Ultraviolet structure of half-maximal supergravity with matter multiplets at two and three loops

Zvi Bern,[1] Scott Davies,[1] and Tristan Dennen[2]

[1]*Department of Physics and Astronomy, University of California at Los Angeles, Los Angeles, California 90095-1547, USA*
[2]*Niels Bohr International Academy and Discovery Center, The Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen, Denmark*

Using the duality between color and kinematics, we construct the two- and three-loop amplitudes of half-maximal supergravity with matter multiplets and show that new divergences occur in $D = 4$ and $D = 5$. In a recent paper, Bossard, Howe and Stelle conjecture the existence of 16-supercharge off-shell harmonic superspaces in order to explain the ultraviolet finiteness of pure half-maximal supergravity with no matter multiplets in $D = 4$ at three loops and in $D = 5$ at two loops. By assuming the required superspace exists in $D = 5$, they argue that no new divergences should occur at two loops even with the addition of Abelian-vector matter multiplets. Up to possible issues with the $SL(2, \mathbb{R})$ global anomaly of the theory, they reach a similar conclusion in $D = 4$ for two and three loops. The divergences we find contradict these predictions based on the existence of the desired off-shell superspaces. Furthermore, our $D = 4$ results are incompatible with the new divergences being due to the anomaly. We find that the two-loop divergences of half-maximal supergravity are directly controlled by the divergences appearing in ordinary nonsupersymmetric Yang-Mills theory coupled to scalars, explaining why half-maximal supergravity develops new divergences when matter multiplets are added. We also provide a list of one- and two-loop counterterms that should be helpful for constraining any future potential explanations of the observed vanishings of divergences in pure half-maximal supergravity.

I. INTRODUCTION

The possibility of finding ultraviolet finite supergravity theories [1] has been reopened in recent years due to the discovery of new unexpected ultraviolet cancellations. Such theories were intensely studied in the late 1970s and early 1980s as possible fundamental theories of gravity, but fell out of favor when the way forward seemed blocked by the likely appearance of nonrenormalizable ultraviolet divergences. (For a review article from that era see, for example, Ref. [2].) At the time it was not possible to definitively determine the divergence structure of supergravity theories because there were no means available for carrying out the required computations. Today thanks to the unitarity method [3] and the recently uncovered duality between color and kinematics [4,5], we have the ability to address this.

Explicit calculations [6,7] show that $\mathcal{N} = 8$ supergravity [8] is finite for dimensions $d < 6/L + 4$, at least through $L = 4$ loops. If one were to extrapolate the observed cancellations, assuming no new ones occur, simple power counting suggests that no divergence can occur in the theory prior to seven loops. Indeed, new detailed studies of the known standard symmetries of $\mathcal{N} = 8$ supergravity demonstrate that no valid counterterms can be found prior to the seventh loop order, but at seven loops a D^8R^4 counterterm can be constructed that appears to obey all known symmetries [9]. An explicit expression for the potential counterterm was written down in Ref. [10]. These facts suggest that $\mathcal{N} = 8$ supergravity diverges at seven loops;

of course, this assumes that all symmetries and structures have been properly taken into account.

When similar arguments are applied to pure half-maximal supergravity [11] at three loops in $D = 4$ and two loops in $D = 5$, counterterms valid under all known symmetries have been found [10]. However, we now know from explicit computations that there are no divergences corresponding to these counterterms [12,13]. In addition, arguments for finiteness in these cases based on string theory have been given in Ref. [14]. At three loops in $D = 4$, there is only one available counterterm in pure $\mathcal{N} = 4$ supergravity [10,15,16], so the fact that its coefficient vanishes implies that the full theory is three-loop finite.

The surprisingly good ultraviolet behavior of pure half-maximal supergravity has led to conjectures to explain its origin. One conjecture is that it is due to a hidden superconformal symmetry [17]. A more controversial conjecture is that the potential counterterms break relevant duality symmetries modified by quantum corrections [18]. A third conjecture is that the duality between color and kinematics leads to cancellation of the ultraviolet infinities in $\mathcal{N} \geq 4$ four-dimensional supergravity by the same mechanism that prevents forbidden loop-level color tensors from appearing in pure nonsupersymmetric Yang-Mills divergences [13].

On the other hand, Bossard *et al.* have given a potential symmetry explanation that would not require any new "miracles" beyond those of supersymmetry and ordinary duality symmetries. By conjecturing the existence of appropriate harmonic superspaces in $D = 4$ and $D = 5$ manifesting all 16 supercharges off shell [15,16], they

have explained the observed ultraviolet cancellations. If true, it would predict that ultraviolet divergences start at four loops in both $D = 4$ and $D = 5$ in pure half-maximal supergravity. (An artifact of dimensional regularization is that there are no three-loop divergences in $D = 5$.) While it is unclear how to construct the conjectured superspaces, one can still deduce consequences by assuming their existence. Following this reasoning, Bossard *et al.* have shown [16] that no new divergences should appear at two loops in $D = 5$ even after adding matter multiplets [19,20], were the desired 16-supercharge superspace to exist. In $D = 4$ the situation is similar, leading us to the issue of whether the anomaly in the rigid $SL(2, \mathbb{R})$ duality symmetry [21] might play a role in the appearance of new divergences in matter-multiplet amplitudes [16]. (We note that the study of matter multiplets in supergravity theories and their divergence properties has a long history [22].)

The predictions of Ref. [16] motivated us to compute the coefficients of two-loop four-point divergences in half-maximal supergravity including vector multiplets in $D = 4, 5, 6$ to definitively demonstrate that there are non-vanishing divergences in all these dimensions, as well as to give their precise form. Our two-loop $D = 5$ result is in direct conflict with the predicted finiteness [16] based on assuming the existence of an off-shell 16-supercharge superspace. We also find a two-loop divergence in $D = 4$ after subtracting the one-loop subdivergences. This two-loop divergence happens to be an iteration of the one-loop divergence, so to remove any potential doubts as to whether there are new divergences, we also calculate the complete set of three-loop divergences in $D = 4$, making it clear that there are indeed new divergences. Furthermore, we find that the explicit two- and three-loop $D = 4$ results are not of the form required had they been due to the anomaly. From Ref. [16], it therefore appears that the desired 16-supercharge superspaces exist in neither $D = 4$ nor $D = 5$.

As discussed in Ref. [13], for one and two loops in half-maximal supergravity, whenever divergences in the coefficients of certain color tensors are forbidden in gauge theory, divergences also cancel from the corresponding half-maximal supergravity amplitudes. In particular, the lack of one-loop divergences in dimensions $D < 8$ in half-maximal supergravity amplitudes with four external states of the graviton multiplet is a direct consequence of the lack of divergences in those terms in gluon amplitudes proportional to the independent one-loop color tensor. Since the four-scalar amplitude of Yang-Mills theory coupled to scalars does contain divergences in terms containing the one-loop color tensor even in $D = 4$, the corresponding four-matter multiplet amplitude of $\mathcal{N} = 4$ supergravity with matter multiplets also diverges. This is in agreement with the result found long ago by Fischler [23] and Fradkin and Tseytlin [24]. At two loops the situation is similar. The four-point amplitudes with pure external-graviton-multiplet states are ultraviolet finite in

$D = 4, 5$ because all corresponding gauge-theory divergences contain only tree-level color tensors. However, because four-scalar amplitudes of gauge theory, both in $D = 4$ and $D = 5$, contain divergences in the coefficients of the independent two-loop color tensors, corresponding two-loop four-matter multiplet amplitudes of half-maximal supergravity must also diverge.

At three and higher loops, the situation is more complicated because loop momenta appear in the maximal super-Yang-Mills duality-satisfying numerators, so the supergravity integrals are no longer the same ones as those appearing in gauge theory. Because of this, a link between the divergences of half-maximal supergravity and those of nonsupersymmetric gauge theory will require nontrivial integral identities and remains speculative [13].

To carry out our investigation, we construct half-maximal supergravity amplitudes via the duality between color and kinematics [4,5]. In this way, gravity loop integrands are obtained from a pair of corresponding gauge-theory loop integrands. The key to this construction is to find a representation where one of the two gauge-theory amplitudes manifestly exhibits the duality between color and kinematics. Here we obtain half-maximal supergravity with matter multiplets from a direct product of maximal super-Yang-Mills theory and nonsupersymmetric Yang-Mills theory with interacting scalars. The required two-loop super-Yang-Mills amplitude in a form where the duality is manifest was given long ago in Refs. [25,26], while the desired form of the three-loop amplitude was given more recently in Ref. [5]. The nonsupersymmetric Yang-Mills theory coupled to n_V scalars is conveniently obtained by dimensionally reducing pure Yang-Mills theory from $D + n_V$ dimensions to D dimensions, matching the construction of half-maximal supergravity with n_V matter multiplets by dimensional reduction of pure half-maximal supergravity [27].

Once we have the integrands for the amplitudes, we need to extract the ultraviolet singularities. The basic procedure for doing so has been long understood [28] and has been applied recently to a variety of supergravity and super-Yang-Mills calculations [7,12,13,29]. Here we will explain in some detail the procedure that we use to extract ultraviolet divergences in the presence of integral-by-integral subdivergences. This procedure was already used in Ref. [12] to demonstrate the ultraviolet finiteness of all three-loop four-point amplitudes of pure $\mathcal{N} = 4$ supergravity in four dimensions.

This paper is organized as follows. In Sec. II, we briefly review some basic facts of the duality between color and kinematics and the double-copy construction of gravity. We also explain the structure of one- and two-loop four-point amplitudes in half-maximal supergravity with n_V Abelian matter multiplets. Then in Sec. III we describe the construction of the integrand and the integration methods used to extract the ultraviolet divergences. We give our

results for the one-loop and two-loop divergences of half-maximal supergravity with matter multiplets in Secs. IV and V, with Sec. V also containing our $D = 4$ three-loop results. Finally we present our conclusions and outlook in Sec. VI. An appendix listing valid counterterms as well as their numerical coefficients is also given.

II. BASIC SETUP

The duality between color and kinematics and the associated gravity double-copy property [4,5] make it straightforward to construct supergravity amplitudes once corresponding gauge-theory amplitudes are arranged into a form that makes the duality manifest. (For a recent review of this duality and its application, see Ref. [30].) While the duality remains a conjecture at loop level, we will use it only for one-, two- and three-loop four-point amplitudes where it is known to hold in maximally supersymmetric Yang-Mills theory. We use it to map out the two- and three-loop divergence structure of half-maximal supergravity with n_V Abelian matter multiplets. We start by first giving a brief summary of the duality before giving a number of formulas that are useful for one- and two-loop amplitudes in half-maximal supergravity [13,31,32].

A. Duality between color and kinematics

The gauge-theory duality between color and kinematics is conveniently described in terms of graphs with only cubic vertices. Using such graphs, any m-point L-loop gauge-theory amplitude with all particles in the color adjoint representation can be written as

$$\mathcal{A}_m^{L\text{-loop}} = i^L g^{m-2+2L} \sum_{\mathcal{S}_m} \sum_j \int \prod_{l=1}^{L} \frac{d^d p_l}{(2\pi)^d} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2}, \quad (2.1)$$

where the sum labeled by j runs over the set of distinct nonisomorphic graphs. Any contact terms in the amplitude can be expressed in terms of graphs with only cubic vertices by multiplying and dividing by appropriate propagators. The product in the denominator runs over all Feynman propagators of graph j. The integrals are over L independent d-dimensional loop momenta. The symmetry factor S_j removes overcounts from the sum over permutations of external legs indicated by $\mathcal{S}_m$ and from internal symmetry factors. The c_j are color factors obtained by dressing every three-vertex with a group-theory structure constant,

$$\tilde{f}^{abc} = i\sqrt{2} f^{abc} = \text{Tr}([T^a, T^b]T^c), \quad (2.2)$$

and n_j are kinematic numerators of graph j depending on momenta, polarizations and spinors. If a superspace formulation is used, n_j can also depend on Grassmann parameters.

The conjectured duality of Refs. [4,5] states that to all loop orders, there exists a form of (super-)Yang-Mills

amplitudes where kinematic numerators satisfy the same algebraic relations as color factors. In these theories, this amounts to imposing the same Jacobi identities on the kinematic numerators as satisfied by adjoint-representation color factors,

$$c_i = c_j - c_k \Rightarrow n_i = n_j - n_k, \quad (2.3)$$

where the indices i, j, k denote the diagram to which the color factors and numerators belong. The basic Jacobi identity is illustrated in Fig. 1 and can be embedded in arbitrary diagrams. The numerator factors are also required to have the same antisymmetry properties as color factors. In general, the duality relations (2.3) work only after appropriate nontrivial rearrangements of the amplitudes.

When a representation of an amplitude is found where the duality (2.3) is made manifest, we can obtain corresponding gravity loop integrands simply by replacing color factors in a gauge-theory amplitude by kinematic numerators of a second gauge-theory amplitude. This gives the double-copy form of corresponding gravity amplitudes [4,5],

$$\mathcal{M}_m^{L\text{-loop}} = i^{L+1} \left(\frac{\kappa}{2}\right)^{m-2+2L} \sum_{\mathcal{S}_m} \sum_j \int \prod_{l=1}^{L} \frac{d^d p_l}{(2\pi)^d} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2}. \quad (2.4)$$

Generalized gauge invariance implies that only one of the two sets of numerators n_j or $\tilde{n}_j$ needs to satisfy the duality relation (2.3) [5,33]. At tree level, the double-copy formula (2.4) encodes the Kawai-Lewellen-Tye (KLT) [34] relations between gravity and gauge-theory amplitudes [4].

In this paper, we will construct amplitudes for half-maximal supergravity with matter multiplets as a double copy of maximally supersymmetric Yang-Mills amplitudes and nonsupersymmetric Yang-Mills amplitudes coupled to interacting scalars. The desired scalars arise from dimensional reduction of pure Yang-Mills theory. For such scalars the conjectured duality holds automatically when it holds in higher-dimensional pure Yang-Mills theory. We will not need duality-satisfying representations of the nonsupersymmetric amplitudes, given that we have them on the maximal super-Yang-Mills side.
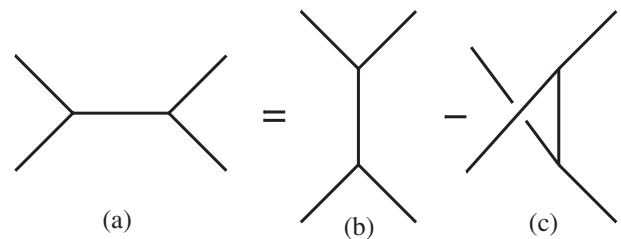

FIG. 1. The basic Jacobi relation for either color or numerator factors given in Eq. (2.3). These three diagrams can be embedded in a larger diagram, including loops.

B. Amplitude relations at one and two loops

As explained in Refs. [13,31,32], the one- and two-loop four-point amplitudes of pure half-maximal supergravity are easily obtained from corresponding amplitudes in non-supersymmetric gauge theory. Here we extend this slightly by noting that the same holds for half-maximal supergravity amplitudes including Abelian-vector multiplets.

The double-copy construction of a half-maximal supergravity amplitude starts by writing the corresponding non-supersymmetric gauge-theory amplitude in a convenient color decomposition, then replacing color factors by super-Yang-Mills numerators that satisfy the duality between color and kinematics. A color-dressed four-point one-loop gauge-theory amplitude with all particles in the adjoint representation can be expressed as [35]

$$\mathcal{A}^{(1)}(1,2,3,4) = g^4[c^{(1)}_{1234}A^{(1)}(1,2,3,4) + c^{(1)}_{1342}A^{(1)}(1,3,4,2)$$
$$+ c^{(1)}_{1423}A^{(1)}(1,4,2,3)]. \quad (2.5)$$

The $c^{(1)}_{1234}$ are the color factors of a box diagram illustrated in Fig. 2, with consecutive external legs (1, 2, 3, 4) and with vertices dressed with structure constants $\tilde{f}^{abc}$ defined in Eq. (2.2). The $A^{(1)}$ are one-loop color-ordered amplitudes [36]. This color decomposition holds just as well whether the external particles are adjoint scalars or gluons and does not depend on supersymmetry.

