

Scalar, electromagnetic, and gravitational perturbations of Kerr-Newman black holes in the slow-rotation limit

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In Einstein-Maxwell theory, according to classic uniqueness theorems, the most general stationary black-hole solution is the axisymmetric Kerr-Newman metric, which is defined by three parameters: mass, spin and electric charge. The radial and angular dependence of gravitational and electromagnetic perturbations in the Kerr-Newman geometry do not seem to be separable. In this paper we circumvent this problem by studying scalar, electromagnetic and gravitational perturbations of Kerr-Newman black holes in the slow-rotation limit. We extend (and provide details of) the analysis presented in a recent Letter [P. Pani, E. Berti, and L. Gualtieri, *Phys. Rev. Lett.* **110**, 241103 (2013)]. Working at linear order in the spin, we present the first detailed derivation of the axial and polar perturbation equations in the gravito-electromagnetic case, and we compute the corresponding quasinormal modes for any value of the electric charge. Our study is the first self-consistent stability analysis of the Kerr-Newman metric, and in principle it can be extended to any order in the small rotation parameter. We find numerical evidence that the axial and polar sectors are isospectral at first order in the spin and speculate on the possible implications of this result.

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I. INTRODUCTION

A. Motivation and background

Classic uniqueness theorems (reviewed in, e.g., [1]) guarantee that the Kerr-Newman (KN) metric [2] describes the most general stationary electrovacuum solution in Einstein-Maxwell theory. The KN line element

$$ds^2 = -dt^2 + \Sigma \left(\frac{dr^2}{\Delta} + d\vartheta^2 \right) + (r^2 + a^2) \sin^2 \vartheta d\varphi^2 + \frac{2Mr - Q^2}{\Sigma} (dt - a \sin^2 \vartheta d\varphi)^2 \quad (1)$$

(where $\Sigma = r^2 + a^2 \cos^2 \vartheta$, $\Delta = r^2 + a^2 - 2Mr + Q^2$) is characterized by three parameters: mass M , angular momentum $J = Ma$ and electromagnetic charge Q . In the neutral case ($Q = 0$) the KN solution reduces to the Kerr metric, whereas in the nonspinning limit ($J = 0$) it reduces to the Reissner-Nordström (RN) metric. When both $Q \neq 0$ and $J \neq 0$ the spacetime is endowed with a magnetic dipole moment, and it has the same gyromagnetic ratio $g = 2$ as the electron [3]. This has led to some speculation that the KN metric could be used as a classical model for elementary particles [4].

Most work in black hole (BH) physics has focused on the Kerr metric, because astrophysical BHs are likely to be electrically neutral. The reason is that a BH of mass M and charge Q will not gravitationally accrete particles of mass m and charge e as long as $eQ > Mm$ (in the natural units $G = c = 1$ used in this paper). Since $m/e \sim 10^{-21}$ for electrons, large BHs will hardly acquire any charge (see, e.g., [5]). Furthermore, in astrophysical environments the electric charge is expected to be shorted out by the surrounding plasma [6].

Notwithstanding the fact that charge is unlikely to play a significant role in astrophysics, the KN metric is a precious theoretical laboratory to investigate the dynamics of Einstein-Maxwell theory in curved spacetime, and the linearized dynamics of test fields on a KN background have been intensively studied in the past.

The KN metric is the only nontrivial, asymptotically flat solution of the Einstein-Maxwell system for which the geodesic and Klein-Gordon equations can be solved by separation of variables [7]. The neutrino [8], massive spin-1/2 [9,10] and Rarita-Schwinger [11] equations in the KN metric are also known to be separable. The separability of fermionic fields is related to the existence of a generalized total angular momentum operator for the Dirac equation in curved spacetime, that satisfies appropriate conservation laws [12,13]. Scalar and Dirac perturbations of a KN BH can therefore be treated using the same general methods that apply to Kerr BHs. In particular, it is straightforward to compute the quasinormal modes (QNMs) of KN

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BHs for these classes of perturbations (see [14–18] for reviews on QNMs). For the KN metric, scalar QNMs were computed in [19] (see also Ref. [20] for a recent generalization to massive and charged scalar QNMs) and Dirac QNMs were computed in [21]. The superradiant instability of massive scalar fields was studied in [22]. Hartman, Song, and Strominger studied the scattering of charged scalars and fermions in near-extremal KN spacetimes [23], complementing earlier work on the so-called Kerr-Newman/conformal field theory (KN/CFT) conjecture [24]. Because the equations are separable, (the absence of) superradiant effects [25] and graybody factors for charged and massive Dirac particles in KN have also been studied extensively [4,26–35]: see [36] for a good overview of work in this field and [37] for a study of Dirac perturbations of KN BHs in anti-de Sitter space.

Much less is known about the gravitational-electromagnetic sector of KN perturbations. The reason is that most methods to compute QNMs, graybody factors and scattering amplitudes (including the continued fraction method [38] and the asymptotic iteration method [39]) rely on the separability of the perturbation equations. Despite several attempts [40–42], no one has succeeded at separating the angular and radial dependence of the gravitational-electromagnetic eigenfunctions. Chandrasekhar’s monograph [43] gives a comprehensive overview of this long-standing unsolved problem.

A relatively small number of papers tried to address the problem of the oscillations and stability of the KN metric under the combined effect of electromagnetic and gravitational perturbations. Mashhoon [44] used the analogy between geodesic stability and QNMs first proposed by Goebel [45] (see also [46]) to estimate the QNMs in the eikonal approximation and to argue that the KN metric should be stable. Dudley and Finley ([40,41], henceforth DF) made a remarkable study of the separability of linear perturbations of the solutions of the Einstein-Maxwell equations found by Plebánski and Demiański [47], which include all vacuum type D solutions in the Petrov classification. Because of nonseparability of the perturbation equations, they could not treat gravito-electromagnetic perturbations in a fully consistent approach. In their work, DF “either keep the geometry fixed and perturb the electric field or, of more interest, keep the electric field fixed and perturb the geometry.” This approach should be appropriate for values of the charge Q at most as large as the perturbations of the spacetime metric. Kokkotas first used the DF equation to compute the fundamental gravitational QNM using Wentzel-Kramers-Brillouin (WKB) methods [48]. Later on, Berti and Kokkotas [19] confirmed that the WKB approximation is reasonably accurate for all values of the dimensionless spin parameter ($\tilde{a} \equiv J/M^2$) and of the charge Q by comparing WKB results to a continued-fraction solution. The main problem of the DF approach is not computational, but physical. The DF

equation does not treat the gravito-electromagnetic coupling in a self-consistent way (for example, it does not reduce to the well-known RN perturbation equations as $J \rightarrow 0$). Therefore it is unclear whether it provides a correct description of gravitational and electromagnetic perturbations of KN BHs.

In a recent Letter [49], we have presented the first self-consistent study of the gravito-electromagnetic perturbations of KN BHs. This paper complements and extends the results of [49], providing details of the derivation of the perturbation equations and a more comprehensive set of numerical results.

Our approach relies on a clear physical approximation, i.e., a slow-rotation expansion of the perturbations of spinning BHs [50,51]. The formalism to address this problem was originally proposed in the context of slowly rotating compact stars [52–55], and it can be extended (at least in principle) to any perturbative order in the small rotation parameter. Within the slow-rotation expansion (which is valid for any value of the BH charge Q) it is possible to estimate truncation errors, e.g., by extending the computation to the next order in rotation or by comparison with cases where a nonperturbative solution is available (such as the case of scalar perturbations considered below).

B. Executive summary

In the remainder of this introduction we provide a short executive summary of our main results, that is also meant as a guide to the structure of the paper.

In Sec. II we review our approach to separate the scalar and the gravito-electromagnetic perturbations of a KN BH in the slow-rotation limit [50,51,53]. We derive the equations describing gravito-electromagnetic oscillations in the slow-rotation approximation by linearizing the Einstein-Maxwell equations with respect to both the amplitude of the oscillation and the BH spin parameter $\tilde{a} \equiv J/M^2$. By expanding the perturbations of the spacetime metric and of the electromagnetic field in tensor spherical harmonics, we obtain a coupled system of differential equations. Our main analytical result is the derivation of two sets of coupled, second-order equations (one for the axial and one for the polar sector, respectively) which fully describe gravito-electromagnetic perturbations of a KN BH to first order in the spin:

$$\hat{D}Z_i^\pm \equiv V_0^{(i,\pm)}Z_i^\pm + m\tilde{a}[V_1^{(i,\pm)}Z_i^\pm + V_2^{(i,\pm)}Z_i^{\pm'}] + m\tilde{a}Q^2[W_1^{(i,\pm)}Z_j^\pm + W_2^{(i,\pm)}Z_j^{\pm'}]. \quad (2)$$

Here $i, j = 1, 2, i \neq j$ and there is no sum over the indices i, j . A prime denotes a derivative with respect to r , and we have introduced the differential operator

$$\hat{D} \equiv \frac{d^2}{dr_*^2} + \omega^2 - F \frac{\ell(\ell+1)}{r^2}, \quad (3)$$

where r_* is the standard tortoise coordinate defined as $dr/dr_* = (r - r_-)(r - r_+)/r^2$, and $r_{\pm} = M \pm \sqrt{M^2 - Q^2}$ are the outer (r_+) and Cauchy (r_-) horizons of a RN BH. The functions Z_i^- and Z_i^+ are linear combinations of axial and polar variables, respectively, and they are also combinations of gravitational and electromagnetic perturbations. A step-by-step derivation of these equations is presented in Appendix A and in a publicly available MATHEMATICA notebook [56]. The axial potentials $V^{(i,-)}$ and $W^{(i,-)}$ are listed in Appendix A 1, while the polar potentials $V^{(i,+)}$ and $W^{(i,+)}$ are so lengthy that we decided to make them available only through the MATHEMATICA notebook [56] in order to save space.

