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Dynamical black holes: Approach to the final state

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Since black holes can be formed through widely varying processes, the horizon structure is highly complicated in the dynamical phase. Nonetheless, as numerical simulations show, the final state appears to be universal, well described by the Kerr geometry. How are all these large and widely varying deviations from the Kerr horizon washed out? To investigate this issue, we introduce a well-suited notion of horizon multipole moments and equations governing their dynamics, thereby providing a coordinate- and slicing-independent framework to investigate the approach to equilibrium. In particular, our flux formulas for multipoles can be used as analytical checks on numerical simulations and, in turn, the simulations could be used to fathom possible universalities in the way black holes approach their final equilibrium.

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I. INTRODUCTION

Black hole uniqueness theorems [1] strongly suggest that late stages of a gravitational collapse or a black hole merger are well described by the Kerr solution. In particular, once the black hole reaches the final equilibrium, its horizon is expected to match the Kerr isolated horizon which can be characterized intrinsically, without reference to the exterior space-time [2]. By now a wide variety of numerical simulations have confirmed this expectation. However, these simulations also bring out the fact that there is great diversity in the structure of the horizon during the preceding dynamical phase. At its formation, the horizon of the final black hole generically exhibits large, time-dependent distortions. Heuristically, its intrinsic geometry appears to have many ‘‘bumps’’ and there is no simple relation between its rotational state and the spin vector of the final black hole. However, in the process of settling down to equilibrium, the Einstein dynamics manages to wash away these apparently large deviations leaving behind the Kerr isolated horizon. How does this come about? Can one provide a precise mathematical description of this approach to equilibrium? Does it carry a clear imprint of general relativity that could perhaps be seen in future gravitational wave observations? The final state is universal. Are there universalities associated also with the *approach* to this final state? Answers to these questions would both provide us a deeper conceptual understanding of the *strong field regime* of general relativity and suggest avenues to test the theory through its specific predictions for the *nonlinear, dynamical* phase of black hole formation.

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However, it is rather difficult to investigate these issues precisely because the dynamical processes of interest occur in the strong field regime of general relativity. Numerical simulations have provided insights but the horizon distortions seen in simulations often refer to components of geometrical tensors in coordinate systems and, more importantly, foliations they use. What one needs is an invariant characterization of the horizon geometry in its dynamical phase. A natural avenue is provided by the horizon multipole moments [3–5] which can be interpreted as the ‘‘source multipole moments’’ of the black hole. However, as we discuss in Secs. II B and IV C, the current definitions are not as well suited to investigate the approach to equilibrium as one would like.

The purpose of this article is to provide multipole moments which are well tailored for this task and provide equations for their dynamical evolution. These moments are just sets of numbers that capture the diffeomorphism invariant content of dynamical and arbitrarily distorted horizon geometries. Their evolution provides a coordinate- and slicing-independent description of how black holes shed the deviations from the Kerr horizon geometry and its spin structure. These equations can be used as nontrivial checks on numerical simulations in the strong field regime and, conversely, numerical solutions of these equations will bring out universalities in the approach to equilibrium, if they exist.

This article is organized as follows. In Sec. II we collect the material on isolated and dynamical horizons that serves as our starting point. Main results are presented in Secs. III and IV which also include a discussion of the relation to other definitions of multipoles in the literature [4,5] and to vortexes and tendexes that have been used to visualize the strong field geometry near black holes [6,7]. For the convenience of computational relativists, in Sec. IV the ideas and equations needed for numerical simulations have been presented in a self-contained fashion. If the goal is only to use these multipoles in numerical simulations, one can skip Sec. III and go directly to IV. In Sec. V we discuss their relation to similar issues that have been explored in the literature, including Price's law [8–10], the close-limit approximation [11,12], and the relation between dynamics at the horizon and at infinity [13–16]. The Appendix collects a few analytical results on the behavior of the key fields on the dynamical horizon H in the passage to equilibrium: which of them diverge, which of them admit finite limits and which of them vanish in the limit and at what rate.

Our conventions are as follows. We use Penrose's abstract index notation. The space-time metric g_{ab} has signature $-, +, +, +$ and curvature tensors are defined by $2\nabla_{[a}\nabla_{b]}k_c = R_{abc}{}^d k_d$, $R_{ac} = R_{abc}{}^b$ and $R = R_{ab}g^{ab}$.

II. QUASILOCAL HORIZONS

This section is divided into two parts. In the first we recall the notions of isolated and dynamical horizons and their basic properties [17–19]. In the second, we summarize the definition of multipoles in the axisymmetric case [3,4]. These quasilocal horizons have had numerous applications, including black hole thermodynamics [19,20], construction of initial data and extraction of physics from numerical simulations [4,21–24], and the definition of *quantum* horizons and analysis of their properties [25,26] in loop quantum gravity.

A. Dynamical and isolated horizons

The notion of event horizons has played a major role in the discussion of black holes. However, it is teleological and "too global" in that one needs the entire space-time evolution before one can locate it. Dynamical and isolated horizons are quasilocal notions which are free from these limitations.[1]

A *dynamical horizon* (DH) H is a three-dimensional spacelike submanifold (possibly with boundary) of space-time $(\mathcal{M}, g_{ab})$, foliated by a family of 2-spheres S such that

(i) each S is marginally trapped—i.e., the expansion $\Theta_{(\ell)}$ of one of the (future directed) null normals ℓ^a to each S vanishes—and

(ii) the expansion $\Theta_{(n)}$ of the other (future directed) null normal is negative.

Heuristically, since H is obtained by "stacking together" marginally trapped surfaces (MTSs), it can be thought of as the boundary of a trapped region of space-time representing a black hole. The area of the MTSs S increases in time, depicting a *dynamical phase* during which the black hole grows as it swallows matter and gravitational waves. Furthermore, Einstein's equations imply that there is a detailed balance law equating the rate of growth of the area radius R_S of any MTS S with the total flux of energy (in matter and gravitational waves) falling into the black hole across S [19].

Given a DH H, one can show that it does not admit any MTS that is not in the foliation. Thus, the foliation by MTSs—the "internal structure" of H—is unique. DHs naturally arise in numerical simulations where one begins with a foliation of space-time and uses efficient algorithms to zero in on the outermost MTSs. A local existence theorem ensures that, given such an MTS, it will "evolve" to a DH (provided certain generic conditions are met) [28,29]. However, DHs are not unique: A space-time region that appears to represent a black hole can carry multiple DHs. Nonetheless, partial uniqueness theorems do exist. In particular they imply that in the numerical relativity constructions, there is a unique DH that asymptotes to the event horizon in the distant future [27]. This is the situation of interest in this paper.

Once the flux of energy across the horizon becomes zero, the horizon becomes isolated. More precisely,

an *isolated horizon* (IH) Δ is a null, three-dimensional submanifold in $(\mathcal{M}, g_{ab})$, topologically $S^2 \times R$ and equipped with a specific null normal $\bar{\ell}^a$ such that

(i) the expansion $\Theta_{(\bar{\ell})}$ of $\bar{\ell}^a$ vanishes;

(ii) $\mathcal{L}_{\bar{\ell}} q_{ab} = 0$; and

(iii) $(\mathcal{L}_{\bar{\ell}} D_a - D_a \mathcal{L}_{\bar{\ell}}) t^a = 0$.

Here q_{ab} is the intrinsic (degenerate) metric on Δ, D the derivative operator induced on Δ by the space-time derivative operator ∇, and t^b is any vector field that is tangential to Δ.[2]

The fields (q_{ab}, D) constitute the *intrinsic geometry* of the IH Δ. By requiring that (q_{ab}, D) be time independent (with respect to the evolution defined by $\bar{\ell}^a$), the notion of an IH extracts from that of Killing horizons just the minimal properties to ensure that the horizon itself is in equilibrium, allowing for dynamical processes to occur arbitrarily close to it [22,30]. The definition ensures that neither matter nor gravitational waves fall across Δ and the area of any 2-sphere cross section of Δ is the same. Event horizons of stationary black holes are simplest examples of IHs [17,18,22–24].

Consider formation of a black hole via gravitational collapse or merger of two compact objects, one or both of which may be black holes. We are primarily interested in the late stage of such processes, when a common DH H

[1]Since our goal is only to convey the main ideas, the discussion will be brief and we will have to gloss over some finer points. For details and precise statements of results and properties, see [18,19,22–24,27].

[2]Note that q_{ab} has signature 0, +, + with $\bar{\ell}^a$ as the degenerate direction; $q_{ab}\bar{\ell}^b = 0$. Condition (ii) implies that ∇ induces a well-defined derivative operator D on Δ. It is automatically satisfied if the stress-energy tensor satisfies a mild version of the dominant energy condition: $-T^a{}_b\ell^b$ is a future directed causal vector everywhere on Δ.

develops and approaches an IH Δ representing the future part of the event horizon of the final black hole. Because of backscattering of gravitational waves, in the exact theory the approach would only be asymptotic. However, in numerical simulations one invariably finds that the backscattering becomes negligible within numerical errors rather soon and H joins on to Δ at some finite time. Therefore, in this paper we will focus on this situation. (The case in which the equilibrium is reached only asymptotically is in fact somewhat simpler [19,31].)

B. Multipole moments: The axisymmetric case

Numerical simulations invariably use convenient choices of coordinates and foliations and these choices vary from one research group to another. Therefore, the task of comparing the final results requires analytical tools to probe and compare distinct horizon geometries in an invariant fashion. Multipole moments provide such a tool. In this subsection we will summarize the situation in the case when the horizons are axisymmetric [3,4,22].

Let us begin with IHs Δ. An IH Δ is said to be *axisymmetric* if it admits a vector field φ^a satisfying $\mathcal{L}_\varphi \bar\ell^a = 0$, $\mathcal{L}_\varphi q_{ab} = 0$, and $(\mathcal{L}_\varphi D_a - D_a \mathcal{L}_\varphi) t^b = 0$ for all vectors t^a tangential to Δ. Thus, diffeomorphisms generated by φ^a on Δ preserve its geometry. These conditions imply that φ^a has an unambiguous projection on the 2-sphere of integral curves of $\bar\ell^a$ which is a rotational Killing field there.

Now, it is known that the diffeomorphism invariant content of the geometry (q_{ab}, D) of Δ is captured in two fields:

(i) the scalar curvature $\mathcal{R}$ of $\tilde q_{ab}$, the induced metric on any 2-sphere cross section S of Δ, and

(ii) the "rotational" 1-form ω_a on Δ defined by $D_a \bar\ell^a = \omega_a \ell^a$ [18,22].

The geometrical relation of these fields is brought out by the Weyl tensor. On any IH, the component Ψ_2 of Weyl curvature is gauge invariant and furthermore we have

$$\Psi_2 = \frac{1}{4}\mathcal{R} + \frac{i}{2}\epsilon^{ab} D_a \omega_b. \tag{2.1}$$

Here ϵ^{ab} is the area bivector on any 2-sphere cross section of Δ (and the right-hand side is independent of the specific choice of the cross section S). Thus, on Δ, the scalar curvature $\mathcal{R}$ is essentially the same as the real part of Ψ_2 while the rotational 1-form is a potential for the imaginary part of Ψ_2. In numerical simulations, one can calculate these fields on Δ. However, it is still not possible to compare the results of two different simulations because the fields live on two different 3-manifolds Δ and there is no natural identification between them. Geometric multipoles are two sets of numbers I_l, L_l, with $l = 0, 1, 2, \ldots$ which capture the entire diffeomorphism invariant content of these fields [3]. Therefore, to compare the results of any two simulations, it suffices to compute these numbers in each simulation and compare them. In practice, it suffices to compare just the first few multipoles.