To obtain the one-loop half-maximal supergravity amplitudes, we simply replace the gauge coupling with the gravitational one and the color factors in Eq. (2.5) with maximal super-Yang-Mills duality-satisfying kinematic numerators [31],

$$c^{(1)}_{ijkl} \rightarrow n^{(1)}_{ijkl}, \qquad g^4 \rightarrow i\left(\frac{\kappa}{2}\right)^4, \quad (2.6)$$

where [37]

$$n^{(1)}_{1234} = n^{(1)}_{1342} = n^{(1)}_{1423} = stA^{\text{tree}}_{Q=16}(1,2,3,4), \quad (2.7)$$

and $A^{\text{tree}}_{Q=16}(1,2,3,4)$ is the four-point tree amplitude of maximal 16-supercharge super-Yang-Mills theory, valid for all states of the theory. (See for example Eq. (2.8) of Ref. [38] for the explicit form of these tree amplitudes in
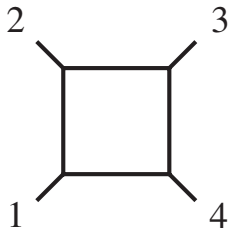

FIG. 2.   The one-loop box diagram. The one-loop color factor $c^{(1)}_{1234}$ is obtained by dressing each vertex with an $\tilde{f}^{abc}$.

$D = 4$.) This gives us a rather simple formula for one-loop four-point amplitudes in half-maximal supergravity [31],

$$\mathcal{M}^{(1)}_{Q=16} = i\left(\frac{\kappa}{2}\right)^4 stA^{\text{tree}}_{Q=16}(1,2,3,4)[A^{(1)}(1,2,3,4)$$
$$+ A^{(1)}(1,3,4,2) + A^{(1)}(1,4,2,3)]. \quad (2.8)$$

This formula is valid for all matter- and graviton-multiplet states of half-maximal supergravity. This simple replacement rule means that the supergravity divergences can be read off directly from the gauge-theory divergences. In particular, we can read off the divergences of half-maximal supergravity with n_V vector multiplets directly from the corresponding divergences of nonsupersymmetric Yang-Mills theory coupled to n_V scalars.

The expression (2.8) automatically satisfies the unitarity cuts if the input gauge-theory amplitudes are correct. This is because once the maximally supersymmetric numerators that satisfy the duality between color and kinematics are used, the cuts necessarily match those obtained by feeding in gravity tree amplitudes obtained by either the double-copy formula or the KLT relations. In addition, this formula has been used [31] to reproduce known expressions [39] for the integrated amplitudes in $\mathcal{N} = 4, 6$ supergravity, when $A^{(1)}$ is taken to represent more general, possibly supersymmetric, gauge-theory amplitudes. It also matches the known expression for $\mathcal{N} = 8$ supergravity [26].

As explained in Ref. [13], we can line up the divergences of supergravity with those appearing in the independent one-loop color tensor of the color basis given in Appendix B of Ref. [40] (see also Ref. [41]). In this color basis we have

$$b^{(1)}_1 \equiv c^{(1)}_{1234}, \qquad c^{(1)}_{1342} = b^{(1)}_1 + \cdots, \qquad c^{(1)}_{1423} = b^{(1)}_1 + \cdots, \quad (2.9)$$

where "$+ \cdots$" represents dropped terms proportional to the tree-level color tensors,

$$b^{(0)}_1 = \tilde{f}^{a_1a_2b}\tilde{f}^{ba_3a_4}, \qquad b^{(0)}_2 = \tilde{f}^{a_2a_3b}\tilde{f}^{ba_4a_1}. \quad (2.10)$$

After expressing all the color factors in the basis (2.9), the gauge-theory amplitude (2.5) can be expressed as

$$\mathcal{A}^{(1)}(1,2,3,4) = g^4b^{(1)}_1[A^{(1)}(1,2,3,4) + A^{(1)}(1,3,4,2)$$
$$+ A^{(1)}(1,4,2,3)] + \cdots. \quad (2.11)$$

In this form, we see that the half-maximal supergravity amplitudes line up with those terms in the nonsupersymmetric gauge-theory amplitudes containing the independent one-loop color tensor.

This remarkable relation between the one-loop half-maximal supergravity amplitudes (2.8) and the parts of gauge-theory amplitudes containing the independent one-loop color tensor (2.11) allows us to obtain the supergravity amplitude simply by converting to a color basis, dropping

the tree-level color factors, and then replacing the one-loop color tensor and gauge coupling,

$$b_1^{(1)} \rightarrow st A^{\mathrm{tree}}_{Q=16}(1,2,3,4), \qquad g^4 \rightarrow i\left(\frac{\kappa}{2}\right)^4. \qquad (2.12)$$

This works as well at the integrated level, so that once we have the gauge-theory divergences, the substitution (2.12) directly gives us the corresponding divergences in one-loop half-maximal supergravity with or without matter multiplets.

The situation is similar for two-loop four-point amplitudes. At two loops any color-dressed gauge-theory amplitude with only adjoint-representation particles can be conveniently written as [31,32,35]

$$\mathcal{A}^{(2)}(1,2,3,4) = g^6 \sum_{x \in \{\mathrm{P,NP}\}} [c^x_{1234} A^x(1,2,3,4)$$
$$+ c^x_{3421} A^x(3,4,2,1) + c^x_{1423} A^x(1,4,2,3)$$
$$+ c^x_{2341} A^x(2,3,4,1) + c^x_{1342} A^x(1,3,4,2)$$
$$+ c^x_{4231} A^x(4,2,3,1)], \qquad (2.13)$$

where the sum runs over the planar and nonplanar contributions. Here c^{P}_{1234} and c^{NP}_{1234} are the color factors obtained by dressing the planar and nonplanar double-box graphs in Fig. 3 with structure constants $\tilde{f}^{abc}$ defined in Eq. (2.2). The A^{P} and A^{NP} are planar and nonplanar partial amplitudes.

To obtain supergravity amplitudes, we replace the gauge coupling with the gravitational one and the color factors in Eq. (2.13) with the super-Yang-Mills numerators,

$$c^{\mathrm{P}}_{ijkl} \rightarrow n^{\mathrm{P}}_{ijkl}, \qquad c^{\mathrm{NP}}_{ijkl} \rightarrow n^{\mathrm{NP}}_{ijkl}, \qquad g^6 \rightarrow i\left(\frac{\kappa}{2}\right)^6, \qquad (2.14)$$

where [25,26]

$$n^x_{1234} = sK, \qquad n^x_{3421} = sK, \qquad n^x_{1423} = tK,$$
$$n^x_{2341} = tK, \qquad n^x_{1342} = uK, \qquad n^x_{4231} = uK, \qquad (2.15)$$

and $x \in \mathrm{P, NP}$. The factor K is the fully crossing-symmetric prefactor,

$$K = st A^{\mathrm{tree}}_{Q=16}(1,2,3,4). \qquad (2.16)$$
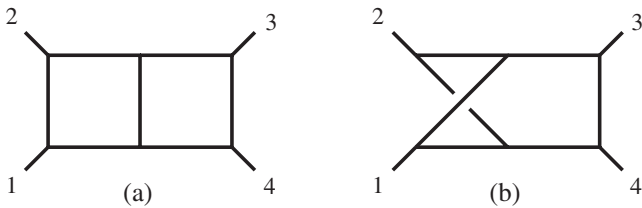
FIG. 3. The (a) planar and (b) nonplanar double-box graphs. The c^{P}_{1234} and c^{NP}_{1234} color factors are obtained by dressing each vertex with an $\tilde{f}^{abc}$.

In this way we immediately obtain the four-point two-loop amplitude of half-maximal supergravity [31,32],

$$\mathcal{M}^{(2)}_{Q=16}(1,2,3,4)$$
$$= i\left(\frac{\kappa}{2}\right)^6 st A^{\mathrm{tree}}_{Q=16}(1,2,3,4) \sum_{x \in \{\mathrm{P,NP}\}} [s(A^x(1,2,3,4)$$
$$+ A^x(3,4,2,1)) + t(A^x(1,4,2,3) + A^x(2,3,4,1))$$
$$+ u(A^x(1,3,4,2) + A^x(4,2,3,1))]. \qquad (2.17)$$

As for one loop, this holds for all states of the graviton or vector multiplets of half-maximal supergravity. A nontrivial check that has been carried out on this formula [32] is that when the appropriate integrated gauge-theory amplitudes [42] are inserted, it correctly reproduces the known infrared singularities of $\mathcal{N} \geq 4$ supergravity theories [43].

We can line up the supergravity amplitude with the contributions proportional to the two independent two-loop color tensors [13],

$$b_1^{(2)} \equiv c^{\mathrm{P}}_{1234}, \qquad b_2^{(2)} \equiv c^{\mathrm{P}}_{2341}, \qquad (2.18)$$

where the other color factors in the amplitude (2.13) can be expressed in terms of these:

$$c^{\mathrm{P}}_{3421} = b_1^{(2)} + \cdots, \qquad c^{\mathrm{P}}_{1423} = b_2^{(2)} + \cdots,$$
$$c^{\mathrm{P}}_{1342} = -b_1^{(2)} - b_2^{(2)} + \cdots, \qquad c^{\mathrm{P}}_{4231} = -b_1^{(2)} - b_2^{(2)} + \cdots, \qquad (2.19)$$

and the "$+ \cdots$" represents dropped terms containing lower-loop color tensors. The nonplanar color factors are the same as the planar ones, up to corrections proportional to lower-loop color tensors:

$$c^{\mathrm{NP}}_{ijkl} = c^{\mathrm{P}}_{ijkl} + \cdots. \qquad (2.20)$$

Substituting these into Eq. (2.13) gives

$$\mathcal{A}^{(2)}(1,2,3,4) = g^6 \sum_{x \in \{\mathrm{P,NP}\}} [b_1^{(2)}(A^x(1,2,3,4) + A^x(3,4,2,1)$$
$$- A^x(1,3,4,2) - A^x(4,2,3,1))$$
$$+ b_2^{(2)}(A^x(1,4,2,3) + A^x(2,3,4,1)$$
$$- A^x(1,3,4,2) - A^x(4,2,3,1)] + \cdots. \qquad (2.21)$$

This lines up with the supergravity expression (2.17) once we replace $u = -s - t$. Comparing Eq. (2.21) to the two-loop supergravity expression shows that we can obtain half-maximal supergravity divergences directly from the gauge-theory ones by going to the color basis (2.19), dropping all one-loop and tree color tensors, and then replacing

$$b_1^{(2)} \rightarrow s^2 t A^{\mathrm{tree}}_{Q=16}(1,2,3,4),$$
$$b_2^{(2)} \rightarrow st^2 A^{\mathrm{tree}}_{Q=16}(1,2,3,4), \qquad g^6 \rightarrow i\left(\frac{\kappa}{2}\right)^6. \qquad (2.22)$$

We note that in general at higher loops, one should not use a color basis to make numerator substitutions because it assumes that internal color sums have been performed, while in the corresponding kinematic numerators the loop momenta are not integrated but held fixed. In our relatively simple one- and two-loop cases, the substitutions (2.12) and (2.22) hold because they happen to be equivalent to making the substitutions prior to switching to a color basis. For carrying out the three-loop calculation of divergences, we instead directly use Eq. (2.4).

III. PROCEDURE FOR COMPUTATION

In this section we give our procedure for constructing the half-maximal supergravity amplitudes and then extracting the ultraviolet divergences.

A. General construction

Using Eqs. (2.8) and (2.17), we obtain one- and two-loop half-maximal supergravity amplitudes directly from non-supersymmetric gauge-theory amplitudes. Because we are interested in cases where no integrated results exist for the amplitudes, we use slightly modified forms where we replace the gauge-theory amplitudes by their Feynman diagrams. Although it may seem inefficient to use Feynman diagrams, in our case it makes little difference because we are interested in ultraviolet divergences in only the relatively small number of contributions that carry the color factors of the box diagrams at one loop and the double-box diagrams at two loops. In addition, we need expressions valid in general dimensions, making it more difficult to use more sophisticated helicity methods.

As we already discussed in Sec. II B, the gauge-theory divergences that feed into half-maximal supergravity divergences are those with color factors depending on the independent color tensor $b_1^{(1)}$ at one loop and the independent color tensors $b_1^{(2)}$ and $b_2^{(2)}$ at two loops. Any Feynman diagram that has a triangle or bubble subgraph, as displayed in Fig. 4, will not contribute to the needed color tensors and will therefore not contribute to the supergravity
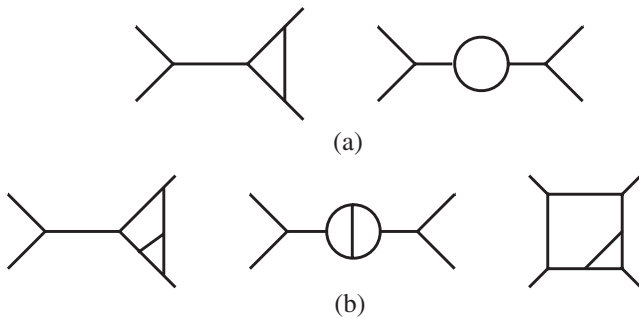

(a)

(b)

FIG. 4.   Diagrams with triangle and bubble subgraphs at (a) one loop and (b) two loops. These do not contribute to terms proportional to the needed color tensors in Yang-Mills and therefore do not contribute to the supergravity divergences.

divergences. One can also see that the antisymmetry of the kinematic part of the vertices will cause these diagrams to cancel in the permutation sum in Eq. (2.8).

This allows us to replace the one-loop amplitudes in Eq. (2.8) with box diagram contributions:

$$\mathcal{M}_{Q=16}^{(1)} = i\left(\frac{\kappa}{2}\right)^4 st A_{Q=16}^{\text{tree}}(1,2,3,4)[B^{(1)}(1,2,3,4) + B^{(1)}(1,3,4,2) + B^{(1)}(1,4,2,3)], \qquad (3.1)$$

where $B^{(1)}(1,2,3,4)$ collects all Feynman-diagram contributions which have the color factor of the box in Fig. 2. This includes the box Feynman diagrams, whether containing scalars, ghosts or gluons, and any terms in diagrams with four-point contact terms carrying the box color factor. The other contributions $B^{(1)}(1,3,4,2)$ and $B^{(1)}(1,4,2,3)$ are similar except the external legs are permuted. Due to the color Jacobi relations, there is an ambiguity in assigning terms to diagrams, but in the end it does not matter because if a term cancels in Eq. (3.1) in one particular arrangement, it will cancel in other arrangements as well.

At two loops the situation is similar. Expressing the gauge-theory amplitudes in Eq. (2.17) in terms of Feynman diagrams, we find that only those diagrammatic contributions that carry the color factor of either the planar or nonplanar double box do not cancel. Keeping these contributions, we have the supergravity amplitude as

$$\mathcal{M}_{Q=16}^{(2)}(1,2,3,4)$$
$$= i\left(\frac{\kappa}{2}\right)^6 st A_{Q=16}^{\text{tree}}(1,2,3,4) \sum_{x\in\{\text{P,NP}\}} [s(B^x(1,2,3,4)$$
$$+ B^x(3,4,2,1)) + t(B^x(1,4,2,3) + B^x(2,3,4,1))$$
$$+ u(B^x(1,3,4,2) + B^x(4,2,3,1))], \qquad (3.2)$$

where $B^{\text{P}}(1,2,3,4)$ are the diagrammatic contributions with the planar double-box color factor shown in Fig. 3(a), and $B^{\text{NP}}(1,2,3,4)$ are the diagrammatic contributions containing the nonplanar double-box color factor in Fig. 3(b). The other nonvanishing contributions carry color factors that are just relabelings of these, while all contributions that do not carry such color factors cancel in Eq. (3.2). As for one loop, the assignment of the terms in each of these contributions is not unique.