We have integrated the coupled system (2) and computed the corresponding eigenfrequencies using two independent methods, which are described in Sec. III (see also Ref. [57] for a review). For any value of Q , our analysis allows us to extract the first-order corrections to the complex QNM frequencies $\omega = \omega_R + i\omega_I$:

$$\omega_R = \omega_R^{(0)} + \tilde{a}m\omega_R^{(1)} + \mathcal{O}(\tilde{a}^2), \quad (4)$$

$$\omega_I = \omega_I^{(0)} + \tilde{a}m\omega_I^{(1)} + \mathcal{O}(\tilde{a}^2), \quad (5)$$

where $\omega_R^{(i)}$ and $\omega_I^{(i)}$ are functions of Q and of the multipolar index ℓ , and the m dependence has been factored out.

Section IV presents our numerical results. We begin by studying scalar perturbations of KN BHs, for which the perturbation equations are separable and QNM frequencies can be computed “exactly” in the Teukolsky formalism

[19]. By comparing QNM frequencies in the slow-rotation approximation to the Teukolsky-based results, we find that the relative error of the slow-rotation approximation is less than 1% as long as $J/J_{\max} \lesssim 0.3$, where J_{\max} is the maximum allowed KN spin for any fixed value of the electric charge.

Figure 1 shows our main numerical results for the fundamental gravito-electromagnetic perturbations with $\ell = 2, 3$, which are the most relevant for gravitational-wave emission (see, e.g., [58]). In each panel we show four curves, corresponding to the axial and polar “gravitational” and “electromagnetic” modes (as defined in the decoupled $Q = 0$ limit: see Sec. IV for details). The zeroth-order terms shown in the small left panels are simply RN QNMs; they agree with continued-fraction solutions of the equations first derived by Zerilli [59], as computed by Leaver [60].

Gravito-electromagnetic perturbations of nonspinning BHs in general relativity have a noteworthy property that was proved by Chandrasekhar [43]: the polar and axial potentials can be written in terms of a superpotential, which implies that the polar and axial QNMs are *isospectral* [17]. *A priori*, there is no reason why such a remarkable property should hold true also for KN BHs.

In addition to computing the QNMs of a KN BH in a full consistent setting for the first time, perhaps the most important result of our numerical study is strong evidence that the axial and polar sectors of KN gravito-electromagnetic perturbations are indeed isospectral to first order in the BH spin. In the inset of Fig. 1 we show the relative difference between the coefficients of axial and polar

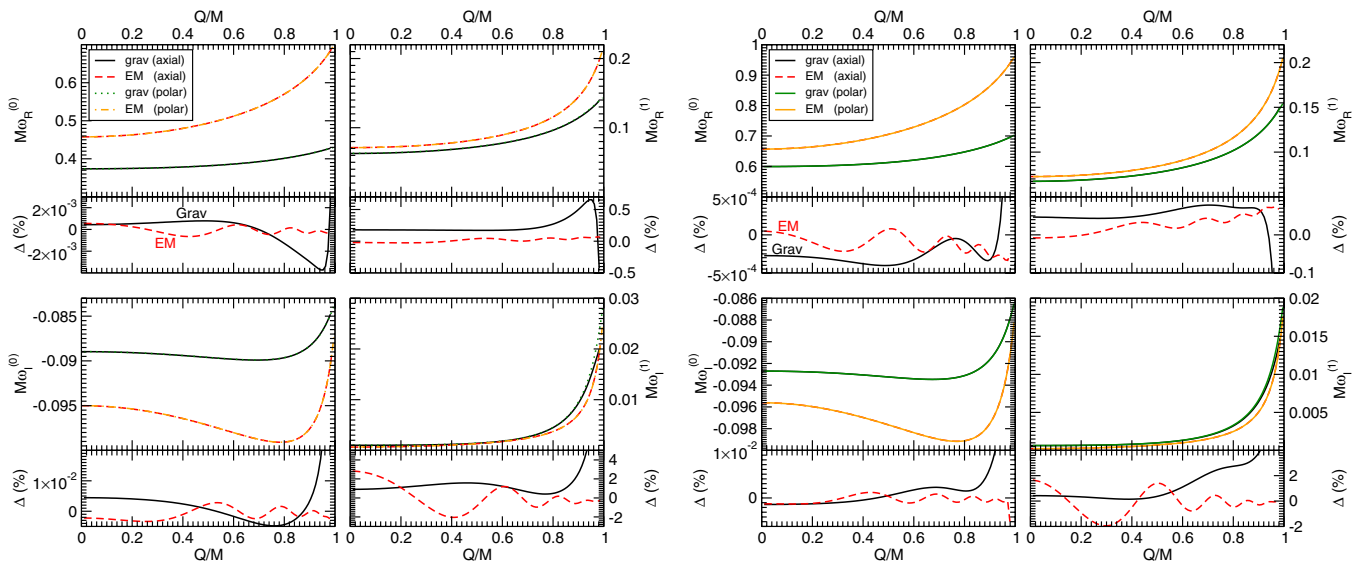


FIG. 1 (color online). Left (reproduced from Ref. [49]): Zeroth-order (small left panels) and first-order (small right panels) terms of the slow-rotation expansion of the KN QNM frequencies [cf. Eqs. (4) and (5)]. All quantities are plotted as a function of Q/M , and they refer to the fundamental mode ($n = 0$) with $\ell = 2$. The lower part of each panel shows the percentage difference between axial and polar quantities: our results are consistent with isospectrality to $\mathcal{O}(0.1\%)$ for the real part and to $\mathcal{O}(1\%)$ for the imaginary part of these modes. Right: The same, but for fundamental modes with $\ell = 3$.

modes as functions of Q : our results are consistent with isospectrality within the numerical errors implicit in our method. This is further discussed in Sec. IV, where we provide additional evidence that higher multipoles (with $\ell > 2$) and higher overtones (for a given ℓ) are also isospectral to first order in rotation.

These numerical results lead us to the tantalizing conjecture that the modes of a KN BH may be isospectral to *any order* in \tilde{a} . We conclude our paper in Sec. V with a brief discussion of this conjecture and of possible ways to put it to the test (see also [49]).

II. FORMALISM

Let us consider Einstein-Maxwell theory:

$$S = \int d^4x \sqrt{-g} (R - F_{\mu\nu} F^{\mu\nu}), \quad (6)$$

where R is the Ricci scalar and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell field strength. The KN metric (1) is the most general stationary electrovacuum solution of this theory. Here and in the following we linearize in the spin parameter $\tilde{a} \equiv J/M^2$, neglecting terms of order $\mathcal{O}(\tilde{a}^2)$. To this order, the KN metric reads

$$\begin{aligned} ds_0^2 &= g_{\mu\nu}^{(0)} dx^\mu dx^\nu \\ &= -F(r) dt^2 + F(r)^{-1} dr^2 + r^2 d^2\Omega \\ &\quad - 2\varpi(r) \sin^2 \vartheta d\varphi dt, \end{aligned} \quad (7)$$

where

$$F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad (8)$$

$$\varpi(r) = \frac{2\tilde{a}M^2}{r} - \frac{\tilde{a}Q^2M}{r^2}, \quad (9)$$

and the background electromagnetic potential is given by

$$A_\mu = \left(\frac{Q}{r}, 0, 0, -\frac{\tilde{a}QM}{r} \sin^2 \vartheta \right). \quad (10)$$

The magnetic field is given by the curl of the three-potential above. Thus, the presence of *both* rotation and charge ($\tilde{a}Q \neq 0$) induces a magnetic field in the (ϑ, φ) directions.

A. Harmonic decomposition

We linearize the metric as

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}, \quad (11)$$

and we decompose the metric perturbations $h_{\mu\nu}$ in the Regge–Wheeler gauge:

$$h_{\mu\nu} = \begin{pmatrix} H_0^\ell Y^\ell & H_1^\ell Y^\ell & h_0^\ell S_\vartheta^\ell & h_0^\ell S_\varphi^\ell \\ H_1^\ell Y^\ell & H_2^\ell Y^\ell & h_1^\ell S_\vartheta^\ell & h_1^\ell S_\varphi^\ell \\ h_0^\ell S_\vartheta^\ell & h_1^\ell S_\vartheta^\ell & r^2 K^\ell Y^\ell & 0 \\ h_0^\ell S_\varphi^\ell & h_1^\ell S_\varphi^\ell & 0 & r^2 K^\ell \sin^2 \vartheta Y^\ell \end{pmatrix}, \quad (12)$$

where $Y^\ell = Y^\ell(\vartheta, \varphi)$ are the ordinary scalar spherical harmonics, $(S_\vartheta^\ell, S_\varphi^\ell) \equiv (-Y_{,\varphi}^\ell / \sin \vartheta, \sin \vartheta Y_{,\vartheta}^\ell)$ are the axial vector harmonics, and $H_{0,1,2}^\ell, h_{0,1}^\ell, K^\ell$ are functions of (t, r) . Here and in the following, a sum over the harmonic indices ℓ and m (with $|m| < \ell$) is implicit. We will append the relevant multipolar index ℓ to any perturbation variable but omit the index m , because in an axisymmetric background the perturbation equations with different values of m are decoupled from each other.

