Consistency of nonminimally coupled $f(R)$ gravity

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Theories with a nonminimal coupling between the space-time curvature and matter fields introduce an extra force due to the nonconservation of the matter energy momentum. In the present work the theoretical consistency of such couplings is studied using a scalar field Lagrangian to model the matter content. The conditions that the coupling does not introduce ghosts, classical instabilities or superluminal propagation of perturbations are derived. These consistency conditions are then employed to rule out or severely restrict the forms of the nonminimal coupling functions considered in the previous literature. For example, a power-law coupling is viable only for sublinear positive power of the curvature scalar.

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I. INTRODUCTION

One of the fundamental issues in gravity is its coupling to the matter fields. In Einstein's general relativity (GR) theory, the equivalence principle dictates the minimal coupling prescription, but despite stringent constraints on its violations at local astrophysical and especially laboratory experiments, there is no compelling reason to take it for granted at all scales and at all times [1]. In fact, high energy physics theories often predict nonminimal couplings of matter to e.g. scalar fields or the gravitational degrees of freedom.

Although the most popular type of nonminimal coupling is the conformal one,[1] for purely gravitational modifications, it is known that $f(R)$ theories represent the only local, metric-based, generally coordinate invariant and stable modifications of gravity [6,7]. Thus, when considering the possibility of a nonminimal gravitational coupling, a natural starting point could be the coupling of a function of the curvature scalar to the matter Lagrangian. The general action for such nonminimally coupled $f(R)$ gravity [8] can be written as[2]

$$S_0 = \int d^4x \sqrt{-g}[f_1(R) + f_2(R)\mathcal{L}_M], \qquad (1)$$

where $f_1(R)$ and $f_2(R)$ are two general (sufficiently regular) functions of the curvature scalar R and $\mathcal{L}_M$ denotes the matter Lagrangian.

Such a theory predicts an extra force due to the nonconservation of the stress energy tensor [8,14]. The resulting nongeodesic motion and some of its astrophysical

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[1]However, more general forms of couplings can be well motivated too; see e.g. [2–5].

[2]See [9–13] for earlier studies of some instances of theories within this class.

implications have been extensively analyzed in [14–23]. Theories of the form (1) have been widely applied to cosmology: dark matter has been proposed as an effect of the extra force on baryonic matter [14,24,25], dark energy has been considered to arise from the nonminimal coupling [26–28], the evolution of cosmological perturbations has been studied [29–31], and inflation was considered in this context, too [32]. Astrophysical and Solar System constraints were investigated in [33–36], and spherically symmetric solutions were derived in [37,38]. Connection of the nonminimal coupling with extra dimensional theories was attempted in [39], and wormholes [40–42], singularities [43], and Gödel-type universes [44] have been constructed. The theory (1) can be considered also in the Palatini formulation [8,45,46]. Furthermore, it was shown to be embedded into the wider framework of C theories [47] and then applied to deduce the post-Newtonian parameters for given models [48]. Some other aspects of the viability of the theory and the effective violations of the energy conditions have already been studied in [49–53]. Finally, it has been understood that different parametrizations of perfect fluid Lagrangians can yield nonequivalent results in the presence of the nonminimal coupling [54–59].

In this paper we study the theoretical consistency of the models (1). We separate out the propagating degrees of freedom in the nonminimally coupled matter sector, which are intricately mixed with the gravitational degrees of freedom due to the nonminimal coupling. The aim is to analyze whether matter fields may become ghostlike or exhibit unstable growth or superluminality. For this purpose we need to specify the matter field explicitly, and for simplicity we employ a massless scalar field. If the coupling is considered universal, there should be no pathologies for any kind of matter, but in any case the constraints derived from the basic example of a scalar will present the necessary conditions for the viability of the coupling function. Moreover, by considering the Lagrangian of a fundamental

field instead of employing an effective phenomenological description of an averaged (perfect or otherwise) fluid, our results cannot be compromised by the aforementioned ambiguities of such effective fluid parametrizations.

We will proceed as follows. In Sec. II we will reformulate the theory in terms of scalar fields and manipulate their action into a convenient form for the consistency analysis. The recipe to uncover the viability of the theory, given the two functions f_1 and f_2, will then be given in Sec. III. In Sec. IV we will apply this recipe for several specific cases proposed in the previous literature. Conclusions and discussions will finally be drawn in Sec. V. In addition, in the Appendix we will show that the presented recipe can be applied to theories with general nonlinear dependence upon the matter Lagrangian as well.

Notation and conventions. In the following we will set $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ with the convention $R_{\mu\nu} \sim +\partial_\alpha \Gamma^\alpha_{\mu\nu}$, $c = 1$ (speed of light) and $M_{\text{Pl}}^2 = 1/(16\pi G)$ (Planck mass).

II. EQUIVALENT ACTIONS AND CONFORMAL TRANSFORMATIONS

In order to find some constraints on the functions $f_1(R)$ and $f_2(R)$ we will start by studying an action which is dynamically equivalent to (1). The dynamical equivalence of action (1) with other gravitational theories, especially scalar-tensor theories, has already been extensively studied [15,33,49]. Our approach will closely follow the one exposed in [33]. Consider the following theory:

$$S = \int d^4x\sqrt{-g}\left[f_1(\alpha) + \left(\frac{\partial f_1(\alpha)}{\partial \alpha} + \frac{\partial f_2(\alpha)}{\partial \alpha}\beta\right)(R - \alpha) + f_2(\alpha)\mathcal{L}_M\right],$$