We have numerically confirmed, using helicity states in four dimensions, that Eq. (3.1) has the correct two-particle unitarity cuts and that Eq. (3.2) has the correct double two-particle cuts. As noted earlier, the one- and two-loop double-copy formulas (3.1) and (3.2) are guaranteed to hold, as long as the input gauge-theory amplitudes have the correct cuts. Nevertheless, this is a nontrivial consistency check to show that we have assembled the contributions correctly.

At three loops the situation is somewhat more complicated. We will follow the construction in Ref. [12], where all four-point three-loop half-maximal pure supergravity

divergences were constructed. Here the construction is identical except that on the nonsupersymmetric gauge-theory side of the double copy we include scalars, giving us supergravity amplitudes including matter multiplets.

B. Dimensional reduction for matter multiplets

Half-maximal supergravity in D dimensions with n_V Abelian matter multiplets is conveniently generated by dimensionally reducing pure half-maximal supergravity from $D + n_V$ dimensions to D dimensions (with $D + n_V \leq 10$) [19,20,27]. This automatically generates half-maximal supergravity with proper interactions between the different vector multiplets. Indeed, in Ref. [19] the Lagrangian of $\mathcal{N} = 4$ supergravity in four dimensions with six vector multiplets is constructed via dimensional reduction of pure $\mathcal{N} = 1$, $D = 10$ supergravity.

This observation makes it straightforward to modify previous computations in pure half-maximal supergravity [12,13] to now include Abelian matter multiplets. Indeed, dimensional reduction is very natural in the double-copy formalism. Under dimensional reduction the number of states is unchanged; in particular, maximal super-Yang-Mills theory is just the dimensional reduction of $\mathcal{N} = 1$,

$D = 10$ super-Yang-Mills theory. Under dimensional reduction from $D + n_V$ dimensions to D dimensions, each gluon carries $D + n_V - 2$ physical states that split into n_V scalars and $D - 2$ gluon states. The tensor product of the states of maximally supersymmetric Yang-Mills theory with a scalar state gives a vector matter multiplet, while the tensor product with a gluon state gives a graviton multiplet. Therefore, tensoring dimensionally reduced nonsupersymmetric Yang-Mills theory with maximal super-Yang-Mills theory yields half-maximal supergravity with vector matter multiplets.

Besides the standard gauge-theory couplings, the scalars generated by dimensional reduction in Yang-Mills theory can interact with other scalars. To determine the appropriate scalar couplings needed for the double-copy construction of half-maximal supergravity with matter multiplets, we simply track the scalar interactions under dimensional reduction. Explicitly, under dimensional reduction we obtain the gauge-theory Lagrangian,

$$\mathcal{L} = \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{ghost}} + \mathcal{L}_{\text{scalar}}, \tag{3.3}$$

where the gluon, ghost, and scalar contributions are

$$\mathcal{L}_{\text{YM}} = -\frac{1}{2}(\partial_\mu A_\nu^a)(\partial^\mu A^{\nu a}) + \frac{ig}{\sqrt{2}}\tilde{f}^{abc}(\partial_\mu A_\nu^a)A^{\mu b}A^{\nu c} + \frac{g^2}{8}\tilde{f}^{abe}\tilde{f}^{ecd}A_\mu^a A_\nu^b A^{\mu c}A^{\nu d},$$

$$\mathcal{L}_{\text{ghost}} = (\partial_\mu \bar{c}^a)(\partial^\mu c^a) - \frac{ig}{\sqrt{2}}\tilde{f}^{abc}(\partial_\mu \bar{c}^a)A^{\mu b}c^c, \tag{3.4}$$

$$\mathcal{L}_{\text{scalar}} = \frac{1}{2}(\partial_\mu \phi_i^a)(\partial^\mu \phi_i^a) - \frac{ig}{\sqrt{2}}\tilde{f}^{abc}(\partial_\mu \phi_i^a)A^{\mu b}\phi_i^c - \frac{g^2}{8}\tilde{f}^{abe}\tilde{f}^{ecd}(2A_\mu^a \phi_i^b A^{\mu c}\phi_i^d - \phi_i^a \phi_j^b \phi_i^c \phi_j^d).$$

We use the Feynman gauge in $\mathcal{L}_{\text{YM}}$, which makes it straightforward to identify the propagators in Eq. (2.1). The scalar Lagrangian $\mathcal{L}_{\text{scalar}}$ is the result of dimensionally reducing $\mathcal{L}_{\text{YM}}$ by separating out the higher-dimensional components of A and ∂ as

$$A_\mu^a \rightarrow (A_\mu^a, \phi_i^a), \qquad A^{\mu a} \rightarrow (A^{\mu a}, -\phi_i^a), \qquad \partial_\mu \rightarrow (\partial_\mu, 0). \tag{3.5}$$

In our metric convention, we have $\phi^{ia} = -\phi_i^a$. Our color factors are rescaled as in Eq. (2.2).

In the bare Lagrangian (3.4), we have normalized the four-scalar interaction to carry the same coupling as the gluons. Of course, under renormalization the coefficient of the four-scalar interaction is no longer locked to the gauge coupling by gauge invariance nor is the color structure locked to the one of Yang-Mills theory. Because the scalars and gluons of this theory have different ultraviolet-divergence structure, the double-copy property implies that amplitudes with external matter multiplets will also behave differently. This has important ramifications for the

divergence structure of the double-copy supergravity theories.

As a practical matter, it is easier to not use $\mathcal{L}_{\text{scalar}}$ explicitly but instead to incorporate the scalars into the gluon Lagrangian $\mathcal{L}_{\text{YM}}$ taken in $D_s = D + n_V$ dimensions but with all momenta restricted to the $(D - 2\epsilon)$-dimensional subspace (where we take D to be an integer and $\epsilon < 0$ for the purposes of determining Lorentz dot products). In a given Feynman diagram, whenever the gluon propagators contract around a loop, we take the circulating states to be in D_s dimensions; in other words, we take $\eta^\mu{}_\mu = D_s$ assuming the contraction is formed only from $\eta_{\mu\nu}$'s explicitly appearing in the Feynman rules. (If a contraction is formed using also an $\eta_{\mu\nu}$ from reducing tensor loop integrals to scalar integrals, then the contraction instead gives $d \equiv D - 2\epsilon$.) In addition, for an external scalar state, say on leg 1, we take the polarization vector to be orthogonal to the $(D - 2\epsilon)$-dimensional subspace where momenta live,

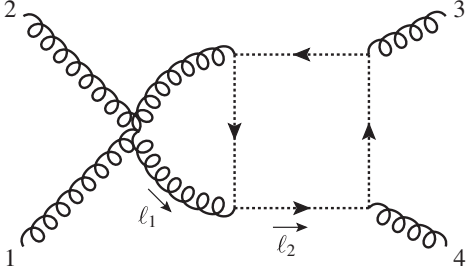$$\varepsilon_{1\mu}^\phi \rightarrow (0, \varepsilon_{1i}), \tag{3.6}$$

FIG. 5. A Feynman diagram appearing in the calculation of the gauge-theory amplitude $\mathcal{A}^{(2)}(1_\phi, 2_\phi, 3_g, 4_g)$ and supergravity amplitude $\mathcal{M}^{(2)}(1_V, 2_V, 3_H, 4_H)$. The closed loop on the right is that of a ghost.

so that it is annihilated whenever it contracts with a momentum vector:

$$\varepsilon_1 \cdot p_i = \varepsilon_1 \cdot \ell_i = 0. \tag{3.7}$$

This is the crucial difference between a scalar and vector contribution, and this difference alters the results so that color tensors that are forbidden in the divergences of gluonic amplitudes can appear in amplitudes with external scalars. The only possible nonvanishing contractions for the scalar polarization vectors are those with the polarization vectors of other external scalars; for example if we desire particle 2 to be another scalar, then $\varepsilon_1 \cdot \varepsilon_2$ can be nonvanishing.

As a concrete example, consider the two-loop ultraviolet divergence in five dimensions of the supergravity amplitude with two legs from a matter multiplet and two legs from a graviton multiplet, $\mathcal{M}^{(2)}(1_V, 2_V, 3_H, 4_H)|_{D=5\,\text{div.}}$, where the subscripts V and H indicate whether a leg is a state from a vector multiplet or a graviton multiplet. As discussed in Sec. II B, the super-Yang-Mills side of the double copy is incorporated by a simple replacement of a color factor with a numerator factor. On the nonsupersymmetric gauge-theory side, we must compute the divergence, $\mathcal{A}^{(2)}(1_\phi, 2_\phi, 3_g, 4_g)|_{D=5\,\text{div.}}$, or more specifically its terms proportional to the independent two-loop color tensors. One contribution to this divergence comes from the Feynman diagram shown in Fig. 5 after dimensional reduction from D_s dimensions.

This diagram involves a contact vertex and a ghost loop, and it has a piece proportional to the color factor c^P_{1234} shown in Fig. 3(a), on which we will focus in this example. In D_s dimensions, the gauge-theory integrand is given by

$$\frac{ig^6}{8} c^P_{1234} \frac{\varepsilon_{1\mu_1}\varepsilon_{2\mu_2}\varepsilon_{3\mu_3}\varepsilon_{4\mu_4}(\ell_2 + p_{12})^{\mu_3}(\ell_2 - p_4)^{\mu_4}(\ell_2 - \ell_1)_{\mu_5}\ell_{2\mu_6}(\eta^{\mu_1\mu_5}\eta^{\mu_2\mu_6} - \eta^{\mu_1\mu_2}\eta^{\mu_5\mu_6})}{(\ell_1)^2(\ell_1 + p_{12})^2(\ell_1 - \ell_2)^2(\ell_2)^2(\ell_2 + p_{12})^2(\ell_2 - p_4)^2}, \tag{3.8}$$

where the p_i are the momenta of the external legs, $p_{12} = p_1 + p_2$ and the ℓ_i are the loop momenta as indicated in Fig. 5. The supergravity integrand is obtained with the simple replacement, $g^6 c^P_{1234} \rightarrow i(\kappa/2)^6 s^2 t A^{\text{tree}}_{Q=16}$. To obtain the amplitude with legs 1 and 2 being identical scalars in gauge theory or vector multiplet states in supergravity, we restrict the momenta to be orthogonal to the polarization vectors ε_1 and ε_2 which live entirely in the $(D_s - D)$-dimensional subspace. In this way their only nonvanishing contraction is $\varepsilon_1 \cdot \varepsilon_2 = -1$ since legs 3 and 4 are gluons and their polarizations live in D-dimensional subspace which is not orthogonal to the momenta. Under this restriction, the sample in Eq. (3.8) becomes

$$I^{\text{sample}}_d = \frac{i}{8} \int \frac{d^d\ell_1}{(2\pi)^d} \frac{d^d\ell_2}{(2\pi)^d} \frac{\varepsilon_3 \cdot (\ell_2 + p_{12})\varepsilon_4 \cdot \ell_2(\ell_2 - \ell_1) \cdot \ell_2}{(\ell_1)^2(\ell_1 + p_{12})^2(\ell_1 - \ell_2)^2(\ell_2)^2(\ell_2 + p_{12})^2(\ell_2 - p_4)^2}, \tag{3.9}$$

where we have integrated over $d = D - 2\epsilon$ with D an integer and not included the $g^6 c^P_{1234}$ prefactor. The remaining task is to evaluate integrals of this type in order to extract their ultraviolet divergences.

C. Series expansion of the integrand

Rather than evaluate integrals with their full momentum dependence, it is advantageous to series expand the integrals to pick up only the desired ultraviolet divergences. To do so we follow the procedure of Ref. [28]. As a first example, consider the two-loop $D = 5$ case. Odd dimensions are a bit simpler at two loops than even dimensions because there are never one-loop subdivergences in dimensional regularization, even integral by integral. Since there are no subdivergences, the $D = 5$ ultraviolet divergence of the integral in Eq. (3.9) begins at $\mathcal{O}(\epsilon^{-1})$ instead of $\mathcal{O}(\epsilon^{-2})$

and is a polynomial in external momenta. Power counting shows this polynomial to be quadratic. We may therefore apply the dimension-counting operator which effectively extracts powers of external momenta from the integral, reducing its degree of divergence:

$$\left(\sum_{i=1}^{4} p_{i\mu} \frac{\partial}{\partial p_{i\mu}}\right) I^{\text{sample}}_{d=5-2\epsilon} = 2 I^{\text{sample}}_{d=5-2\epsilon} + \mathcal{O}(\epsilon^0). \tag{3.10}$$

We use this observation to repeatedly extract powers of external momenta from the integral, until eventually we are left with logarithmically divergent integrals whose divergences no longer depend on the external momenta. Explicitly, after the first application of Eq. (3.10), we are left with

$$2I^{\text{sample}}_{d=5-2\epsilon} + \mathcal{O}(\epsilon^0) = \frac{i}{8} \int \frac{d^{5-2\epsilon}\ell_1}{(2\pi)^{5-2\epsilon}} \frac{d^{5-2\epsilon}\ell_2}{(2\pi)^{5-2\epsilon}} \frac{\varepsilon_4 \cdot \ell_2 (\ell_2 - \ell_1) \cdot \ell_2}{(\ell_1)^2 (\ell_1 + p_{12})^2 (\ell_1 - \ell_2)^2 (\ell_2)^2 (\ell_2 + p_{12})^2 (\ell_2 - p_4)^2}$$

$$\times \left\{ \varepsilon_3 \cdot p_{12} - 2\varepsilon_3 \cdot (\ell_2 + p_{12}) \left(\frac{p_{12} \cdot \ell_1 + s_{12}}{(\ell_1 + p_{12})^2} + \frac{p_{12} \cdot \ell_2 + s_{12}}{(\ell_2 + p_{12})^2} - \frac{p_4 \cdot \ell_2}{(\ell_2 - p_4)^2} \right) \right\}. \quad (3.11)$$

We see that the quadratically divergent term in Eq. (3.9) has been eliminated in favor of linearly and logarithmically divergent integrals, along with some ultraviolet-finite terms which we ignore. One more application of Eq. (3.10) yields, after some rearrangement and dropping of finite pieces, the result,

$$I^{\text{sample}}_{d=5-2\epsilon} + \mathcal{O}(\epsilon^0) = \frac{i}{8} \int \frac{d^{5-2\epsilon}\ell_1}{(2\pi)^{5-2\epsilon}} \frac{d^{5-2\epsilon}\ell_2}{(2\pi)^{5-2\epsilon}} \frac{\varepsilon_4 \cdot \ell_2 (\ell_2 - \ell_1) \cdot \ell_2}{(\ell_1)^2 (\ell_1 + p_{12})^2 (\ell_1 - \ell_2)^2 (\ell_2)^2 (\ell_2 + p_{12})^2 (\ell_2 - p_4)^2}$$

$$\times \left\{ -\varepsilon_3 \cdot p_{12} X - \varepsilon_3 \cdot \ell_2 \left(\frac{s_{12}}{(\ell_1 + p_{12})^2} + \frac{s_{12}}{(\ell_2 + p_{12})^2} \right) \right.$$

$$\left. + \frac{1}{2} \varepsilon_3 \cdot \ell_2 \left(\frac{(2p_{12} \cdot \ell_1)^2}{(\ell_1 + p_{12})^4} + \frac{(2p_{12} \cdot \ell_2)^2}{(\ell_2 + p_{12})^4} + \frac{(2p_4 \cdot \ell_2)^2}{(\ell_2 - p_4)^4} + X^2 \right) \right\}, \quad (3.12)$$

where we have defined the quantity,

$$X = \frac{2p_{12} \cdot \ell_1}{(\ell_1 + p_{12})^2} + \frac{2p_{12} \cdot \ell_2}{(\ell_2 + p_{12}^2)} - \frac{2p_4 \cdot \ell_2}{(\ell_2 - p_4)^2}. \quad (3.13)$$

At this point, the sample integral is purely logarithmically divergent, and the polynomial dependence of its divergence is manifest. We can now freely alter the dependence in the propagators on external momenta without worrying about affecting the divergence. In particular, we can take $p_i \to 0$ in the propagators. As a result, we see that what we have done to the original integral is equivalent to making the propagator replacements,

$$\frac{1}{(\ell - p)^2} \to \frac{1}{\ell^2} \sum_{n=1}^{\infty} \left(\frac{2\ell \cdot p - p^2}{\ell^2} \right)^n, \quad (3.14)$$

and retaining only the logarithmically divergent terms.