We expand the electromagnetic potential as follows [61]:

$$\delta A_\mu(t, r, \vartheta, \varphi) = \begin{bmatrix} 0 \\ 0 \\ u_{(4)}^\ell S_b^\ell / \Lambda \end{bmatrix} + \begin{bmatrix} u_{(1)}^\ell Y^\ell / r \\ u_{(2)}^\ell Y^\ell / (rF) \\ u_{(3)}^\ell Y_b^\ell / \Lambda \end{bmatrix}, \quad (13)$$

where $\Lambda = \ell(\ell + 1)$, $b = (\vartheta, \varphi)$, $Y_b^\ell = (Y_{,\vartheta}^\ell, Y_{,\varphi}^\ell)$ are the polar vector harmonics, and $u_{(1,2,3,4)}^\ell$ are functions of (t, r) . Inserting the harmonic expansion of the metric perturbations (12) and of the Maxwell field (13) into the linearized Einstein equations we find the equations for the perturbation functions to linear order in \tilde{a} . The latter naturally separate into three groups [50,51,53,57]. By denoting the linearized Einstein equations as $\delta \mathcal{E}_{\mu\nu} = 0$, the first group is formed by the equations $\delta \mathcal{E}_{(I)} = 0$, where $I = 0, 1, 2, 3$ is shorthand notation for (tt) , (tr) , (rr) and $(+)$, respectively, and we have defined $\delta \mathcal{E}_{(+)} \equiv \delta \mathcal{E}_{\vartheta\vartheta} + \delta \mathcal{E}_{\varphi\varphi} / \sin^2 \vartheta$. The equations can be cast in the form

$$\delta \mathcal{E}_{(I)} \equiv (A_\ell^{(I)} + \tilde{A}_\ell^{(I)} \cos \vartheta) Y^\ell + B_\ell^{(I)} \sin \vartheta Y_{,\vartheta}^\ell + C_\ell^{(I)} Y_{,\varphi}^\ell = 0, \quad (14)$$

where $A_\ell^{(I)}, \tilde{A}_\ell^{(I)}, B_\ell^{(I)}, C_\ell^{(I)}$ are combinations of the perturbation functions $\{H_{0,1,2}^\ell, h_{0,1}^\ell, K^\ell, u_{(1,2,3,4)}^\ell\}$. The second group is formed by the equations $\delta \mathcal{E}_{L\vartheta} = 0$, with $L = 0, 1$ corresponding to the t, r components, respectively. They can be cast in the form

$$\begin{aligned} \delta \mathcal{E}_{(L\vartheta)} \equiv & (\alpha_\ell^{(L)} + \tilde{\alpha}_\ell^{(L)} \cos \vartheta) Y_{,\vartheta}^\ell - (\beta_\ell^{(L)} + \tilde{\beta}_\ell^{(L)} \cos \vartheta) \frac{Y_{,\varphi}^\ell}{\sin \vartheta} \\ & + \eta_\ell^{(L)} \sin \vartheta Y^\ell + \xi_\ell^{(L)} X^\ell + \chi_\ell^{(L)} \sin \vartheta W^\ell = 0, \end{aligned} \quad (15)$$

$$\begin{aligned} \delta\mathcal{E}_{(L\varphi)} \equiv & (\beta_\ell^{(L)} + \tilde{\beta}_\ell^{(L)} \cos \vartheta) Y_{,\vartheta}^\ell + (\alpha_\ell^{(L)} + \tilde{\alpha}_\ell^{(L)} \cos \vartheta) \frac{Y_{,\varphi}^\ell}{\sin \vartheta} \\ & + \zeta_\ell^{(L)} \sin \vartheta Y^\ell + \chi_\ell^{(L)} X^\ell - \xi_\ell^{(L)} \sin \vartheta W^\ell = 0, \end{aligned} \quad (16)$$

where $\alpha_\ell^{(L)}, \tilde{\alpha}_\ell^{(L)}, \beta_\ell^{(L)}, \tilde{\beta}_\ell^{(L)}, \eta_\ell^{(L)}, \xi_\ell^{(L)}, \zeta_\ell^{(L)}, \chi_\ell^{(L)}$ are combinations of the perturbation functions. The third group is (defining $\delta\mathcal{E}_{(-)} \equiv \delta\mathcal{E}_{\vartheta\vartheta} - \delta\mathcal{E}_{\varphi\varphi}/\sin^2 \vartheta$)

$$\delta\mathcal{E}_{(\vartheta\varphi)} \equiv f_\ell \sin \vartheta \partial_\vartheta Y^\ell + g_\ell Y_{,\varphi}^\ell + s_\ell \frac{X^\ell}{\sin \vartheta} + t_\ell W^\ell = 0, \quad (17)$$

$$\delta\mathcal{E}_{(-)} \equiv g_\ell \sin \vartheta \partial_\vartheta Y^\ell - f_\ell Y_{,\varphi}^\ell - t_\ell \frac{X^\ell}{\sin \vartheta} + s_\ell W^\ell = 0, \quad (18)$$

where we have defined

$$X^\ell \equiv 2(Y_{,\vartheta\varphi}^\ell - \cot \vartheta Y_{,\varphi}^\ell), \quad (19)$$

$$W^\ell \equiv Y_{,\vartheta\vartheta}^\ell - \cot \vartheta Y_{,\vartheta}^\ell - \frac{Y_{,\varphi\varphi}^\ell}{\sin^2 \vartheta}. \quad (20)$$

The linearized Maxwell equations can be also recast in a similar form. The t and r components belong to the first group and can be arranged in the form of Eq. (14) with $I = 4, 5$, respectively. The ϑ and φ components belong to the second group and can be written as in Eqs. (15) and (16) with $L = 2$. The coefficients can be split into two sets:

$$\text{Polar: } A_\ell^{(I)}, C_\ell^{(I)}, \alpha_\ell^{(L)}, \tilde{\beta}_\ell^{(L)}, \zeta_\ell^{(L)}, \xi_\ell^{(L)}, s_\ell, f_\ell,$$

$$\text{Axial: } \tilde{A}_\ell^{(I)}, B_\ell^{(I)}, \beta_\ell^{(L)}, \tilde{\alpha}_\ell^{(L)}, \eta_\ell^{(L)}, \chi_\ell^{(L)}, t_\ell, g_\ell,$$

where henceforth the indices $I = 0, \dots, 5$ and $L = 0, 1, 2$ account for both the Einstein and Maxwell equations. The explicit form of the coefficients is not particularly illuminating, and it is given in a publicly available MATHEMATICA notebook [56]. The crucial point is to recognize that the coefficients above are only functions of t and r , so that the entire angular dependence enters the field equations only in specific combinations related to spherical harmonics of various type.

B. Separation of the angular dependence

The separation of the angular dependence of Einstein's equations for a slowly rotating star was performed in Ref. [53]. The procedure has been extended to general slowly rotating BH solutions in Refs. [50,51] (see also [57]); we refer the reader to those papers for details. Using the orthogonality properties of scalar spherical harmonics, from Eq. (14) we get

$$\begin{aligned} A_\ell^{(I)} + imC_\ell^{(I)} + \mathcal{Q}_\ell[\tilde{A}_{\ell-1}^{(I)} + (\ell-1)B_{\ell-1}^{(I)}] \\ + \mathcal{Q}_{\ell+1}[\tilde{A}_{\ell+1}^{(I)} - (\ell+2)B_{\ell+1}^{(I)}] = 0, \end{aligned} \quad (21)$$

where we have defined $\mathcal{Q}_\ell^2 = (\ell^2 - m^2)/(4\ell^2 - 1)$. From the orthogonality of vector spherical harmonics and using Eqs. (15) and (16) we get

$$\begin{aligned} \Lambda\alpha_\ell^{(L)} + im[(\ell-1)(\ell+2)\xi_\ell^{(L)} - \tilde{\beta}_\ell^{(L)} - \zeta_\ell^{(L)}] \\ + \mathcal{Q}_\ell(\ell+1)[(\ell-2)(\ell-1)\chi_{\ell-1}^{(L)} + (\ell-1)\tilde{\alpha}_{\ell-1}^{(L)} - \eta_{\ell-1}^{(L)}] \\ - \mathcal{Q}_{\ell+1}\ell[(\ell+2)(\ell+3)\chi_{\ell+1}^{(L)} - (\ell+2)\tilde{\alpha}_{\ell+1}^{(L)} - \eta_{\ell+1}^{(L)}] = 0, \end{aligned} \quad (22)$$

$$\begin{aligned} \Lambda\beta_\ell^{(L)} + im[(\ell-1)(\ell+2)\chi_\ell^{(L)} + \tilde{\alpha}_\ell^{(L)} + \eta_\ell^{(L)}] \\ - \mathcal{Q}_\ell(\ell+1)[(\ell-2)(\ell-1)\xi_{\ell-1}^{(L)} - (\ell-1)\tilde{\beta}_{\ell-1}^{(L)} + \zeta_{\ell-1}^{(L)}] \\ + \mathcal{Q}_{\ell+1}\ell[(\ell+2)(\ell+3)\xi_{\ell+1}^{(L)} + (\ell+2)\tilde{\beta}_{\ell+1}^{(L)} + \zeta_{\ell+1}^{(L)}] = 0. \end{aligned} \quad (23)$$

Finally, the orthogonality of tensor spherical harmonics applied to Eqs. (17) and (18) yields

$$0 = \Lambda s_\ell - imf_\ell - \mathcal{Q}_\ell(\ell+1)g_{\ell-1} + \mathcal{Q}_{\ell+1}\ell g_{\ell+1}, \quad (24)$$

$$0 = \Lambda t_\ell + img_\ell - \mathcal{Q}_\ell(\ell+1)f_{\ell-1} + \mathcal{Q}_{\ell+1}\ell f_{\ell+1}. \quad (25)$$

Summarizing, once we fix the value of m , truncating the expansion in ℓ to a value ℓ_{\max} , the index ℓ can have $n_\ell = \ell_{\max} - |m| + 1$ possible values; our separation procedure in the slow-rotation limit yields a system of $14n_\ell$ coupled, ordinary differential equations (ten for the gravitational sector and four for the Maxwell sector for each ℓ), given in Eqs. (21)–(25). Their explicit form is available online [56].

C. First-order corrections to the eigenvalue equations

In the case of slow rotation ($\tilde{a} \ll 1$), a Laporte-like selection rule couples perturbations with harmonic index ℓ with those having opposite parity and harmonic index $\ell \pm 1$ [50–53]. This is a consequence of using a basis of spherical harmonics in a nonspherical background. However, as discussed in detail in [51,54,62], the couplings to the $\ell \pm 1$ terms do not contribute to the QNM spectrum to first order in \tilde{a} . For this reason, we shall neglect these terms in the following. In this way we get a system of 14 coupled differential equations for any chosen value of ℓ .

Once the couplings among different ℓ 's are neglected, we can simply fix a value of ℓ (and in order to simplify the notation we will drop the index ℓ from all perturbation functions). We perform a Fourier decomposition by assuming that all perturbations have a time dependence $\sim e^{-i\omega t}$.