(2)

where α and β are two scalar fields and f_1 and f_2 are the same functions appearing in (1). This theory is dynamically equivalent to (1); i.e. the classical equations of motion are the same. To prove this we take the variation of (2) with respect to α and β, which yields the following field equations:

$$\left(\frac{\partial^2 f_1}{\partial \alpha^2} + \frac{\partial^2 f_2}{\partial \alpha^2}\beta\right)(R - \alpha) + \frac{\partial f_2}{\partial \alpha}(\mathcal{L}_M - \beta) = 0, \quad (3)$$

$$\frac{\partial f_2}{\partial \alpha}(R - \alpha) = 0. \quad (4)$$

Provided $f_2 \neq \text{const}$, which represents the minimal coupling, the unique solution of these equations is

$$\alpha = R \quad \text{and} \quad \beta = \mathcal{L}_M. \quad (5)$$

If we now substitute back solution (5) into action (2) we obtain exactly action (1), implying that the two actions are indeed dynamically equivalent. In general, the second scalar field β is unnecessary and we could consider action (2)

with only one auxiliary field α. However, the forthcoming analysis is much simplified in the two-field approach, which moreover can be straightforwardly applied to more general Lagrangians such as $f(R, \mathcal{L}_M)$ (see the Appendix). For these reasons, considering two independent auxiliary fields in action (2) is more suitable for our purposes.

At this point we define two new scalar fields as

$$\phi_1 = \frac{\partial f_1(\alpha)}{\partial \alpha} + \frac{\partial f_2(\alpha)}{\partial \alpha}\beta \quad \text{and} \quad \phi_2 = f_2(\alpha), \quad (6)$$

which allows us to rewrite action (2) as

$$S = \int d^4x\sqrt{-g}[\phi_1 R + \phi_2 \mathcal{L}_M + V(\phi_1, \phi_2)], \quad (7)$$

with

$$V(\phi_1, \phi_2) = f_1(\alpha) - \phi_1\alpha, \quad (8)$$

where α has to be interpreted as a function depending on ϕ_2 through (6). Action (7) represents a Brans-Dicke theory with a vanishing kinetic term and a matter sector coupled to a second scalar field which interacts with the Brans-Dicke field through the potential V.

We now want to integrate out the scalar field ϕ_2. To do this we consider the variation of (7) with respect to ϕ_2, which yields

$$\mathcal{L}_M + \frac{\partial V}{\partial \phi_2} = 0. \quad (9)$$

We need a solution of this equation for ϕ_2. Our analysis only works when Eq. (9) admits such a solution. If this is not the case, one has to follow other ways in order to constrain the original theory (1). In any case, as we will see in Sec. IV, a solution to (9) can always be found for a wide class of theories, implying that our method is quite generally applicable. Assume then that

$$\phi_2^* = \Sigma(\phi_1, \mathcal{L}_M) \quad (10)$$

is a solution of (9) with Σ a function of ϕ_1 and $\mathcal{L}_M$. Plugging (10) back into action (7) gives

$$S = \int d^4x\sqrt{-g}[\phi_1 R + P(\mathcal{L}_M, \phi_1)], \quad (11)$$

where

$$P(\mathcal{L}_M, \phi_1) = \mathcal{L}_M\Sigma + V(\phi_1, \Sigma). \quad (12)$$

We can now perform a conformal transformation in order to switch from the Jordan to the Einstein frame. Consider the following conformally related metric,

$$\tilde{g}_{\mu\nu} = 16\pi G\phi_1 g_{\mu\nu}, \quad (13)$$

and redefine ϕ_1 as

$$\phi_1 = \phi_0 \exp\left(\frac{\sqrt{G}}{16\pi\sqrt{3}}\tilde{\phi}_1\right), \quad (14)$$

where ϕ_0 is a constant. With the help of this transformation we can rewrite action (11) as

$$S = \int d^4x\sqrt{\tilde{g}}\left[\frac{R}{16\pi G} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}_1\partial_\nu\tilde{\phi}_1 + \tilde{P}(\mathcal{L}_M, \tilde{\phi}_1)\right],$$
(15)

where

$$\tilde{P}(\mathcal{L}_M, \tilde{\phi}_1) = \frac{1}{\phi_1^2}P(\mathcal{L}_M, \phi_1),$$
(16)

with ϕ_1 to be interpreted as a function of $\tilde{\phi}_1$ and $\mathcal{L}_M$ depending on $\tilde{g}_{\mu\nu}$ through $g_{\mu\nu}$. Action (15) is nothing but Einstein gravity with a scalar field ϕ_1 which interacts with all the matter degrees of freedom. This scalar field is obviously regular, its kinetic term being of the canonical type. However, the matter Lagrangian appears in the theory in a nontrivial manner, and the stability of its fields is not guaranteed.

III. CONSISTENCY CONDITIONS FOR NONMINIMALLY COUPLED $f(R)$ GRAVITY

According to the spirit of the original nonminimally coupled $f(R)$ theory given by (1), one can in principle consider any kind of matter Lagrangian. In other words, the theory must be consistent for every choice of matter we want to couple to gravity. In order to continue our analysis we are thus free to choose any matter Lagrangian at our convenience. In what follows we will consider $\mathcal{L}_M$ to be the Lagrangian of a matter scalar field ϕ_M:

$$\mathcal{L}_M \equiv X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi_M\partial_\nu\phi_M.$$
(17)

In principle we could have added any potential for the scalar field ϕ_M, but the following analysis would not have to be altered; thus we choose to work with only the kinetic term. Note that assuming $\partial_\mu\phi_M$ to be timelike implies that $X > 0$. The choice (17) is probably the simplest one we can consider, and it will allow us to put some constraints on the functions f_1 and f_2 when a particular model is chosen. However, since we are restricting our study to a specific choice of the matter Lagrangian, the results will only be necessary conditions and more restrictive constraints could be found with different choices. In any case, since the original theory (1) must work for every matter Lagrangian, any constraint we obtain on the functions f_1 and f_2 must be satisfied *in primis* by every model. In other words, we will be able to exclude the theories which do not satisfy such constraints, but we cannot assure that other models are ultimately viable. A complete analysis is impossible for obvious reasons (one should study all the possible matter Lagrangians), and each theory has to be considered separately if one wants to check its consistency in depth. A scalar field ensures a sufficiently simple but still fundamental matter sector. Other possibilities do not

share these properties: a point particle Lagrangian cannot be regarded as fundamental, while Yang-Mills or spinor Lagrangians would complicate the analysis because of technical reasons. The study of these and other fundamental fields will be left for future works.