Taking $p_i \to 0$ in the propagators makes the integrals much simpler, but then they no longer have a scale and technically vanish in dimensional regularization. To correct this, we must reintroduce a scale. A computationally convenient choice is to give all of the propagators a uniform mass m. This makes the integral well defined and more tractable:

$$I^{\text{sample}}_{d=5-2\epsilon} + \mathcal{O}(\epsilon^0) = \frac{i}{8} \int \frac{d^{5-2\epsilon}\ell_1}{(2\pi)^{5-2\epsilon}} \frac{d^{5-2\epsilon}\ell_2}{(2\pi)^{5-2\epsilon}} \frac{\varepsilon_4 \cdot \ell_2 (\ell_2 - \ell_1) \cdot \ell_2}{(\ell_1^2 - m^2)^2 ((\ell_1 - \ell_2)^2 - m^2)(\ell_2^2 - m^2)^3}$$

$$\times \left\{ -\varepsilon_3 \cdot p_{12} \tilde{X} - \varepsilon_3 \cdot \ell_2 \left(\frac{s_{12}}{\ell_1^2 - m^2} + \frac{s_{12}}{\ell_2^2 - m^2} \right) \right.$$

$$\left. + \frac{1}{2} \varepsilon_3 \cdot \ell_2 \left(\frac{(2p_{12} \cdot \ell_1)^2}{(\ell_1^2 - m^2)^2} + \frac{(2p_{12} \cdot \ell_2)^2}{(\ell_2^2 - m^2)^2} + \frac{(2p_4 \cdot \ell_2)^2}{(\ell_2^2 - m^2)^2} + \tilde{X}^2 \right) \right\}, \quad (3.15)$$

with

$$\tilde{X} = \frac{2p_{12} \cdot \ell_1}{\ell_1^2 - m^2} + \frac{2p_{12} \cdot \ell_2}{\ell_2^2 - m^2} - \frac{2p_4 \cdot \ell_2}{\ell_2^2 - m^2}. \quad (3.16)$$

D. Tensor reduction

The next step in the analysis of $I^{\text{sample}}_{d=5-2\epsilon}$ is to simplify the tensor numerators. This can be handled straightforwardly using Lorentz invariance, as recently discussed in, for example, Ref. [40]. Consider the terms in Eq. (3.15) proportional to

$$I^{\mu_1 \mu_2 \mu_3 \mu_4}_{\text{tensor}} = \int \frac{d^{5-2\epsilon}\ell_1}{(2\pi)^{5-2\epsilon}} \frac{d^{5-2\epsilon}\ell_2}{(2\pi)^{5-2\epsilon}} \frac{\ell_2^{\mu_1} \ell_2^{\mu_2} \ell_1^{\mu_3} \ell_1^{\mu_4} (\ell_2 - \ell_1) \cdot \ell_2}{(\ell_1^2 - m^2)^4 ((\ell_1 - \ell_2)^2 - m^2)(\ell_2^2 - m^2)^3}. \quad (3.17)$$

This is a rank-4 tensor integral, but because no dependence on the external momenta remains, it must evaluate to a linear combination of products of metric tensors, as nothing else is available:

$$I^{\mu_1 \mu_2 \mu_3 \mu_4}_{\text{tensor}} = \alpha_1 \eta^{\mu_1 \mu_2} \eta^{\mu_3 \mu_4} + \alpha_2 \eta^{\mu_1 \mu_3} \eta^{\mu_2 \mu_4} + \alpha_3 \eta^{\mu_1 \mu_4} \eta^{\mu_2 \mu_3}. \quad (3.18)$$

The particular integrand of $I_{\text{tensor}}^{\mu_1\mu_2\mu_3\mu_4}$ enforces a symmetry between $\mu_1 \leftrightarrow \mu_2$ and $\mu_3 \leftrightarrow \mu_4$, so that $\alpha_2 = \alpha_3$, but we will ignore such optimizations here. Instead, we contract the indices of Eq. (3.18) in all possible ways to obtain the following system of three equations:

$$\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}I_{\text{tensor}}^{\mu_1\mu_2\mu_3\mu_4}$$
$$= \alpha_1(5-2\epsilon)^2 + \alpha_2(5-2\epsilon) + \alpha_3(5-2\epsilon),$$

$$\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4}I_{\text{tensor}}^{\mu_1\mu_2\mu_3\mu_4}$$
$$= \alpha_1(5-2\epsilon) + \alpha_2(5-2\epsilon)^2 + \alpha_3(5-2\epsilon),$$

$$\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3}I_{\text{tensor}}^{\mu_1\mu_2\mu_3\mu_4}$$
$$= \alpha_1(5-2\epsilon) + \alpha_2(5-2\epsilon) + \alpha_3(5-2\epsilon)^2. \quad (3.19)$$

The left-hand sides of these equations are scalar, single-scale vacuum integrals, which are amenable to direct integration. So to evaluate $I_{\text{tensor}}^{\mu_1\mu_2\mu_3\mu_4}$, we just need to invert these equations and solve for α_i. The same idea works as well for higher-rank tensors, allowing us to reduce all the tensor integrals to scalar integrals.

After performing the tensor reduction, it is useful to cancel as many propagators as possible using numerator replacements,

$$\ell_1^2 \to (\ell_1^2 - m^2) + m^2, \qquad \ell_2^2 \to (\ell_2^2 - m^2) + m^2,$$

$$\ell_1 \cdot \ell_2 \to -\frac{1}{2}(((\ell_1 - \ell_2)^2 - m^2)$$
$$- (\ell_1^2 - m^2) - (\ell_2^2 - m^2) - m^2). \quad (3.20)$$

In this way the divergence of the integral $I_{d=5-2\epsilon}^{\text{sample}}$ becomes a linear combination of scalar, single-scale vacuum integrals of the form,

$$\int \frac{d^{5-2\epsilon}\ell_1}{(2\pi)^{5-2\epsilon}} \frac{d^{5-2\epsilon}\ell_2}{(2\pi)^{5-2\epsilon}}$$
$$\times \frac{1}{(\ell_1^2-m^2)^{a_1}((\ell_1-\ell_2)^2-m^2)^{a_2}(\ell_2^2-m^2)^{a_3}}, \quad (3.21)$$

with a_i integers.

E. Scalar integral evaluation

After reducing $I_{d=5-2\epsilon}^{\text{sample}}$ to scalar integrals of the form (3.21), we must evaluate the integrals. We first reduce them to a basis using integration by parts as implemented in FIRE [44]. In all dimensions considered here, our basis consists of two scalar vacuum integrals:

$$I_1 = \int \frac{d^d\ell_1}{(2\pi)^d} \frac{d^d\ell_2}{(2\pi)^d} \frac{1}{(\ell_1^2-m^2)(\ell_2^2-m^2)},$$

$$I_2 = \int \frac{d^d\ell_1}{(2\pi)^d} \frac{d^d\ell_2}{(2\pi)^d} \frac{1}{(\ell_1^2-m^2)((\ell_1-\ell_2)^2-m^2)(\ell_2^2-m^2)}. \quad (3.22)$$

The first integral is simply a product of two easily evaluated one-loop integrals. We evaluate the second integral using the code MB [45] that implements Mellin-Barnes integration [46]. The results are collected in Table I, where an overall prefactor of $1/(4\pi)^d$ has been removed for simplicity. For the cases considered here, we need the basis integrals through order $1/\epsilon$. Using these results for the scalar integrals completes the evaluation of $I_{d=5-2\epsilon}^{\text{sample}}$ and other similar two-loop integrals prior to the subtraction of subdivergences.

F. Subdivergences

The example $I_{d=5-2\epsilon}^{\text{sample}}$ is special in that it has no subdivergences. More generally, subdivergences occur and can greatly complicate the analysis. To deal with this, we follow the basic approach of Ref. [28]. If we alter the previous example to be in six dimensions instead of five dimensions, then we see by power counting that $I_{d=6-2\epsilon}^{\text{sample}}$ in Eq. (3.9) has one-loop subdivergences in both the ℓ_1 and ℓ_2 integrals. There is also a third subloop that could in principle have a divergence—the loop parametrized by $\ell_1 + \ell_2$—but it turns out to be finite in I_d^{sample} for $d < 8$.

The presence of subdivergences means that $I_{d=6-2\epsilon}^{\text{sample}}$ begins at $\mathcal{O}(\epsilon^{-2})$, and Eq. (3.10) will need to be modified. One possible way to modify it is that we need to keep factors of ϵ that can strike a $1/\epsilon^2$:

$$\left(\sum_{i=1}^{4} p_{i\mu} \frac{\partial}{\partial p_{i\mu}}\right) I_{d=6-2\epsilon}^{\text{sample}} = (4-4\epsilon) I_{d=6-2\epsilon}^{\text{sample}}, \quad (3.23)$$

which holds to all orders in ϵ. However, we cannot simply disregard terms that are naively finite by overall power counting because they may still contain subdivergences that would contribute at $\mathcal{O}(\epsilon^{-1})$. In addition, after 4 powers of external momenta have been extracted, leaving only logarithmically divergent integrals and pure-subdivergence integrals, we cannot set $p_i \to 0$ or add masses in the propagators without affecting the $\mathcal{O}(\epsilon^{-1})$ term. In fact, the results would depend on the details used to regulate infrared singularities generated by the momentum expansion.

TABLE I. The basis integrals in $d = 4-2\epsilon$, $5-2\epsilon$, $6-2\epsilon$ required for the two-loop Yang-Mills and gravity divergence computations, valid through $\mathcal{O}(1/\epsilon)$. A factor of $1/(4\pi)^d$ has been dropped from the results in the table.

Integral	$d = 4 - 2\epsilon$	$d = 5 - 2\epsilon$	$d = 6 - 2\epsilon$
I_1	$-\frac{1}{\epsilon^2}(m^2)^{2-2\epsilon}e^{2(1-\gamma_{\text{E}})\epsilon} + \mathcal{O}(\epsilon^0)$	$\mathcal{O}(\epsilon^0)$	$-\frac{1}{4\epsilon^2}(m^2)^{4-2\epsilon}e^{(3-2\gamma_{\text{E}})\epsilon} + \mathcal{O}(\epsilon^0)$
I_2	$-\frac{3}{2\epsilon^2}(m^2)^{1-2\epsilon}e^{(3-2\gamma_{\text{E}})\epsilon} + \mathcal{O}(\epsilon^0)$	$\frac{\pi}{2\epsilon}(m^2)^2 + \mathcal{O}(\epsilon^0)$	$-\frac{5}{8\epsilon^2}(m^2)^{3-2\epsilon}e^{(113/30-2\gamma_{\text{E}})\epsilon} + \mathcal{O}(\epsilon^0)$

For these reasons, we instead work with subtracted divergences, which we denote as $S[\mathcal{I}]$. A subtracted divergence of an integral is the integral's divergence in dimensional regularization with all of its subdivergences subtracted off:

$$S\left[\int \prod_{i=1}^{L} \frac{d^d \ell_i}{(2\pi)^d} I(\ell_1, \ldots, \ell_L)\right]$$
$$= \mathrm{Div}\left[\int \prod_{i=1}^{L} \frac{d^d \ell_i}{(2\pi)^d} I(\ell_1, \ldots, \ell_L)\right] - \sum_{l=1}^{L-1} \sum_{\substack{l\text{-loop} \\ \text{subintegrals}}} \mathrm{Div}\left[\int \prod_{i=l+1}^{L} \frac{d^d \tilde{\ell}_i}{(2\pi)^d} S\left[\int \prod_{j=1}^{l} \frac{d^d \tilde{\ell}_j}{(2\pi)^d} I(\tilde{\ell}_1, \ldots, \tilde{\ell}_L)\right]\right]. \quad (3.24)$$

Here, Div indicates the divergent part of the integral [i.e. its value through $\mathcal{O}(\epsilon^{-1})$], and $\tilde{\ell}_i$ is a reparametrization of the integral such that a particular l-loop subintegral is parametrized by $\tilde{\ell}_1$ through $\tilde{\ell}_l$. This definition can be thought of as adding counterterm diagrams integral by integral to remove their subdivergences. It has the nice property that $S[\mathcal{I}]$ is a polynomial in external momenta, so we can extract all of the external dependence of the sample integral, in this case of quartic order, with

$$\left(\sum_{i=1}^{4} p_{i\mu} \frac{\partial}{\partial p_{i\mu}}\right) S[\mathcal{I}_{d=6-2\epsilon}^{\mathrm{sample}}] = 4 S[\mathcal{I}_{d=6-2\epsilon}^{\mathrm{sample}}], \quad (3.25)$$

and then freely set $p_i \to 0$ and introduce masses into the propagators. In contrast to Eq. (3.23), we have a 4 instead of a $(4 - 4\epsilon)$ on the right-hand side because the subtractions remove the source of the additional terms. In the case that $S[\cdots]$ is a subintegral [as in the second line of Eq. (3.24)], the remaining loop momenta and any introduced masses should also be treated as external variables, so that the dimension-counting operator is instead

$$\sum_{i=1}^{4} p_{i\mu} \frac{\partial}{\partial p_{i\mu}} + \sum_{i=l+1}^{L} \tilde{\ell}_{i\mu} \frac{\partial}{\partial \tilde{\ell}_{i\mu}} + 2m^2 \frac{\partial}{\partial m^2}. \quad (3.26)$$

After the subtractions are taken into account, the tensor integrals can be simplified as before to obtain a final answer consisting of a linear combination of scalar single-scale vacuum integrals.

IV. ONE-LOOP DIVERGENCES

As a warm up before turning to two and three loops, we determine the one-loop four-point divergences of half-maximal supergravity with n_V matter multiplets in $D = 4$, 6, 8. (We do not consider $D = 5, 7$ because there are no divergences in dimensional regularization in odd dimensions for odd loop orders.) We confirm the appearance of divergences in four-matter amplitudes found long ago by Fischler [23] and by Fradkin and Tseytlin [24]. We also illustrate the connection of the supergravity divergences to those of four-scalar amplitudes in nonsupersymmetric gauge theory, as noted in Ref. [13].

We start by presenting the divergences in Yang-Mills theory coupled to scalars that are proportional to the one-loop color tensor $b_1^{(1)}$. The half-maximal supergravity divergences are then obtained by replacing the color factor

with the $\mathcal{N} = 4$ super-Yang-Mills duality-satisfying numerator, as given in Eq. (2.12). We collect the counterterms corresponding to the supergravity divergences with external gravitons and matter vectors in the Appendix. In $D = 4$ the divergences and counterterms all carry an $SO(6) \times SO(n_V)$ symmetry. The $SO(6)$ is just the symmetry of the vectors of the graviton multiplet and is inherited from the R symmetry of the scalars of $\mathcal{N} = 4$ super-Yang-Mills theory. The $SO(n_V)$ symmetry is a reflection of the fact that all matter multiplets are equivalent.