As we explicitly show in Appendix A, axial and polar perturbations decouple, and Eqs. (21)–(25) can be reduced to two coupled, second-order differential equations

(one pair for each parity), that we have already written down in the introduction [Eq. (2)]. These equations display the same symmetries as the master equations for a RN BH [43], and indeed they exactly reduce to those equations in the nonrotating case. We remark in passing that the DF equations previously used to investigate gravito-electromagnetic perturbations of the KN metric do not satisfy this requirement. Equation (2) is the main result of this paper. It contains two first-order corrections in the rotational parameter \tilde{a} . The first term couples the functions Z_j^\pm with the same Z_j^\pm . It is responsible for a Zeeman-like splitting of the eigenfrequencies, which breaks the degeneracy in the azimuthal index m . The second line in Eq. (2) is more interesting. First, it is proportional to the combination $\tilde{a}Q^2$, so it is vanishing when the BH is nonspinning or uncharged. Furthermore, this term couples the function Z_1^+ with the function Z_2^+ , and the function Z_1^- with the function Z_2^- .

We remark that this coupling disappears when $\tilde{a} = 0$, but this does not mean that gravitational and electromagnetic perturbations decouple in that limit: whenever $Q \neq 0$, the functions Z_j^\pm are combinations of electromagnetic and gravitational perturbations, which are then coupled. Electromagnetic and gravitational perturbations only decouple when $Q = 0$ (e.g., for Schwarzschild or Kerr BHs) because the functions Z_1^\pm describe pure gravitational perturbations, the functions Z_2^\pm describe pure electromagnetic perturbations, and the coupling term in (2) vanishes (see, e.g., Ref. [43]).

Despite the complicated form of Eq. (2), the asymptotic behavior is simple. The asymptotic solutions at the horizon and at infinity are plane waves, namely,

$$Z_j^\pm(r) \sim \begin{cases} e^{i\omega r_*}, & r \rightarrow \infty, \\ e^{-i(\omega - m\Omega_H)r_*}, & r \rightarrow r_+. \end{cases} \quad (26)$$

As shown by the behavior above, near the BH horizon we obtain the typical frame-dragging effect: a static observer at infinity would see a wave whose wave number is

$$k_H = \omega - m\Omega_H \sim \omega - \frac{\tilde{a}m}{M(1 + \tilde{a}_{\max}^2)} + \mathcal{O}(\tilde{a}^3), \quad (27)$$

where we have assumed $\omega M \ll \tilde{a}$, and

$$\tilde{a}_{\max} \equiv J_{\max}/M^2 = \sqrt{1 - (Q/M)^2} \quad (28)$$

is the maximum spin parameter of a KN BH and $\Omega_H = -\lim_{r \rightarrow r_+} g_{t\varphi}/g_{\varphi\varphi}$ is the angular velocity at the horizon of locally nonrotating observers to first order in \tilde{a} . In the small-charge limit, $k_H \sim \omega - \tilde{a}m/(4M) - \tilde{a}mQ^2/(8M^3) + \mathcal{O}[(Q/M)^4]$.

According to Eq. (26), if $\omega < m\Omega_H$, an observer at infinity would see waves coming out of the horizon. This corresponds to extraction of energy from a spinning BH, resulting in superradiant amplification of the wave [25]. Therefore it is not necessary to solve the linearized field

equations in order to show that superradiance in a KN spacetime occurs also for gravito-electromagnetic perturbations, similarly to the scalar case. A note of caution is necessary in this regard: as discussed in Ref. [51], we must include *second-order* terms in the expansion for a consistent treatment of superradiance in the slow-rotation approximation. The reason is that the superradiance condition implies $\omega < \Omega_H \sim \mathcal{O}(\tilde{a})$; therefore, the ω^2 term in the wave equation becomes of the same order as $\mathcal{O}(\tilde{a}^2)$ terms. However the results of Ref. [51] show that (at least in some specific cases) first- and second-order results are in qualitative (and sometimes in remarkably good quantitative) agreement even in the superradiant regime.

III. QUASINORMAL MODE CALCULATION FOR COUPLED SYSTEMS

After imposing physically motivated boundary conditions at the horizon and at infinity, Eq. (2) forms an eigenvalue problem for the frequency ω . Robust numerical methods to solve this class of coupled eigenvalue problems have recently been extended to BH spacetimes (cf. [57] for a review).

We have solved the field equations (2) by two independent techniques: a matrix-valued continued-fraction method and direct integration [51,61].

A. Matrix-valued continued fractions

In order to reduce Eq. (2) to a matrix-valued recurrence relation, we use the ansatz

$$Z_j^\pm(r) = \frac{r_+(r_+ - r_-)^{-4iM\omega - 1}}{r} e^{-2i\omega r_+ (r - r_-)^{1+2iM\omega}} \times e^{i\omega r} z^{\frac{-ik_H r_*^2}{r_+ - r_-}} \sum_n a_n^\pm j z^n, \quad (29)$$

where $z = (r - r_+)/(r - r_-)$ and the \pm superscripts (not to be confused with the subscripts of the inner and outer horizon, r_\pm) refer to the polar case [“plus” superscript in Eq. (2)] and to the axial case [“minus” superscript in Eq. (2)], respectively. In the axial case we obtain a ten-term matrix-valued recurrence relation, whereas in the polar case we obtain a 12-term matrix-valued recurrence relation. They can be reduced to three-term recurrence relations using a matrix analog of Gaussian elimination [57]. Finally, the problem of finding the QNMs of the system is reduced to finding the complex roots of a determinant obtained from a nested series of inverted matrices [57].

B. Direct integration

In alternative, a direct integration technique can be shown to work very well, as long as the modes’ real part is sufficiently larger than the imaginary part. This condition is typically satisfied by the fundamental mode and (to a lesser extent) by the first overtone. In this direct

integration approach, the system (2) is integrated numerically from the horizon out to infinity. At the horizon we set the solution equal to some high-order series expansion of the form

$$Z_j^\pm \sim e^{-ik_H r_*} \sum_n b_n^{\pm,j} (r - r_+)^n, \quad r \rightarrow r_+, \quad (30)$$

where the coefficients $b_n^{\pm,j}$ ($n > 0$, $j = 1, 2$) can be computed in terms of $b_0^{\pm,i}$ by solving the near-horizon equations order by order. Two independent solutions are obtained by choosing the orthonormal bases $(b_0^{\pm,1}, b_0^{\pm,2}) = (1, 0)$ and $(b_0^{\pm,1}, b_0^{\pm,2}) = (0, 1)$. At infinity the generic behavior of the solutions reads

$$Z_j^\pm \sim B_j^\pm e^{i\omega r_*} \sum_n \frac{c_n^{\pm,j}}{r^n} + C_j^\pm e^{-i\omega r_*} \sum_n \frac{d_n^{\pm,j}}{r^n}, \quad r \rightarrow \infty. \quad (31)$$

The QNM boundary conditions correspond to imposing $C_j^\pm = 0$, i.e., purely outgoing waves at infinity. The eigenfrequencies are computed as the complex roots of (see, e.g., [55])

$$\det(\mathbf{S}^\pm) \equiv \det \begin{pmatrix} C_1^{\pm,1} & C_1^{\pm,2} \\ C_2^{\pm,1} & C_2^{\pm,2} \end{pmatrix}, \quad (32)$$

where the superscripts denote a particular vector of the basis; i.e., $C_j^{\pm,1}$ are related to $(b_0^{\pm,1}, b_0^{\pm,2}) = (1, 0)$ and $C_j^{\pm,2}$ are related to $(a_0^{\pm,1}, a_0^{\pm,2}) = (0, 1)$. For any given frequency we integrate the system (2) twice and construct the matrix \mathbf{S}^\pm . The QNMs are then obtained by imposing $\det(\mathbf{S}^\pm) = 0$, which can be achieved via a shooting method [57].

IV. NUMERICAL RESULTS

A. Scalar quasinormal modes in the slow-rotation limit

As a test of the slow-rotation approximation, we have computed the scalar QNMs of a KN BH to first order in \tilde{a} . Since these modes can be computed exactly in the Teukolsky formalism (e.g., via continued fractions [19]), we can use these ‘‘exact’’ results to estimate the errors introduced by the slow-rotation approximation. For any stationary and axisymmetric spacetime, the scalar modes at first order in the angular momentum are governed by a master equation [51]. In the special case of a background given by Eq. (7), the master equation reduces to

$$\frac{d^2 \psi}{dr_*^2} + \left[\omega^2 - \frac{2m\omega\varpi(r)}{r^2} - F \left(\frac{\ell(\ell+1)}{r^2} + \frac{F'}{r} \right) \right] \psi = 0. \quad (33)$$

The corresponding eigenvalue problem can be solved with standard continued-fraction techniques [19,60,63]. Using the ansatz (29) for ψ yields a five-term recurrence relation, whose coefficients are listed in Appendix B. The five-term recurrence relation can be reduced to a three-term relation via standard Gaussian elimination.

In Ref. [49] we computed the relative error of the slow-rotation approximation with respect to the exact result. We found a near-universal behavior of the percentage errors as functions of $J/J_{\max} = \tilde{a}/\tilde{a}_{\max}$, where \tilde{a}_{\max} is the maximum allowed value of the spin parameter [corresponding to an extremal KN BH with the given charge Q ; cf. Eq. (28)] for all values of Q . This near universality suggests that the parameter \tilde{a}_{\max} [which appears explicitly in the QNM boundary conditions (27)] plays a fundamental role in our perturbative scheme (see also the discussion in Sec. IV B). In other words, the slow-rotation approximation is accurate *only far from extremality*, i.e., when

$$\tilde{a} \ll \tilde{a}_{\max} = \sqrt{1 - \left(\frac{Q}{M}\right)^2} < 1. \quad (34)$$

This condition is obviously stronger than the condition one may expect to apply *a priori*, i.e., $\tilde{a} \ll 1$, and it implies that the slow-rotation approximation will break down even for small values of the spin as $Q \rightarrow M$. *A posteriori*, this result is not too surprising, and it is consistent with the requirement that the perturbation parameter \tilde{a} must be smaller than *any dimensionless quantity* characterizing the background spacetime.

