Nonexistence of the final first integral in the Zipoy-Voorhees space-time

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We show that the geodesic motion in the Zipoy-Voorhees space-time is not Liouville integrable, in that there does not exist an additional first integral meromorphic in the phase-space variables.

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I. INTRODUCTION

The question of the integrability of the test particle motion in the Zipoy-Voorhees metric has recently attracted some attention, with both numerical [1,2] and analytical investigations [3]. The authors of Ref. [3] were able to exclude the existence of some polynomial first integrals, but they argue that some weaker form of integrability might take place taking into account the results of Ref. [1]. On the other hand, the results of Ref. [2] indicate chaotic behavior of the system, but the region where that happens is very small when compared to the phase space dominated by invariant tori, and the integration was performed with the Runge-Kutta (R-K) method of the fifth order only. Since it is known [4,5] that integrable systems can exhibit numerical chaos (particularly for the R-K method), the results of Ref. [2] should be taken cautiously. Our own numerical integration produced a Poincaré section visibly shifted from the one in Ref. [2] (see the end of Sec. IV), and since we used a more accurate method, it poses the question of whether the picture would be further deformed as the precision was increased. In other words, to decide on the integrability of the problem, a rigorous mathematical analysis is required rather than numerical simulations.

The physical problem and its significance are the same as in the classical paper by Carter [6]—that the existence of an additional first integral in the Kerr space-time makes the problem completely integrable. Carter's integral is not generated by a Killing vector, so it is not a usual symmetry of the manifold, but it is quadratic in momenta, which has important consequences. Such integrals translate into the separability of the Hamilton-Jacobi equation and d'Alembertian [7], which in turn appears in the Teukolsky [8] equation. That is to say, both the classical problem of particle motion in this space-time and the linear perturbation equations governing the gravitational waves and potentially quantum equations in that background become considerably easier to solve. This fact is also used in numerical approaches, when trying to determine possible spectra of gravitational radiation in anticipation of the observed data [9].

It is then natural to analyze other space-times which could serve as models of compact objects, and the stationary axisymmetric ones are one direction to explore. However, despite some numerical evidence [1], we find that the particular Zipoy-Voorhees metric with the parameter $\delta = 2$ is not integrable. To be more precise, we consider the motion of a test particle as a Hamiltonian system with n degrees of freedom and ask for the existence of an additional constant of motion I_n that would yield *Liouvillian integrability* with respect to the canonical Poisson bracket $\{\cdot,\cdot\}$. That is, for all first integrals I_k we would have $\{I_k, I_l\} = 0$, where the Hamiltonian is included as $H = I_1$, and $I_2, \ldots, I_{n-1}$ are also already known. It turns out that no such first integral can be found in the class of meromorphic functions, and we will use the differential Galois theory to prove that. Recall that a function is called meromorphic when its singularities (if it has any) are just poles, so by allowing first integrals that are potentially singular at some points of the phase space we are considering a fairly wide class of functions.

The reason for using this particular theory is that it gives the strongest known necessary conditions for the integrability of dynamical systems. It was used for proving the nonintegrability of the hardest problems of classical mechanics, like the three-body problem [10,11], which had been open for centuries. For an accessible overview of applications, see Ref. [12].

II. FORMULATION OF THE PROBLEM

The Zipoy-Voorhees metric under consideration is given by

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$$ds^2 = -\left(\frac{x-1}{x+1}\right)^2 dt^2 + \frac{(x+1)^3(1-y^2)}{x-1}\,d\phi^2 + \frac{(x^2-1)^2(x+1)^4}{(x^2-y^2)^3}\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right), \quad (1)$$

where x, y and ϕ form the prolate spheroidal coordinates. Instead of working directly with the geodesic equations, we take the Hamiltonian approach with

$$\begin{aligned} H &= \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta \\ &= -\frac{(x+1)^2}{2(x-1)^2} p_0^2 + \frac{(x^2-y^2)^3}{2(x-1)(x+1)^5} p_1^2 \\ &\quad + \frac{(x^2-y^2)^3(1-y^2)}{2(x-1)^2(x+1)^6} p_2^2 + \frac{x-1}{2(x+1)^3(1-y^2)} p_3^2, \end{aligned} \quad (2)$$