A. Four dimensions

As discussed in Ref. [13], in $D = 4$ the renormalizability of gauge theory ensures that four-point divergences involving external gluons must be proportional to tree-level amplitudes. This means that divergences proportional to the one-loop color tensor vanish:

$$\mathcal{A}^{(1)}(1_g, 2_g, 3_g, 4_g)|_{D=4 \text{ div.}} = 0 + \cdots,$$
$$\mathcal{A}^{(1)}(1_g, 2_g, 3_\phi, 4_\phi)|_{D=4 \text{ div.}} = 0 + \cdots, \quad (4.1)$$

where the label g or ϕ indicates that an external leg is a gluon or scalar, respectively, and as before, "$+\cdots$" signifies that we dropped divergences proportional to the tree color tensor.

On the other hand, renormalizability does not protect divergences in the four-scalar amplitude because operators of the form $b_1^{(1),abcd} \phi^a \phi^b \phi^c \phi^d$ are perfectly valid counterterms. Carrying out the computation, we find that for four identical external scalar states the divergence is

$$\mathcal{A}^{(1)}(1_\phi, 2_\phi, 3_\phi, 4_\phi)|_{D=4 \text{ div.}}$$
$$= \frac{i}{\epsilon} \frac{1}{(4\pi)^2} g^4 b_1^{(1)} \frac{3(D_s - 2)}{2} + \cdots, \quad (4.2)$$

where $D_s - 4$ is the number of distinct real scalars that circulate in the loop. In this case, for consistency we should take the state-counting parameter $D_s \geq 5$ so that we have at least one scalar state. Taking the state-counting parameter to be an integer which leaves the number of gluon states (for each color) at their four-dimensional values is equivalent to using the four-dimensional helicity scheme [47].

For a pair of distinct external scalars, we find the divergence,

$$\mathcal{A}^{(1)}(1_{\phi_1}, 2_{\phi_1}, 3_{\phi_2}, 4_{\phi_2})|_{D=4 \text{ div.}}$$
$$= \frac{i}{\epsilon} \frac{1}{(4\pi)^2} g^4 b_1^{(1)} \frac{D_s - 2}{2} + \cdots. \qquad (4.3)$$

As expected, the case with identical scalars follows from the one with distinct scalars by summing over the three distinct permutations corresponding to the distinct ways of connecting the external scalar legs. The number of scalar states circulating in the loop is again given by $D_s - 4$, so for consistency we should take $D_s \geq 6$ to have two scalars. In both Eqs. (4.2) and (4.3), the terms containing D_s, and therefore those due to scalar states in the loop, arise from contact diagrams of the form displayed in Fig. 6(a). We will find this useful in Sec. V for understanding the structure of the two-loop divergences in $D = 4$.

Using the double-copy replacement (2.12) for the color factor in terms of the $\mathcal{N} = 4$ super-Yang-Mills duality-satisfying numerator, we obtain the corresponding divergences in $\mathcal{N} = 4$ supergravity with $n_V = D_s - 4$ matter multiplets:

$$\mathcal{M}^{(1)}(1_H, 2_H, 3_H, 4_H)|_{D=4 \text{ div.}} = 0,$$
$$\mathcal{M}^{(1)}(1_H, 2_H, 3_V, 4_V)|_{D=4 \text{ div.}} = 0,$$
$$\mathcal{M}^{(1)}(1_V, 2_V, 3_V, 4_V)|_{D=4 \text{ div.}}$$
$$= -\frac{1}{\epsilon} \frac{1}{(4\pi)^2} \left(\frac{\kappa}{2}\right)^4 st A_{Q=16}^{\text{tree}} \frac{3(D_s - 2)}{2},$$
$$\mathcal{M}^{(1)}(1_{V_1}, 2_{V_1}, 3_{V_2}, 4_{V_2})|_{D=4 \text{ div.}}$$
$$= -\frac{1}{\epsilon} \frac{1}{(4\pi)^2} \left(\frac{\kappa}{2}\right)^4 st A_{Q=16}^{\text{tree}} \frac{D_s - 2}{2}, \qquad (4.4)$$

where, as noted in Sec. III B, the label H indicates that a leg is a state of the graviton multiplet while a subscript V indicates that the leg is a state of a vector multiplet. The cases with subscripts V_1 and V_2 indicate that the legs belong to distinct vector multiplets. Cases with an odd number of external matter-multiplet legs vanish trivially. The total number of matter vector multiplets is given by $n_V = D_s - 4$, and the supersymmetric prefactor $A_{Q=16}^{\text{tree}}$ automatically incorporates all valid external states in both the vector and graviton multiplets. It is interesting to note that the contribution to the divergence from the matter multiplet in the loop is proportional to that of the
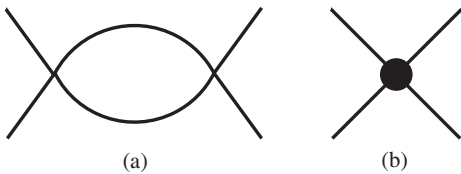
FIG. 6. The divergences containing a one-loop color tensor and a factor D_s all arise from diagrams of the form (a). The total divergence subtraction is shown in (b), where the large dot signifies a local subtraction.

graviton multiplet, and that the result diverges for any number of vector multiplets. For consistency, we must have $n_V \geq 1$ for the cases with all matter belonging to the same matter multiplet and $n_V \geq 2$ for the case where the two pairs of external states belong to different matter multiplets.

B. Six dimensions

As already discussed in Ref. [13], the only available F^3 Yang-Mills counterterm for external gluons generates amplitudes with color tensors proportional to the tree-level color tensors, a fact that is unaltered with the addition of scalars to the theory. Thus we immediately have that the part of the divergence proportional to the one-loop color tensor vanishes:

$$\mathcal{A}^{(1)}(1_g, 2_g, 3_g, 4_g) = 0 + \cdots. \qquad (4.5)$$

For four identical external scalars, the Yang-Mills counterterm involving the one-loop color tensor of the form $D^2 \phi^4$ vanishes because by crossing symmetry, it needs to be proportional to $s + t + u = 0$. One might worry about an interference of the crossing properties of the color and the kinematics, but the independent one-loop color tensor can be put into a fully crossing-symmetric form plus terms proportional to tree color factors:

$$b_1^{(1)} = \frac{1}{4!} \sum_\sigma c_{\sigma(1),\sigma(2),\sigma(3),\sigma(4)}^{(1)} + \cdots, \qquad (4.6)$$

where $c_{ijkl}^{(1)}$ is a one-loop box color factor and σ runs over all 4! permutations of the external legs. Therefore for trivial symmetry reasons there is no divergence in terms containing the one-loop color tensor when all four external scalars are identical:

$$\mathcal{A}^{(1)}(1_\phi, 2_\phi, 3_\phi, 4_\phi)|_{D=6 \text{ div.}} = 0 + \cdots. \qquad (4.7)$$

For the case of two pairs of nonidentical scalars, the amplitude no longer has the full crossing symmetry and hence the divergence no longer vanishes from simple symmetry considerations. Instead we find

$$\mathcal{A}^{(1)}(1_{\phi_1}, 2_{\phi_1}, 3_{\phi_2}, 4_{\phi_2})|_{D=6 \text{ div.}}$$
$$= -\frac{i}{\epsilon} \frac{1}{(4\pi)^3} g^4 b_1^{(1)} \frac{26 - D_s}{12} s + \cdots. \qquad (4.8)$$

Finally, the two-scalar two-gluon divergence proportional to the one-loop color tensor is

$$\mathcal{A}^{(1)}(1_g, 2_g, 3_\phi, 4_\phi)|_{D=6 \text{ div.}}$$
$$= \frac{i}{\epsilon} \frac{1}{(4\pi)^3} g^4 b_1^{(1)} \frac{26 - D_s}{24} (\varepsilon_1 \cdot \varepsilon_2 s - 2k_1 \cdot \varepsilon_2 k_2 \cdot \varepsilon_1) + \cdots. \qquad (4.9)$$

Substituting the color factor with the kinematic numerator (2.12) in Eqs. (4.5) and (4.7)–(4.9) immediately gives us

the half-maximal supergravity divergences for cases including external states from vector multiplets:

$$\mathcal{M}^{(1)}(1_H, 2_H, 3_H, 4_H)|_{D=6\,\text{div.}} = 0,$$

$$\mathcal{M}^{(1)}(1_V, 2_V, 3_V, 4_V)|_{D=6\,\text{div.}} = 0,$$

$$\mathcal{M}^{(1)}(1_{V_1}, 2_{V_1}, 3_{V_2}, 4_{V_2})|_{D=6\,\text{div.}}$$
$$= \frac{1}{\epsilon}\frac{1}{(4\pi)^3}\left(\frac{\kappa}{2}\right)^4 st A^{\text{tree}}_{Q=16}\frac{26-D_s}{12}s,$$

$$\mathcal{M}^{(1)}(1_H, 2_H, 3_V, 4_V)|_{D=6\,\text{div.}}$$
$$= -\frac{1}{\epsilon}\frac{1}{(4\pi)^3}\left(\frac{\kappa}{2}\right)^4 st A^{\text{tree}}_{Q=16}\frac{26-D_s}{24}$$
$$\times (\varepsilon_1\cdot\varepsilon_2 s - 2k_1\cdot\varepsilon_2 k_2\cdot\varepsilon_1). \quad (4.10)$$

C. Eight dimensions

In eight dimensions at one loop, nonsupersymmetric Yang-Mills theory has an F^4 divergence containing a one-loop color tensor. Therefore, the corresponding half-maximal supergravity diverges at one loop [13]. The explicit value of the divergences for four external graviton multiplets is given in Eq. (3.19) of Ref. [13], with the number of vector supermultiplets given by $n_V = D_s - 8$; the pure supergravity divergence was first computed in Ref. [48]. Yang-Mills operators generating divergences for external scalars in $D = 8$, specifically $D^2\phi^2 F^2$ and $D^4\phi^4$, can also be contracted with one-loop color tensors. Thus it is no surprise that cases with external matter multiplets also diverge in half-maximal supergravity.

For four identical scalars in Yang-Mills theory, the divergence proportional to the one-loop color tensor is

$$\mathcal{A}^{(1)}(1_\phi, 2_\phi, 3_\phi, 4_\phi)|_{D=8\,\text{div.}}$$
$$= \frac{i}{\epsilon}\frac{1}{(4\pi)^4}g^4 b_1^{(1)}\frac{D_s+18}{120}(s^2+t^2+u^2)+\cdots. \quad (4.11)$$

For two pairs of distinct external scalars, we have the gauge-theory divergence,

$$\mathcal{A}^{(1)}(1_{\phi_1}, 2_{\phi_1}, 3_{\phi_2}, 4_{\phi_2})|_{D=8\,\text{div.}}$$
$$= \frac{i}{\epsilon}\frac{1}{(4\pi)^4}g^4 b_1^{(1)}\frac{(D_s-2)s^2-40tu}{120}+\cdots, \quad (4.12)$$

while the two-scalar two-gluon divergence is

$$\mathcal{A}^{(1)}(1_g, 2_g, 3_\phi, 4_\phi)|_{D=8\,\text{div.}}$$
$$= \frac{i}{\epsilon}\frac{1}{(4\pi)^4}g^4 b_1^{(1)}\frac{1}{180}[(D_s-2)s(2k_1\cdot\varepsilon_2 k_2\cdot\varepsilon_1 - \varepsilon_1\cdot\varepsilon_2 s)$$
$$+ 60(2k_3\cdot\varepsilon_1 k_4\cdot\varepsilon_2 t + 2k_4\cdot\varepsilon_1 k_3\cdot\varepsilon_2 u$$
$$+ \varepsilon_1\cdot\varepsilon_2 tu]+\cdots. \quad (4.13)$$

In these eight-dimensional expressions the number of real scalars circulating in the loops is $D_s - 8$.

As before, we obtain the corresponding half-maximal supergravity divergences with $n_V = D_s - 8$ matter multiplets by substituting the color factors in the Yang-Mills expressions with the kinematic numerator (2.12):

$$\mathcal{M}^{(1)}(1_V, 2_V, 3_V, 4_V)|_{D=8\,\text{div.}} = -\frac{1}{\epsilon}\frac{1}{(4\pi)^4}\left(\frac{\kappa}{2}\right)^4 st A^{\text{tree}}_{Q=16}\frac{D_s+18}{120}(s^2+t^2+u^2),$$

$$\mathcal{M}^{(1)}(1_{V_1}, 2_{V_1}, 3_{V_2}, 4_{V_2})|_{D=8\,\text{div.}} = -\frac{1}{\epsilon}\frac{1}{(4\pi)^4}\left(\frac{\kappa}{2}\right)^4 st A^{\text{tree}}_{Q=16}\frac{(D_s-2)s^2-40tu}{120},$$

$$\mathcal{M}^{(1)}(1_H, 2_H, 3_V, 4_V)|_{D=8\,\text{div.}} = -\frac{1}{\epsilon}\frac{1}{(4\pi)^4}\left(\frac{\kappa}{2}\right)^4 st A^{\text{tree}}_{Q=16}\frac{1}{180}[(D_s-2)s(2k_1\cdot\varepsilon_2 k_2\cdot\varepsilon_1 - \varepsilon_1\cdot\varepsilon_2 s)$$
$$+ 60(2k_3\cdot\varepsilon_1 k_4\cdot\varepsilon_2 t + 2k_4\cdot\varepsilon_1 k_3\cdot\varepsilon_2 u + \varepsilon_1\cdot\varepsilon_2 tu)]. \quad (4.14)$$

Combined with the result in Ref. [13], this gives the complete set of four-point divergences in $D = 8$ for any external states, whether in the graviton multiplet or in a vector multiplet.

V. TWO- AND THREE-LOOP DIVERGENCES

In this section we systematically list out the two-loop four-point divergences of half-maximal supergravity with external matter multiplets in $D = 4, 5, 6$ along with the divergences of corresponding nonsupersymmetric gauge theory that control them. In all our expressions we always subtract subdivergences. As it turns out, in $D = 4$ the divergence appears to be of a form where it is an iteration

of the one-loop divergence, so to conclusively demonstrate that new divergences occur, we also present the three-loop divergences. Two-loop supergravity counterterms are provided in the Appendix. As at one loop all divergences and counterterms in $D = 4$ carry a manifest $SO(6)\times SO(n_V)$ symmetry.

A. Four dimensions

In $D = 4$, renormalizability dictates that gauge-theory counterterms for the four-gluon divergence and the two-gluon two-scalar divergence contain only tree-level color tensors. Thus from simple renormalizability considerations, we have [13]

$$\mathcal{A}^{(2)}(1_g, 2_g, 3_g, 4_g)|_{D=4 \text{ div.}} = 0 + \cdots,$$
$$\mathcal{A}^{(2)}(1_g, 2_g, 3_\phi, 4_\phi)|_{D=4 \text{ div.}} = 0 + \cdots, \tag{5.1}$$

where "$+\cdots$" refers to dropped terms that contain tree and one-loop color tensors. As noted in Sec. II B, the dropped terms are not needed for converting to supergravity.