According to Ref. [49], the slow-rotation approximation is accurate within 1% as long as $J/J_{\max} \lesssim 0.3$, and it is still accurate within 3% for $J/J_{\max} \lesssim 0.5$.

B. Gravito-electromagnetic quasinormal modes at first order in rotation

We have computed the fundamental mode and the first overtone of the gravito-electromagnetic QNMs of a KN BH in the axial and polar sectors for $\ell = 2$ and $\ell = 3$. The numerical solution of both the axial and polar perturbation equations (2) is extremely challenging, because their explicit form is lengthy and complicated.

We have computed the axial modes using both the matrix continued-fraction method and direct integration. The results agree extremely well, so that the two methods validate each other. In the polar case the equations are too complicated to be cast in a tractable continued-fraction form, even using algebraic manipulation software like MATHEMATICA. In this case the only viable technique turned out to be direct integration. Even the task of numerical integration is challenging due to the complexity of the field equations, but reasonably accurate results can be obtained by using high-order series expansions at the horizon and at infinity and by requiring better numerical precision in the integration. The difficulties we met in integrating the first-order equations suggest that a second-order analysis, while possible in principle, may be very challenging from the algebraic and computational standpoints.

For any value of Q , our analysis allows us to extract the first-order corrections to the QNM spectrum, as defined in Eqs. (4) and (5). At first order the m dependence can be

factored out, so the calculation is complete once we know the functions $\omega_R^{(1)}$ and $\omega_I^{(1)}$. Furthermore, we remark that first-order corrections vanish identically when $m = 0$.

We find four families of modes, two of which are associated to axial perturbations, while the other two are associated to polar perturbations. In the $Q = 0$ limit the two families with a given parity reduce to the gravitational and electromagnetic modes of a Kerr BH. Therefore, with a slight abuse of notation, we will dub the two families “gravitational” and “electromagnetic” also in the general case. It should be stressed that, when $Q \neq 0$, oscillations involving any of these modes excite both electromagnetic and gravitational perturbations.

We also mention that we have carried out a further check of our results. We have extracted $\mathcal{O}(\tilde{a})$ corrections of the QNMs from a fit of the “full” numerical solution of the Teukolsky equations in the Kerr background, and we have verified that this procedure matches our results for $Q = 0$ within a relative error $\lesssim 0.1\%$ for $\omega_R^{(1)}(Q = 0)$. The error is only a few times larger for $\omega_I^{(1)}(Q = 0)$.

1. Isospectrality

For generic values of ℓ , gravito-electromagnetic perturbations of nonspinning BHs in general relativity are isospectral [17,43]. The left panels of Fig. 1 are fully consistent with isospectrality within our numerical accuracy, and this is a nontrivial consistency check of our method.¹

A priori, there is no reason to expect that such a remarkable property should hold true also for KN BHs. Isospectrality is easily broken: for example it is well known that polar and axial modes are *not* isospectral (even for nonrotating BHs) if the cosmological constant is nonzero [64–66], if the underlying theory is not general relativity [67,68], or in higher dimensions [17].

Our numerics provide strong evidence that the gravito-electromagnetic modes of KN BHs are isospectral to first order in the angular momentum. This was shown in Fig. 1 for the fundamental mode with $\ell = 2$ and $\ell = 3$. In the insets of Fig. 1 we show the relative difference between the coefficients of axial and polar modes as functions of Q : our results are consistent with isospectrality within the numerical accuracy of the direct integration method.

We believe that deviations from isospectrality are of a purely numerical nature, being almost entirely due to the intrinsic errors of the direct integration to compute QNMs. This issue was discussed in detail in Ref. [49]. Here we simply add the observation that, as shown in Fig. 1, the fundamental mode with $\ell = 3$ (for which the direct integration is more accurate) is numerically closer to

isospectrality than the fundamental mode with $\ell = 2$. We have checked that deviations from isospectrality decrease as ℓ grows.

2. Fitting formulas at first order in rotation

Polar and axial modes are the same to first order within our numerical accuracy, but the equations for axial modes, which are listed explicitly in Eq. (2) and Appendix A 1 of this paper, are much simpler. We carried out a more extensive QNM calculation working in the axial case, where our results can be verified using two independent methods. Because of isospectrality, these results cover the whole QNM spectrum of slowly rotating KN BHs.

We found that the zeroth- and first-order terms in Eqs. (4) and (5) shown in Fig. 1 are well fitted by functions of the form

$$M\omega_{R,I}^{(0,1)} = f_0 + f_1y + f_2y^2 + f_3y^3 + f_4y^4, \quad (35)$$

where we have defined a parameter $y = 1 - \tilde{a}_{\max} = 1 - \sqrt{1 - Q^2/M^2}$ which is in one-to-one correspondence with Q (such that $y \in [0, 1]$ as $Q \in [0, M]$) but is better suited for fitting. As discussed in Sec. IVA, \tilde{a}_{\max} seems to be the most appropriate dimensionless quantity to normalize our perturbative parameter; in some sense, the parameter y measures the “distance from extremality” of the KN metric. The ℓ - and n -dependent fitting coefficients f_i of the functions $\omega_R^{(0,1)}$ and $\omega_I^{(0,1)}$ for a selected subset of gravitational and electromagnetic modes are listed in Table I, which extends a similar table in [49].

Our QNM calculations in the slow-rotation approximation can be seen as an empirical confirmation of the stability of the KN metric. We have looked for unstable modes in the region $0 < Q < M$, $J \ll J_{\max}$ and for $\ell = 2, 3, 4$ and we found none. This confirms early arguments by Mashhoon, who used calculations in the eikonal limit to make a case for the stability of the KN metric [44]. Notice however that Mashhoon’s results apply only to perturbation modes with $\ell \gg 1$ and they rely on a geodesic analogy, rather than on a self-consistent treatment of the perturbation equations. In this sense, our findings provide the first self-consistent stability analysis of the KN metric.

C. Comparison with the quasinormal modes of the Dudley-Finley equation

Finally, we can compare our results against the DF equation [40,41] to quantify the regime of validity of both approximation schemes. The results of this comparison for the fundamental gravitational mode with $\ell = m = 2$ are shown in Fig. 2.

The DF equation reduces to the Teukolsky equation for Kerr BHs in the limit $Q \rightarrow 0$. However, it does *not* reduce to the RN Regge-Wheeler/Zerilli equations when $\tilde{a} \rightarrow 0$ [19]. On the other hand, our slow-rotation approximation is valid for any Q , but we must impose the condition

¹To the best of our knowledge, in the published literature there are no studies checking isospectrality for RN BHs by an explicit calculation of polar QNMs. This is an interesting by-product of our analysis.

TABLE I. Coefficients of the fit (35) for the real and imaginary parts of a selected subset of gravito-electromagnetic modes. The values (ℓ, n) correspond to the multipolar index and the overtone number, respectively. Fundamental modes correspond to $n = 0$. We denote by $s = 1$ and $s = 2$ the modes that in the decoupled $Q \rightarrow 0$ limit are electromagnetic and gravitational in the Kerr background, respectively. The fits (35) reproduce the data to within 1% for $\omega_I^{(1)}$ and to within 0.1% for the other quantities for any $Q \leq 0.95M$.

	(ℓ, n, s)	f_0	f_1	f_2	f_3	f_4
$\omega_R^{(0)}$	(2,0,1)	0.4576	0.2659	0.0118	0.1228	-0.1382
$\omega_R^{(1)}$	(2,0,1)	0.0712	0.0769	0.0596	0.0727	-0.0216
$\omega_I^{(0)}$	(2,0,1)	-0.0950	-0.0184	0.0137	0.0132	0.0107
$\omega_I^{(1)}$	(2,0,1)	0.0007	0.0043	0.0060	-0.0089	0.0366
$\omega_R^{(0)}$	(2,0,2)	0.3737	0.0525	0.0607	-0.0463	-0.0070
$\omega_R^{(1)}$	(2,0,2)	0.0628	0.0676	0.0209	0.0823	-0.0810
$\omega_I^{(0)}$	(2,0,2)	-0.0890	-0.0055	0.0024	0.0214	-0.0084
$\omega_I^{(1)}$	(2,0,2)	0.0010	0.0014	0.0091	0.0174	0.0145
$\omega_R^{(0)}$	(2,1,1)	0.4365	0.2793	0.0125	0.1399	-0.1637
$\omega_R^{(1)}$	(2,1,1)	0.0780	0.0785	0.0588	0.0776	-0.0277
$\omega_I^{(0)}$	(2,1,1)	-0.2907	-0.0515	0.0438	0.0364	0.0363
$\omega_I^{(1)}$	(2,1,1)	0.0043	0.0138	0.0164	-0.0230	0.1062
$\omega_R^{(0)}$	(2,1,2)	0.3467	0.0546	0.0709	-0.0292	-0.0433
$\omega_R^{(1)}$	(2,1,2)	0.0717	0.0764	0.0020	0.1959	-0.2213
$\omega_I^{(0)}$	(2,1,2)	-0.2739	-0.0157	0.0099	0.0668	-0.0239
$\omega_I^{(1)}$	(2,1,2)	0.0065	0.0070	0.0360	0.0254	0.0905
$\omega_R^{(0)}$	(3,0,1)	0.6569	0.3684	-0.0820	0.2851	-0.2574
$\omega_R^{(1)}$	(3,0,1)	0.0726	0.0768	0.0595	0.0617	-0.0259
$\omega_I^{(0)}$	(3,0,1)	-0.0956	-0.0177	0.0178	0.0074	0.0106
$\omega_I^{(1)}$	(3,0,1)	0.0002	0.0032	0.0027	-0.0002	0.0216
$\omega_R^{(0)}$	(3,0,2)	0.5994	0.0790	0.1734	-0.2019	0.0700
$\omega_R^{(1)}$	(3,0,2)	0.0673	0.0693	0.0211	0.0791	-0.0677
$\omega_I^{(0)}$	(3,0,2)	-0.0927	-0.0043	-0.0013	0.0292	-0.0130
$\omega_I^{(1)}$	(3,0,2)	0.0006	0.0014	0.0084	0.0058	0.0122
$\omega_R^{(0)}$	(3,1,1)	0.6418	0.3782	-0.0863	0.3104	-0.2875
$\omega_R^{(1)}$	(3,1,1)	0.0760	0.0786	0.0511	0.0924	-0.0576
$\omega_I^{(0)}$	(3,1,1)	-0.2897	-0.0509	0.0508	0.0333	0.0222
$\omega_I^{(1)}$	(3,1,1)	0.0015	0.0098	0.0096	-0.0050	0.0693
$\omega_R^{(0)}$	(3,1,2)	0.5826	0.0819	0.1752	-0.1753	0.0304
$\omega_R^{(1)}$	(3,1,2)	0.0713	0.0736	0.0108	0.1287	-0.1282
$\omega_I^{(0)}$	(3,1,2)	-0.2813	-0.0123	-0.0050	0.0972	-0.0472
$\omega_I^{(1)}$	(3,1,2)	0.0029	0.0059	0.0194	0.0374	0.0231

$J \ll J_{\max}$. In the region $J \ll J_{\max}$, the slow-rotation approximation can be used to quantify the errors introduced by the DF equation for any value of Q .