With the matter Lagrangian choice (17) $\tilde{P}$ becomes, in general, a function of X and $\tilde{\phi}_1$. The kinetic term for $\tilde{\phi}_1$ is canonical and separated from $\tilde{P}$, meaning that all the dependence on derivatives of fields inside $\tilde{P}$ is contained in X. Such types of theories are known in cosmological contexts as k-essence models [60–63]. They usually depend on a scalar field which enters the action in a noncanonical way. The action for k-essence is commonly written as

$$S_k = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} + p(X_\phi, \phi)\right],$$
(18)

where p is a general function of a scalar ϕ and $X_\phi = -1/2(\partial\phi)^2$.

Because of their applications to dark energy and inflation, these theories have been well studied and several results have already been discovered. In particular, we are interested in the stability conditions one must impose on the function p in order for the model to be physically viable. These can be obtained by requiring the positivity of both the energy density of the scalar field and the speed of sound at which the perturbations propagate in the scalar fluid.[3] The required conditions are given by [62,63]

$$\epsilon(\phi) = 2X_\phi p_{,X_\phi} - p > 0,$$
(19)

$$c_s^2(\phi) = \frac{p_{,X_\phi}}{\epsilon_{,X_\phi}} = \frac{p_{,X_\phi}}{2X_\phi p_{,X_\phi X_\phi} + p_{,X_\phi}} \geq 0,$$
(20)

where $_{,X_\phi}$ denotes differentiation with respect to X_ϕ. Here $\epsilon(\phi)$ is the energy density of the scalar field, while $c_s(\phi)$ is the speed of sound. Conditions (19) and (20) have to be satisfied by every physically viable model. If they do not hold, the theory cannot be employed at macroscopic scales. If we further require that the perturbations do not propagate faster than the speed of light, we should impose the further condition that $c_s(\phi) \leq 1$.

We now transfer conditions (19) and (20) to our theory given by action (15). In order for this theory to be consistent we require that

$$\tilde{\epsilon}(\phi_M, \phi_1) = 2\tilde{X}\tilde{P}_{,\tilde{X}} - \tilde{P} > 0,$$
(21)

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[3]Technically this is defined as the ratio of the pressure perturbation and the density perturbation evaluated in the rest frame of the field. For a canonical scalar field the speed of sound is identically unity; in general, it can depend on time, and it differs from the so-called adiabatic speed of sound that is the ratio of the time derivative of the background pressure and the background energy density.

$$1 \geq \tilde{c}_s^2(\phi_M, \phi_1) = \frac{\tilde{P}_{,\tilde{X}}}{2\tilde{X}\tilde{P}_{,\tilde{X}\tilde{X}} + \tilde{P}_{,\tilde{X}}} \geq 0, \qquad (22)$$

where $\tilde{X}$ denotes the kinetic term of the Einstein frame, and it is related to X by

$$\tilde{X} = \frac{X}{16\pi G \phi_1}. \qquad (23)$$

The theory can thus be constrained by requiring that the conditions (21) and (22) are satisfied in the Einstein frame. However, since we know the relations between X and $\tilde{X}$ and between P and $\tilde{P}$, we can find how the conditions (21) and (22) are written in terms of Jordan frame quantities. Using (16) and (23) we can easily obtain

$$\tilde{\epsilon} = \frac{1}{\phi_1^2}\epsilon \quad \text{and} \quad \tilde{c}_s^2 = c_s^2, \qquad (24)$$

where

$$\epsilon = 2XP_{,X} - P \quad \text{and} \quad c_s^2 = \frac{P_{,X}}{2XP_{,XX} + P_{,X}} \qquad (25)$$

are the energy density and speed of sound in the Jordan frame. Thus the two consistency conditions (21) and (22) can conveniently be translated into

$$\epsilon > 0 \quad \text{and} \quad 1 \geq c_s^2 \geq 0, \qquad (26)$$

which can be computed directly from the Jordan frame, avoiding the passage to the Einstein frame. As we will see, once a particular $f(R)$ model is chosen in action (1), the conditions (26) will permit us to constrain the free parameters of the theory.

At this point we conveniently summarize the whole recipe one has to follow in order to find consistency conditions on nonminimally coupled $f(R)$ gravity:
  (1) Identify the two functions $f_1(R)$ and $f_2(R)$ in action (1).
  (2) Compute ϕ_1, ϕ_2 and $V(\phi_1, \phi_2)$ from definitions (6) and (8).
  (3) Find a solution ϕ_2^* of Eq. (9) (if possible).
  (4) Evaluate $P(\mathcal{L}_M, \phi_1)$ from (12) with $\mathcal{L}_M = X$.
  (5) Constrain the free parameters of the theory using conditions (26).
In the remaining part of the paper we will apply these instructions to some nonminimally coupled $f(R)$ models. This will also explain how this procedure works in practice and how it can be employed to constrain the free parameters of the theory.

IV. CONSISTENCY OF SPECIFIC MODELS

In this section we will consider some particular models where the functions $f_1(R)$ and $f_2(R)$ of action (1) are assumed to be of some specific kind. Some of these theories will be considered for their simplicity, others because

of their relevance for cosmological and astrophysical applications.