As already discussed in Ref. [13], renormalizability considerations do not protect four-scalar divergences from containing higher-loop color tensors. However, it turns out that when the four scalars are identical, the divergences proportional to the two-loop color tensors cancel because of a color identity. This happens when all the scalars are identical because the two-loop color tensors appear fully symmetrized in their indices, i.e. as

$$\phi^a \phi^b \phi^c \phi^d (c^{Pabcd}_{1234} + c^{Pabcd}_{3421} + c^{Pabcd}_{1423} + c^{Pabcd}_{2341}$$
$$+ c^{Pabcd}_{1342} + c^{Pabcd}_{4231}) = 0 + \cdots. \tag{5.2}$$

By reexpressing these color factors in the basis (2.19), we immediately see that the two-loop color tensors $b_1^{(2)}$ and $b_2^{(2)}$ cancel out, so there can be no divergence containing these color tensors when the four scalars are identical:

$$\mathcal{A}^{(2)}(1_\phi, 2_\phi, 3_\phi, 4_\phi)|_{D=4 \text{ div.}} = 0 + \cdots. \tag{5.3}$$

If the scalars are not all identical, the previous symmetry argument no longer applies. Indeed, we find that gauge-theory amplitudes with nonidentical external scalars are divergent. The nonvanishing contribution to the two-loop four-scalar divergence with a distinct pair of scalars that contains an independent two-loop color tensor is

$$\mathcal{A}^{(2)}(1_{\phi_1}, 2_{\phi_1}, 3_{\phi_2}, 4_{\phi_2})|_{D=4 \text{ div.}}$$
$$= \frac{i}{\epsilon^2} \frac{1}{(4\pi)^4} g^6 b_1^{(2)} \frac{(D_s - 2)^2}{4} + \cdots, \tag{5.4}$$

where the one-loop subdivergences have all been subtracted. Since the number of scalars is $D_s - 4$, the divergence does not vanish for any (positive) number of scalar fields. In this case the t-channel basis color tensor $b_2^{(2)}$ is absent.

A curious feature of the divergence in Eq. (5.4) is that it does not contain a $1/\epsilon$ divergence but only a $1/\epsilon^2$ divergence. This may be understood straightforwardly for terms proportional to the square of the state-counting parameter D_s. Prior to subtractions, the only diagrams that give contributions that contain both a factor of D_s^2

and a two-loop color tensor are those of the form in Fig. 7(a). Without this configuration it is not possible to get two factors of D_s, each of which come from contracting a Lorentz index around an independent loop. A second source of these terms is the subtraction diagrams where one factor of D_s comes from the one-loop subtraction and the second comes from a loop, as illustrated in Fig. 7(b). The structure of the result follows from the fact that each of the contributing loops is of the form,

$$\frac{a}{\epsilon} + b, \tag{5.5}$$

where a and b are parameters that depend on the external momenta. The two-loop diagram [Fig. 7(a)] contains two such loops and is just the square of this:

$$V^{(a)} = \left(\frac{a}{\epsilon} + b\right)\left(\frac{a}{\epsilon} + b\right), \tag{5.6}$$

times a prefactor. The subtraction terms in Fig. 7(b) are of the form,

$$V^{(b)} = -\frac{a}{\epsilon}\left(\frac{a}{\epsilon} + b\right) - \left(\frac{a}{\epsilon} + b\right)\frac{a}{\epsilon}, \tag{5.7}$$

times the same prefactor. Combining the direct terms (5.6) with the subtraction terms (5.7) flips the sign of the $1/\epsilon^2$ terms and cancels the $1/\epsilon$ terms, as given in Eq. (5.4). The terms that are subleading in D_s are more complicated because other diagrams contribute. Once all pieces are added together, we find that all $1/\epsilon$ terms cancel for the divergence, as for the D_s^2 terms. The fact that a supergravity theory can display this iterative structure is highly nontrivial and presumably inherited from nonsupersymmetric Yang-Mills theory through the relatively simple relation between the corresponding amplitudes. This property appears to be special to two loops. Since the supergravity divergence is inherited from the gauge-theory one, the corresponding divergences will also not have $1/\epsilon$ contributions.

Converting the lack of gauge-theory divergences in Eqs. (5.1) and (5.3) to supergravity divergences using the double-copy substitution in Eq. (2.22) gives us the following finiteness results:

$$\mathcal{M}^{(2)}(1_H, 2_H, 3_H, 4_H)|_{D=4 \text{ div.}} = 0,$$
$$\mathcal{M}^{(2)}(1_H, 2_H, 3_V, 4_V)|_{D=4 \text{ div.}} = 0, \tag{5.8}$$
$$\mathcal{M}^{(2)}(1_V, 2_V, 3_V, 4_V)|_{D=4 \text{ div.}} = 0.$$

These hold for all external states in the respective multiplets, independent of the number of matter multiplets added to the theory. The vanishing of the divergences in the four
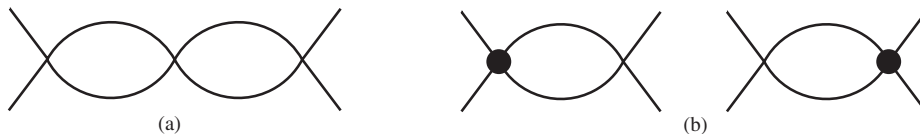

FIG. 7.   The divergences containing a two-loop color tensor and a factor of D_s^2 all arise from diagrams of the form (a). The diagrams subtracting the one-loop subdivergences are shown in (b).

identical-matter-multiplet case can also be seen from its symmetry. From dimensional analysis, after extracting the crossing-symmetric factor of $stA_{Q=16}^{\text{tree}}$, there is an additional factor of s which can appear only in the crossing-symmetric form $s + t + u = 0$, implying the vanishing of the divergence. We can think of this cancellation as ''accidental,'' similar to the vanishing of one-loop divergences in pure Einstein gravity. Interestingly, all the above finite results exhibit cancellation separately in both the unsubtracted pieces and in the subtractions when a uniform mass infrared regulator is used. This is true even when one-loop divergences imply the presence of subdivergences in higher-loop amplitudes. Another notable case where this happens is three-loop $\mathcal{N} = 4$ supergravity in four dimensions with internal matter [12].

Finally, applying the substitution rule (2.22) to the gauge-theory amplitude with a pair of distinct scalars gives the nonvanishing divergence for a four-point supergravity

amplitude with external states from a pair of distinct vector multiplets,

$$\mathcal{M}^{(2)}(1_{V_1}, 2_{V_1}, 3_{V_2}, 4_{V_2})|_{D=4 \text{ div.}}$$
$$= -\frac{1}{\epsilon^2} \frac{1}{(4\pi)^4} \left(\frac{\kappa}{2}\right)^6 s^2 t A_{Q=16}^{\text{tree}} \frac{(D_s - 2)^2}{4}, \quad (5.9)$$

where the state-counting parameter takes the value, $D_s = n_V + D$, with $D = 4$ in the four-dimensional helicity scheme [47]. Since the divergence is for a pair of distinct matter multiplets, for consistency we need $n_V \geq 2$.

The fact that in $D = 4$ the two-loop divergence (5.9) is an iteration of the one-loop divergence raises the question[1] of whether it might follow from the one-loop divergences in Eq. (4.4). To definitively settle any such potential question, we have also computed the complete set of three-loop $\mathcal{N} = 4$ supergravity divergences in $D = 4$. Our results for these divergences are

$$\mathcal{M}^{(3)}(1_H, 2_H, 3_H, 4_H)|_{D=4 \text{ div.}} = 0, \qquad \mathcal{M}^{(3)}(1_H, 2_H, 3_V, 4_V)|_{D=4 \text{ div.}} = 0,$$

$$\mathcal{M}^{(3)}(1_V, 2_V, 3_V, 4_V)|_{D=4 \text{ div.}} = -\frac{1}{(4\pi)^6} \left(\frac{\kappa}{2}\right)^8 (s^2 + t^2 + u^2) st A_{Q=16}^{(0)} \frac{(D_s - 2)^2}{4} \left(\frac{D_s - 2}{2\epsilon^3} - \frac{1}{\epsilon^2} + \frac{1}{\epsilon}\right),$$

$$\mathcal{M}^{(3)}(1_{V_1}, 2_{V_1}, 3_{V_2}, 4_{V_2})|_{D=4 \text{ div.}} = -\frac{1}{(4\pi)^6} \left(\frac{\kappa}{2}\right)^8 st A_{Q=16}^{(0)} \left[\frac{1}{\epsilon^3} \frac{(D_s - 2)^2}{8} ((D_s - 4)s^2 - 4tu) + \frac{1}{\epsilon^2} \frac{(D_s - 2)^2}{2} tu \right.$$

$$\left. + \frac{1}{\epsilon} \left(\frac{(D_s - 2)^2}{2} (s^2 + tu) - (7D_s - 38)(s^2 + 2tu)\zeta_3\right)\right], \quad (5.10)$$

where the number of vector multiplets is $n_V = D_s - 4$ and all subdivergences have been subtracted, as usual. The vanishing of divergences when all four external states are from the graviton multiplet was shown in Ref. [12]. The vanishing of divergences when two external states are from the graviton multiplet and two from a matter multiplet is new. We leave the comparison of these divergences to those of nonsupersymmetric Yang-Mills coupled to scalars to future studies.

The divergences (5.10) are not of a form where they can be induced by the one-loop divergences, settling any potential issues on whether these are new divergences. In particular, we cannot obtain a ζ_3 from a one-loop divergence. If we ignore, for the moment, any potential issues with the $SL(2, \mathbb{R})$ duality anomaly, this result contradicts the expected finiteness [16], had an off-shell superspace manifesting all 16 supercharges existed [15].

Now consider the $SL(2, \mathbb{R})$ duality anomaly. One may wonder if it can somehow be responsible for the divergences in Eqs. (5.9) and (5.10), since it can prevent use of the duality symmetry to rule out a counterterm [16]. However, these divergences are not of the proper form had they been

due to the anomaly. Anomalies are associated with a ''0/0,'' or more precisely in dimensional regularization, contributions of $\mathcal{O}(\epsilon)$ that violate a symmetry and can give an $\mathcal{O}(\epsilon^0)$ contribution when they hit a $1/\epsilon$ divergence. Indeed, this is how the anomaly enters into one-loop amplitudes [49]. At two loops these finite one-loop terms could feed in to give at most a $1/\epsilon$ divergence, and we would have found that at two loops the divergences would contain no $1/\epsilon^2$ term and at three loops no $1/\epsilon^3$ terms. In addition, the amplitudes containing the divergences are all inert under the anomalous $U(1)$, using the helicity counting rules of Ref. [49]. These features are incompatible with the anomaly being the source of the $D = 4$ divergences. We therefore conclude that our results are inconsistent with the existence of a 16-supercharge off-shell superspace in $D = 4$.

B. Five dimensions

The two-loop gluon amplitudes of five-dimensional gauge theory coupled to scalars have divergences due to an F^3 operator. This operator generates divergences containing only tree-level color factors, and hence no two-loop color tensors are present [13]. Thus, the four-gluon divergence is given by

$$\mathcal{A}^{(2)}(1_g, 2_g, 3_g, 4_g)|_{D=5 \text{ div.}} = 0 + \cdots. \quad (5.11)$$

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[1]We thank G. Bossard, K. Stelle and P. Howe for raising this question.

However, when we have external adjoint scalars, counterterms involving the two-loop color tensors exist. The available counterterms involving two-loop color tensors at two loops in five dimensions are similar to those involving one-loop color tensors at one loop in six dimensions. For four external scalars, we have $D^2\phi^4$. For identical scalars, the two-loop gauge-theory divergence is

$$\mathcal{A}^{(2)}(1_\phi, 2_\phi, 3_\phi, 4_\phi)|_{D=5\,\mathrm{div.}}$$
$$= \frac{i}{\epsilon}\frac{1}{(4\pi)^5}g^6\frac{(10-D_s)\pi}{3}(b_1^{(2)}(u-s)+b_2^{(2)}(u-t)) + \cdots. \tag{5.12}$$

For distinct external scalars, the divergence containing two-loop color tensors is

$$\mathcal{A}^{(2)}(1_{\phi_1}, 2_{\phi_1}, 3_{\phi_2}, 4_{\phi_2})|_{D=5\,\mathrm{div.}}$$
$$= \frac{i}{\epsilon}\frac{1}{(4\pi)^5}g^6\frac{(10-D_s)\pi}{6}\left(b_1^{(2)}(t-3s)+b_2^{(2)}(t-u)\right) + \cdots. \tag{5.13}$$

Finally for the two-scalar two-gluon divergence, we have a $\phi^2 F^2$ counterterm. The divergence corresponding to this operator is

$$\mathcal{A}^{(2)}(1_g, 2_g, 3_\phi, 4_\phi)|_{D=5\,\mathrm{div.}}$$
$$= \frac{i}{\epsilon}\frac{1}{(4\pi)^5}g^6 b_1^{(2)}\frac{(10-D_s)\pi}{6}(\varepsilon_1\cdot\varepsilon_2 s - 2k_1\cdot\varepsilon_2 k_2\cdot\varepsilon_1)$$
$$+\cdots. \tag{5.14}$$

As in $D=4$, we can convert these results to those of half-maximal supergravity by replacing the color tensors with the kinematic numerators in Eq. (2.22) to yield the supergravity divergences:

$$\mathcal{M}^{(2)}(1_H, 2_H, 3_H, 4_H)|_{D=5\,\mathrm{div.}} = 0,$$
$$\mathcal{M}^{(2)}(1_V, 2_V, 3_V, 4_V)|_{D=5\,\mathrm{div.}}$$
$$= \frac{1}{\epsilon}\frac{1}{(4\pi)^5}\left(\frac{\kappa}{2}\right)^6 st A_{Q=16}^{\mathrm{tree}}\frac{(10-D_s)\pi}{3}(s^2+t^2+u^2),$$
$$\mathcal{M}^{(2)}(1_{V_1}, 2_{V_1}, 3_{V_2}, 4_{V_2})|_{D=5\,\mathrm{div.}}$$
$$= \frac{1}{\epsilon}\frac{1}{(4\pi)^5}\left(\frac{\kappa}{2}\right)^6 st A_{Q=16}^{\mathrm{tree}}\frac{(10-D_s)\pi}{6}(3s^2+2tu),$$
$$\mathcal{M}^{(2)}(1_H, 2_H, 3_V, 4_V)|_{D=5\,\mathrm{div.}}$$
$$= -\frac{1}{\epsilon}\frac{1}{(4\pi)^5}\left(\frac{\kappa}{2}\right)^6 s^2 t A_{Q=16}^{\mathrm{tree}}\frac{(10-D_s)\pi}{6}$$
$$\times (\varepsilon_1\cdot\varepsilon_2 s - 2k_1\cdot\varepsilon_2 k_2\cdot\varepsilon_1), \tag{5.15}$$

where the number of matter multiplets is $n_V = D_s - 5$. An interesting feature of these divergences is that they

vanish if the number of vector supermultiplets is $n_V = 5$, which corresponds to the theory of $\mathcal{N}=1$, $D=10$ supergravity dimensionally reduced to $D=5$. It would be interesting to know if these cancellations for five vector multiplets persist to higher-loop orders. It is noteworthy that a linearized superspace exists for this theory in $D=10$ [50].

The divergent results in Eq. (5.15) are in direct conflict with the predictions of Ref. [16], under the assumption [15] that there is a harmonic superspace manifesting all 16 supercharges off shell. On the other hand, the two-loop ultraviolet finiteness of $D=5$ pure half-maximal supergravity is a direct consequence of the duality between color and kinematics and the general structure of corresponding gauge-theory divergences, so from this vantage point there is no mystery.

C. Six dimensions

Half-maximal supergravity is divergent in six dimensions with or without matter in the loop [13]. This can be understood through the presence of an F^4 counterterm in pure Yang-Mills theory that contains the independent two-loop color tensors. The divergence was given in Ref. [13], so we do not reproduce it here. One fact of interest is that there is no $1/\epsilon^2$ term in the divergence for pure half-maximal supergravity (no matter in the loop), which is consistent with expectations based on the lack of one-loop divergences in pure half-maximal supergravity in six dimensions.

