichel, and W. J. Zakrzewski, *Phys. Lett. A* **357**, 1 (2006); *Phys. Lett. B* **650**, 203 (2007); A. V. Galajinsky, *Phys. Lett. B* **675**, 116 (2009); A. Bagchi and R. Gopakumar, *J. High Energy Phys.* **07** (2009) 037; **08** (2010) 004; P. A. Horvathy and P. M. Zhang, *Eur. Phys. J. C* **65**, 607 (2010); V. K. Dobrev, H.-D. Doebner, and C. Mrugalla, *Rep. Math. Phys.* **39**, 201 (1997); *J. Phys. A* **29**, 5909 (1996); *Mod. Phys. Lett. A* **14**, 1113 (1999); M. Henkel and J. Unterberger, *Nucl. Phys.* **B660**, 407 (2003); S. Fedoruk, P. Kosiński, J. Lukierski, and P. Maślanka, *Phys. Lett. B* **699**, 129 (2011); S. Fedoruk, E. Ivanov, and J. Lukierski, *Phys. Rev. D* **83**, 085013 (2011); J. Lukierski, *Phys. At. Nucl.* **75**, 1256 (2012); I. Masterov, *J. Math. Phys. (N.Y.)* **53**, 072904 (2012).
 [7] A. V. Galajinsky and I. Masterov, *Phys. Lett. B* **702**, 265 (2011).
 [8] J. Gomis and K. Kamimura, *Phys. Rev. D* **85**, 045023 (2012).
 [9] D. Martelli and Y. Tachikawa, *J. High Energy Phys.* **05** (2010) 091.
 [10] K. Andrzejewski and J. Gonera, *Phys. Rev. D* **86**, 065009 (2012).
 [11] A. Kirillov, *Elements of the Theory of Representations* (Springer, New York, 1976); J. M. Soriau, *Structure of Dynamical Systems: A Symplectic View of Physics* (Birkhauser, Boston, 1997); V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer, New York, 1989); R. Giachetti, *Riv. Nuovo Cimento Soc. Ital. Fis.* **4**, 1 (1981).
 [12] M. Ostrogradski, *Mem. Acad. Sci. St. Petersburg VI* **4**, 385 (1850).
 [13] M. Perroud, *Helv. Phys. Acta* **50**, 233 (1977).
 [14] C. Duval and P. A. Horvathy, *J. Phys. A* **44**, 335203 (2011).
 [15] P. J. Sally, *Mem. Am. Math. Soc.* **69** (1967).
 [16] S. Lang, $SL_2(\mathbb{R})$, *Graduate Texts in Mathematics* (Springer, New York, 1985), Vol. 105.
 [17] L. Puhansky, *Math. Ann.* **156**, 96 (1964).
 [18] A. V. Galajinsky and I. Masterov, *Nucl. Phys.* **B866**, 212 (2013).