where the canonical coordinates are

$$q_0 = t, \qquad q_1 = x, \qquad q_2 = y, \qquad q_3 = \phi. \quad (3)$$

The equations then read

$$\begin{cases} \frac{dq_i}{d\tau} = \frac{\partial H}{\partial p_i}, \\ \frac{dp_i}{d\tau} = -\frac{\partial H}{\partial q_i}, \end{cases} \quad (4)$$

with $i = 0, 1, 2, 3$, and the normalization of four-velocities gives the value of the (conserved) Hamiltonian to be $H = -\frac{1}{2}\mu^2$. The new time parameter is the rescaled proper time $\mu\tau = s$, which allows us to include the zero geodesics for photons without introducing another affine parameter but with simply $\mu = 0$.

Since the metric has two Killing vector fields ∂_t and ∂_ϕ, the two momenta p_0 and p_3 are conserved. Together with the Hamiltonian they provide three first integrals. The question then is whether there exists one more first integral that would make the system Liouville integrable. To answer this question, we employ the differential Galois approach to integrability. More specifically, we use the main theorem of the Morales-Ramis theory [13].

Theorem 1.—If a complex Hamiltonian system is completely integrable with meromorphic first integrals, then the identity component of the differential Galois group of the variational and the normal variational equations along any nonconstant particular solution of this system is Abelian.

III. THEORETICAL SETTING

Let us try to explain the involved mathematics somewhat. For detailed exposition of the differential Galois theory, the reader is referred to Refs. [14,15]. The Morales-Ramis theory is exposed in Refs. [13,16], and a short introduction with application to another relativistic system can be found in Ref. [17].

To describe the differential Galois approach to the integrability, we consider a general system of differential equations

$$\frac{du}{d\tau} = f(u), \qquad u = (u_1, \ldots, u_m). \quad (5)$$

We assume that the right-hand sides

$$f(u) = (f_1(u), \ldots, f_m(u)),$$

are meromorphic in the considered domain. Let $\varphi(\tau)$ be a nonequilibrium solution of this system. Then the variational equations (VEs) along this solution have the form

$$\frac{d\xi}{d\tau} = A(\tau)\xi, \qquad A(\tau) = \frac{\partial f}{\partial u}(\varphi(\tau)). \quad (6)$$

It is not difficult to prove that if the original system has an analytic first integral $I(u)$, then the variational equations have a time-dependent first integral $I^\circ(\tau, \xi)$ which is polynomial in ξ. Similarly, one can show that if $I(u)$ is a meromorphic first integral, then the variational equations (6) have a first integral $I^\circ(\tau, \xi)$ which is rational in ξ. The Ziglin lemma (see p. 64 in Ref. [16]) says that if the system in Eq. (5) has $1 \le k < m$ functionally independent first integrals $I_j(u)$, $j = 1, \ldots, k$, then the variational equations in Eq. (6) have the same number of functionally independent first integrals $I_j{}^\circ(\tau, \xi)$ which are rational functions of ξ.

In the considered theory, time is assumed to be a complex variable, and for complex $\tau \in \mathbb{C}$, the solution $\varphi(\tau)$ can have singularities. Assume that $\tau_0 \in \mathbb{C}$ is not a singular point of $\varphi(\tau)$. Then in a neighborhood of τ_0 there exist m linearly independent solutions of the variational equations (6). They are the columns of the fundamental matrix $\Xi(\tau)$ of the system [Eq. (6)]. This matrix can be analytically continued along an arbitrary path σ on the complex plane avoiding the singularities of the solution $\varphi(\tau)$. Assume that σ is such a closed path, or loop, with the base point τ_0. Let $\hat{\Xi}(\tau)$ be a continuation of $\Xi(\tau)$. Solutions of a system of n linear equations form a linear n-dimensional space. Thus, in a neighborhood of τ_0, each column of $\hat{\Xi}(\tau)$ is a linear combination of columns of $\Xi(\tau)$. We can write this fact in the form $\hat{\Xi}(\tau) = \Xi(\tau)M_\sigma$, where M_σ is a complex nonsingular matrix, i.e., $M_\sigma \in \mathrm{GL}(m, \mathbb{C})$. In fact, the matrix M_σ depends only on the homotopy class $[\sigma]$ of the loop. Taking all loops with the base point τ_0, we obtain a group of matrices $\mathcal{M} \subset \mathrm{GL}(m, \mathbb{C})$ which is called the monodromy group of Eq. (6).