The key idea behind the definition of multipoles is the following. Given an axisymmetric metric q_{ab} on a 2-sphere S, one can construct a *canonical* round 2-sphere metric $\mathring{q}_{ab}$ on S together with a preferred rotational Killing field [3]. This structure in turn provides canonical weighting functions $Y_{l,m}$, the spherical harmonics of $\mathring{q}_{ab}$. The multipoles are now defined as

$$I_{l,m} - i L_{l,m} := \oint_S \left[\frac{1}{4}\mathcal{R} + \frac{i}{2}\epsilon^{ab} D_a \omega_b\right] Y_{l,m} d^2V \tag{2.2}$$

$$\equiv \oint_S \Psi_2 Y_{l,m} d^2 V, \tag{2.3}$$

where the integral is performed on *any* 2-sphere cross section S of Δ and d^2V is the volume element on S. Of course, because the horizon geometry is axisymmetric, only the $m = 0$ multipole moments are nonvanishing. Furthermore, $I_{0,0}$ is just 1/4th the Gauss invariant, $I_{0,0} = 2\pi$, and $L_{0,0}$ vanishes. Therefore only the $l \neq 0$ moments are nontrivial.

Since each step in the construction is diffeomorphism covariant—none involved introduction of a structure other than the given axisymmetric IH—the final numbers are diffeomorphism invariant. A given axisymmetric horizon geometry yields these numbers and, conversely, given the numbers that arise from an axisymmetric horizon geometry, one can reconstruct that geometry up to an overall diffeomorphism. Finally, by a simple rescaling of these geometrical multipoles, one can obtain the *mass and spin multipoles* associated with the horizon. Since these refer only to the horizon without any reference to the exterior space-time region, they represent the *source multipoles* associated with the black hole itself. Indeed, as explained in [3], the construction suggests that one can assign a "surface mass density" $\rho_\Delta = -(1/8\pi) M_\Delta \mathcal{R}$ and a "surface spin current" $j^\Delta_a = (1/8\pi G)\omega_a$ to the isolated horizon Δ, where M_Δ is the total mass of M_Δ. By contrast, the multipole moments defined at infinity represent "field multipoles" which include contributions not only from the black hole but also from the exterior gravitational field (and matter, if any). In the Newtonian theory, the two sets agree. But because of its non-Abelian character, in general relativity gravity sources gravity. Therefore the two moments differ. For the mass quadrupole in Kerr space-time, for example, the difference increases with spin and is of the order of 40% near extremality $a \sim m$ [3].

What about dynamical horizons H? The diffeomorphism invariant content of the intrinsic geometry of any MTS S is again encoded in the scalar curvature $\mathcal{R}$ of S, while the role played by the rotational 1-form is now played by $\tilde\omega_a := \tilde q_a{}^b K_{bc} \hat r^c$, where $\tilde q_{ab}$ is the intrinsic 2-metric on S, $\hat r^c$ is the unit (spacelike) normal to S within

H and K_{ab} is the extrinsic curvature of H in space-time.[3] Using the same motivation as on IHs, one can introduce an effective "mass surface density" and an "angular spin current density" on any MTS S of the DH H and they are given by $\rho_S = -(1/8\pi) M_S \mathcal{R}$ and $j_a^S = (1/8\pi G)\tilde{\omega}_a$, where $\mathcal{R}$ is the scalar curvature of the 2-metric $\tilde{q}_{ab}$ on S [4,22]. Therefore, in the numerical relativity literature the definition (2.2) has been recast in terms of these fields:

$$I_{l,m}[S] - iL_{l,m}[S] := \oint_S \left[\frac{1}{4}\mathcal{R} + \frac{i}{2}\epsilon^{ab}\tilde{D}_a\tilde{\omega}_b\right] Y_{l,m} d^2V,$$
$$\text{where } \tilde{\omega}_a = \tilde{q}_a{}^b K_{bc}\hat{r}^c, \tag{2.4}$$

and taken over to assign multipole moments $I_{l,0}$, $L_{l,0}$ with each marginally trapped surface S in the foliation [4]. (Note that, whenever there is possible ambiguity, we use a tilde over symbols that refer to two-dimensional fields on the MTSs.)

On a DH, these multipole moments change in time, capturing the "intrinsic" dynamics of the black hole, encapsulated in the horizon geometry. However, to implement this strategy, one has to find an axial symmetry φ^a on each S. There *are* efficient numerical algorithms to locate this required axial Killing field φ^a, if it exists [21,32–35]. However, as one might expect, the DH formed in a gravitational collapse or a black hole merger generically fails to be even approximately axisymmetric except at very late time when the geometry is already close to that of the Kerr IH. Therefore the strategy is not well suited to study how the horizon loses its "hair" in its approach to the final Kerr state. Indeed, in the dynamical phase one expects the black hole spin, for example, to change not only in magnitude but also in direction, while the axisymmetry assumption forces the angular $l = 1$ momentum moment to have only the "z component." More generally, one would expect most moments to have nonzero values for $m \neq 0$ and it is of significant interest to see how dynamics of general relativity forces the black hole to shed them as it approaches equilibrium. To probe this issue, in Secs. III and IV we will generalize the framework by going beyond axisymmetry in a manner that is well suited to understanding the passage to equilibrium. We will also comment on the relation of this strategy to another approach [5] that has been proposed in the literature.

[3]This follows from the following considerations involving the "Weingarten map." On an IH Δ, the 1-form ω_a that features in (2.2) is the pullback to a 2-sphere cross section S of Δ of the 1-form $-(1/2)\bar{n}_b D_a \bar{\ell}^b \equiv -(1/2)\bar{n}_b \underleftarrow{\nabla}_a \bar{\ell}^b$, where $\underleftarrow{\nabla}$ is the pullback to Δ of the space-time connection. On a DH, the 1-form $\tilde{\omega}_a := \tilde{q}_a{}^b K_{bc}\hat{r}^c$ is given by the pullback to MTSs S of $-(1/2)n_b \underleftarrow{\nabla}_a \hat{r}^b$, where $\underleftarrow{\nabla}_a$ is the pullback to H of the space-time connection and $\hat{\tau}^b$ is the unit timelike normal to S. As in [19], we use the conventions $\ell^a n_a = -2 = \bar{\ell}^a \bar{n}_a$.

1. Remarks

1. In recent years, there has been considerable interest in using the Kerr multipoles to test the no-hair theorems of general relativity through gravitational wave signals. Much of this analysis is based on some key ideas introduced by Ryan [36] using signals arising from a compact object orbiting around a supermassive black hole. The strategy is to express the metric of the supermassive black hole at the location of the compact object as an expansion, with the Geroch-Hansen field multipoles at infinity as coefficients [37–39]. However, it would seem that the expansion of the space-time metric in terms of the source multipoles that characterize the horizon geometry would provide a more accurate route to mapping the Kerr geometry, unless the orbiting compact object is truly in the asymptotic region, very far from the central black hole. If it is closer, then expanding the space-time metric "outward" starting from the horizon [17], rather than "inward" from infinity, should require far fewer terms to attain the desired accuracy. There is also a conceptual advantage that one would only need to assume vacuum equations in the region between the two bodies.

2. The simple relation (2.1) between the fields $\mathcal{R}$ and ω_a and the Weyl curvature component Ψ_2 on IHs is modified on a DH. We now have

$$\mathcal{R} = (4\,\mathrm{Re}\Psi_2 - \tilde{q}^{ab}\tilde{q}^{cd}\sigma^{(\ell)}{}_{ac}\sigma^{(n)}_{bd}), \tag{2.5}$$

$$2\epsilon^{ab}\tilde{D}_a\tilde{\omega}_b = (4\,\mathrm{Im}\Psi_2 + \epsilon^{ab}\tilde{q}^{cd}\sigma^{(\ell)}_{ac}\sigma^{(n)}_{bd}), \tag{2.6}$$

where $\sigma^{(n)}_{ab}$ and $\sigma^{(\ell)}_{ab}$ are the shears associated with the null normals ℓ^a and n^a to the MTSs S and $\tilde{q}^{ab}$ is the metric on S. Therefore, on a DH, multipoles are no longer determined by Ψ_2 alone. (When the horizon becomes isolated, $\sigma^{(\ell)}_{ab}$ vanishes and the extra term drops out.)

III. MULTIPOLE MOMENTS OF GENERAL QUASILOCAL HORIZONS

In this section we present the conceptual strategy which allows us to define multipole moments on general, nonaxisymmetric horizons and track their time evolution. The material is divided into four parts. In the first, we introduce the main idea behind the generalization to nonaxisymmetric contexts; in the second, we execute this strategy; in the third, we present the generalized multipoles; and, in the fourth, we present "balance laws" that dictate the dynamics of multipole moments.

A. Main ideas

The underlying strategy is the same for both sets of geometric moments $I_{l,m}$ and $L_{l,m}$. We will first describe it in detail for the geometric spin moments $L_{l,m}$ and then summarize the situation for the $I_{l,m}$. In the first part of the discussion, we will consider the isolated and dynamical horizons simultaneously. For IHs, S can be any cross section (or the 2-sphere of the null generators $\bar{\ell}^a$) of Δ while for DHs, S can be any MTS.

Let us first integrate the expression (2.4) for $L_{l,m}$ by parts to obtain

$$L_{l,m}[S] = -\frac{1}{2}\sqrt{\frac{2l+1}{4\pi}}R^{-2}\oint_S \varphi^a_{l,m}\tilde{\omega}_a d^2V,$$
$$\text{where } \varphi^a_{l,m} = \sqrt{\frac{4\pi}{2l+1}}R^2\epsilon^{ab}D_bY_{l,m}, \tag{3.1}$$

where, as before, R is the area radius of S and we have introduced certain normalization factors for later convenience. Note that the $\varphi^a_{l,m}$ are all divergence-free on S and, furthermore, they provide a complete basis on the space of divergence-free vectors. Therefore $L_{l,m}$ can be thought of as providing a linear map from a basis of divergence-free vector fields on S to reals. In this respect, there is a structural similarity between multipole moments on Δ or H and "conserved" charges at null infinity, which can be regarded as linear maps from the generators of the Bondi-Metzner-Sachs (BMS) group to the reals [40–43]. With multipoles, the divergence-free vector fields play the role of infinitesimal symmetries. This conceptual parallel will be useful in our discussion.

In the axisymmetric case, we have a symmetry vector field φ^a and only $L_{l,0}$ are nonzero. In the language of vector fields these correspond to moments associated with the $\varphi^a_{l,0}$ satisfying $\mathcal{L}_\varphi \varphi^a_{l,0} = 0$. In the literature one often sets $Y_{1,0} = \sqrt{3/4\pi}\zeta$. Then $Y_{l,0}$ are all essentially just the Legendre polynomials in ζ; $Y_{l,0} = \sqrt{(2l+1)/4\pi}P_l(\zeta)$. The function ζ is singled out by the axial Killing field: $\varphi^a = R^2\epsilon^{ab}\tilde{D}_b\zeta \equiv \varphi^a_{1,0}$ (whence $\varphi^a = \varphi^a_{1,0}$). On a general horizon, the major obstacle has been that we do not have access to this route; without axisymmetry, there is no preferred ζ on S and hence we do not have the required basis $Y_{l,m}$ of functions.