A. $f_1(R) \propto R$ and $f_2(R) \propto R^\gamma$

The first model we study is motivated by its relative simplicity and will be utilized as a working example. We will consider a theory specified by the functions

$$f_1(R) = \frac{R}{16\pi G} \quad \text{and} \quad f_2(R) = AR^\gamma, \qquad (27)$$

where A and γ are two free real parameters. The first function represents nothing but the standard Einstein-Hilbert Lagrangian for GR. Similar to several models already studied in the literature, this specific choice has the scope of only modifying the minimal coupling with matter and leaving the pure gravity sector unchanged. The theory reduces to GR in the limit $\gamma \to 0$ and $A \to 1$, and if we want to study only physically motivated modifications of gravity, we cannot consider negative values for A where the gravitational force would be repulsive on all matter fields. In what follows we will thus set $A > 0$. Even though the function $f_2(R)$ is usually taken to be $f_2(R) = 1 + AR^\gamma$ for physical applications of the theory (e.g. [16,24,26,29]), we first study the model (27), both because of its simplicity and because it has already been analyzed in other works (e.g. [53]). The case $f_2(R) = 1 + AR^\gamma$ will be studied Sec. IV B.

Having identified the two shape functions of the theory, the next step we have to take is to compute the fields ϕ_1, ϕ_2 and the potential $V(\phi_1, \phi_2)$. These are given by

$$\phi_1 = M_{\mathrm{Pl}}^2 + \gamma A \alpha^{\gamma-1}\beta, \qquad \phi_2 = A\alpha^\gamma,$$
$$V = (M_{\mathrm{Pl}}^2 - \phi_1)\left(\frac{\phi_2}{A}\right)^{1/\gamma}, \qquad (28)$$

where $M_{\mathrm{Pl}}^2 = 1/(16\pi G)$ is the Planck mass. At this point we must find a solution to Eq. (9), which in this case reads

$$\mathcal{L}_M + (M_{\mathrm{Pl}}^2 - \phi_1)\frac{1}{\gamma}\frac{1}{A^{1/\gamma}}\phi_2^{(1-\gamma)/\gamma} = 0. \qquad (29)$$

Provided $\gamma \neq 1$, which will be analyzed later, we can easily find the solution as

$$\phi_2^* = \Sigma(\mathcal{L}_M, \phi_1) = \left(\frac{\gamma A^{1/\gamma}\mathcal{L}_M}{\phi_1 - M_{\mathrm{Pl}}^2}\right)^{\gamma/(1-\gamma)}. \qquad (30)$$

Next we compute $P(\mathcal{L}_M, \phi_1)$ using (12). This is given by

$$P(\mathcal{L}_M, \phi_1) = (1-\gamma)A^{1/(1-\gamma)}\left(\frac{\gamma}{\phi_1 - M_{\mathrm{Pl}}^2}\right)^{\gamma/(1-\gamma)}\mathcal{L}_M^{1/(1-\gamma)}. \qquad (31)$$

Finally, after setting $\mathcal{L}_M = X$, we are ready to compute the consistency conditions (26) which will constrain the range of possible values of γ. We first calculate the sound speed, which turns out as

$$c_s^2 = \frac{1-\gamma}{1+\gamma}. \tag{32}$$

For absence of instabilities we thus require that

$$c_s^2 \geq 0 \Leftrightarrow -1 \leq \gamma \leq 1. \tag{33}$$

This tells us that values of γ for which $|\gamma| \geq 1$ are forbidden in such a theory. This is expected since small deviations from GR, for which $\gamma = 0$, are usually more viable under a phenomenological point of view. If we further wish to avoid superluminal signals, we obtain the constraint

$$0 \leq c_s^2 \leq 1 \Leftrightarrow 0 \leq \gamma \leq 1. \tag{34}$$

The constraint coming from the positivity of the energy density gives us

$$\epsilon = \frac{1+\gamma}{1-\gamma} P > 0, \tag{35}$$

which combined with (32) leaves us with

$$P > 0. \tag{36}$$

Since $A > 0$, $\mathcal{L}_M = X > 0$ and $|\gamma| < 1$, the only object which remains to be checked in (31) is $\gamma/(\phi_1 - M_{\text{Pl}}^2)$. From the definition of ϕ_1 (28) we have

$$\frac{\gamma}{\phi_1 - M_{\text{Pl}}^2} = \frac{1}{A\alpha^{\gamma-1}\beta}, \tag{37}$$

whose left-hand side is determined by the right-hand side (RHS) which, in the GR limit $\gamma \to 0$, $A \to 1$ and $\phi_1 \to M_{\text{Pl}}^2$, becomes

$$\left.\frac{\gamma}{\phi_1 - M_{\text{Pl}}^2}\right|_{\text{GR}} \to \frac{\alpha}{\beta} \approx \frac{R}{\mathcal{L}_M}, \tag{38}$$

where in the last equality the solution (5) of the equations of motion has been used. We can then realize that the fraction $\gamma \to 0$ over $\phi_1 - M_{\text{Pl}}^2 \to 0$ controls the relative sign between R and $\mathcal{L}_M$. Since in GR this is nothing but the Planck mass M_{Pl}, it is natural to assume that this also remains positive for a physically viable theory. In this manner there are no inconsistencies in the definition of P (31), and the positive energy condition is automatically satisfied.

This model is thus always unphysical whenever $|\gamma| > 1$ and presents superluminal propagation unless $0 < \gamma < 1$. Employing our recipe we managed to impose a constraint on the range of the possible values of the free parameter γ.