One can show that if $I^\circ(\tau, \xi)$ is a first integral of Eq. (6), then $I^\circ(\tau, \xi) = I^\circ(\tau, M\xi)$ for an arbitrary $M \in \mathcal{M}$, and for an arbitrary τ from a neighborhood of τ_0. In other words, if the original system has a meromorphic first integral, then the monodromy group has a rational invariant. Hence, if the system possesses a big number of first integrals, then the monodromy group of variational equations cannot be too big because it has a large number of independent rational invariants. This observation can be transformed into an effective tool if we restrict our attention to Hamiltonian systems and the integrability in the Liouville sense (complete integrability). The above facts are the basic ideas of the elegant Ziglin theory [18,19]. The problem in applying this theory is that the monodromy group is known for a very limited number of equations.

At the end of the previous century, the Ziglin theory found a nice generalization. It was developed by Baider, Churchill, Morales, Ramis, Rod, Simó, and Singer; see Refs. [13,16,20] and references therein. In the context of Hamiltonian systems it is called the Morales-Ramis theory, and in some sense, it is an algebraic version of the Ziglin theory. It formulates the necessary conditions for the integrability in terms of the differential Galois group $\mathcal{G} \subset \mathrm{GL}(m, \mathbb{C})$ of the variational equations. It is known that it is a linear algebraic group and that it contains the the monodromy group. By definition, it is a subgroup of $\mathrm{GL}(m, \mathbb{C})$ which preserves all polynomial relations between solutions of the considered system, see Ref. [21]; and for a wide class of equations it is generated by $\mathcal{M}$. The differential Galois group can serve for a study of integrability problems on the same footing as the monodromy group. Namely, first integrals of Eq. (6) give rational invariants of $\mathcal{G}$.

If the considered linear system is Hamiltonian, then necessarily $m = 2n$, and groups $\mathcal{M}$ and $\mathcal{G}$ are subgroups of the symplectic group $\mathrm{Sp}(2n, \mathbb{C})$. It can also be shown that the differential Galois group $\mathcal{G}$ is a Lie group. If the system is completely integrable with n meromorphic first integrals, then $\mathcal{G}$ has n commuting rational invariants. The key lemma (see p. 72 in Ref. [16]) states that if the above is the case, then the Lie algebra of $\mathcal{G}$ is Abelian. This means exactly that the identity component of $\mathcal{G}$ is Abelian.

Determination of the differential Galois group is a difficult task. Fortunately, in the context of integrability, we need to know only if its identity component is Abelian. If it is not Abelian, then the system is nonintegrable. If we find that a subsystem of the VEs has a non-Abelian identity component of the differential Galois group, then conclusions are the same. This is why, in practice, we always try to distinguish a subsystem of the VEs. It is easy to notice that $\psi(t) = f(\varphi(t))$ is a solution of Eq. (6). Using it, we can reduce the dimension of the VEs by 1. If the system [Eq. (5)] is Hamiltonian, then first we restrict it to the energy level of the particular solution. In effect, in Hamiltonian context we can easily distinguish a subsystem of variational equations of dimension $2(n-1)$, which are called the normal variational equations (NVEs).

The difficulty of investigation of the differential Galois group of NVEs depends, among other things, on the form of its matrix of coefficients, and so also on the functional form of particular solution. Quite often, by an introduction of a new independent variable $z = z(\tau)$ we can transform the NVEs to a system with rational coefficients

$$\frac{\mathrm{d}}{\mathrm{d}z}\xi = B(z)\xi \qquad B(z) = [b_{i,j}(z)], \qquad b_{i,j}(z) \in \mathbb{C}(z). \tag{7}$$

The set of rational functions $\mathbb{C}(z)$ is a field, and equipped with the usual differentiation it becomes a differential field. Solutions of a system with rational coefficients are typically not rational. The smallest differential field containing all solutions of Eq. (7) is called the Picard-Vessiot extension of $\mathbb{C}(z)$. The differential Galois group $\mathcal{G}$ of Eq. (7) tells us how complicated its solutions are, i.e., if the equations are solvable. Here solvability means that all solutions can be obtained from a rational function by a finite number of integrations, exponentiation and algebraic operations [14]. This category of functions, called Liouvillian, includes all elementary functions, as well as some transcendental, such as the logarithm or elliptic integrals, and is commonly referred to as the ‘‘closed-form’’ or ‘‘explicit’’ solutions. The following classical result connects the group structure of $\mathcal{G}$ with the form of the solutions.