The first step in the generalization is just to forego the preferred basis and use (3.1) to associate multipole moments L_ϕ with *any* divergence-free vector field ϕ^a on S:

$$L_\phi[S] = -\frac{1}{2}\oint_S \phi^a\tilde{\omega}_a d^2V, \quad \text{where } \mathcal{L}_\phi\epsilon^{ab} = 0. \tag{3.2}$$

But since the vector fields ϕ^a are defined separately on each S, we need a prescription to identify vector fields that lie on *different* cross sections S. Otherwise, we would not be able to compare multipoles associated with two different cross sections: On an IH the definition would be ambiguous and on a DH we would not be able to study the evolution of multipoles.

On IHs the required identification is easy to achieve: Consider the diffeomorphism generated by the appropriate (possibly angle-dependent) multiple of the vector field $\bar{\ell}^a$ that maps the first cross section S_1 to the second S_2. This natural map—the analog of the BMS supertranslation at null infinity—sends divergence-free vector fields on S_1 to divergence-free vector fields on S_2. With this identification between divergence-free vector fields, it follows that multipole moments are independent of the choice of the cross section S. Equivalently, we can use the 2-sphere of generators $\bar{\ell}^a$ of Δ for S in (3.2). This simpler procedure makes it manifest that the multipoles L_ϕ are properties of the IH as a whole.

On DHs, on the other hand, the geometry and hence the multipoles evolve in time and we need to follow the analog of the first procedure. Now S can be any one of the MTSs. Therefore, we need to construct a *dynamical vector field* X^a on H that provides a natural identification between the leaves of the foliation provided by MTSs. Motions along X^a will then be interpreted as "time evolution." We need this vector field X^a to have the following four properties:

(i) The one-parameter family of diffeomorphisms generated by X^a on H should preserve the foliation by MTSs.

(ii) It should provide an isomorphism between the space of divergence-free vector fields on any S to that of divergence-free vector fields on its image.

(iii) X^a should be constructed covariantly, using only that structure which is already available on general dynamical horizons without any symmetry.

(iv) If the DH is axisymmetric, diffeomorphisms generated by X^a should preserve the symmetry vector field φ^a. As we will see this will guarantee that the multipole moments given by the more general construction—that does not refer to axisymmetry at all—do reduce to the multipoles used in the literature in the axisymmetric case [4].

We will show that one can select, in a diffeomorphism covariant fashion, a class of vectors fields X^a satisfying these properties on any DH and multipoles are insensitive to the choice of X^a within this class.

B. Determining the dynamical vector field X^a

Since we already have a natural foliation by MTSs, any dynamical vector field X^a on H can be decomposed into a part that is orthogonal to the foliation and a part that is tangential: $X^a = N\hat{r}^a + N^a$, where, as before, $\hat{r}^a$ is the unit normal to each leaf of the foliation. Because X^a must map every MTS to some other MTS, the "lapse" N is severely restricted. To write out the restriction explicitly, let us introduce a coordinate v on H such that the leaves of the foliation are given by $v = \text{const}$. Then $N = C(q^{ab}D_a v D_b v)^{-1/2}$, where C is a constant and q^{ab} the inverse of the intrinsic $+, +, +$ metric q_{ab} on H. Without loss of generality, we can set $C = 1$ making v the affine parameter of the vector field X^a. This choice of lapse is denoted by $2b$ in the literature. Thus, we have

$$X^a = 2b\hat{r}^a + N^a, \quad \text{where} \quad 2b = |Dv|^{-\frac{1}{2}}, \tag{3.3}$$

and it now remains to determine the "shift" N^a. We will now show that the shift is also naturally fixed by our requirements.

We will first describe the strategy. The dynamical horizon is naturally equipped with a 2-form ϵ_{ab} that serves as the area 2-form on each MTS: $\epsilon_{ab} = \epsilon_{abc}\hat{r}^c$, where ϵ_{abc} is the volume 3-form on H. Had $\mathcal{L}_{2b\hat{r}}\epsilon_{ab}$ been zero, $2b\hat{r}^a$ would have mapped divergence-free vector fields on any S to divergence-free vector fields on its image and we could just set $X^a = 2b\hat{r}^a$, i.e., choose the shift N^a to be zero. But this strategy is not viable because $\mathcal{L}_{2b\hat{r}}\epsilon_{ab} = 2b\tilde{K}\epsilon_{ab}$, where $\tilde{K}$ is the trace of the extrinsic curvature of the MTS S within H which is necessarily nonzero because the area of these MTSs increases in time. The idea is to remedy this "problem" with an appropriate choice of the shift N^a. But no matter which shift N^a we use, we will not be able to compensate for the entire term $2b\tilde{K}$: Since $\mathcal{L}_N\epsilon_{ab} = (\tilde{D}_aN^a)\epsilon_{ab}$, even with a judicious choice of the shift N^a, we can only remove the *purely inhomogeneous* part

$$2b\tilde{K} - (1/4\pi R^2)\oint_S 2b\tilde{K}d^2V = 2b\tilde{K} - 2\dot{R}/R \qquad (3.4)$$

of $2b\tilde{K}$, where the area of each MTS is given by $4\pi R^2$, and the "dot" denotes the derivative with respect to v. Then, although $\mathcal{L}_X\epsilon_{ab}$ will not vanish, it will be of the form $f(v)\epsilon_{ab}$, where $f(v) = 2\dot{R}/R$. Clearly, this is the best one can hope for, given the fact that the area of the MTSs changes with v. But since v is constant on any S, this is sufficient to guarantee that the diffeomorphisms generated by X^a will map divergence-free vector fields on any MTS S to divergence-free vector fields on its image.

Let us now implement this strategy. First, we construct a unique function g on each S such that

$$\tilde{q}^{ab}\tilde{D}_a\tilde{D}_bg = -(2b\tilde{K} - 2\dot{R}/R) \quad \text{and} \quad \oint_S g d^2V = 0, \qquad (3.5)$$

where, as before, the tilde quantities refer to the intrinsic 2-geometry of each MTS. The existence of the solution to the first equation is guaranteed because its right-hand side integrates out to zero and the second equation makes the solution unique by removing the freedom to add a constant to g. We then set

$$N^a = \tilde{q}^{ab}\tilde{D}_bg, \quad \text{so that} \quad \tilde{D}_aN^a = -(2b\tilde{K} - 2\dot{R}/R), \qquad (3.6)$$

so that $X^a = 2b\hat{r}^a + N^a$ satisfies $\mathcal{L}_X\epsilon_{ab} = (2\dot{R}/R)\epsilon_{ab}$ or $\mathcal{L}_XR^{-2}\epsilon_{ab} = 0$. Note that, since $\tilde{K} = -(1/2)b\Theta_{(\bar{n})}$, and $\bar{n}^a$ is smooth on all of M, $b\tilde{K}$ vanishes in the limit $v \rightarrow v_o$. Since $\dot{R}$ also vanishes, it follows from (3.5) that g and hence the shift N^a vanishes on S_o and X^a joins on smoothly with $\bar{\ell}^a$ there.

By its construction, X^a satisfies the first three of our four requirements: It is constructed covariantly, preserves the foliation, and maps divergence-free vector fields on any S to divergence-free vector fields on its image. It turns out that it also satisfies the fourth requirement. To see this, let us suppose the DH is axisymmetric with an axial symmetry vector field φ^a. Then, since our construction of X^a uses only the horizon geometry, it follows that $\mathcal{L}_\varphi X^a = 0$. Therefore the diffeomorphisms generated by X^a map the axisymmetry vector field φ^a on any given MTS S to the axisymmetry vector field φ^a on its image. We will see in Sec. III C that this implies that, in the axisymmetric case, the multipole defined using this general strategy coincide with those defined using axisymmetry as in [4].

Finally, what would happen if we replace the coordinate v labeling the MTSs by $v' = f(v)$, where f is a monotonic function of v? It is straightforward to check that $X^a \mapsto X'^a = \dot{f}^{-1}X^a$. A vector field ϕ^a which is everywhere tangential to the MTSs and divergence-free on them satisfies $\mathcal{L}_X\phi^a = 0$ if and only if it satisfies $\mathcal{L}_{X'}\phi^a = 0$. Therefore, the "permissible" divergence-free vector fields ϕ^a selected by X^a are the same as those selected by X'^a, whence the multipoles $L_\phi[S]$ of Eq. (3.2) are also the same.

C. Generalized multipoles

We can now readily combine the results of the last two subsections to define the generalized geometric spin multipoles. We first introduce a vector field $X^a = 2b\hat{r}^a + N^a$ on H, where $2b$ is given by (3.3) and N^a by (3.6) and (3.5). Using it, we can single out the *admissible* weighting fields ϕ^a: A vector field ϕ^a on H which is tangential to every MTS, and divergence-free on it, is an admissible weighting field if $\mathcal{L}_X\phi^a = 0$. Note that every admissible vector field can be obtained simply by fixing a MTS $\bar{S}$, and a divergence-free vector field $\bar{\phi}^a$ thereon, and Lie dragging it along X^a. Given an admissible weighting field ϕ^a and a MTS S, we now define the spin multipole moments $L_\phi[S]$ following Eq. (3.2):

$$L_\phi[S] = -\frac{1}{2}\oint_S \phi^a\tilde{\omega}_a d^2V. \qquad (3.7)$$

By varying S we can study the dynamical evolution of these multipoles. Our weighting fields ϕ^a are "time independent" in the sense that $\mathcal{L}_X\phi^a = 0$. Therefore the multipole moments $L_\phi[S]$ derive their time dependence solely from the time dependence of the horizon geometry encoded in $\tilde{\omega}_a$ and the 2-sphere volume element.

Next, let us discuss the extension of the second set of multipoles, $I_{l,m}$, from axisymmetric horizons to generic ones. For this we first note that any metric 2-sphere S admits an Abelian $U(1)$ connection Γ_a whose curvature 2-form is determined by the scalar curvature $\mathcal{R}$ of the metric: $\tilde{D}_{[a}\Gamma_{b]} = (\mathcal{R}/4)\epsilon_{ab}$, where ϵ_{ab} is the area 2-form of S. Therefore, one can think of repeating the above procedure, and defining the other set of multipoles simply by replacing the 1-form $\tilde{\omega}_a$ by Γ_a in (3.2).

However, there is a subtlety. While $\tilde{\omega}_a$ is defined globally on the 2-spheres S, the connection 1-form Γ_a

has to be defined in patches: Since $\oint_S \tilde{D}_{[a}\Gamma_{b]}\epsilon^{ab}d^2V = (1/2)\oint_S \mathcal{R}d^2V = 4\pi$, the connection 1-form is globally defined only on the nontrivial $U(1)$ bundle over S^2 with the first Chern class.[4] But we can just fix a fiducial connection $\mathring{\Gamma}_a$ which is compatible with a round 2-sphere metric $\mathring{q}_{ab}$ whose area 2-form is the same as that of the given physical metric q_{ab} on S. Then $C_a := \Gamma_a - \mathring{\Gamma}_a$ *is globally defined* on S with the property that $\tilde{D}_{[a}C_{b]} = (1/4)(\mathcal{R} - \mathring{R})\epsilon_{ab}$, where $\mathring{R} = 2/R^2$, where R is the area radius of S. Then, for each permissible vector field ϕ^a on any MTS S of the horizon, we can set

$$I_\phi[S] = \frac{1}{2}\oint_S \phi^a C_a d^2V \quad \text{for any } \phi^a \text{ such that } \mathcal{L}_\phi \epsilon_{ab} = 0. \tag{3.8}$$

Although $\mathring{\Gamma}_a$ is arbitrary, because each ϕ^a is divergence-free, the integral is in fact independent of the choice of $\mathring{\Gamma}_a$ because $\oint_S \mathring{R}d^2V = \oint_S \mathcal{R}d^2V = 8\pi$, the Gauss invariant of a 2-sphere.