1. Remarks on the linear coupling and viable (and otherwise) generalizations of the theory

It is interesting to stop to consider briefly the special case of the linear coupling $\gamma = 1$ because of its particular simplicity (phenomenological studies have also claimed the linear coupling may be observationally viable in some regimes [34]) and because it has been excluded from the analysis above. In this particular case, it is

possible to easily make contact with the Horndeski theories and thus propose viable generalizations of the theory.

Here the ghost condition (35) appears to become ill defined at the limit of linear coupling. However, by comparing with the Horndeski theories we know that a term $\sim R(\partial\phi)^2$ in the action entails a ghost.[4] Thus $\gamma = 1$ is ruled out. If one instead considers the full Horndeski term $\sim R(\partial\phi)^2 - 2[(\Box\phi)^2 - \phi_{;\mu\nu}\phi^{;\mu\nu}]$, the higher derivatives in the equations of motion are canceled and the energy associated with the fluctuations of the field can remain bounded from below despite the appearance of the higher derivatives in the action. Thus, modifying the coupling in this way would cure the negative-energy pathology.

Written in terms of the perfect fluid parametrization, where one identifies the properly normalized four-velocity as $u_\mu = \phi_{,\mu}/\sqrt{2X}$, the coupling would look rather clumsy,

$$R\mathcal{L}_M + \frac{1}{2}\mathcal{L}_M[(u\cdot\nabla\log\mathcal{L}_M)^2 + (\nabla\log\mathcal{L}_M)^2]$$
$$+ 2\mathcal{L}_M[(\nabla\cdot u)(u\cdot\log\mathcal{L}_M) + (\nabla\cdot u)^2 - (\nabla u)^2]. \tag{39}$$

A lesson here is at least that when writing down nonminimally coupled theories, especially with extra derivatives (either the ∇'s acting directly on the matter Lagrangian or the derivatives implicit in the curvature scalar), one has to consider very well-specified classes of theories in order to avoid the introduction of unphysical pathologies. This is extremely useful since it permits one to efficiently constrain the otherwise infinite theory space and to immediately exclude models such as e.g. those considered in[5] [68–71].

B. $f_1(R) \propto R$ and $f_2(R) = 1 + (R/R_0)^\gamma$

In this section we study a model similar to the one already analyzed in Sec. IV A. The nonminimal coupling is again characterized by a power-law function, but now the matter Lagrangian also appears with its canonical terms:

$$f_1(R) = \frac{R}{16\pi G} \quad \text{and} \quad f_2(R) = 1 + \left(\frac{R}{R_0}\right)^\gamma, \tag{40}$$

where R_0 and γ are the parameters of the model. The GR limit is given by $R/R_0 \ll 1$ but the limit $\gamma \to 0$ also reduces the theory to standard GR after a redefinition of

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[4]Such couplings were already studied by Amendola in [9]. The more recent nonminimally coupled three-form model [64] might also have a ghost; as in the scalar field dual description the three-form field invariant is the kinetic term of the scalar.

[5]Some of these and many other studies start with an action involving the stress energy tensor (contracted with the metric or some other tensor). It is difficult to see how this could make sense conceptually, as the stress energy tensor is fundamentally *derived* from the action. However, it is possible to consider theories constructed through recursive definitions for the couplings of the stress energy tensor [65,66], thus entailing a similar loopy structure that emerges in C theories [47] and nonlocal gravity [67].

the gravitational constant. Though this model is mathematically almost the same as the one in Sec. IVA, physically it presents more interesting features because the deviations from standard GR can be better parametrized. For this reason it has been studied more intensively throughout the literature [16,24,26,29].

The scalar fields (6) and the potential (8) are now

$$\phi_1 = M_{\text{Pl}}^2 + \frac{\gamma}{R_0^\gamma}\alpha^{\gamma-1}\beta, \qquad \phi_2 = 1 + \left(\frac{\alpha}{R_0}\right)^\gamma,$$

$$V = R_0(M_{\text{Pl}}^2 - \phi_1)(\phi_2 - 1)^{1/\gamma}, \tag{41}$$

and solving Eq. (9) eventually yields

$$P(X, \phi_1) = X + (1 - \gamma)X^{1/(1-\gamma)}\left(\frac{\gamma}{R_0(\phi_1 - M_{\text{Pl}}^2)}\right)^{\gamma/(1-\gamma)}. \tag{42}$$

As expected, this is, apart from a different definition of the parameters, the same as (31) but with a canonical term added.

Before evaluating conditions (26) we give an argument for the positiveness of

$$F_\gamma(\phi_1) = \left[\frac{\gamma}{R_0(\phi_1 - M_{\text{Pl}}^2)}\right]^{\gamma/(1-\gamma)}. \tag{43}$$

If we require $R_0 > 0$ for consistency, then from the definition of ϕ_1 (41) we obtain

$$\frac{\phi_1 - M_{\text{Pl}}^2}{\gamma} = \frac{\beta}{R_0}\left(\frac{\alpha}{R_0}\right)^{\gamma-1} \approx \frac{\mathcal{L}_M}{R_0}\left(\frac{R}{R_0}\right)^{\gamma-1}, \tag{44}$$

where in the last equality the solution (5) of the equations of motion has been used. In this case the GR limit $R/R_0 \ll 1$ is not very insightful since it just forces $\phi_1 \approx M_{\text{Pl}}^2$ as it has to be. However, as we noticed above, the limit $\gamma \to 0$ also reduces the theory to GR. In this limit the left-hand side of (44) is indeterminate, while the right-hand side yields again $\mathcal{L}_M/R$. This implies that in the GR limit the left-hand side is nothing but the relative sign between R and $\mathcal{L}_M$ or, in other words, the gravitational constant. Since in GR this constant is positive it is again natural to assume that physically motivated deviations from GR also leave this constant positive. In what follows we can thus assume that $F_\gamma(\phi_1) > 0$.