Theorem 2.— System (7) is solvable, i.e., all its solutions are Liouvillian, if and only if the the identity component of its differential Galois group is solvable.

The connection of this theorem with integrability is the following. If it is possible to show that either the NVEs, or a subsystem of the NVEs are not solvable, then the identity component of their differential Galois group is not solvable, and so, in particular it is not Abelian. Thus, by Theorem 1, the system is not integrable. The question of whether a given system with rational coefficients is solvable can be resolved completely for a system of two equations (or one equation of second order). In this case there is an effective algorithm by Kovacic for finding the Liouvillian solutions [22]. This algorithm gives a definite answer, and if Liouvillian solutions exist it provides their analytical form. There exist a similar, almost complete algorithm for systems of three equations and some partial results for systems of four equations.

IV. PROOF OF NONINTEGRABILITY

The plan of attack is thus to look for particular solutions for which the NVE system has a block structure so that a two-dimensional subsystem can be separated. We then rewrite it as a second-order linear differential equation with rational coefficients and apply the Kovacic algorithm to see if it has any Liouvillian solutions. Note that the system has no external parameters, and only the values of particular first integrals enter as internal parameters. They are synonymous with initial conditions, so that if we manage to find just one solution, for particular values of μ, p_0 and p_3, such that the respective NVE is unsolvable, we will have proven that there cannot exist another first integral over the whole phase space.

It might so happen, unlike in the Carter case, that the system exhibits some particular invariant set on which there exists an additional integral. For example, one could have $\dot{I}_4 = H$, which would mean that I_4 is conserved on the zero-energy hypersurface $\mu = 0$, which is clearly a physically distinguished case. We will then have to look for particular solutions on those sets to make the results even more restrictive than just the lack of a global first integral.

The obvious particular solution to look at is a particle moving along a straight line, through the center in the equatorial plane, which in prolate coordinates means $y = 0$ and $p_3 = 0$. The nontrivial equations then read

$$\begin{aligned} \frac{\mathrm{d}t}{\mathrm{d}\tau} &= -\frac{(x+1)^2 p_0}{(x-1)^2}, \qquad \frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{p_1 x^6}{(x-1)(x+1)^5}, \\ \frac{\mathrm{d}p_1}{\mathrm{d}\tau} &= p_1^2 \frac{x^5(3-2x)}{(x+1)^6(x-1)^2} - p_0^2\frac{2(x+1)}{(x-1)^3}. \end{aligned} \tag{8}$$

Or, upon rescaling time by

$$\mathrm{d}\tau = \frac{(x-1)^2(x+1)^3}{x^3}\,\mathrm{d}u, \tag{9}$$

we have

$$\dot{x} = \frac{p_1 x^3(x-1)}{(x+1)^2}, \qquad \dot{p}_1 = p_1^2\frac{x^2(3-2x)}{(x+1)^3} - p_0^2\frac{2(x+1)^4}{x^3(x-1)}, \tag{10}$$

where the dot denotes differentiation with respect to u, and we have omitted the first equation, as the other two do not depend on t. This two-dimensional subsystem defines the particular solution around which we will construct the NVEs as mentioned before. The conservation of the Hamiltonian now reads

$$-\frac{1}{2}\mu^2 = -\frac{p_0^2(x+1)^8 + p_1^2 x^6(1-x^2)}{2(x-1)^2(x+1)^6}, \tag{11}$$

which together with the equation for $\dot{x}$ yields

$$\dot{x}^2 = (x^2-1)(p_0^2(x+1)^2 - \mu^2(x-1)^2), \tag{12}$$

so that $x(u)$ is expressible by the Jacobi elliptic functions. This fact is important, as we will change the independent variable from u to x, which is permissible (does not change the identity component of the Galois group) only if the function $x(u)$ defines a finite cover of the complex plane [15].