Let us summarize. Given a generic DH H we have introduced a family of vector fields X^a, unique up to a rescaling by a function that is constant on each MTS. The definition of this family is covariant and constructive: Given *any* DH, one can construct this family using only the structure that is already available. The diffeomorphisms generated by any of these X^a preserve the foliation by MTSs. We then defined *permissible weighting fields* ϕ^a on H; each ϕ^a is time independent, tangential to each MTS and divergence-free on it. *This family of* ϕ^a *refers only to the geometric structure that is naturally available on* H. They generalize the weighting functions $Y_{l,m}$ used on axisymmetric horizons. Given a permissible weighting field ϕ^a, we use (3.2) and (3.8) to define geometric multipoles $I_\phi[S]$, $L_\phi[S]$ on any MTS S. By varying S we track its time development.

What if the DH under consideration is axisymmetric? Then, as we saw in Sec. III B, the axial symmetry field φ^a is guaranteed to be time independent, i.e., Lie dragged by X^a. Now, by construction, $\mathcal{L}_X R^2\epsilon^{ab} = 0$ and, since $\varphi^a = R^2\epsilon^{ab}\tilde{D}_b\zeta$ with ζ satisfying $\oint_S \zeta d^2V = 0$, it follows that $\mathcal{L}_X\zeta = 0$. Therefore, the vector fields $\varphi^a_{l,0} := R^2\epsilon^{ab}\tilde{D}_b P_l(\zeta)$ are all permissible in our general setting. In this setting, they define multipoles via (3.8) and (3.7). From (3.2) it is clear that this general definition agrees with the definition introduced in [4]. Put differently, in the axisymmetric case, the function ζ defined separately on each cross section using the axial symmetry field φ^a is automatically time independent, i.e., satisfies $\mathcal{L}_X\zeta = 0$ in the language of our general setting. Therefore, with the identification $\phi^a = \varphi^a_{l,0}$, the multipoles $L_{l,0}[S]$ defined in the axisymmetric case (2.4) coincide with the multipoles $L_\phi[S]$ defined by the more general procedure, that does not refer to axisymmetry at all.

D. Balance laws

On the DH, we have balance laws which express the difference between the area radius (and in the axisymmetric case also spin) associated with two different MTSs S_1 and S_2 and flux of energy (and angular momentum) across the portion ΔH of the DH bounded by S_1 and S_2 [19,22]:

$$\frac{R_2 - R_1}{2G} = \int_{\Delta H} |dR| T_{ab}\hat{r}^a\ell^b d^3V + \frac{1}{16\pi G}\int_{\Delta H} |dR|(|\sigma^{(\ell)}|^2 + 2|\zeta|^2)d^3V, \tag{3.9}$$

where as before $\sigma^{(\ell)}_{ab}$ is the shear of the outward pointing null normal ℓ^a to the MTSs and R is the area radius of the MTSs, and where $|dR| = (q^{ab}D_aRD_bR)^{1/2}$ and the vector field ζ^a tangential to each S is defined by

$$\zeta^a := \tilde{q}^{ab}\hat{r}^c\nabla_c\ell_b = \tilde{\omega}^a + \tilde{D}^a \ln|dR|. \tag{3.10}$$

For the horizon spin, we have [19,22]

$$J^\varphi[S_2] - J^\varphi[S_1] = -\int_{\Delta H}\left(T_{ab}\hat{r}^a\varphi^b + \frac{1}{16\pi G}(K^{ab} - Kq^{ab})\mathcal{L}_\varphi q_{ab}\right)d^3V. \tag{3.11}$$

These two balance laws follow directly from Einstein's equations. On the conceptual side, they are significant because (unlike, say, Hawking's area theorem for event horizons) they provide a detailed link between the changes of physical quantities defined on S_2 and S_1 and energy and angular momentum fluxes across the portion ΔH bounded by them. In this respect, they are completely analogous to the balance laws for the Bondi energy momentum and angular momentum at null infinity. On the practical side, because the quantities that appear in the integrand of the right-hand side can be calculated independently of those that appear on the left side of these equations, these balance laws can serve as internal checks on accuracy of numerical simulations. We will now show that there are balance laws associated with multipole moments *that share all these features.*

As in Sec. III C, let us begin with the spin multipoles. Note first that, apart from overall constants that are needed for dimensional reasons, the spin multipole moment $L_\phi[S]$

[4]In the language that is more familiar in the numerical relativity literature, we have a connection that acts on the complex dyad m^a, $\bar{m}^a$ on S which is orthonormal in the sense $\tilde{q}_{ab}m^am^b = 0$ and $\tilde{q}_{ab}m^a\bar{m}^b = 1$: $\tilde{D}_am_b = i\Gamma_am_b$. The $U(1)$ gauge freedom corresponds to the local rotations of the dyad via $m^a \mapsto e^{i\theta}m^a$, where θ is a function on S. Neither the dyad nor the connection is globally defined on S. But we can define them in patches and in the overlap region the two sets are related by a gauge transformation, $m'^a = e^{i\theta}m^a$ and $\Gamma'_a = \Gamma_a - i\tilde{D}_a\theta$.

of Eq. (3.7) is obtained simply by replacing the axial Killing vector φ^a in the definition of the horizon spin [19,22]:

$$J^{\varphi}[S] := -\frac{1}{8\pi G}\oint_S \tilde{\omega}_a \varphi^a d^2V, \qquad (3.12)$$

with *any* permissible divergence-free vector field ϕ^a. Therefore the balance law (3.11) readily generalizes to

$$L_\phi[S_2] - L_\phi[S_1] = -\int_{\Delta H}\Big(4\pi G T_{ab}\hat{\tau}^a\phi^b + \frac{1}{4}(K^{ab} - Kq^{ab})\mathcal{L}_\phi q_{ab}\Big)d^3V. \qquad (3.13)$$

This generalization also has a direct analog at null infinity, where one can introduce balance laws not just for the 4-momentum and angular momentum but for charges associated with *any* of the generators of the infinite dimensional BMS Lie algebra [40–43]. Finally, we can also obtain a differential balance law directly from the definition (3.2) of the spin multipole moments:

$$\frac{dL_\phi}{dv} = -\frac{1}{2}\oint_S \mathcal{L}_X(\tilde{\omega}_c\phi^c\epsilon_{ab}) = -\frac{1}{2}\oint_S[\mathcal{L}_X(\tilde{\omega}_a)\phi^a + 2(\dot{R}/R)\tilde{\omega}_a\phi^a]d^2V, \qquad (3.14)$$

where we have used the fact that ϕ^a is a permissible weighting field. The structure of the right-hand side of this equation is quite analogous to that associated with the "BMS fluxes" at null infinity [40–43].

As one might expect from Sec. III C, the situation is the same for the other set of moments, $I_\phi[S]$. One only has to replace $\tilde{\omega}_a$ with $-C_a = -(\Gamma_a - \mathring{\Gamma}_a)$:

$$\frac{dI_\phi}{dv} = \frac{1}{2}\oint_S[\mathcal{L}_X(C_a)\phi^a + 2(\dot{R}/R)C_a\phi^a]d^2V. \qquad (3.15)$$

IV. APPROACH TO AN AXISYMMETRIC ISOLATED HORIZON

In this section we will consider the physically interesting situation in which a *generic* DH settles down to an axisymmetric IH. Then, on the IH portion we can introduce a convenient basis $\varphi^a_{l,m}$ of divergence-free vectors using the basis functions $Y_{l,m}$ made available by axisymmetry. By transporting them along the canonical vector field X^a to the DH portion we will obtain a convenient basis also on the DH portion, thereby converting the multipoles I_ϕ, L_ϕ defined in Sec. III to *a set of numbers* $I_{l,m}$, $L_{l,m}$ also on the DH. These can be readily evaluated in numerical simulations of black hole formation to study the approach to equilibrium.

This section is of direct interest to numerical relativity because (i) one expects the final IH in physical situations to be the Kerr IH and therefore axisymmetric; (ii) these moments are better suited to unravel universalities, if any, in the approach to equilibrium; and (iii) in most circumstances it would suffice to track just the first few multipoles. Therefore, for convenience of this readership, we have attempted to make this section self-contained.

A. The setting

Let us begin with a brief summary of the notation, collecting in one place the terminology used to denote the numerous fields that feature this analysis. Consider a quasilocal horizon M with two parts: a DH H in the past that is joined on to an IH at a 2-sphere cross section S_o (see Fig. 1). We will assume that M is a three-dimensional, C^{k+1} submanifold of space-time and the space-time metric g_{ab} is C^k with $k \geq 2$. We will denote by q_{ab} the pullback of g_{ab} to M; thus q_{ab} has signature +, +, + on the portion H of M and 0, +, + on the portion Δ. The space-time connection ∇_a induces a natural connection on M which we denote by D_a. It satisfies $D_a q_{bc} = 0$ on all of M.

The (future pointing) null normal to the IH Δ will be denoted by $\bar{\ell}^a$. The second (also future pointing) null normal to S_o will be denoted by $\bar{n}^a$. As noted below, these null vector fields admit natural smooth extensions to H and everywhere on M we choose them to satisfy the normalization $\bar{\ell}^a\bar{n}_a = -2$. Finally, we define a 1-form ω_a on all of M via $t^a\omega_a = -(1/2)\bar{n}_b t^a\nabla_a\bar{\ell}^b$ for all t^a tangential to

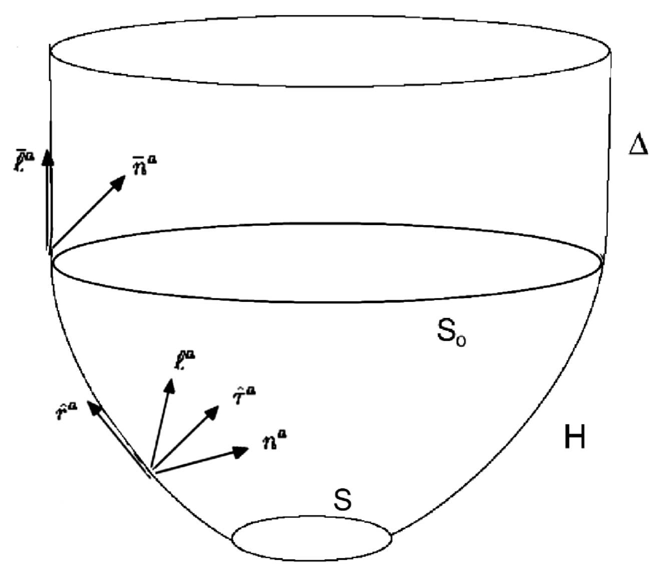

FIG. 1. A quasilocal horizon M. The past portion of M consists of a dynamical horizon H: This portion is spacelike and foliated by marginally trapped surfaces S. $\hat{\tau}^a$ is the unit timelike normal to H and $\hat{r}^a$ the unit spacelike normal within H to the foliation. Although H is spacelike, motions along $\hat{r}^a$ can be regarded as time evolution with respect to observers at infinity. H joins on to an isolated horizon Δ in the future, representing the equilibrium state of the black hole. Δ is null, endowed with a preferred null normal $\bar{\ell}^a$. The transition from H to Δ occurs at S_o.

M. ω_a represents the rotational 1-form on Δ while, as discussed below, on H it equals the $\tilde{\omega}_a$ defined in (2.4) modulo a gradient which drops out of the expression of multipole moments. We will denote the axial symmetry vector field on the IH by φ^a. But we *do not* assume that the DH is axisymmetric. It is allowed to have arbitrary distortions in its intrinsic and extrinsic geometry.