The four-point Yang-Mills divergence for two scalars and two gluons is given by a $D^2\phi^2 F^2$ operator. We do not give the divergence or the associated two-external-matter four-point supergravity divergence, but we do mention the presence of a factor of $26-D_s$ multiplying the $1/\epsilon^2$ piece in both. This is again consistent with the one-loop subdivergence.

For four external scalars, counterterms of the form $D^4\phi^4$ are valid. The corresponding divergence involving the two-loop color tensors is given by

$$\mathcal{A}^{(2)}(1_\phi, 2_\phi, 3_\phi, 4_\phi)|_{D=6\,\mathrm{div.}}$$
$$= i\frac{1}{(4\pi)^6}g^6\frac{1}{144}\left(\frac{(D_s-6)(26-D_s)}{\epsilon^2} - \frac{13D_s+142}{3\epsilon}\right)$$
$$\times (b_1^{(2)}t(s-u)+b_2^{(2)}s(t-u)) + \cdots. \tag{5.16}$$

We do not present the Yang-Mills divergence for two different scalars because it is somewhat complicated and not particularly enlightening. We do note that once again a factor of $26-D_s$ in the $1/\epsilon^2$ pieces multiplies each color tensor.

Applying the substitution rule (2.22) to the gauge-theory results, we have for half-maximal supergravity divergences with external matter states:

$$\mathcal{M}^{(2)}(1_V, 2_V, 3_V, 4_V)|_{D=6\;\mathrm{div.}} = \frac{1}{(4\pi)^6}\left(\frac{\kappa}{2}\right)^6 s^2 t^2 u A^{\mathrm{tree}}_{Q=16}\frac{1}{48}\left(\frac{(D_s-6)(26-D_s)}{\epsilon^2} - \frac{13D_s + 142}{3\epsilon}\right),$$

$$\mathcal{M}^{(2)}(1_{V_1}, 2_{V_1}, 3_{V_2}, 4_{V_2})|_{D=6\;\mathrm{div.}} = \frac{1}{(4\pi)^6}\left(\frac{\kappa}{2}\right)^6 s^2 t A^{\mathrm{tree}}_{Q=16}\frac{1}{144}\left[\frac{26-D_s}{\epsilon^2}((D_s-6)s^2 - 5(s^2+t^2+u^2))\right.$$

$$\left. - \frac{1}{3\epsilon}\left((13D_s+142)s^2 + \frac{13D_s-578}{2}(s^2+t^2+u^2)\right)\right]. \tag{5.17}$$

As always, the result when the four external states are from a single vector multiplet can be obtained from the result when the states are from a distinct pair of vector multiplets simply by summing over the independent external permutations. The complete set of counterterms for external gravitons and matter vectors may be found in the Appendix.

VI. CONCLUSIONS AND OUTLOOK

In this paper we mapped out the one- and two-loop four-point divergences of half-maximal supergravity including Abelian-vector matter multiplets in various dimensions. In particular, we showed that half-maximal supergravity with matter multiplets does contain ultraviolet divergences in $D = 4, 5, 6$. We also worked out the four-point divergences at three loops in $D = 4$ to conclusively show that new divergences do occur in $D = 4$. The $D = 4$ theory has long been known to be divergent at one loop [23,24], which we confirmed here as well. Our one- and two-loop results are summarized in Table II, which shows a schematic form of the counterterms of half-maximal supergravity, as well as those of nonsupersymmetric gauge theory involving the color tensors that control the gravity divergences. (We do not include odd dimensions at one loop because those are automatically finite when using dimensional regularization.)

Bossard *et al.* recently conjectured [15] the existence of 16-supercharge linearly realized harmonic superspaces in $D = 4$ and in $D = 5$ in order to explain the finiteness of pure half-maximal supergravity at three loops in $D = 4$ and at two loops in $D = 5$ [12–14]. Such superspaces have

the appealing feature that no new "miracles" would be required to explain the observed finiteness. Very recently they argued [16] that if the conjectured superspace were to exist in $D = 5$, then there would be no new two-loop divergences even when matter multiplets are added to the theory. In $D = 4$ the situation is similar except for the appearance of an anomaly [21] in the rigid $SL(2, \mathbb{R})$ duality symmetry. However, we found that the calculated divergences in $D = 4$ are not compatible with them being been due to the anomaly. The results of the present paper then show that there are new divergences in all these cases, contradicting the predictions had the desired superspaces existed in $D = 4$ and $D = 5$.

We emphasize that there is no mystery in the half-maximal supergravity divergence structure from the vantage point of the duality between color and kinematics. At one and two loops, it shows in a direct way why amplitudes with external matter can diverge when the purely external-graviton-multiplet case does not, linked to the well-understood divergences of nonsupersymmetric gauge theory. In addition, it gives us the means to precisely determine the coefficients of the divergences.

The one- and two-loop cases analyzed in this paper are especially simple because the maximal super-Yang-Mills numerators used in the double-copy construction are independent of loop momenta. For higher loops the situation was more complex to analyze because loop momenta enter into the super-Yang-Mills numerators, altering the form of the integrals compared to those of nonsupersymmetric gauge theory. Nevertheless, as suggested in Ref. [13], we expect the divergences of half-maximal supergravity to be related to the divergence structure of

TABLE II.   A schematic table of the counterterms of half-maximal supergravity with matter multiplets in various dimensions at one and two loops, together with corresponding gauge-theory counterterms with the appropriate color tensors. The displayed supergravity counterterms are for gravitons and vector matter multiplets. For the gauge-theory case they are for the gluons and scalars of the theory.

Amplitude	One Loop			Two Loops		
	$D=4$	$D=6$	$D=8$	$D=4$	$D=5$	$D=6$
$\mathcal{A}^{(L)}(1_g, 2_g, 3_g, 4_g)$	Finite	Finite	F^4	Finite	Finite	F^4
$\mathcal{M}^{(L)}(1_H, 2_H, 3_H, 4_H)$	Finite	Finite	R^4	Finite	Finite	D^2R^4
$\mathcal{A}^{(L)}(1_g, 2_g, 3_\phi, 4_\phi)$	Finite	ϕ^2F^2	$D^2\phi^2F^2$	Finite	ϕ^2F^2	$D^2\phi^2F^2$
$\mathcal{M}^{(L)}(1_H, 2_H, 3_V, 4_V)$	Finite	F^2R^2	$D^2F^2R^2$	Finite	$D^2F^2R^2$	$D^4F^2R^2$
$\mathcal{A}^{(L)}(1_\phi, 2_\phi, 3_\phi, 4_\phi)$	ϕ^4	Finite	$D^4\phi^4$	Finite	$D^2\phi^4$	$D^4\phi^4$
$\mathcal{M}^{(L)}(1_V, 2_V, 3_V, 4_V)$	F^4	Finite	D^4F^4	Finite	D^4F^4	D^6F^4
$\mathcal{A}^{(L)}(1_{\phi_1}, 2_{\phi_1}, 3_{\phi_2}, 4_{\phi_2})$	ϕ^4	$D^2\phi^4$	$D^4\phi^4$	ϕ^4	$D^2\phi^4$	$D^4\phi^4$
$\mathcal{M}^{(L)}(1_{V_1}, 2_{V_1}, 3_{V_2}, 4_{V_2})$	F^4	D^2F^4	D^4F^4	D^2F^4	D^4F^4	D^6F^4

corresponding nonsupersymmetric gauge-theory amplitudes. We look forward to new calculations that will shed further light on the origin of the remarkably good ultraviolet behavior of pure supergravity theories with 16 or more supercharges.

APPENDIX: SUPERGRAVITY COUNTERTERMS

Any potential symmetry explanation of the vanishings of potential divergences of supergravity must also properly give the allowed counterterms in detail. For this purpose, in this appendix, we give counterterms corresponding to our calculated one- and two-loop divergences in half-maximal supergravity coupled to $n_{\rm V} = D_s - D$ matter vector multiplets in various dimensions. For each divergence, we give a counterterm for a particular field content, focusing on external graviton and Abelian-vector matter states. These can in turn be supersymmetrized using the supersymmetric form of the divergences given Secs. IV and V, but we do not do so here. The fact that this theory diverges at one loop in $D = 4$ has been known since the early days of supergravity [23,24]. Here we map out the full set of counterterms for four external gravitons or Abelian-vector matter states at one and two loops.

For notational simplicity, we define contractions of field strengths as

$$(FF) \equiv F_{\mu\nu}F^{\mu\nu}, \quad (FFFF) \equiv F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}. \quad \text{(A1)}$$

We also allow derivatives in this notation, for example,

$$(D_{\alpha\beta}FFD^\alpha FD^\beta F) \equiv (D_\alpha D_\beta F_{\mu\nu})F^{\nu\rho}(D^\alpha F_{\rho\sigma})(D^\beta F^{\sigma\mu}), \quad \text{(A2)}$$

where we have also introduced the notation, $D_{\mu\nu} \equiv D_\mu D_\nu$. For $F^2 R^2$-type operators, when no indices are written, the first two indices of one Riemann tensor are understood to be contracted with the first two indices of the other Riemann tensor, while the last two indices of each obey the relations of the field strengths in Eq. (A1):

$$(RR) \equiv R_{\mu\nu\lambda\gamma}R^{\mu\nu\lambda\gamma},$$
$$(RF)(RF) \equiv R_{\mu\nu\lambda\gamma}F^{\lambda\gamma}R^{\mu\nu}{}_{\delta\kappa}F^{\delta\kappa}, \quad \text{(A3)}$$
$$(RFRF) \equiv R_{\mu\nu\lambda\gamma}F^{\gamma\delta}R^{\mu\nu}{}_{\delta\kappa}F^{\kappa\lambda}.$$

In cases where the first two indices of a Riemann tensor are not contracted with the other one, we will write them out explicitly:

$$(R_{\mu\nu}R^\mu{}_\rho D^\nu F D^\rho F) \equiv R_{\mu\nu\lambda\gamma}R^\mu{}_\rho{}^{\gamma\delta}(D^\nu F_{\delta\kappa})(D^\rho F^{\kappa\lambda}). \quad \text{(A4)}$$

All R's in the counterterm operators refer to Riemann tensors and should not be confused with the Ricci tensor or Ricci scalar, despite the notation. The indices in R^4-type operators are written out explicitly with the understanding that derivatives act only on the tensor that they immediately precede, e.g. $D_\alpha R_{\mu\nu\lambda\gamma}R^{\mu\nu\lambda\gamma} \equiv (D_\alpha R_{\mu\nu\lambda\gamma})R^{\mu\nu\lambda\gamma}$.

Since the duality-satisfying numerators of maximally supersymmetric Yang-Mills amplitudes are independent of loop momenta at one and two loops, we exploit the double-copy property to construct our counterterm operators, as was done in Ref. [13]. The four-point one-loop duality-satisfying numerator for maximal super-Yang-Mills theory is given by a contraction of field strengths,

$$F^4 = -2\left[(F_1 F_2 F_3 F_4) - \frac{1}{4}(F_1 F_2)(F_3 F_4) + \text{cyclic}(2, 3, 4)\right], \quad \text{(A5)}$$

while the two-loop numerator $s^2 t A^{\rm tree}_{Q=16}$ is given by

$$D^2 F^4 = 4\left[(D_\alpha F_1 D^\alpha F_2 F_3 F_4) + (D_\alpha F_1 F_3 F_4 D^\alpha F_2) \right.$$
$$+ (D_\alpha F_1 F_4 D^\alpha F_2 F_3) - \frac{1}{4}(D_\alpha F_1 D^\alpha F_2)(F_3 F_4)$$
$$\left. - \frac{1}{4}(D_\alpha F_1 F_3)(D^\alpha F_2 F_4) - \frac{1}{4}(D_\alpha F_1 F_4)(D^\alpha F_2 F_3)\right], \quad \text{(A6)}$$

where the labels on the field strengths in these cases indicate the corresponding external legs. We use these expressions as replacements for the color factors in operators generating the nonsupersymmetric Yang-Mills divergences. We then associate products of Yang-Mills objects with gravity objects:

$$\phi_i F_{i\mu\nu} \rightarrow F_{i\mu\nu}, \qquad F_{i\mu\nu}F_{i\rho\sigma} \rightarrow -2R_{i\mu\nu\rho\sigma}. \quad \text{(A7)}$$

At the linearized level, the products of Yang-Mills objects and the gravity object each have the same contribution to the amplitude (see Ref. [13] for more detail).

For example, the nonsupersymmetric Yang-Mills divergence for four identical scalars at one loop in four dimensions involving the one-loop color tensor (4.2) is generated by

$$\frac{1}{\epsilon}\frac{1}{(4\pi)^2}g^4\frac{3(D_s-2)}{2}\frac{1}{4!}b_1^{(1)abcd}\phi^a\phi^b\phi^c\phi^d. \quad\text{(A8)}$$

Substituting Eq. (A5) for $b_1^{(1)}$ and using Eq. (A7), we have for the operator generating the single-matter-vector divergence in half-maximal supergravity,

$$\frac{1}{\epsilon}\frac{1}{(4\pi)^2}\left(\frac{\kappa}{2}\right)^4\frac{3(D_s-2)}{8}\left((FFFF)-\frac{1}{4}(FF)^2\right). \quad\text{(A9)}$$

We will not provide the input Yang-Mills operators as they do not generate the full divergences, but only generate the pieces proportional to the color tensors of interest. Nevertheless, the double-copy construction is evident in the contraction structure of the indices in the gravity counterterms, where Lorentz indices can be separated according to the gauge theory to which they belong. For

each dimension and loop order, we provide counterterms for the following four-point half-maximal supergravity amplitudes:

(i) four external gravitons with matter included in the loop,

(ii) two external gravitons and two external vector matter states,

(iii) four external vector matter states belonging to the same multiplet,

(iv) four external vector matter states belonging to two different multiplets. In this case the subscript labels on the field strengths indicate the matter multiplet to which the vector state belongs; these expressions are also valid for $i=j$, returning the counterterm for a single multiplet up to terms that vanish on shell.