The variational equations along this solution separate so that the NVEs read

$$\dot{\xi}_1 = \frac{x^3}{(1+x)^3}\xi_2, \qquad \dot{\xi}_2 = 3p_1^2\frac{x(x-1)}{(x+1)^2}\xi_1, \tag{13}$$

where the variations ξ correspond to the perturbations of variables y and p_2. This is another step of the reduction mentioned in the previous section—the particular solution only has x and p_1 components, and the NVEs only have components in the orthogonal directions of y and p_2.

Introducing a new dependent variable

$$\xi = \frac{p_0^{1/2}x^{5/2}(x-1)^{1/4}}{(x+1)^{5/4}(p_0^2(x+1)^2 - \mu^2(x-1)^2)^{1/4}}\xi_2, \tag{14}$$

and taking x as the new independent variable, the NVEs can be brought to the standard form of

$$\xi''(x) = r(x)\xi(x), \tag{15}$$

with the rational coefficient r

$$r(x) := \frac{R(x)}{4x^2(x^2-1)^2(p_0^2(x+1)^2 - \mu^2(x-1)^2)^2}, \tag{16}$$

where R is the following polynomial:

$$\begin{aligned} R(x) = {}& p_0^4(34x^2 - 40x + 3)(x+1)^4 \\ & - 6p_0^2\mu^2(6x^2 - 10x + 1)(x^2-1)^2 \\ & + \mu^4(22x^2 - 20x + 3)(x-1)^4. \end{aligned} \tag{17}$$

Since for all physical particles we have $p_0 \neq 0$, all the others parameters can be rescaled by it:

$$\mu \to \mu/p_0, \qquad p_3 \to p_3/p_0, \tag{18}$$

which we use in what follows.

As is customary, we will use the same notation as in Kovacic's paper, adhering exactly to the steps and cases of the algorithm [22]. We note that a linear equation like Eq. (15) has local solutions in some neighborhood of a singularity $x_\star$ of $r(x)$, which take the form

$$\xi = (x - x_\star)^\alpha g(x - x_\star), \tag{19}$$

where g is analytic at zero, $g(0) \neq 0$, and α is called the characteristic exponent. The algorithm checks if it is possible to construct a global solution, which, in the simplest case, is of the form

$$\xi = Pe^{\int \omega \mathrm{d}x} \tag{20}$$

for a polynomial $P(x)$ and rational $\omega(x)$. The degree of P is then linked with the exponents, and that provides preliminary restrictions on the parameters' values and integrability.

The application of the algorithm itself is straightforward, and the only complication is that the singularities and exponents might depend on parameters. Fortunately, there are only several special values of μ that influence the outcome, and we outline the general steps in the two subsections below. For details, the reader is referred to Ref. [22], and another version of the algorithm, as applied to the dynamical system of the Bianchi VIII cosmology, can be found in Ref. [23].

A. General $r(x)$

The poles of $r(x)$ are

$$\left\{-1, 0, 1, \frac{\mu-1}{\mu+1}, \frac{\mu+1}{\mu-1}\right\}, \tag{21}$$

and for all of them to be different we must have $\mu^2 \neq 1$ and $\mu \neq 0$. All are of order 2, and the order at infinity is 4, so that we need to check all the cases of the algorithm.

In case 1, the characteristic exponents $\alpha_c^{\pm}$ of Eq. (15) form the following set:

$$\left\{(0,1), \left(\frac{9}{4}, -\frac{5}{4}\right), \left(\frac{3}{2}, -\frac{1}{2}\right), \left(\frac{3}{4}, \frac{1}{4}\right), \left(\frac{5}{4}, -\frac{1}{4}\right), \left(\frac{5}{4}, -\frac{1}{4}\right)\right\}, \tag{22}$$

where the first pair corresponds to ∞, and the combinations

$$d = \alpha_{\infty}^{\pm} - \sum_{c,s} \alpha_c^s \tag{23}$$

give only nine non-negative integers (not all distinct) as possible degrees of the appropriate polynomial P, which enters into the solution of Eq. (15). However, the respective test solutions of the form as in Eq. (20) require that $\mu = 0$, and have to be discarded so that this case cannot hold.