On the dynamical horizon H, we will denote the unit normal to H in the space-time manifold $(\mathcal{M}, g_{ab})$ by $\hat{\tau}^a$ and the unit normal to the MTSs S within H by $\hat{r}^a$. Then $\ell^a = \hat{\tau}^a + \hat{r}^a$ and $n^a = \hat{\tau}^a - \hat{r}^a$ are the two null normals to the MTSs, with $\ell^a n_a = -2$. Let the MTSs be the level surfaces of a C^{k+1} function v. We will assume that S_o is the uniform limit of MTSs and is thus labeled by $v = v_o$. One can continue the foliation in the future on the IH such that v is the affine parameter of the null normal $\bar{\ell}^a$. We will do so. On H we set $2b = |dv|^{-1} = (q^{ab}D_a v D_b v)^{-1/2}$. The area radius of the horizon cross sections will be denoted by R; R increases monotonically with v and remains constant on Δ. The extrinsic curvature of H within space-time $(\mathcal{M}, g_{ab})$ is denoted by K_{ab}. The intrinsic 2-metric on the MTSs S is denoted by $\tilde{q}_{ab}$ and its derivative operator by $\tilde{D}_a$. More generally, if there is an ambiguity in the notation, we use a tilde to denote fields that are intrinsic to the MTSs.

Since M is spacelike in the past and null in the future, the transition at S_o is somewhat subtle. Let us collect the basic facts (from [19,31] and the Appendix) that are needed in the analysis of multipoles.

(i) b^2 admits a C^k limit to S_o and vanishes there.

(ii) On H we have $\bar{\ell}^a = b\ell^a$ and $\bar{n}^a = b^{-1}n^b$. Thus, while ℓ^a is well defined on H it diverges at S_o and cannot be extended to Δ. $(\bar{\ell}^a, \bar{n}^a)$, on the other hand, are smooth on all of M.

(iii) The vector field $V^a := 2b\hat{r}^a$ on H admits a smooth extension to Δ and equals $\bar{\ell}^a$ on Δ. On all of M, V^a can be regarded as an evolution vector field with zero shift. Indeed, since $V^a D_a v = 1$ everywhere on M, v serves as the affine parameter of V^a. Finally, $b^2 = V^a \bar{\ell}^b g_{ab}$.

(iv) If we set $\dot{R} = dR/dv$, then both $\dot{R}$ and b are nonzero on H but vanish on S_o and remain zero on Δ. The field $b_o^2 := b^2/\dot{R}$ is nonzero and smooth on S_o.

(v) The rotational 1-form ω_a which is well defined everywhere on M and the 1-form $\tilde{\omega}_a := \tilde{q}_a{}^b K_{bc}\hat{r}^c$ defined on H are related by $\omega_a = \tilde{\omega}_a - \tilde{D}_a \ln b_o$.

B. Steps for numerical simulations

The general multipole moments defined in Sec. III are somewhat abstract: Given any MTS S, the $I_\phi[S]$, $L_\phi[S]$ can be regarded as linear mappings from permissible divergence-free vector fields ϕ^a on H to real numbers. As noted in the beginning of this section, in physical situations we expect H to join on to an *axisymmetric* isolated horizon in the future, in fact the Kerr IH. We can exploit this extra structure by first locating a preferred basis $\varphi^a_{l,m}$ of divergence-free ϕ^a on Δ and then dragging it along the preferred dynamical vector field X^a [of (3.3), spelled out again below] to the DH portion H of M. Put differently, we can now define the weighting functions $Y_{l,m}$ on the axisymmetric IH and drag them down to H along X^a, making them explicitly time independent. Given this basis of weighting functions, one can now replace the multipole moments $I_\phi[S]$ and $L_\phi[S]$ on H with just a set of numbers $I_{l,m}[S]$, $L_{l,m}[S]$ which are well suited to study, in an invariant manner, how the black hole reaches its equilibrium in any one numerical simulation. Furthermore, now one can also compare the results of two *different* simulations since one just has to compare numbers $I_{l,m}[S]$, $L_{l,m}[S]$ associated with the MTSs S with the same area. In practice the first few moments are likely to contain the most interesting information on passage to equilibrium.

Consider, then, a numerical simulations of a black hole formation. The world tube of MTSs found after a common horizon forms provides us with the 3-manifold M of Fig. 1. To extract multipole moments, one has to carry out the following steps.

(i) In the portion H of M on which the area of the MTSs increases monotonically, calculate the following quantities: (a) the 3-metric q_{ab}, (b) the intrinsic 2-metric $\tilde{q}_{ab}$, (c) the area radius R of each MTS S, so that the area of S is $4\pi R^2$, (d) the unit normal $\hat{r}^a$ to each S, and (e) the trace of the extrinsic curvature $\tilde{K}$ of each S within H.[5]

(ii) Find the 1-form $\tilde{\omega}_a := \tilde{q}_a{}^b K_{bc}\hat{r}^c$ on each MTS S. This is the "seed" that will generate the (geometric) spin moments $L_{l,m}$. Find the scalar curvature $\mathcal{R}$ of the metric $\tilde{q}_{ab}$ on each MTS S, which will serve as the seed for the (geometric) mass moments $I_{l,m}$. Taking the required second derivatives may introduce undesirable numerical errors (see, however, [34]). If so, it may be more convenient to introduce a complex orthonormal dyad m^a, $\bar{m}^a$ on each S and calculate the so-called "spin connection" Γ_a via $\tilde{D}_a m_b =: i\Gamma_a m_b$. This 1-form Γ_a can also serve as the seed to calculate the second set of moments $I_{l,m}$.

(iii) Now introduce a coordinate v on M such that the MTSs are the $v = $ const surfaces and the vector field $V^a = |dv|^{-1}\hat{r}^a \equiv 2b\hat{r}^a$ smoothly becomes the null normal $\bar{\ell}^a$ to the IH in the future region of M.

[5] In numerical simulations, one solves the initial value problem using a one-parameter family of Cauchy surfaces Σ_t and locates the outermost marginally trapped surface S_t on each Σ_t. The DH H is the world tube of these 2-surfaces. Therefore, fields which are naturally available refer to Σ_t and S_t and some extra steps are necessary to extract the fields such as q_{ab} and $\hat{r}^a$ we need here. These are described in Sec. III of [4]; see in particular Eqs. (3.3), (3.5), (3.9), and (3.13) in that section.

(iv) In the next step, construct the dynamical vector field X^a (that will be used to transport the weighting functions $Y_{l,m}$ from the IH to the DH). On each MTS S, find the function g via (3.5):

$$\tilde{q}^{ab}\tilde{D}_a\tilde{D}_b g = -(2b\tilde{K} - 2\dot{R}/R) \quad \text{and} \quad \oint_S g d^2V = 0 \tag{4.1}$$

and define the shift field N^a via $N^a = \tilde{q}^{ab}\tilde{D}_b g$. Then the dynamical vector field X^a is given by $X^a = V^a + N^a$. (As we approach the IH, g and N^a tend to zero and X^a joins on smoothly with $\bar{\ell}^a$.)

(v) On the 2-surface S_o where the DH H joins on to the IH Δ (or anywhere to its future), find the axial Killing field φ^a, e.g., using the algorithm described in [21,32–35]. In practice, one would expect the geometry to become axisymmetric within numerical errors already at late times on the DH and one can then find φ^a on that MTS without having to locate S_o. On the MTS S on which φ^a is found, by the standard procedure developed in [4] using [3], find the basis functions $Y_{l,m}$ (defined by the canonical "round" metric determined by φ^a and the $\tilde{q}_{ab}$ on that MTS).

(vi) Drag these weighting functions to any MTS S of interest via $\mathcal{L}_X Y_{lm} = 0$. Construct the multipole moments $L_{l,m}$ on that S using (3.2):

$$L_{l,m} = -\frac{1}{2}\oint_S (\epsilon^{ab}\tilde{D}_b Y_{l,m})\tilde{\omega}_a d^2V = -\frac{1}{2}\oint_S (\epsilon^{ab}\tilde{D}_b Y_{l,m})\omega_a d^2V. \tag{4.2}$$

Thus, one has to evaluate either the 1-form $\tilde{\omega}_a$ that refers to the extrinsic curvature K_{ab} of H in the four-dimensional space-time, *or* the rotational 1-form ω_a that refers to $\bar{\ell}^a$ and $\bar{n}^a$, whichever is numerically easier. Next consider the moments $I_{l,m}$. For $l = 0$, we have $I_{0,0} = 2\pi$, a topological invariant. For $l \neq 0$, we again have two avenues, given in the following two equivalent definitions:

$$I_{l,m} := \frac{1}{4}\oint_S \mathcal{R} Y_{l,m} d^2V = \frac{1}{2}\oint_S (\epsilon^{ab}\tilde{D}_b Y_{l,m})(\Gamma_a - \mathring{\Gamma}_a) d^2V, \tag{4.3}$$

where $\mathring{\Gamma}_a$ is a fiducial connection; we can set $\mathring{\Gamma}_a = -(1/R^2)\cos\theta\partial_a\phi$ in the coordinates used to express the $Y_{l,m}$. The second form may be more helpful if there are large numerical errors in computing the scalar curvature $\mathcal{R}$. Finally, the mass and spin multipoles $M_{l,m}$ and $J_{l,m}$ can be constructed by multiplying these geometric multipoles with appropriate dimensionful factors [3,4,22].

This six-step procedure enables one to compute the geometric multipole moments and study their evolution during the highly dynamical phase immediately after the formation of the common horizon. Computing these moments in examples is likely to bring out patterns in the way black holes shed their hair and approach the final equilibrium state, which in turn may enable one to uncover any universalities this process may have. In particular, on each MTS S the procedure provides a *spin vector* since generically $L_{1,m}$ will be nonzero even when $m \neq 0$. The "direction" of the spin vector can change during the dynamical phase and the black hole would shed the x and the y components of this spin vector entirely as it reaches equilibrium. Does this process simply vary from case to case, depending strongly on the structure of the common horizon at its birth, or is there some underlying law that relates it to, say, the angular momentum radiated away to null infinity?

Note that all the moments are anchored in the structure provided by the final equilibrium state of the black hole. The change in the mass dipole, for example, tells us how the black hole loses its 3-momentum *with respect to its final equilibrium state*. In fact a natural "home" for the multipoles is provided by the tangent space at the point i^+ at future timelike infinity: The $I_{l,m}$ (or $L_{l,m}$), for example, can be naturally regarded as constituting an lth-rank, trace-free, symmetric tensor in the tangent of i^+, all of whose indices are orthogonal to the final Bondi 4-momentum of the black hole.

Finally, as discussed in Sec. III D, there are balance laws that bring out the fact that the multipoles evolve in time in response to fluxes of physical fields across the DH H. In Sec. III D we considered the multipole moments weighted by permissible divergence-free vectors ϕ^a. We now have a preferred basis $\phi^a_{l,m}$ constructed from spherical harmonics $Y_{l,m}$, given in Eq. (3.1). Therefore we can rewrite the balance laws using the $Y_{l,m}$ as weighting fields. Given two MTSs S_1 and S_2, the difference between the spin multipoles associated with them can be expressed in terms of a flux across the portion ΔH of H, bounded by S_1 and S_2:

$$L_{l,m}[S_2] - L_{l,m}[S_1] = -\int_{\Delta H}\Big(4\pi G T_{ab}\hat{\tau}^a\epsilon^{bc}D_c Y_{l,m} + \frac{1}{2}(K^{ab} - Kq^{ab})D_a(\epsilon_{bc}D^c Y_{l,m})\Big)d^3V. \tag{4.4}$$

Similarly, for the $I_{l,m}$, we have the balance law:

$$I_{l,m}[S_2] - I_{l,m}[S_1] = \int_{\Delta H} \frac{|dR|}{2R} (|\sigma^{(\ell)}|^2 + 2|\zeta|^2 + 16\pi G T_{ab} \hat{\tau}^a \ell^b) Y_{l,m} d^3V + \int_{\Delta H} |dR| \left(\frac{1}{4} Y_{l,m} \partial_R \mathcal{R} + \frac{1}{R} \zeta^a \partial_a Y_{l,m} \right) d^3V. \quad (4.5)$$

On the IH, the flux integral on the right vanishes identically and the multipoles are conserved. On the DH portion, on the other hand, these balance laws could provide useful checks for numerics since the left and right sides refer to entirely different fields and they are equal only when Einstein's equations are satisfied.