Figure 2 shows that the deviations between the DF and slow-rotation calculations vanish when $Q \ll M$ and $J \ll J_{\max}$, i.e., in the region where the assumptions underlying both approximations are consistent. As we increase Q we observe an increasing deviation of the DF modes with

respect to the slow-rotation calculation. The offset increases with Q , and it is nearly constant for any Q in the region $J \ll J_{\max}$: the errors introduced by the DF approximation in the nonrotating case do not increase much when (a small amount of) rotation is included. The deviations do increase for larger values of J , but in that regime the slow-rotation approximation is not reliable anymore.

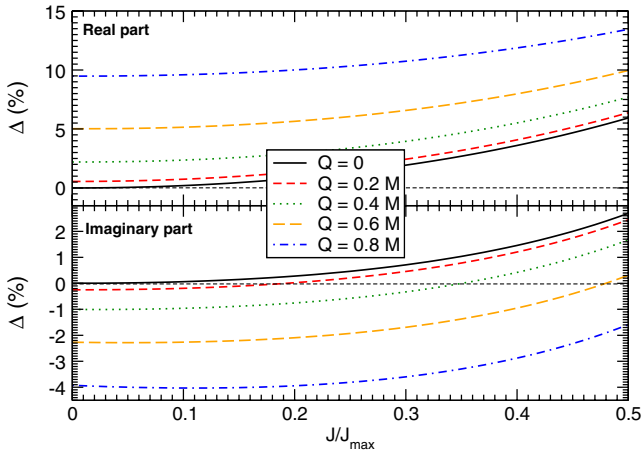


FIG. 2 (color online). Percentage deviation of QNM frequencies in the slow-rotation approximation with respect to the DF equation, for the $\ell = m = 2$ fundamental gravitational mode. As expected, the two approximations agree with each other when $Q \rightarrow 0$, but they deviate from each other when $\tilde{a} \rightarrow 0$. The discrepancy between the two approximations is nearly constant as long as $J \ll J_{\max}$, confirming that the DF equation is not very accurate in that regime when $Q \neq 0$ (cf. [19]).

V. CONCLUSIONS

In this paper and the accompanying Letter [49] we have presented the first fully consistent analysis of the gravito-electromagnetic QNMs of the KN metric. Working in a slow-rotation approximation, the long-standing problem of nonseparability of the perturbation equations can be evaded. We have computed the gravito-electromagnetic QNMs to first order in the BH spin and provided fitting formulas for the fundamental mode and first overtone with $\ell = 2, 3$.

Furthermore, our numerical study of gravito-electromagnetic perturbations shows strong numerical evidence for the isospectrality of polar and axial gravito-electromagnetic perturbations of KN black holes at linear order in rotation. It would be interesting to understand whether isospectrality holds exactly, at all orders in rotation. An important extension of our work in this direction is to include second-order effects. The causal structure of a spinning metric starts differing from the nonspinning case at second order in the angular momentum (e.g., changes in the horizon location and in the ergoregion are of second order); thus, if isospectrality holds true also at second order, there is no fundamental reason to believe that it is broken at higher order.

If isospectrality is an exact property of KN BHs, extensions of the polar and axial field equations (2) to *any* order in the spin should be related to each other by some transformation that leaves the QNM spectrum invariant. Even at linear order in rotation isospectrality is a highly nontrivial property, in view of the mixing of gravitational and electromagnetic perturbations. Hopefully our work will stimulate further study to prove (or disprove) the conjecture that

isospectrality is an exact property of the KN spacetime. This conjecture may be verified using a brute-force extension of our work to higher orders in rotation, numerical time evolutions along the lines of [69–71], or (ideally) an analytical proof, perhaps similar to Chandrasekhar’s proof in the nonrotating case (see also Ref. [72]).

In Ref. [49] we have presented some possible extensions of our work. If isospectrality turns out to be valid for any value of the angular momentum, an interesting avenue of research is to understand whether such property has implications in the context of the KN/CFT conjecture [23,24], which predicts that the QNMs of the near-horizon KN geometry correspond to the poles of the retarded Green’s function of the dual chiral CFT [73].

Some further interesting applications concern nonasymptotically flat spacetimes. Even for nonrotating RN (anti-)de Sitter BHs isospectrality is known to be partially broken, depending on the relative size of the BH and (anti-)de Sitter horizon radii [64–66,74]. The slow-rotation approximation may be used to understand whether similar considerations also apply to Kerr-Newman (anti-)de Sitter. Indeed, our approach can be easily extended to include a nonvanishing cosmological constant. In the context of the AdS/CFT correspondence [75], the QNMs of a KN BH are dual to thermal states of a CFT living in a rotating Einstein universe [76,77].

Finally, certain KN-AdS BHs embedded in $N = 2$ four-dimensional supergravity preserve half of the supersymmetry [78,79]. This is analogous to the case of asymptotically flat, extremal RN BHs, for which supersymmetry implies a remarkable property of the QNMs: electromagnetic perturbations with multipolar index ℓ are isospectral with gravitational perturbations with index $\ell + 1$ [80]. Using this property, it is possible to prove that the one-loop corrections to the BH entropy cancel [81]. It would be interesting to understand whether such supersymmetry implies a similar property for the supersymmetric KN-AdS BH solutions found in Refs. [78,79]. We note that such solutions can be slowly rotating, so our framework can be directly applied to this interesting problem.

We hope that our paper will stimulate further work in these, and possibly other, directions.

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APPENDIX A: DERIVATION OF THE PERTURBATION EQUATIONS

In this Appendix we derive the first-order equations (2) for the axial and polar gravito-electromagnetic perturbations of a KN BH. Various intermediate steps are presented in a supplementary MATHEMATICA notebook [56].

1. Axial sector

As discussed in the main text and in Ref. [51], the coupling between perturbations with different parity and different harmonic index ℓ does not contribute to the QNM to first order in \tilde{a} . Neglecting the couplings to $\ell \pm 1$ terms, the axial sector is fully described by four equations:

$$0 = \Lambda\beta_\ell^{(L)} + im[(\ell - 1)(\ell + 2)\chi_\ell^{(L)} + \tilde{\alpha}_\ell^{(L)} + \eta_\ell^{(L)}], \quad (\text{A1})$$

$$0 = \Lambda t_\ell + img_\ell, \quad (\text{A2})$$

where $L = 0, 1, 2$. The choice $L = 2$ refers to one of the Maxwell equations, and $\{\beta_\ell^{(L)}, \chi_\ell^{(L)}, \tilde{\alpha}_\ell^{(L)}, \eta_\ell^{(L)}, t_\ell, g_\ell\}$ are combinations of the perturbation functions $\{h_{0,1}^\ell, u_{(4)}^\ell\}$, whose explicit form can be found online [56]. Actually, Einstein's equations imply Maxwell's equations, and only three out of the four equations above are independent. This can be easily verified as a consistency check of our approach. The three independent equations can be solved for the functions h_0^ℓ, h_1^ℓ and $u_{(4)}^\ell$. We define the Regge-Wheeler function Ψ^ℓ as

$$\Psi^\ell = \frac{Fh_1^\ell}{r}. \quad (\text{A3})$$

Then, from Eq. (A1) with $L = 1$, we get

$$h_0^{\ell'} = \frac{2F(\Lambda r\omega h_0^\ell - 2Q\omega u_{(4)}^\ell) + i\Lambda r(F(\Lambda - 2) - r^2\omega^2)\Psi^\ell}{F\Lambda r^2\omega} - \frac{im\tilde{a}M^2}{F\Lambda^2Mr^4\omega^2}[12iF\Lambda M\omega h_0^\ell + 4iF^2\Lambda Q\omega u_{(4)}^\ell + (F - 1)\Lambda^2r(F(\Lambda - 2) + r^2\omega^2)\Psi^\ell + 8iF^2Qr\omega u_{(4)}^{\ell'}].$$

Replacing this equation in the remaining Eq. (A1) with $L = 0$ and $L = 2$ we get a system for $u_{(4)}^\ell$ and Ψ^ℓ only. At first order in \tilde{a} , the system contains third derivatives of $u_{(4)}^\ell$. Within our perturbative scheme, the latter can be eliminated by using the zeroth-order perturbation equations. The result is a system of coupled, second-order equations for $u_{(4)}^\ell$ and Ψ^ℓ . In order to decouple the system at zeroth order, we define a linear combination of two new functions $Z_i^{(-)}$ ($i = 1, 2$), such that

$$u_{(4)}^\ell = \alpha_{11}Z_1^{(-)} + \alpha_{12}Z_2^{(-)}, \quad (\text{A4})$$

$$\Psi^\ell = \alpha_{21}Z_1^{(-)} + \alpha_{22}Z_2^{(-)}, \quad (\text{A5})$$

where $\alpha_{21} = 4i\alpha_{11}\omega Q/(\Lambda q_1)$, $\alpha_{22} = 4i\alpha_{12}\omega Q/(\Lambda q_2)$, and we have defined

$$q_{1,2} = 3M \pm \sqrt{9M^2 + 4(\Lambda - 2)Q^2}, \quad (\text{A6})$$

such that $q_1q_2 = -4Q^2(\Lambda - 2)$ and $q_1 + q_2 = 6M$. The constants α_{11} and α_{12} can be set equal to unity without loss of generality.