In case 2, the families of exponents E_c are

$$\{(0,2,4), (9,2,-5), (6,2,-2), (3,2,1), (5,2,-1), (5,2,-1)\}, \tag{24}$$

which in turn give 131 possible integer degrees for the appropriate polynomial. Checking them one by one, we find that they require $\mu = \pm 1$ in order to form a solution, so that this case can be discarded as well under the current assumptions.

In case 3, the families E_c contain $6 \times 13 = 78$ numbers, which make 4,826,809 combinations for d, out of which 230,856 are non-negative integers. We thus first resort to checking for the presence of logarithms in the solutions, which would prevent this case [15].

The only poles with integer difference in the exponents are 0 and ∞. Using the Frobenius method [24], we get the two independent solutions around zero

$$\begin{aligned} v_1 &= x^{3/2}\left(1 + \frac{5(\mu^2-2)}{3(1-\mu^2)}x + \frac{23\mu^4 - 38\mu^2 + 65}{12(1-\mu^2)^2}x^2 + \cdots\right), \\ v_2 &= x^{-1/2}\left(\frac{1}{9}(\mu^2-1) + \frac{5}{9}(\mu^2-2)x + \cdots\right) \\ &\quad + (5-\mu^2)\ln(x) v_1. \end{aligned} \tag{25}$$

As can be seen, the logarithm is present when $\mu^2 \neq 5$, and since the solutions around ∞ do not have logarithms at all, the only possibility for case 3 left here is with $\mu^2 = 5$.

B. Special subcases

In order to exclude the special energy hypersurfaces $\mu = 0$, $\mu^2 = 1$ and $\mu^2 = 5$, we have to resort to a more general particular solution, namely one with $p_3 \neq 0$. As already mentioned, it is enough to find one solution for each such surface, and that means we can take a specific value of p_3. The corresponding NVEs will only have numeric coefficients, and checking for their Liouvillian solutions is much easier, for it suffices to use one of available implemented routines, for example the "kovacicsols" of the symbolic system MAPLE.

The solution will also be expressible by (hyper)elliptic function as defined by the Hamiltonian constraint

$$\dot{x}^2 = \frac{(x-1)(x+1)^5 - (x-1)^4 p_3^2 - (x^2-1)^3 - \mu^2}{(x+1)^2}, \tag{26}$$

and the counterparts of the NVEs given in Eq. (13) will read

$$\dot{\xi}_1 = \frac{x^3}{(1+x)^3}\xi_2, \qquad \dot{\xi}_2 = \frac{(x-1)(3p_1^2 x^4 - (x^2-1)^2 p_3^2)}{x^3(x+1)^2}\xi_1. \tag{27}$$

We then proceed exactly as above, taking x as the independent variable and reducing the system to one equation of the form $\xi_2'' = r\xi_2$. For each hypersurface in question, the value of $p_3 = 1$ leads to NVEs that are not solvable with Liouvillian functions. This finishes the proof for all possible levels of the Hamiltonian.

To further illustrate the complexity of this system, we have also obtained a Poincaré section for the cross plane $y = 0$, shown in Fig. 1. The numerical integrator was based on the Bulirsch-Stoer modified midpoint scheme with Richardson extrapolation. We note that the special solution defined by Eq. (12) lies entirely in the plane $y = 0$ and is a trajectory beginning and ending at the singularity, so it does not contribute to the section. It also lies outside the visible chaotic region, which is confined to a very small subset of the phase space, as mentioned in Ref. [2].