1. Remarks

1. Throughout this analysis we have restricted ourselves to the dynamical horizon H of the final black hole. Suppose we begin with two widely separated black holes which coalesce. Before the merger, we would have two distinct DHs, say, H_1 and H_2. In the distant *past* these would join on to two distinct IHs Δ_1 and Δ_2, each of which would be well modeled by a Kerr IH and hence axisymmetric. Therefore, using the procedure described in this section, on *each* of these two quasilocal horizons, one would be able to define multipole moments $I_{l,m}$ and $L_{l,m}$ separately, where the required weighting functions $Y_{l,m}$ would now be transported from Δ_1 to H_1 and from Δ_2 to H_2. In particular, one would be able to study the evolution of the spin $L_{1,m}$ of each individual black hole. However, at present the DH framework cannot describe the merger phenomenon simply because $H_1 \cup H_2 \cup H$ is not a DH. Therefore, there is no simple relation between the two sets of multipoles prior to the merger and the set of multipoles after the merger.

2. Nonetheless, using the structure available at null infinity one can discuss *global* balance laws. Recall first that the total Arnowitt-Deser-Misner energy-momentum is well defined at the point i^o at spatial infinity [44], or, equivalently, in the distant past of the future null infinity $\mathcal{I}^+$ [45]. Denote it by P^a_{initial}. (Note that $P^a_{\text{initial}} \neq P^a_1 + P^a_2$ in general, e.g., because of the potential energy in the system.) Similarly, in the distant future, the mass monopole of the IH determines the final Bondi 4-momentum P^a_{final}. Both can be thought of as living in the four-dimensional vector space dual to the space of BMS translations. Therefore, it is meaningful to consider their difference $P^a_{\text{initial}} - P^a_{\text{final}}$ and this is precisely the Bondi 4-momentum radiated across $\mathcal{I}^+$ in the dynamical coalescence for which we have an independent formula [45].

The situation with angular momentum is similar but more subtle. The total (Lorentz) angular momentum of the system M^{ab}_{initial} is a well-defined mapping [42,43] from the Lorentz Lie algebra of the BMS Lie algebra, picked out by the fact that the Bondi news goes to zero as one approaches i^o [46,47]. Again, M^{ab}_{initial} is not simply related to $S^a_1 + S^a_2$, e.g., because it also contains a contribution due to the orbital motion. The final angular momentum M^{ab}_{final}, on the other hand, *is* determined entirely by the final spin of H because in the distant future we only have a single black hole. However, it refers to a distinct Lorentz sub-Lie algebra of the BMS Lie algebra now selected by the fact that the Bondi news goes to zero in the distant future. (The two Lorentz subalgebras agree only in the special circumstance in which the integral of the Bondi news along every generator of $\mathcal{I}^+$ vanishes [46,47].) Therefore it is not meaningful to take the difference $M^{ab}_{\text{initial}} - M^{ab}_{\text{final}}$. Rather, in place of M^{ab}_{initial}, we have to consider the angular momentum $\bar{M}^{ab}_{\text{initial}}$, again evaluated in the distant past of $\mathcal{I}^+$ but associated with the Lorentz subgroup picked out by the Kerr geometry in the distant future. This $\bar{M}^{ab}_{\text{initial}}$ is well defined but *not* the same as M^{ab}_{initial} even conceptually. The difference $\bar{M}^{ab}_{\text{initial}} - M^{ab}_{\text{final}}$ is well defined because both quantities now refer to the *same* Lorentz subgroup of the BMS group. Furthermore, by the balance laws [42,43], this is precisely the angular momentum (associated with the common Lorentz group) radiated across $\mathcal{I}^+$.

To summarize, the balance laws are meaningful both for the 4-momentum and angular momentum, although in the case of angular momentum, to compare "apples with apples," we have to drag the weight functions corresponding to the canonical Lorentz group in the distant future of $\mathcal{I}^+$ to distant past. Thus, there is no simple relation between the initial spins S_1 and S_2 of the individual black holes, the final spin S of the common black hole and the angular momentum radiated away across $\mathcal{I}^+$. For the 4-momentum, we do have a balance law relating P^a_{final}, P^a_{initial} and the 4-momentum radiated away across $\mathcal{I}^+$. However, unless $P^a_{\text{initial}} \approx P^a_1 + P^a_2$, there is no simple relation between the initial 4-momenta of individual black holes and the 4-momentum of the single, final black hole.

C. Comparisons

We will conclude Sec. IV with a discussion of the relation of this construction with similar ideas in the literature.

As we showed in Sec. III, ours is a genuine generalization of the definition [4] used in cases when the DH is axisymmetric. The generalization is both technically nontrivial and conceptually important because in the early stage of the postmerger phase, the DH is generally very far from being axisymmetric. We allowed the DH to be generic and assumed axisymmetry only for the IH representing the final equilibrium. Nonetheless, if the entire quasilocal horizon is axisymmetric as in [4], then on any MTS S our weighting functions $Y_{l,m}$ coincide with those determined intrinsically on S using the restriction of the axial symmetry φ^a to S.

There is another generalization in the literature, due to Owen [5]. That definition has the nontrivial feature that, while it uses only the DH portion of M without reference the final IH as in Sec. III, the multipoles are a set of numbers as in Sec. IV B. This is achieved using a construction that is local to each MTS S of the DH. In particular, the weighting functions used in [5] are eigenfunctions of certain elliptic operators constructed entirely from the geometry of the MTS; one does not transport them from a final axisymmetric state. For the mass moments, the elliptic operator is just the intrinsic Laplacian (determined by the physical metric $\tilde{q}_{ab}$) but for the spin moments a different, fourth-order elliptic operator is used to ensure that, if the DH *is* axisymmetric, the general procedure provides the well-established spin vector. These multipoles are distinct from ours and, generically, in the axisymmetric case they are different also from the multipoles introduced in [4].

The main differences from our definition are the following. First, while we use the same weighting functions for both sets of multipoles, Owen used different weighting functions. The second and more important difference is that we transport the weighting functions by dragging them from the final equilibrium configuration so that they are constant along the dynamical vector field X^a. By contrast, Owen's weighting functions are determined by the local, time-varying geometry. Owen's construction has the advantage of being "local in time," i.e., being covariant with respect to the geometry of each individual MTS. Our procedure is covariant only with respect to the geometry of the quasilocal horizon M as a whole. On the other hand, because our weighting functions on any MTS are "the same" as those in the final equilibrium state, our multipoles directly capture the dynamics of the horizon geometry encoded in $\mathcal{R}$ and $d\tilde{\omega}$ in the passage to equilibrium; one compares apples with apples.

To clarify this issue of time dependence, it is useful to recall the conceptual parallel between the definition of multipoles on a DH and that of the "BMS charges" at null infinity [40–43] we used in the remark at the end of Sec. IV B. The BMS charges are integrals over 2-sphere cross sections of null infinity of seed physical fields, weighted by functions that refer to the BMS symmetry corresponding to the charge. (In this analogy, the cross sections of null infinity play the role of the MTS S on the DH, the seed physical fields correspond to our $\mathcal{R}$, $d\tilde{\omega}$ on S, and the weighting functions to the $Y_{l,m}$ used here.) In the BMS case, given any cross section of null infinity, using its intrinsic geometry, one can find weighting functions corresponding to a specific Lorentz sub-Lie algebra of the BMS Lie algebra and construct six charges that represent the Lorenz-angular momentum at (the retarded instant of time represented by) that cross section. However, generically, different cross sections select *different* Lorentz sub-Lie algebras of the BMS Lie algebra and therefore it is not meaningful to compare the resulting Lorentz charges on one cross section to that on another. To compare apples with apples, one has to use the *same* Lorentz subgroup of the BMS group. This is achieved by appropriately *transporting* the generators (or weighting functions) corresponding to the Lorentz subgroup used on the first cross section to the second cross section and carrying out the 2-sphere integral with these *transported* generators which, in general, are distinct from those determined intrinsically by that cross section. Thus, the notion of the "same" Lorentz sub-Lie algebra refers to the structure of the three-dimensional null infinity as a whole; it cannot be captured by working locally on each cross section. And it is only when the same Lorentz generators are used that the change between the two sets of Lorentz charges refers to the change in the same *physical* quantities. There is no "contamination" due to a change in the weighting function itself, which would have occurred if we had used the generators selected by each cross section separately.

On quasilocal horizons, our procedure embodies this spirit in that our transport of weighting fields $Y_{l,m}$ from the final isolated horizons Δ to the dynamical horizon H is analogous to the transport of the Lorentz generators which is necessary for comparisons. Therefore, our multipoles $I_{l,m}[S]$, $L_{l,m}[S]$ on any MTS S of the DH can be meaningfully compared to those in the final equilibrium state. They are thus well adapted to meet the goal of this paper: capturing the physics of dynamics that makes the black hole shed its hair in its *approach to* equilibrium. Owen's goal was different. The focus there was to investigate the structure of the final state itself and the analysis provided evidence that it is Kerr. To meet that goal, it is not necessary to transport the weighting fields.

Finally, over the last two years there has been notable interest in numerical simulations whose goal is to visualize the strong field regime around black holes in terms of the so-called "tendex and vortex lines" [6,7]. The idea is to repeat the strategy that has been so successful in electrodynamics where pictorial representations of the magnetic lines of force often provide good intuition for the complicated dynamics, e.g., in problems involving neutron stars. In the case of black holes, the gravitational lines of force are obtained using the eigendirections of the electric and magnetic parts of the Weyl tensor:

$$E_{ab} = C_{acbd}\tilde{\tau}^c\tilde{\tau}^d \quad \text{and} \quad B_{ab} = {}^{\star}C_{acbd}\tilde{\tau}^c\tilde{\tau}^d = \frac{1}{2}\epsilon_{ac}{}^{pq}C_{pqbd}\tilde{\tau}^c\tilde{\tau}^d, \tag{4.6}$$

with respect to a space-time foliation to which $\tilde{\tau}^a$ is the unit timelike normal field. In the Kerr space-time, one can use natural foliations, the lines cross the MTSs, and their visual properties provide intuition for physical effects of the near-horizon, strong gravitational field. These images are also useful when one considers perturbations around Kerr. However, in a truly nonlinear, dynamical situation, e.g., at the formation of the common horizon during generic black hole collisions, there are no natural space-time foliations. Since the lines of force are tied to foliation choices that are made by extrapolating one's intuition based on the stationary Kerr geometry (and perturbative dynamics thereon) these visual images cannot be used to draw reliable conclusions about the *physics* of dynamical processes in the strong field, near-horizon geometry. Multipole moments introduced in this paper serve a complementary role. In particular, it would be instructive to develop programs to visualize the distortions in the geometry and the angular momentum content of the dynamical horizon membrane. In Kerr space-times, the geometrical intuition provided by these visualizations would not be as rich as that provided by vortexes and tendexes where the lines of force extend beyond the horizon, all the way to infinity. But in the strong field and highly dynamical regime, the intuition these multipoles provide would capture a more accurate depiction of the actual, invariant physics.