Replacing the linear combinations above into the equations for $u_{(4)}^\ell$ and Ψ^ℓ and solving for $Z_1^{(-)}$ and $Z_2^{(-)}$, we obtain Eq. (2) with a “minus” superscript.

The potentials appearing in the axial sector of the perturbation equations (2) read

$$V_0^{(i,-)} = \frac{F}{r^3} \left[-q_j + \frac{4Q^2}{r} \right], \quad (\text{A7})$$

$$\begin{aligned} V_1^{(i,-)} = & -\frac{M}{\Lambda(q_i - q_j)r^{11}\omega} [4Q^6(10q_iq_j - 12Mr + (-18q_j + \Lambda(q_i + 8q_j))r + \lambda r^2) \\ & - 2Mq_jr^3(r - 2M)(r((18 + \Lambda)q_i + 12(\Lambda - 3)r) - 6M(7q_i + (-14 + 5\Lambda)r)) \\ & + r^7(2Mq_iq_j - 4\Lambda M(q_i - q_j)r - q_iq_jr)\omega^2 + Q^4r(192M^2r - 4M(50q_iq_j + (-92q_j + \Lambda(4q_i + 39q_j))r \\ & + 4(6 + \lambda)r^2) + r(72 + \Lambda)q_iq_j + 8\Lambda q_i r + 4r(2(-16 + 7\Lambda)q_j + \lambda r + (6 + \Lambda)r^3\omega^2))] \\ & + Q^2r^2(-192M^3r + 4M^2(81q_iq_j + (-154q_j + \Lambda(4q_i + 61q_j))r + 4(12 + \lambda)r^2) + r^2(8(3\Lambda - 7)q_jr \\ & + 2r^3(-\Lambda q_j + 2(6 + \Lambda)r)\omega^2 + q_i(q_j(32 + \Lambda - r^2\omega^2) + 2\Lambda r(2 + r^2\omega^2))) - 4Mr((55 + \Lambda)q_iq_j \\ & + 4\Lambda q_i r + r((-102 + 41\Lambda)q_j + 2r(6 + 6r^2\omega^2 + \Lambda(-2 + \Lambda + r^2\omega^2))))], \end{aligned} \quad (\text{A8})$$

$$V_2^{(i,-)} = \frac{2MF^2}{\Lambda(q_i - q_j)r^6\omega} (Q^2(q_i(5q_j - 4r) + 4(6M + (\Lambda - 1)q_j)r) + q_j r(3q_i r - 2M(4q_i + 3\Lambda r))), \quad (\text{A9})$$

$$W_1^{(i,-)} = \frac{4(\Lambda - 2)MF}{\Lambda(q_i - q_j)q_j^2 r^9 \omega} [4Q^4(10q_j^2 + 9(\Lambda - 2)q_j r + r(\lambda r - 12M)) + q_j r^2(2M(r(-(18 + \Lambda)q_j - 12(\Lambda - 3)r) + 6M(7q_j + (-14 + 5\Lambda)r)) - q_j r^4 \omega^2) + Q^2 r(96M^2 r + q_j r((32 + \Lambda)q_j + 28(\Lambda - 2)r) + 4M(-30q_j^2 + (56 - 25\Lambda)q_j r - 2(6 + \lambda)r^2) + 4(6 + \Lambda)r^5 \omega^2)], \quad (\text{A10})$$

$$W_2^{(i,-)} = \frac{8(\Lambda - 2)MF^2}{\Lambda(q_i - q_j)q_j^2 r^6 \omega} (q_j r(8Mq_j + 6\Lambda M r - 3q_j r) + Q^2(-5q_j^2 - 4(6M + (\Lambda - 2)q_j)r)), \quad (\text{A11})$$

where $q_{i,j}$ ($i, j = 1, 2$ and $i \neq j$) are defined as in Eq. (A6), and $\lambda \equiv \Lambda(\ell + 2)(\ell - 1)$.

2. Polar sector

The equations governing the polar sector are more involved. They can be obtained by following Zerilli's original derivation of the gravitational and electromagnetic perturbations of a RN BH [59], extended to first order in the BH spin.

Polar perturbations are fully described by the equations

$$0 = A_\ell^{(I)} + imC_\ell^{(I)}, \quad (\text{A12})$$

$$0 = \Lambda\alpha_\ell^{(L)} + im[(\ell - 1)(\ell + 2)\xi_\ell^{(L)} - \tilde{\beta}_\ell^{(L)} - \zeta_\ell^{(L)}], \quad (\text{A13})$$

$$0 = \Lambda s_\ell - imf_\ell, \quad (\text{A14})$$

where $I = 0, \dots, 5$ and $L = 0, 1, 2$, and $\{A_\ell^{(I)}, C_\ell^{(I)}, \alpha_\ell^{(L)}, \xi_\ell^{(L)}, \tilde{\beta}_\ell^{(L)}, \zeta_\ell^{(L)}, s_\ell, f_\ell\}$ are combinations of the perturbation functions $\{H_{0,1,2}^\ell, K^\ell, u_{(1,2,3)}^\ell\}$, whose explicit form can be found online [56]. Actually, only seven out of the ten equations above are independent and they can be solved for the seven polar functions: $H_0^\ell, H_1^\ell, H_2^\ell, K^\ell, u_{(i)}^\ell$ ($i = 1, 2, 3$).

The key point of Zerilli's calculation is to use perturbations of the field strength $F_{\mu\nu}$, rather than the electromagnetic potential A_μ , as dynamical variables. Following [59], we define

$$\delta F_{\mu\nu} \equiv f_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu. \quad (\text{A15})$$

In the polar sector, we fix the gauge by requiring $u_{(3)}^\ell = 0$; the remaining components are related to $f_{\mu\nu}$ in the following way:

$$u_{(1)}^\ell = r\tilde{f}_{02}, \quad (\text{A16})$$

$$u_{(2)}^\ell = F(r)\tilde{f}_{12}, \quad (\text{A17})$$

$$u_{(1)}^{\ell'} = r\tilde{f}_{01} + \tilde{f}_{02} - ir\omega\tilde{f}_{02}, \quad (\text{A18})$$

where $-\tilde{f}_{\mu\nu}$ denotes the angle-independent part of $f_{\mu\nu}$ (note that our definition differs from Zerilli's definition by a minus sign). The equation

$$\tilde{f}_{01} = \tilde{f}'_{02} + i\omega\tilde{f}_{12} \quad (\text{A19})$$

is automatically satisfied due to Eq. (A15).

First, we solve Eq. (A14) for H_2^ℓ and substitute the result into the remaining equations. Then, Eq. (A12) with $I = 5$ and Eq. (A13) with $L = 2$ can be solved for \tilde{f}_{01} and \tilde{f}_{02} . The solutions can be replaced into Eq. (A19), which takes the form of a second-order differential equation for \tilde{f}_{12} with source terms linear in the polar gravitational function $H_0^\ell, H_1^\ell, K^\ell$ and in their first derivatives.

The equations for the gravitational sector can be obtained by solving Eq. (A12) with $I = 1$ and Eq. (A13) with $L = 0, 1$ for $H_0^{\ell'}, H_1^{\ell'}$ and $K^{\ell'}$. Similarly to the axial sector, second derivatives of these perturbation functions (which appear at first order in \bar{a}) can be eliminated using the zeroth-order equations. By substituting the solution into Eq. (A12) with $I = 2$, one can solve for the function H_0^ℓ and eliminate it from the remaining equations. As a result of this procedure we obtain a system of coupled equations

$$\mathbf{Y}' + \mathbf{U}\mathbf{Y} = 0, \quad (\text{A20})$$

where $\mathbf{Y} = (H_1^\ell, K^\ell, \tilde{f}_{12}, \tilde{f}'_{12})$ and \mathbf{U} is a matrix. The system above collectively denotes two first-order equations for the gravitational perturbations H_0^ℓ and K^ℓ and the second-order equation for \tilde{f}_{12} , which has been separated into two first-order equations for \tilde{f}_{12} and \tilde{f}'_{12} .

As shown by Zerilli [59], at zeroth order in the BH spin the two equations for the gravitational perturbations H_0^ℓ and K^ℓ can be reduced to a single, second-order equation. We describe the procedure here, extending it to first order in the BH spin. The perturbations H_0^ℓ and K^ℓ satisfy the schematic matrix-valued equation

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{S}, \quad (\text{A21})$$

where \mathbf{A} is a matrix, $\mathbf{y} = (K^\ell, H_1^\ell/\omega)$ and $\mathbf{S} = (S_1, S_2)$ is a source term, which depends on \tilde{f}_{12} and its derivatives only.

The idea is to find a transformation $\mathbf{y} = \mathbf{F}\hat{\mathbf{y}}$ to a new pair of functions $\hat{\mathbf{y}} = (\hat{\Psi}_1, \hat{\Psi}_2)$ such that

$$\frac{d\hat{\Psi}_1}{d\hat{r}} = G_1\hat{\Psi}_1 + (1+W)\hat{\Psi}_2 + \hat{S}_1, \quad (\text{A22})$$

$$\frac{d\hat{\Psi}_2}{d\hat{r}} = -(\omega^2 - V - V_1)\hat{\Psi}_1 + G_2\hat{\Psi}_2 + \hat{S}_2, \quad (\text{A23})$$

where \hat{r} is a new variable defined by $dr/d\hat{r} = n(r)$. Indeed, assuming the relations above, we can solve Eq. (A22) for $\hat{\Psi}_2$ and substitute it into Eq. (A23). We obtain a single, second-order equation for $\hat{\Psi}_1$:

$$\frac{d^2\hat{\Psi}_1}{d\hat{r}^2} + \hat{U}\frac{d\hat{\Psi}_1}{d\hat{r}} + \hat{V}\hat{\Psi}_1 = \hat{S}, \quad (\text{A24})$$

where, to first order,

$$\hat{U} = -\frac{G_1 + G_2 + FW'}{F'F}, \quad (\text{A25})$$

$$\hat{V} = (\omega^2 - V)(1+W) - V_1 - FG'_1, \quad (\text{A26})$$

$$\hat{S} = (1+W)\hat{S}_2 + F\hat{S}'_1 - \hat{S}_1(G_2 + FW'). \quad (\text{A27})$$

Note that G_i , V_1 and W are first-order quantities in rotation, which are absent in the nonrotating case discussed in Ref. [59].