We can now compute the energy and speed of sound from (25), obtaining

$$\epsilon = X + (1 + \gamma)F_\gamma(\phi_1)X^{1/(1-\gamma)}, \tag{45}$$

$$c_s^2 = \frac{1 + F_\gamma(\phi_1)X^{\gamma/(1-\gamma)}}{1 + \frac{1+\gamma}{1-\gamma}F_\gamma(\phi_1)X^{\gamma/(1-\gamma)}}. \tag{46}$$

In general, it is now more complicated to understand for which values of γ the conditions (26) are not satisfied. However, such conditions have to be true for every value of the kinetic energy X. Requiring that the conditions (26) are satisfied both when $X \gg 1$ and $X \ll 1$ provides the following constraint

$$-1 \leq \gamma \leq 1, \tag{47}$$

which must be true in order for the theory to be viable. Furthermore, excluding superluminal propagation cuts out the possibility of a negative exponent, and the constraint tightens to $0 \leq \gamma < 1$. Note that this is the same result we have obtained in (33). This does not come as a surprise since in one of the regimes $X \gg 1$ or $X \ll 1$ the canonical kinetic term of the scalar field is always negligible in comparison with the nonminimal coupling term. The theory effectively becomes the same as the one analyzed in Sec. IVA.

Since the inclusion of the canonical matter term inside the gravitational action just corresponds to the redefinition $P(X, \phi_1) \mapsto X + P(X, \phi_1)$ in our analysis, we can conjecture that all conditions valid for a theory $f_2(R) = f^*(R)$ are also true for a theory $f_2(R) = 1 + f^*(R)$, with $f^*(R)$ a general function. This is because in the appropriate limit of the kinetic energy of the matter scalar field, the canonical matter term will always be negligible if compared to the nonminimal coupled term.

C. $f_1(R) \propto R$ and $f_2(R) \propto e^{\lambda R}$

Another simple model to analyze is the exponential nonminimal coupling described by

$$f_1(R) = \frac{R}{16\pi G} \quad \text{and} \quad f_2(R) = Ae^{\lambda R}, \tag{48}$$

where λ and A are two free parameters. The GR limit is now characterized by $\lambda \to 0$ and $A \to 1$, which suggests that we consider $A > 0$ for consistency. The two scalar fields (6) and the potential (8) are now given by

$$\phi_1 = M_{\text{Pl}}^2 + \lambda Ae^{\lambda\alpha}\beta, \qquad \phi_2 = Ae^{\lambda\alpha},$$

$$V = \frac{M_{\text{Pl}}^2 - \phi_1}{\lambda}\log\left(\frac{\phi_2}{A}\right), \tag{49}$$

and Eq. (9) reads

$$\mathcal{L}_M + (M_{\text{Pl}}^2 - \phi_1)\frac{1}{\lambda\phi_2} = 0, \tag{50}$$

with the solution

$$\phi_2^* = \frac{M_{\text{Pl}}^2 - \phi_1}{\lambda\mathcal{L}_M}. \tag{51}$$

At this point one can easily compute $P(X, \phi_1)$ to be

$$P(X, \phi_1) = \frac{M_{\text{Pl}}^2 - \phi_1}{\lambda}\left(1 - \log\left[\frac{A\lambda X}{M_{\text{Pl}}^2 - \phi_1}\right]\right). \tag{52}$$

Finally, it is enough to evaluate the condition on the sound speed in order to realize that this model is immediately ruled out. In fact, from (25) we find

$$c_s^2 = -1, \qquad (53)$$

which is obviously never positive.

To conclude, we have found that the nonminimal coupled $f(R)$ model characterized by (48) is never physically viable. It is thus not possible to employ such a theory in order to study consistent modifications of gravity at large scales. Of course the case $\lambda = 0$ corresponding to GR does not present any problem and is indeed excluded from this analysis. Finally, we mention that the model $f_2(R) = 1 + A \exp(\lambda R)$ is also ruled out by our analysis. In fact, in the appropriate limit of kinetic energy this theory would become identical to the one we just ruled out. Since we require that a nonminimally coupled theory is physically viable for every value of the matter Lagrangian one considers, adding a canonical matter term to such a model will not prevent these instabilities.

D. $f_1(R) = \frac{R}{16\pi G} + A R^n$ and $f_2(R) \propto R^n$

Finally, an example where the function f_1 is different from the usual Einstein-Hilbert term will be provided in this section. We will generalize the power-law model of Sec. IVA by adding a similar self-interacting power-law term to the gravitational sector. The two functions in (1) will then be chosen to be

$$f_1(R) = \frac{R}{16\pi G} + A R^n \quad \text{and} \quad f_2(R) = B R^n, \quad (54)$$

where A, B and n are free parameters. In general, in accordance with other works [51], we have considered different exponents for the power-law terms in the two functions. However, setting them to coincide enormously simplifies the analysis of this section, and since our scope here is to present a simple example where the gravitational Lagrangian differs from the Einstein-Hilbert term, we will only consider the functions (54) in the following. Note that the GR limit is achieved for $A \to 0$, $B \to 1$ and $n \to 0$.