V. DISCUSSION

There is growing evidence that, in general relativity, the final equilibrium state of black hole horizons is extremely well approximated by the Kerr horizon. However, immediately after its formation, the common horizon that surrounds all matter and individual black holes is highly dynamical and its evolution varies from case to case. In this paper we have introduced multipole moments to gain physical insights into the strong field dynamical processes that efficiently smoothen all the distortions, leading to an universal final geometry.

We presented two sets of ideas. The first, discussed in Sec. III, is most useful on DHs H. It associates with each MTS S of H multipole moments $I_\phi[S]$, $L_\phi[S]$, where the weighting functions ϕ are a set of time-independent vector fields ϕ^a which are tangential to each S and divergence-free on them. On any given DH, the evolution of these multipoles captures the dynamics of the transition to equilibrium in a coordinate- and slicing-independent fashion. The second idea, presented in Sec. IV, is applicable only in a setting in which the DH is joined on to an *axisymmetric* IH in the future. However, from physical considerations, this is not a genuine restriction because, as we just noted, one expects the final equilibrium state to be the Kerr IH. In this case, one can introduce a convenient basis $\varphi^a_{l,m}$ in the space of weighting fields ϕ^a, labeled by spherical harmonics $Y_{l,m}$ that are determined on H in an invariant fashion by the future axisymmetric structure. Consequently, now the multipole moments on any MTS S are *just a set of numbers* $I_{l,m}[S]$, $L_{l,m}[S]$. As we saw in Sec. IV B, their definitions are well suited for numerical simulations. Not only can one use them to monitor dynamics on any one DH, but they also enable one to *compare* results of distinct simulations. (This is not possible with $I_\phi[S]$, $L_\phi[S]$ because one does not have a canonical identification between the divergence-free vector fields ϕ^a on the two DHs obtained in two distinct simulations.) Also, these multipoles provide tools to physically interpret the dynamical process. For example, $L_{1,m}[S]$ provides a well-defined notion of the *spin vector* during the dynamical phase. Tracking the evolution of the direction of the spin is likely to provide new insights. More generally, by explicitly evaluating a few low l multipoles and monitoring their evolution, numerical simulations should be able to find any patterns or universalities in the manner black holes shed their hair. We also provided formulas for fluxes of these multipoles. Since they are strict consequences of field equations, they can serve as analytic checks on numerics in the strong field and highly dynamical regimes.

This framework is well suited to analyze a number of issues. Recall first that, over the years, perturbative investigations have provided strong indications that the passage to equilibrium may have some universal features. In particular, Price's law and increasing evidence in its favor [8–10], the success of the close limit approximation of Price and Pullin [11,12], and the universality of quasinormal ringing [9] all suggest that, although the strong field dynamics after the formation of a common horizon is highly nonlinear, it has a deep underlying simplicity. However, to date the investigations in the strong field limit have been restricted to spherical symmetry [10]. In this case, there is no gravitational radiation, the DH has no hair, and its dynamics is rather simple and fully understood [48]. The central questions concerning the dynamical processes that wash away the distortions and nontrivial angular momentum structure of the DH simply do not arise. Therefore, numerical studies of the time evolution of multipoles $I_{l,m}[S]$, $L_{l,m}[S]$ in the general case, far removed from spherical symmetry, could lead to fresh and interesting insights. As the DH reaches its equilibrium, is there a correlation between the rate at which it sheds its multipoles and, say, Price's law? For example, recent numerical simulations suggest that the end point of the collision of two spinning black holes can be a Schwarzschild black hole [49]. In this case the DH would have to lose *all* its multipoles except the mass monopole. Is there a pattern to how they are lost? Is it the case, as one would intuitively expect, that the high l multipoles die quickly while the low l are dissipated more slowly? Is there in fact a *quantitative*, universal behavior? Another example is provided by the "antikick" that is associated with the postmerger phase of dynamics of binary black holes [14]. There are general arguments to suggest that it should be possible to account for this phenomenon in terms of the behavior of the mass monopole and dipole of the DH that forms after coalescence. Again, calculation of these moments and investigating their dynamics are likely to provide new physical insights.

The physical process involved in the manner equilibrium is reached is not directly intuitive because the DH lies *inside* the event horizon. Consequently, it does *not* radiate away its multipoles to infinity. Rather, distortions in the geometry and the angular momentum structure of the DH are washed out by the radiation that falls *into* the black hole. But it appears that there is a *correlation* between what falls *into* the DH and what gets radiated away to null infinity. The qualitative picture is that there is some radiation in the potential just outside the event horizon, some of which falls into the black hole and the rest escapes to infinity "remembering" the way it was correlated. At first this scenario can seem rather far-fetched because it is difficult to imagine processes responsible for this memory retention. But the paradigm is supported by several recent simulations [13–16]. Multipole moments defined here should help further develop these ideas.

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APPENDIX: LIMITING BEHAVIOR OF PHYSICAL FIELDS

In this Appendix we sketch the limiting behavior of various fields on the DH H, as one approaches the transition 2-surface S_o that joins H with a nonextremal IH Δ. These limits were used in Secs. III and IV. They also provide guidance for numerical simulations in that they separate fields which are likely to be easier to evaluate numerically from those that would be challenging because they involve ratios of quantities, both of which vanish or diverge in the limit. Finally, this discussion of the limiting behavior should be helpful for further analytical work on the approach to equilibrium.

Our notation is the same as in the main paper; see, e.g., Sec. IV A.

1. The intrinsic and the extrinsic geometry of the DH

Since the DH H is foliated by MTSs S, it is natural to decompose the intrinsic metric q_{ab} and the extrinsic curvature of H as follows:

$$q_{ab} = \tilde{q}_{ab} + \hat{r}_a\hat{r}_b, \tag{A1}$$

$$K_{ab} = A\tilde{q}_{ab} + S_{ab} + 2\tilde{\omega}_{(a}\hat{r}_{b)} + B\hat{r}_a\hat{r}_b, \tag{A2}$$

where, as in the main text, $\hat{r}_a$ is the unit normal to the MTSs S, $\tilde{q}_{ab}$ is the intrinsic 2-metric on each S, S_{ab} is a symmetric trace-free tensor field on S and $\tilde{\omega}_a$ is a 1-form field on S. We will investigate the limiting behavior of these fields as we approach the limiting MTS S_o that joins H to an IH Δ.

As in the main text, let us introduce a time coordinate v on the entire quasilocal horizon M such that the MTSs are the level surfaces of v and $v = v_o$ on S_o. Thus the portion $v < v_o$ on M corresponds to the DH and the portion $v > v_o$ corresponds to the IH. We are interested in the behavior of various geometric fields as v approaches v_o from below. We will assume that on the IH portion of M, v is the affine parameter of the null normal field $\bar{\ell}^a$, i.e., $\bar{\ell}^a\partial_a v = 1$. Given such a function v, there is a unique vector field V^a on M such that (i) on the DH, V^a is normal to each MTS S and (ii) satisfies $V^a\partial_a v = 1$ on all of M. Thus, V^a is a smooth extension of $\bar{\ell}^a$ on Δ to all of M. On H, V^a is proportional to $\hat{r}^a$:

$$V^a = |dv|^{-1}\hat{r}^a =: 2b\hat{r}^a, \quad \text{with} \quad 2b = \dot{R}|dR|^{-1}, \tag{A3}$$

where a "dot" will denote the derivative with respect to v. It then follows that $V\cdot V = 4b^2$. Since V^a is smooth and coincides with $\bar{\ell}^a$ on Δ, we conclude that b^2 is smooth, vanishes at $v = v_o$, and remains zero for $v > v_o$.

Since the function b features in the relation between the natural null normals ℓ^a, n^a adapted to H and the natural null normals $\bar{\ell}^a$, $\bar{n}^a$ adapted to Δ, its limiting behavior dictates that of several fields. Let us therefore make a small detour to specify the "rate" at which b vanishes as we approach $v = v_o$ from below. Note first that the rate of change of area A_S of a MTS S on H can be expressed as $\dot{A}_S = \oint_S \mathcal{L}_V(\epsilon_{ab})$. Using the identity $\mathcal{L}_V\epsilon_{ab} = -b^2\Theta_{(\bar{n})}\epsilon_{ab}$, and expressing $\dot{A}_S$ in terms of the rate of change $\dot{R}$ of the area radius, we obtain $8\pi R\dot{R} = -\oint_{S_v} b^2\Theta_{(\bar{n})}d^2V$. Therefore,

$$\lim_{v\to v_o}\oint_{S_v}\frac{b^2}{\dot{R}}\Theta_{(\bar{n})}d^2V = -8\pi R_o, \tag{A4}$$

where R_o is the area radius of S_o. Now, the integrand in (A4) is strictly negative for $v < v_o$ and $\Theta_{(\bar{n})}$ has a well-defined limit $\Theta^{(o)}_{(\bar{n})}$ on S_o. Let us assume that we are in a generic case and the limit is nonzero (a condition satisfied on the Kerr isolated horizon). Then it follows, e.g., by Taylor expansion of fields in v, that

$$b_o = b(\dot{R})^{-\frac{1}{2}} \tag{A5}$$

is a well-defined function on H admitting a regular nonvanishing limit to S_o. We can thus conclude that b^2 vanishes at the same rate as $\dot{R}$: $b^2 \sim \dot{R}b_o^2$ as v tends to v_o. As an example, in the Vaidya collapse, if one uses for v the standard ingoing Eddington-Finkelstein coordinate, then $b_o = 1/\sqrt{2}$.

Let us return to the expression (A1) of the metric q_{ab} on M. Since b vanishes as v tends to v_o, and V^a joins on smoothly with $\bar{\ell}^a$ on Δ at $v = v_o$, it follows from Eq. (A3) that $\hat{r}^a$ diverges on S_o. On the other hand, since

$$\hat{r}_a = 2b\partial_a v \tag{A6}$$

on H, we conclude that $\hat{r}_a$ vanishes at S_o. Finally, the 2-metric $\tilde{q}_{ab}$ smoothly approaches the intrinsic metric at S_o.

Next, let us consider the expression (A2) of the extrinsic curvature K_{ab} of H. Since $V^a = 2b\hat{r}^a = b(\ell^a - n^a)$ on H, and V^a joins on smoothly with $\bar{\ell}^a$ on Δ, it follows that we can smoothly extend $\bar{\ell}^a$ and $\bar{n}^a$ from Δ to H via

$$\bar{\ell}^a := b\ell^a \quad \text{and} \quad \bar{n}^a := b^{-1}n^a \tag{A7}$$

(where we have used the fact that these null vector fields are normalized via $\ell \cdot n = -2 = \bar{\ell} \cdot \bar{n}$). Now the part S_{ab} of K_{ab} in Eq. (A2) is related to the shear tensors of these null vector fields:

$$S_{ab} = \frac{1}{2}(\sigma^{(\ell)}_{ab} + \sigma^{(n)}_{ab}) \equiv \frac{1}{2}(\sigma^{(\ell)}_{ab} + b\sigma^{(\bar{n})}). \tag{A8}$$

Since $\bar{n}^a$ is smooth on all of M, on S_o we have $S_{ab} = (1/2)\sigma^{(\ell)}_{ab}$. Now, on the DHs, we have the following identity that arises directly from the constraint equations on H [19]:

$$\frac{1}{2G} = \oint_S \left[\frac{1}{16\pi G}(|\sigma^{(\ell)}|^2 + 2|\zeta|^2) + T_{ab}\hat{\tau}^a \ell^b\right] d^2V \tag{A9}$$

on any MTS S, where ζ^a is a vector field tangential to S, given by Eq. (3.10):

$$\zeta^a := \tilde{q}^{ab}\hat{r}^c \nabla_c \ell_b = \tilde{\omega}^a + \tilde{D}^a \ln|dR|. \tag{A10}$$

Since each term in the integrand of (A9) is positive definite, by Taylor expanding the fields in v we conclude that S_{ab} admits a regular limit to S_o.