1. One loop

a. Four dimensions

$$C^{(1),D=4}_{(R,R,R,R)}=0, \qquad C^{(1),D=4}_{(R,R,F,F)}=0, \qquad C^{(1),D=4}_{(F,F,F,F)}=-\frac{1}{\epsilon}\frac{1}{(4\pi)^2}\left(\frac{\kappa}{2}\right)^4\frac{3(D_s-2)}{8}\left((FFFF)-\frac{1}{4}(FF)^2\right),$$

$$C^{(1),D=4}_{(F_i,F_i,F_j,F_j)}=-\frac{1}{\epsilon}\frac{1}{(4\pi)^2}\left(\frac{\kappa}{2}\right)^4\frac{D_s-2}{4}\left((F_iF_iF_jF_j)+\frac{1}{2}(F_iF_jF_iF_j)-\frac{1}{8}(F_iF_i)(F_jF_j)-\frac{1}{4}(F_iF_j)^2\right). \tag{A10}$$

b. Six dimensions

$$C^{(1),D=6}_{(R,R,R,R)}=0, \qquad C^{(1),D=6}_{(R,R,F,F)}=\frac{1}{\epsilon}\frac{1}{(4\pi)^3}\left(\frac{\kappa}{2}\right)^2\frac{26-D_s}{24}\left((RRFF)+\frac{1}{2}(RFRF)-\frac{1}{8}(RR)(FF)-\frac{1}{4}(RF)(RF)\right),$$

$$C^{(1),D=6}_{(F,F,F,F)}=0, \qquad C^{(1),D=6}_{(F_i,F_i,F_j,F_j)}=-\frac{1}{\epsilon}\frac{1}{(4\pi)^3}\left(\frac{\kappa}{2}\right)^4\frac{26-D_s}{12}\left((D_\alpha F_iD^\alpha F_iF_jF_j)+\frac{1}{2}(D_\alpha F_iF_jD^\alpha F_iF_j)\right.$$
$$\left.-\frac{1}{8}(D_\alpha F_iD^\alpha F_i)(F_jF_j)-\frac{1}{4}(D_\alpha F_iF_j)(D^\alpha F_iF_j)\right). \tag{A11}$$

c. Eight dimensions

$$C^{(1),D=8}_{(R,R,R,R)}=-\frac{1}{\epsilon}\frac{1}{(4\pi)^4}\frac{1}{11520}\left[16(238+D_s)\left(\frac{1}{2}R_{\mu\nu\lambda\gamma}R^{\nu\rho\gamma\delta}R_{\rho\sigma\delta\kappa}R^{\sigma\mu\kappa\lambda}+R_{\mu\nu\lambda\gamma}R^{\nu\rho}_{\ \ \nu\rho\delta\kappa}R^{\rho\sigma\gamma\delta}_{\rho\sigma}R^{\sigma\mu\kappa\lambda}\right)\right.$$
$$+5(50-D_s)\left(\frac{1}{2}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})^2+R_{\mu\nu\lambda\gamma}R^{\mu\nu}_{\ \ \delta\kappa}R_{\rho\sigma}^{\ \ \lambda\gamma}R^{\rho\sigma\delta\kappa}\right)$$
$$\left.-16(122-D_s)\left(R_{\mu\nu\lambda\gamma}R^{\nu\rho\lambda\gamma}R_{\rho\sigma\delta\kappa}R^{\sigma\mu\delta\kappa}+\frac{1}{2}R_{\mu\nu\lambda\gamma}R^{\nu\rho}_{\ \ \delta\kappa}R_{\rho\sigma}^{\ \ \lambda\gamma}R^{\sigma\mu\delta\kappa}\right)\right],$$

$$C^{(1),D=8}_{(R,R,F,F)}=\frac{1}{\epsilon}\frac{1}{(4\pi)^4}\left(\frac{\kappa}{2}\right)^2\frac{1}{90}\left[(D_s-32)\left((RRD_\alpha FD^\alpha F)+\frac{1}{2}(RD_\alpha FRD^\alpha F)-\frac{1}{8}(RR)(D_\alpha FD^\alpha F)-\frac{1}{4}(RD_\alpha F)(RD^\alpha F)\right)\right.$$
$$+60\left((R_{\mu\nu}R^\mu_{\ \rho}D^\nu FD^\rho F)+(R_{\mu\nu}R^\mu_{\ \rho}D^\rho FD^\nu F)+(R_{\mu\nu}D^\nu FR^\mu_{\ \rho}D^\rho F)-\frac{1}{4}(R_{\mu\nu}R^\mu_{\ \rho})(D^\nu FD^\rho F)\right.$$
$$\left.\left.-\frac{1}{4}(R_{\mu\nu}D^\nu F)(R^\mu_{\ \rho}D^\rho F)-\frac{1}{4}(R_{\mu\nu}D_\rho F)(R^{\mu\rho}D^\nu F)\right)\right],$$

$$C^{(1),D=8}_{(F,F,F,F)} = -\frac{1}{\epsilon}\frac{1}{(4\pi)^4}\left(\frac{\kappa}{2}\right)^4\frac{D_s+18}{60}\left((D_\alpha F D^\alpha F D_\beta F D^\beta F) + \frac{1}{2}(D_\alpha F D_\beta F D^\alpha F D^\beta F) - \frac{1}{8}(D_\alpha F D^\alpha F)^2\right.$$
$$\left. - \frac{1}{4}(D_\alpha F D_\beta F)(D^\alpha F D^\beta F)\right),$$

$$C^{(1),D=8}_{(F_i,F_i,F_j,F_j)} = -\frac{1}{\epsilon}\frac{1}{(4\pi)^4}\left(\frac{\kappa}{2}\right)^4\frac{1}{60}\left[(D_s-22)\left((D_\alpha F_i D^\alpha F_i D_\beta F_j D^\beta F_j) + \frac{1}{2}(D_\alpha F_i D_\beta F_j D^\alpha F_i D^\beta F_j)\right.\right.$$
$$\left. - \frac{1}{8}(D_\alpha F_i D^\alpha F_i)(D_\beta F_j D^\beta F_j) - \frac{1}{4}(D_\alpha F_i D_\beta F_j)(D^\alpha F_i D^\beta F_j)\right) + 20\left((D_\alpha F_i D_\beta F_i D^\alpha F_j D^\beta F_j)\right.$$
$$+ (D_\alpha F_i D^\alpha F_j D_\beta F_j D^\beta F_i) + (D_\alpha F_i D_\beta F_j D^\beta F_i D^\alpha F_j)$$
$$\left.\left. - \frac{1}{4}(D_\alpha F_i D_\beta F_i)(D^\alpha F_j D^\beta F_j) - \frac{1}{4}(D_\alpha F_i D^\alpha F_j)^2 - \frac{1}{4}(D_\alpha F_i D_\beta F_j)(D^\beta F_i D^\alpha F_j)\right)\right]. \tag{A12}$$

2. Two loops
a. Four dimensions

$$C^{(2),D=4}_{(R,R,R,R)} = 0, \qquad C^{(2),D=4}_{(R,R,F,F)} = 0, \qquad C^{(2),D=4}_{(F,F,F,F)} = 0,$$

$$C^{(2),D=4}_{(F_i,F_i,F_j,F_j)} = \frac{1}{\epsilon^2}\frac{1}{(4\pi)^4}\left(\frac{\kappa}{2}\right)^6\frac{(D_s-2)^2}{4}\left((D_\alpha F_i D^\alpha F_i F_j F_j) + \frac{1}{2}(D_\alpha F_i F_j D^\alpha F_i F_j) - \frac{1}{8}(D_\alpha F_i D^\alpha F_i)(F_j F_j)\right.$$
$$\left. - \frac{1}{4}(D_\alpha F_i F_j)(D^\alpha F_i F_j)\right). \tag{A13}$$

b. Five dimensions

$$C^{(2),D=5}_{(R,R,R,R)} = 0,$$

$$C^{(2),D=5}_{(R,R,F,F)} = -\frac{1}{\epsilon}\frac{1}{(4\pi)^5}\left(\frac{\kappa}{2}\right)^4\frac{(10-D_s)\pi}{3}\left((RRD_\alpha F D^\alpha F) + \frac{1}{2}(RD_\alpha F R D^\alpha F) - \frac{1}{8}(RR)(D_\alpha F D^\alpha F) - \frac{1}{4}(RD_\alpha F)(RD^\alpha F)\right),$$

$$C^{(2),D=5}_{(F,F,F,F)} = \frac{1}{\epsilon}\frac{1}{(4\pi)^5}\left(\frac{\kappa}{2}\right)^6\frac{2(10-D_s)\pi}{3}\left((D_\alpha F D^\alpha F D_\beta F D^\beta F) + \frac{1}{2}(D_\alpha F D_\beta F D^\alpha F D^\beta F) - \frac{1}{8}(D_\alpha F D^\alpha F)^2\right.$$
$$\left. - \frac{1}{4}(D_\alpha F D_\beta F)(D^\alpha F D^\beta F)\right),$$

$$C^{(2),D=5}_{(F_i,F_i,F_j,F_j)} = \frac{1}{\epsilon}\frac{1}{(4\pi)^5}\left(\frac{\kappa}{2}\right)^6\frac{2(10-D_s)\pi}{3}\left(\frac{3}{2}(D_\alpha F_i D^\alpha F_i D_\beta F_j D^\beta F_j) + \frac{3}{4}(D_\alpha F_i D_\beta F_j D^\alpha F_i D^\beta F_j)\right.$$
$$+ (D_{\alpha\beta}F_i F_i D^\alpha F_j D^\beta F_j) + \frac{1}{2}(D_{\alpha\beta}F_i D^\alpha F_j F_i D^\beta F_j) - \frac{3}{16}(D_\alpha F_i D^\alpha F_i)(D_\beta F_j D^\beta F_j)$$
$$\left. - \frac{3}{8}(D_\alpha F_i D_\beta F_j)(D^\alpha F_i D^\beta F_j) - \frac{1}{8}(D_{\alpha\beta}F_i F_i)(D^\alpha F_j D^\beta F_j) - \frac{1}{4}(D_{\alpha\beta}F_i D^\alpha F_j)(F_i D^\beta F_j)\right). \tag{A14}$$

c. Six dimensions

$$C^{(2),D=6}_{(R,R,R,R)} = -\frac{1}{(4\pi)^6}\left(\frac{\kappa}{2}\right)^2\left[\left(\frac{(D_s-6)(26-D_s)}{576\epsilon^2} - \frac{734-19D_s}{864\epsilon}\right)\left(D_\alpha R_{\mu\nu\lambda\gamma}D^\alpha R^{\mu\nu\gamma\delta}R_{\rho\sigma\delta\kappa}R^{\rho\sigma\kappa\lambda}\right.\right.$$
$$\left. + \frac{1}{2}D_\alpha R_{\mu\nu\lambda\gamma}R_{\rho\sigma}{}^{\gamma\delta}D^\alpha R^{\mu\nu}{}_{\delta\kappa}R^{\rho\sigma\kappa\lambda} - \frac{1}{8}(D_\alpha R_{\mu\nu\lambda\gamma})^2(R_{\rho\sigma\delta\kappa})^2 - \frac{1}{4}(D_\alpha R_{\mu\nu\lambda\gamma}R_{\rho\sigma}{}^{\lambda\gamma})^2\right)$$
$$- \frac{26-D_s}{18\epsilon}\left(D_\alpha R_{\mu\nu\lambda\gamma}D^\alpha R_{\rho\sigma}{}^{\gamma\delta}R^{\nu\rho}{}_{\delta\kappa}R^{\sigma\mu\kappa\lambda} + \frac{1}{2}D_\alpha R_{\mu\nu\lambda\gamma}R^{\nu\rho\gamma\delta}D^\alpha R_{\rho\sigma\delta\kappa}R^{\sigma\mu\kappa\lambda}\right.$$
$$\left.\left. - \frac{1}{8}D_\alpha R_{\mu\nu\lambda\gamma}R_{\rho\sigma}{}^{\gamma\delta}D^\alpha R^{\mu\nu}{}_{\delta\kappa}R^{\rho\sigma\kappa\lambda} - \frac{1}{4}D_\alpha R_{\mu\nu\lambda\gamma}R^{\mu\nu\gamma\delta}D^\alpha R_{\rho\sigma\delta\kappa}R^{\rho\sigma\kappa\lambda}\right)\right],$$

$$
\begin{aligned}
C^{(2),D=6}_{(R,R,F,F)} = &-\frac{1}{(4\pi)^6}\left(\frac{\kappa}{2}\right)^4\frac{1}{144}\Bigg[\left(\frac{(26-D_s)(27-2D_s)}{\epsilon^2}+\frac{25(D_s+22)}{6\epsilon}\right)\left(D_\alpha R D^\alpha R D_\beta F D^\beta F\right)+\frac{1}{2}\left(D_\alpha R D_\beta F D^\alpha R D^\beta F\right) \\
&-\frac{1}{8}(D_\alpha R D^\alpha R)(D_\beta F D^\beta F)-\frac{1}{4}(D_\alpha R D_\beta F)(D^\alpha R D^\beta F)\Bigg)+\left(\frac{10(26-D_s)}{\epsilon^2}-\frac{734-19D_s}{3\epsilon}\right) \\
&\times\Bigg(\left(D_\alpha R_{\mu\nu} D^\alpha R^\mu{}_\rho D^\nu F D^\rho F\right)+\left(D_\alpha R_{\mu\nu} D^\alpha R^\mu{}_\rho D^\rho F D^\nu F\right)+\left(D_\alpha R_{\mu\nu} D^\nu F D^\alpha R^\mu{}_\rho D^\rho F\right) \\
&-\frac{1}{4}(D_\alpha R_{\mu\nu} D^\alpha R^\mu{}_\rho)(D^\nu F D^\rho F)-\frac{1}{4}(D_\alpha R_{\mu\nu} D^\nu F)(D^\alpha R^\mu{}_\rho D^\rho F)-\frac{1}{4}(D_\alpha R_{\mu\nu} D_\rho F)(D^\alpha R^{\mu\rho} D^\nu F)\Bigg) \\
&+\left(\frac{16(26-D_s)}{\epsilon^2}-\frac{8(D_s+22)}{3\epsilon}\right)\Big(R_{\mu\nu}D_\alpha R^\mu{}_\rho D^{\alpha\nu}F D^\rho F)+(R_{\mu\nu}D_\alpha R^\mu{}_\rho D^\rho F D^{\alpha\nu}F)+(R_{\mu\nu}D_\alpha{}^\nu F D^\alpha R^\mu{}_\rho D^\rho F\Big) \\
&-\frac{1}{4}(R_{\mu\nu}D_\alpha R^\mu{}_\rho)(D^{\alpha\nu}F D^\rho F)-\frac{1}{4}(R_{\mu\nu}D_\alpha{}^\nu F)(D^\alpha R^\mu{}_\rho D^\rho F)-\frac{1}{4}(R_{\mu\nu}D_\rho F)(D_\alpha R^{\mu\rho}D^{\alpha\nu}F)\Bigg)\Bigg], \\
C^{(2),D=6}_{(F,F,F,F)} = &-\frac{1}{(4\pi)^6}\left(\frac{\kappa}{2}\right)^6\frac{1}{24}\Bigg(\frac{(D_s-6)(26-D_s)}{\epsilon^2}-\frac{13D_s+142}{3\epsilon}\Bigg)\left(D_{\alpha\beta}F D^\alpha{}_\gamma F D^{\beta\gamma}F F\right)-\frac{1}{4}(D_{\alpha\beta}F D^\alpha{}_\gamma F)(D^{\beta\gamma}F F)\Bigg), \\
C^{(2),D=6}_{(F_i,F_i,F_j,F_j)} = &\frac{1}{(4\pi)^6}\left(\frac{\kappa}{2}\right)^6\frac{1}{36}\Bigg[\left(\frac{(26-D_s)(16-D_s)}{\epsilon^2}-\frac{2(218-13D_s)}{3\epsilon}\right)\left(D_{\alpha\beta}F_i D^{\alpha\beta}F_i D_\gamma F_j D^\gamma F_j\right) \\
&+\frac{1}{2}(D_{\alpha\beta}F_i D_\gamma F_j D^{\alpha\beta}F_i D^\gamma F_j)-\frac{1}{8}(D_{\alpha\beta}F_i D^{\alpha\beta}F_i)(D_\gamma F_j D^\gamma F_j)-\frac{1}{4}(D_{\alpha\beta}F_i D_\gamma F_j)(D^{\alpha\beta}F_i D^\gamma F_j)\Bigg) \\
&-\left(\frac{10(26-D_s)}{\epsilon^2}-\frac{578-13D_s}{3\epsilon}\right)\Big(D_{\alpha\beta}F_i D^\alpha{}_\gamma F_i D^{\beta\gamma}F_j F_j)+\frac{1}{2}(D_{\alpha\beta}F_i D^\beta{}_\gamma F_j D^{\alpha\gamma}F_i F_j\Big) \\
&-\frac{1}{8}(D_{\alpha\beta}F_i D^\alpha{}_\gamma F_i)(D^{\beta\gamma}F_j F_j)-\frac{1}{4}(D_{\alpha\beta}F_i D^\beta{}_\gamma F_j)(D^{\alpha\gamma}F_i F_j)\Bigg)\Bigg].
\end{aligned}
\tag{A15}
$$

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