In the present case the two scalar fields defined in (6) can be computed to be

$$\phi_1 = M_{\rm Pl}^2 + n\alpha^{n-1}(A + B\beta) \quad \text{and} \quad \phi_2 = B\alpha^n, \quad (55)$$

while the potential (8) is given by

$$V(\phi_1, \phi_2) = (M_{\rm Pl}^2 - \phi_1)\left(\frac{\phi_2}{B}\right)^{1/n} + \frac{A}{B}\phi_2. \quad (56)$$

At this point we can find Eq. (9) as

$$\mathcal{L}_M + \frac{1}{n} B^{-1/n}(M_{\rm Pl}^2 - \phi_1)\phi_2^{(1-n)/n} + \frac{A}{B} = 0, \quad (57)$$

with the solution

$$\phi_2^* = \Sigma(\mathcal{L}_M, \phi_1) = B\left[\frac{nB(\mathcal{L}_M + A/B)}{\phi_1 - M_{\rm Pl}^2}\right]^{n/(1-n)}. \quad (58)$$

Equation (57) would be much more difficult to solve analytically if the two exponents in the functions (54) were different. One can now find $P(X, \phi_1)$ to be

$$P(X, \phi_1) = B(1-n)\left(\frac{nB}{\phi_1 - M_{\rm Pl}}\right)^{n/(1-n)}\left(X + \frac{A}{B}\right)^{1/(1-n)}. \quad (59)$$

The speed of sound will then be

$$\frac{1}{c_s^2} = 1 + \frac{2n}{(1-n)}\frac{X}{(X + A/B)}. \quad (60)$$

Since this must be positive for every value of X, when $X \gg 1$ we obtain the condition

$$-1 \leq n \leq 1, \qquad (61)$$

which is the same as the one we obtained in Sec. IVA. Again, the superluminality constraint also excludes negative exponents, and we are left with $0 \leq n \leq 1$. On the other hand, the positivity of the energy density becomes

$$\epsilon = B(1-n)\left(\frac{nB}{\phi_1 - M_{\rm Pl}}\right)^{n/(1-n)}\left(X + \frac{A}{B}\right)^{n/(1-n)}$$
$$\times \left(X\frac{1+n}{1-n} - \frac{A}{B}\right) > 0. \quad (62)$$

To check the positivity of this expression we need to discuss all the terms appearing in it. Due to (61) the term $(1-n)$ is always positive. Moreover, we can assume that the third term is also positive because of the following argument. From the definitions (55) we have

$$\frac{\phi_1 - M_{\rm Pl}^2}{nB} = \left(\beta + \frac{A}{B}\right)\alpha^{n-1}, \quad (63)$$

which in the GR limit reduces to

$$\frac{\phi_1 - M_{\rm Pl}^2}{nB}\bigg|_{\rm GR} = \frac{\beta}{\alpha} \approx \frac{\mathcal{L}_M}{R}, \quad (64)$$

where in the last equality the solution (5) has been used. Again, since in GR this reduces to the relative sign between the gravitational and matter sectors, which is basically nothing but the gravitational constant, it is natural to assume that for physically justified modifications of gravity this value also remains positive. Moreover, we notice that

$$\frac{\phi_1 - M_{\rm Pl}^2}{nB}\left(\beta + \frac{A}{B}\right)^{-1} = \alpha^{n-1} \approx R^{n-1}. \quad (65)$$

The RHS of this equation is always positive because the theory we are dealing with is well defined only for values $R > 0$. If $R < 0$ the two functions (54) are ill defined and a different definition should be employed. In any case the RHS of (65) would always remain positive, and thus we can safely assume that the fourth term in (62) is also bigger than zero.

The only terms that remain to check in (62) are the first and the last one. If A and B have different signs the last term is always positive, while if $A/B > 0$ it can be positive or negative depending on the value of X. Since we require the energy to be positive for all values of X, we are led to consider only the case $A/B < 0$. However, the first term in (62) is just B, and we must assume $B > 0$ if we want $\epsilon > 0$.

This in turn implies $A < 0$. Note that the assumption $B > 0$ is quite natural if we require the theory to be physically interesting since in the GR limit we must have $B = 1$.

Finally, we comment on the fact that we could have chosen $f_2(R) = 1 + BR^n$. As before, this redefinition just complicates the whole analysis but does not change the final results. Note that in this case the requirement $B > 0$ has nothing to do with the GR limit of the theory because this is $B \to 0$ now. Then the $B > 0$ condition is indeed a constraint arising from the result of our analysis and has nothing to do with the physical features of the theory.

V. DISCUSSION AND CONCLUSION

The purpose of the present work is to constrain nonminimally coupled $f(R)$ gravity. In order to achieve this goal, the dynamically equivalent representation of the theory in terms of multiscalar-tensor theories has been employed. By choosing the matter Lagrangian to be a simple scalar field, it has been possible to compute the effective energy density and the speed of sound of adiabatic perturbations. Finally, to ensure that the nonminimal couplings of the theory do not introduce instabilities, it was eventually required that these quantities be positive, and to exclude superluminal propagation of perturbations, the sound speed was bounded below unity. The whole procedure has given rise to a recipe, summarized at the end of Sec. III, which can be adopted to constrain possible models of nonminimally coupled $f(R)$ gravity. In order to better explain how this recipe works, some specific models have been studied in Sec. IV. The main results of this analysis are schematized in Table I. As it is clear from the table, the parameters of the various models have been constrained to lie in specific ranges.

In Secs. IV A and IV B a power-law nonminimal coupling has been considered. This kind of model is particularly popular in the literature [16,24,26,29], both because it is sufficiently easy to handle and because it produces results comparable with observations. The results of Sec. IV B are consistent with [50], which poses doubts on the viability of a linear Ricci coupling to matter, though for astrophysical applications such a coupling can be preferable [34]. We argued that the linear coupling is excluded, and proposed a viable generalization of the theory. In any case, for dark matter and dark energy

modeling is commonly considered an exponent $\gamma < 0$ [24,26], since one wants the effects of the nonminimal coupling to be relevant at late times when the Ricci scalar is small. However, such models allow superluminal propagation and thus, apparently, violations of causality. The galactic rotation curves have been explained by employing $\gamma = -1/3$ or $\gamma = -1$ [24], while for dark energy values $\gamma < -1$ better fit the observations [26]. Since from the analysis of Sec. IV B a constraint $\gamma \geq 0$ emerges, it seems that a power-law nonminimally coupled $f(R)$ model can never represent a suitable late time theoretical description for dark matter or dark energy.