Next, consider the term $\tilde{\omega}_a$ in the expansion (A2) of K_{ab}. It is easy to check that $\tilde{\omega}_a = -(1/2)\tilde{q}_a{}^b n^c \nabla_b \ell_c$. On the other hand we also have the corresponding 1-form ω_a associated with the barred null vectors $\bar{\ell}^a$, $\bar{n}^a$, namely, $\omega_a = -(1/2)\tilde{q}_a{}^b \bar{n}^c \nabla_b \bar{\ell}_c$, which is well defined on all of M. On H, the two are related by

$$\omega_a = \tilde{\omega}_a + \tilde{D}_b \ln b. \tag{A11}$$

Recall, further, that $b = b_o\sqrt{\dot{R}}$, where b_o has a well-defined limit to S_o which is nowhere zero. Since $\dot{R}$ is constant on any MTS S, on H we can rewrite (A11) as

$$\omega_a = \tilde{\omega}_a + \tilde{D}_b \ln b_o, \tag{A12}$$

which shows that $\tilde{\omega}_a$ admits a well-defined limit to S_o.

Finally, let us examine the coefficients A and B in the expression (A2) of the extrinsic curvature. We have

$$A = \frac{1}{2}\tilde{q}^{ab}\nabla_a \hat{\tau}_b \quad \text{and} \quad B = \hat{r}^a \hat{r}^b \nabla_a \hat{\tau}_b. \tag{A13}$$

Writing $\hat{\tau}^a$ in terms of the null normals and using the fact that each S is a MTS, we find $A = \Theta_{(n)}/4 = b\Theta_{(\bar{n})}/4$, and so $A \to 0$ as $v \to v_o$. To explore the limiting behavior of B, let us rewrite it as

$$B = \frac{1}{2b}(\kappa_V - V^a \partial_a \ln b), \quad \text{where } \kappa_V := -\frac{1}{2}\bar{n}_b V^a \nabla_a V^b. \tag{A14}$$

Note that κ_V is the surface gravity on DHs [19,31] which, at S_o, becomes the surface gravity $\kappa_{\bar{\ell}}$ of the IH which is positive because of our assumption that Δ is a nonextremal IH. Thus, κ_V has a well-defined limit to S_o. However, because of the overall $1/b$ factor, for B to have a well-defined limit, $V^a\partial_a \ln b = \dot{b}/b$ must approach $\kappa_{\bar{\ell}} > 0$ at a suitable rate. But this would imply $b \sim \exp \kappa_{\bar{\ell}}(v - v_o)$ as v approaches v_o which is impossible since $b = 0$ on S_o. Thus, B diverges in the limit as the DH approaches equilibrium.

This concludes the discussion of the limiting behavior of q_{ab} and K_{ab} as $v \to v_o$ from below. In Eq. (A1), $\hat{r}_a$ tends to zero and $\tilde{q}_{ab}$ has a well-defined limit which equals the intrinsic metric on S_o induced by the IH structure. In Eq. (A2), A tends to zero, and S_{ab} and $\tilde{\omega}_a$ have well-defined limits. However, B diverges in the limit. This implies in particular that the trace $K = q^{ab}K_{ab}$ of the DH also diverges as we approach the isolated horizon.

The divergence of K has the following important consequence. Since the dynamical horizons are spacelike, one can use them as partial Cauchy surfaces for the initial value problem of Einstein's equations. If one could find a general solution to the constraint equations for (H, q_{ab}, K_{ab}), one would have a *complete* description of all DHs that could ever arise in the formation of a black hole. In the spherically symmetric case, thanks to the systematic analysis of [48], this problem has been solved and the initial value equations have been reduced to a single, second-order linear "master equation." As a result, one can locally construct *general* spherically symmetric space-times admitting a DH and also locate the spherical DH in any given spherically symmetric space-time [48]. It is tempting to try to extend this analysis to general dynamical horizons. But because the diffeomorphism and the Hamiltonian constraints are coupled in a complicated fashion in the general setting, the standard strategy to solve initial value constraints is to first decouple them by assuming constancy of the trace K of the extrinsic curvature K_{ab}. However, because K in fact diverges as one approaches S_o, unfortunately this strategy cannot be used to solve the initial value problem for general DHs that approach equilibrium. It would be very interesting to devise another strategy by exploiting the fact that the initial data we seek are very special, in that the 3-manifold H admits a foliation by MTSs.

2. Constraint equations

We will conclude our discussion of the behavior of fields on H as H approaches equilibrium by listing a few consequences of the field equations.

On the DH, by projecting the constraint equations along and orthogonal to the MTSs and using $2\hat{r}^a = \ell^a - n^a$ the initial value equations can be written as

$$2G_{ab}\hat{r}^a\ell^b = 16\pi G T_{ab}\hat{r}^a\ell^b, \tag{A15}$$

$$2G_{ab}\hat{r}^a n^b = 16\pi G T_{ab} n^a\ell^b, \tag{A16}$$

$$G_{bc}\hat{r}^b\tilde{q}^c{}_a = 8\pi G T_{bc}\hat{r}^b\tilde{q}^c{}_a. \tag{A17}$$

Equation (A15) implies [19]

$$\mathcal{R} - |\sigma^{(\ell)}|^2 - 2|\zeta|^2 + 2\tilde{D}_a\zeta^a = 16\pi G T_{ab}\hat{r}^a\ell^b, \tag{A18}$$

and by integrating this equation on any MTS S we obtain Eq. (A9) which, as we have already noted, implies that $\sigma^{(\ell)}_{ab}$, ζ^a and $T_{ab}\hat{r}^a\ell^b$ have well-defined limits as we approach $v = v_o$ from below. Therefore $\mathcal{R}$ also has a well-defined limit and, as one would expect, the limit is just the scalar curvature of the 2-metric $\tilde{q}_{ab}$ on S_o. Furthermore, (A9) implies that the limiting values of $\sigma^{(\ell)}_{ab}$, ζ^a and $T_{ab}\hat{r}^a\ell^b$ cannot all vanish. In fact, if the IH Δ that H approaches is generic in the precise sense spelled out in [18]—and this is in particular the case if it is the Kerr IH—then one can prove a stronger result: $\sigma^{(\ell)}_{ab}$ and $T_{ab}\ell^a\tau^b$ cannot both vanish [50]. On the other hand, the energy flux across any MTS is dictated by these fields and that across any 2-sphere cross section of Δ is zero. But there is no conflict because, even if $\sigma^{(\ell)}_{ab}$ and $T_{ab}\ell^a\tau^b$ cannot both vanish on S_o, the energy flux across S_o *does vanish* because it is given by [19]

$$E_{\rm flux}[S] = \oint_S |dR|\left[\frac{1}{16\pi G}(|\sigma^{(\ell)}|^2 + 2|\zeta|^2) + T_{ab}\hat{r}^a\ell^b\right] d^2V \tag{A19}$$

for any MTS S and $|dR|$ vanishes in the limit.

Let us now turn to Eq. (A16). By expressing the fields in terms of those which have manifestly well-defined limits as $v \to v_o$, we obtain

$$-\frac{b^2}{2}\Theta^2_{(\bar{n})} + V^a\partial_a\Theta_{(\bar{n})} + \kappa_V\Theta_{(\bar{n})} - b^2|\sigma^{(\bar{n})}|^2 - 2\tilde{D}_a\omega^a - 2|\omega|^2 + \mathcal{R} = 16\pi G T_{ab}\hat{r}^a n^b, \tag{A20}$$

which reduces to

$$\mathcal{L}_{\bar{\ell}}(\Theta_{(\bar{n})}) + \kappa_{\bar{\ell}}\Theta_{(\bar{n})} - 2\tilde{D}_a\omega^a - 2|\omega|^2 + \mathcal{R} = 8\pi G T_{ab}\bar{\ell}^a\bar{n}^a, \tag{A21}$$

at S_o. This is precisely one of the field equations on the IH side. Thus, the field equation under consideration is *automatically* continuous across the transition surface. It does not further constrain the limiting behavior of geometrical fields as the horizon attains equilibrium.

Finally let us examine the projection (A17) of the vector constraint into the MTSs. Again, we can express all fields in terms of those which are manifestly smooth at S_o to obtain

$$-\frac{1}{4b}\tilde{D}_a(b^2\Theta_{(\bar{n})}) - \frac{1}{2}b\Theta_{(\bar{n})}\zeta_a + \frac{1}{b}\tilde{D}^c(bS_{ac}) + \frac{1}{2b}(\mathcal{L}_V\omega_a - \tilde{D}_a\kappa_V) = 8\pi G T_{bc}\hat{r}^b\tilde{q}^c{}_a. \tag{A22}$$

The limit of this equation is somewhat subtle since it contains quotients of vanishing quantities. Moving these terms to the right side and taking the limit, we obtain

$$\frac{1}{b_o}\tilde{D}^c(b_o\sigma^{(\ell)}_{ac})|_{v=0} = \lim_{v\to 0}\left\{\frac{1}{b}\left[8\pi G T_{bc}\bar{\ell}^b\tilde{q}^c{}_a - (\mathcal{L}_V\omega_a - \tilde{D}_a\kappa_V)\right]\right\}. \tag{A23}$$

Since the left side is well defined on S_o, we conclude the numerator on the right side must vanish in the limit. This is in complete agreement with the equations on the IH side, which tell us that each term in the numerator vanishes identically on the entire IH. What we learn from Eq. (A23) is that the numerator on the right side must vanish at a rate equal to or faster than b.

We will conclude with an observation pertaining to the physically most interesting case in which *vacuum* equations hold on the IH Δ. Then, if the IH horizon is generic, our discussion of Eq. (A15) implies that $\sigma^{(\ell)}_{ab}$ must be nonzero on S_o. This in turn implies that the left side of (A23) is necessarily nonzero.[6] Therefore we conclude that $(\mathcal{L}_V\omega_a - \tilde{D}_a\kappa_V)$ goes to zero as $\sqrt{R}$. Consider now the case of a C^k transition, that is, when M is C^{k+1} and the space-time metric is C^k. The vector field $\bar{\ell}^a$ on M is then C^k, which implies that $(\mathcal{L}_V\omega_a - \tilde{D}_a\kappa_V)$ is C^{k-1}. Therefore, in local coordinates, it vanishes as $\sim v^k$ or faster. Similarly, b^2 is C^k and so $\dot{R} \sim v^n$ with $n \geq k-1$. But the condition that the ratio in (A23) is finite implies that actually $n \geq 2k$. Thus b and R are smoother than what one might initially expect.

[6]For symmetric trace-free tensors A_{ab} on S, $\tilde{D}^bA_{ab} = 0 \Rightarrow A_{ab} = 0$. This follows from the fact that there are no harmonic 1-forms on 2-spheres. Since A_{ab} can be expanded as $A_{ab} = \tilde{D}_aX_b + \tilde{D}_bX_a - (\tilde{D}_cX^c)q_{ab}$ for some X_a, if $\tilde{D}^aA_{ab} = 0$, we have $\tilde{D}^bA_{ab} = (d + d^\dagger)^2X_a = 0$. This implies X_a—and therefore A_{ab}—must vanish.

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