Let us now find the explicit form of the transformation. Using Eq. (A21), we get

$$\frac{d\hat{\mathbf{y}}}{d\hat{r}} = n(r)\mathbf{F}^{-1}\left[\mathbf{A}\mathbf{F} - \frac{d\mathbf{F}}{dr}\right] + n(r)\mathbf{F}^{-1}\mathbf{S}. \quad (\text{A28})$$

Therefore, the transformation matrix \mathbf{F} and the remaining functions must satisfy

$$n(r)\mathbf{F}^{-1}\left[\mathbf{A}\mathbf{F} - \frac{d\mathbf{F}}{dr}\right] = \begin{pmatrix} G_1 & 1+W \\ -\omega^2 + V + V_1 & G_2 \end{pmatrix}. \quad (\text{A29})$$

and $\hat{\mathbf{S}} = n\mathbf{F}^{-1}\mathbf{S}$. The equation above can be solved perturbatively. At zeroth order it provides a system of four equations that can be uniquely solved for the elements of the matrix \mathbf{F} , for $n(r)$ and for the potential V . We get

$$\begin{aligned} \mathbf{F}_{11} &= [2r^3(r(6M + (\Lambda - 2)r) - 4Q^2)]^{-1} \\ &\quad \times [16Q^4 - 4Q^2r(11M + (\Lambda - 4)r) \\ &\quad + r^2(24M^2 + 6(\Lambda - 2)Mr + \lambda r^2)], \\ \mathbf{F}_{12} &= 1, \end{aligned}$$

$$\mathbf{F}_{21} = -i\left[1 + \frac{Q^2 - Mr}{r^2F} + \frac{8Q^2 - 6Mr}{r[6M + (\Lambda - 2)r] - 4Q^2}\right],$$

$$\mathbf{F}_{22} = -\frac{ir}{F},$$

and

$$\begin{aligned} V(r) &= \frac{F}{r^2}(r(6M + (\Lambda - 2)r) - 4Q^2)^{-2} \\ &\quad \times [-32Q^6 + 24Q^4r(6M + (\Lambda - 2)r) \\ &\quad + 8Q^2r^2(3(\Lambda - 2)r^2 - 27M^2 - 2(4\Lambda - 11)Mr) \\ &\quad + r^3(72M^3 + 36(\Lambda - 2)M^2r + 6(\Lambda - 2)^2Mr^2 \\ &\quad + \Lambda(\Lambda - 2)^2r^3)], \end{aligned} \quad (\text{A30})$$

$$n(r) = F(r). \quad (\text{A31})$$

Note that, by virtue of the field equations, the coordinate \hat{r} is the standard tortoise coordinate. The equations above correct some typos in Ref. [59]. At first order, we can solve the four equations (A29) for the variables G_1 , G_2 , W and V_1 . The lengthy form of these solutions is presented online [56].

With the transformation at hand, we can compute the explicit form of Eq. (A24). The source term \hat{S} explicitly depends on the second derivatives of \tilde{f}_{12} and, upon substitution of the perturbation equation for \tilde{f}_{12} , it depends on K^ℓ and H_1^ℓ . This dependence can be eliminated using the definitions

$$K^\ell = \mathbf{F}_{11}\hat{\Psi}_1 + \frac{\mathbf{F}_{12}}{1+W}[F\hat{\Psi}'_1 - G_1\hat{\Psi}_1 - \hat{S}_1],$$

$$H_1^\ell = \omega\mathbf{F}_{21}\hat{\Psi}_1 + \frac{\omega\mathbf{F}_{22}}{1+W}[F\hat{\Psi}'_1 - G_1\hat{\Psi}_1 - \hat{S}_1],$$

which allows us to write K^ℓ and H_1^ℓ in terms of $\hat{\Psi}_1$ and its first derivative only. When inserted into Eq. (A24), the equations above introduce extra coefficients in front of $d\hat{\Psi}_1/dr_*$ and $\hat{\Psi}_1$. The final result is an equation of the same form as Eq. (A24), with coefficients given in [56].

To summarize, we have obtained two second-order equations for the functions $\hat{\Psi}_1$ and \tilde{f}_{12} which describe gravitational and electromagnetic perturbations, respectively. At zeroth order in rotation, these equations can be decoupled [43,82] by introducing the functions $Z_i^{(\pm)}$ such that

$$\hat{\Psi}_1 = B_{11}Z_1^{(-)} + B_{12}Z_2^{(+)}, \quad (\text{A32})$$

$$\tilde{f}_{\text{EM}} = B_{21}Z_1^{(+)} + B_{22}Z_2^{(+)}, \quad (\text{A33})$$

where B_{ij} are functions of r and we defined $\tilde{f}_{\text{EM}} \equiv F\tilde{f}_{12}$. It is straightforward to verify that with the choice

$$\frac{B_{11}}{\alpha} + q_2 = \frac{B_{12}}{\beta} + q_1 = \frac{4Q^2}{r}, \quad (\text{A34})$$

$$\frac{B_{21}}{\alpha} = \frac{B_{22}}{\beta} = -\frac{8iQ}{\omega}, \quad (\text{A35})$$

the final set of equations takes the form of Eq. (2) with a ‘‘plus’’ superscript; i.e., the equations are decoupled at zeroth order (but coupled at first order) in \tilde{a} . In the

equations above, α and β are constants that can be set to unity without loss of generality. Our equations reduce to those obtained by Chandrasekhar [43] in the nonspinning case, as they should. Let us stress again that the derivation sketched in this Appendix and in [56] corrects some typos in Ref. [59].

The potentials appearing in the perturbation equations (2) of the polar sector are very lengthy [56], and their practical use may be limited. However, since—according to our numerical evidence—axial and polar modes are isospectral to linear order in \tilde{a} , the potentials describing axial perturbations (explicitly listed in Appendix A 1) are sufficient to compute the entire QNM spectrum of slowly rotating KN BHs.

APPENDIX B: OF THE RECURRENCE RELATION FOR SCALAR QNMS OF A KN BH

Using the same ansatz as in Eq. (29) and (33) reduces to a five-term recurrence relation

$$\begin{aligned}\alpha_0 a_1 + \beta_0 a_0 &= 0, & n=0, \\ \alpha_1 a_2 + \beta_1 a_1 + \gamma_1 a_0 &= 0, & n=1, \\ \alpha_2 a_3 + \beta_2 a_2 + \gamma_2 a_1 + \delta_2 a_0 &= 0, & n=2, \\ \alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} + \delta_n a_{n-2} + \rho_n a_{n-3} &= 0, & n>2,\end{aligned}$$

whose coefficients to first order in \tilde{a} , using Leaver's $2M = 1$ unit convention, read

$$\alpha_n = (1+n)\Delta_Q(1+\Delta_Q)^2(2(1+n)\Delta_Q - i(1+\Delta_Q)^2\omega) + 2im\tilde{a}(1+n)\Delta_Q(1+\Delta_Q)^2, \quad (\text{B1})$$

$$\begin{aligned}\beta_n &= \Delta_Q(1+\Delta_Q)(-2\Delta_Q(-1+4n^2+3\Delta_Q+\ell(1+\ell)(1+\Delta_Q)+n(-2+6\Delta_Q)) \\ &\quad + i(1+\Delta_Q)^2(-1+5\Delta_Q+4n(1+\Delta_Q))\omega + 2(1+\Delta_Q)^4\omega^2) \\ &\quad - 2m\tilde{a}\Delta_Q(1+\Delta_Q)(i(4n+3\Delta_Q-1)+(1+\Delta_Q)(2+\Delta_Q)\omega),\end{aligned} \quad (\text{B2})$$

$$\begin{aligned}\gamma_n &= \Delta_Q(2\Delta_Q(6n(\Delta_Q-1)(3+\Delta_Q)-2n^2(\Delta_Q^2-3)-(\Delta_Q-1)(11-7\Delta_Q+2\ell(1+\ell)(1+\Delta_Q))) \\ &\quad + i(1+\Delta_Q)^2(9+6\Delta_Q-15\Delta_Q^2+2n(-3+\Delta_Q(-6+5\Delta_Q)))\omega + 2(1+\Delta_Q)^4(-3+2\Delta_Q)\omega^2) \\ &\quad + 2m\tilde{a}\Delta_Q(3i(\Delta_Q-1)(3+\Delta_Q)-2in(\Delta_Q^2-3)-2(1+\Delta_Q)(\Delta_Q^2-3)\omega),\end{aligned} \quad (\text{B3})$$

$$\begin{aligned}\delta_n &= (\Delta_Q-1)\Delta_Q(2\Delta_Q(29+4n^2+\ell+\ell^2-(21+\ell+\ell^2)\Delta_Q+n(-22+6\Delta_Q)) \\ &\quad + i(1+\Delta_Q)(11+(38-17\Delta_Q)\Delta_Q+4n(-1+(-4+\Delta_Q)\Delta_Q))\omega \\ &\quad + 2(\Delta_Q-3)(1+\Delta_Q)^3\omega^2) + 2m\tilde{a}(\Delta_Q-1)\Delta_Q(-11i+4in+6\omega+\Delta_Q(3i-(\Delta_Q-3)\omega)),\end{aligned} \quad (\text{B4})$$

$$\rho_n = (\Delta_Q-1)^2\Delta_Q(-4+n-2i\omega)(2(-4+n)\Delta_Q - i(1+\Delta_Q)^2\omega) + 2im\tilde{a}(\Delta_Q-1)^2\Delta_Q(-4+n-2i\omega), \quad (\text{B5})$$

where $\Delta_Q = \sqrt{1-4Q^2}$.

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