In Sec. IV C an exponential nonminimal coupling has been probed. Despite its simplicity it has been shown to always lead to a negative speed of sound for adiabatic perturbations. The model has thus been ruled out by the considerations presented in this work and should not be considered for physical applications. As it is shown in Table I, the only possible value for the parameter λ is zero, which corresponds to nothing but GR. Finally, in Sec. IV D, a model where the gravitational sector is also modified has been considered. The results for this case are similar to the ones obtained in Sec. IV A, with the addition that the parameters A and B have to always be negative and positive, respectively.

To conclude, the analysis exposed in these pages aims at presenting and explaining a simple procedure one can utilize in order to find constraints on specific models of nonminimally coupled $f(R)$ gravity. Though for complicated models this can be practically impossible to carry out, for sufficiently simple cases, such as the ones commonly studied in the literature and considered in Sec. IV, it becomes a useful tool to check the phenomenological viability and rule out unphysical theories. Having in mind the possibility of employing such theories in astrophysical and cosmological applications, it is convenient to have an instrument that is able to determine the consistency of the model with which one chooses to work.

APPENDIX: THE SCALAR-TENSOR REPRESENTATION OF $f(R, \mathcal{L}_M)$ GRAVITY

In this appendix we will show that an equivalent scalar-tensor representation also exists for more general theories[6] identified by the extended action

$$S_0 = \int d^4x \sqrt{-g} f(R, \mathcal{L}_M), \tag{A1}$$

where now f is an arbitrary function of both the Ricci scalar R and the matter Lagrangian $\mathcal{L}_M$. Such models have

TABLE I. Constraints on parameters of some specific models of action (1) arising from the analysis presented in Sec. IV.

$f_1(R)$	$f_2(R)$	Stable if	Stable & causal if
$\frac{R}{16\pi G}$	$\frac{AR^\gamma}{1 + AR^\gamma}$	$-1 \leq \gamma < 1$	$0 \leq \gamma < 1$
$\frac{R}{16\pi G}$	$\frac{A \exp(\lambda R)}{1 + A \exp(\lambda R)}$	$\lambda = 0$	$\lambda = 0$
$\frac{R}{16\pi G} + AR^n$	$\frac{BR^n}{1 + BR^n}$	$-1 \leq n < 1$ $A < 0$ and $B > 0$	$0 \leq n < 1$ $A < 0$ and $B > 0$

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[6]For remarks on other generalizations, recall Sec. IV A 1.

been proposed in [72] as a maximal extension of non-minimally coupled $f(R)$ gravity and have been further studied in [73–75]. In what follows action (A1) will be proven to be dynamically equivalent to action (7), with different definitions of the scalar fields ϕ_1, ϕ_2 and the potential $V(\phi_1, \phi_2)$. Once this equivalence has been defined, the procedure following (7) can be identically repeated, and the general conclusions we found for non-minimally coupled $f(R)$ gravity will also hold for $f(R, \mathcal{L}_M)$ gravity. The only crucial difference will be in the definitions of the scalar fields, which in turn will determine the physical differences between the various models.

Consider again another action given by

$$S = \int d^4x \sqrt{-g} \left[f(\alpha, \beta) + \frac{\partial f}{\partial \alpha}(R - \alpha) + \frac{\partial f}{\partial \beta}(\mathcal{L}_M - \beta) \right],$$
(A2)

where α and β are two scalar fields. The variation of this action with respect to the two scalar fields gives

$$\frac{\partial^2 f}{\partial \alpha^2}(R - \alpha) + \frac{\partial^2 f}{\partial \alpha \partial \beta}(\mathcal{L}_M - \beta) = 0,$$
(A3)

$$\frac{\partial^2 f}{\partial \alpha \partial \beta}(R - \alpha) + \frac{\partial^2 f}{\partial \beta^2}(\mathcal{L}_M - \beta) = 0.$$
(A4)

Given that the determinant of this system of equations is nonvanishing, i.e. assuming that

$$\frac{\partial^2 f}{\partial \alpha^2} \frac{\partial^2 f}{\partial \beta^2} \neq \left(\frac{\partial^2 f}{\partial \alpha \partial \beta} \right)^2,$$
(A5)

the unique solution is

$$\alpha = R \quad \text{and} \quad \beta = \mathcal{L}_M.$$
(A6)

It is now easy to realize that substituting this solution back into (A2) immediately produces action (A1), implying that the two theories are dynamically equivalent.

At this point we define two new scalar fields and a potential as

$$\phi_1 = \frac{\partial f}{\partial \alpha}, \qquad \phi_2 = \frac{\partial f}{\partial \beta},$$
$$V(\phi_1, \phi_2) = f(\alpha, \beta) - \alpha \phi_1 - \beta \phi_2,$$
(A7)

where one must consider α and β as functions of ϕ_1 and ϕ_2 in the potential V. In terms of these new variables action (A2) can be rewritten as

$$S = \int d^4x \sqrt{-g} [\phi_1 R + \phi_2 \mathcal{L}_M + V(\phi_1, \phi_2)],$$
(A8)

which coincides with (7) except for the definitions (A7), which differ from (6) and (8).

To conclude, we have proven that the extended theory defined by action (A1) can equally be mapped into a scalar-tensor theory described by action (7). The only differences arise in the definitions of the scalar fields ϕ_1, ϕ_2 and the potential V, which generalize the corresponding definitions in nonminimally coupled $f(R)$ gravity. Because of this result the same analysis we performed in the main body of the paper could be identically carried out for $f(R, \mathcal{L}_M)$ gravity.

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