V. GENERAL METRIC

The above results carry, to some extent, to the general Zipoy-Voorhees metric given by

$$\begin{aligned} ds^2 = -\left(\frac{x-1}{x+1}\right)^{\delta} dt^2 + \left(\frac{x+1}{x-1}\right)^{\delta}\Big((x^2-1)(1-y^2)d\phi^2 \\ + \left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2}(x^2-y^2)\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right)\Big), \end{aligned} \tag{28}$$

where $\delta \in \mathbb{R}$. The main problem that arises for arbitrary δ is that the special solution might no longer be a (hyper)elliptic function, because the Hamiltonian now gives

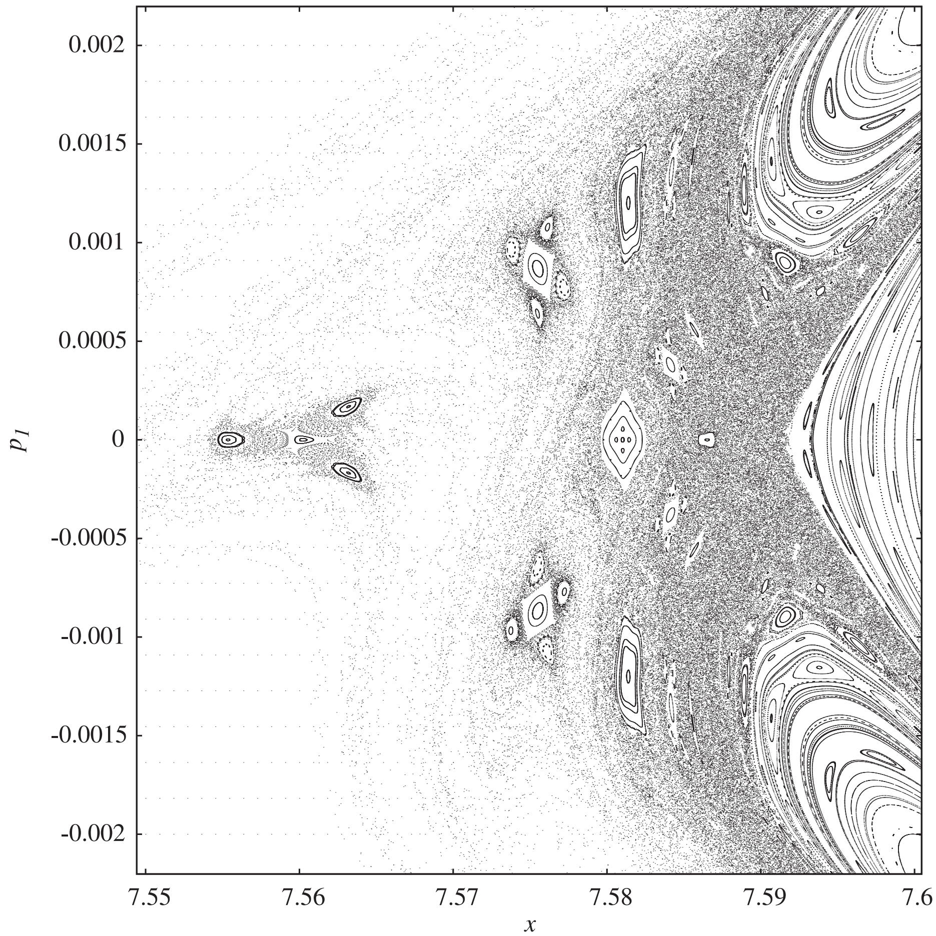

FIG. 1. Poincare section for the system in Eq. (2) at $y = 0$. The parameter values were $p_0 = 0.95$, $p_3 = 3$, $\mu = 1$.

$$\dot{x}^2 = \frac{1}{x^2}\left(1 - \frac{1}{x^2}\right)^{-\delta^2}(x+1)^{-2\delta}((x+1)^{2\delta}(x^2-1)p_0^2 - (x-1)^{2\delta}p_3^2 - (x^2-1)^{\delta+1}\mu^2), \tag{29}$$

so the right-hand side is not necessarily a polynomial or rational function. Accordingly, the rationalization of the NVEs might not preserve the identity component of the differential Galois group. However, when δ is rational we can still proceed by taking a new dependent variable to be

$$w := \frac{x+1}{x-1}, \tag{30}$$

as this leads to the normal form [Eq. (15)] which involves only integral powers of w and w^δ. Assuming then that $\delta = p/q$, we can make the NVEs rational by taking $w^{1/q}$ as the new variable if need be. Unfortunately, the number of poles (and their values) now depends on p and q, so the Kovacic algorithm has to be applied to each δ separately, but for each of them it is as straightforward as above to use the MAPLE package, once suitable numeric values of the parameters have been chosen. For example, we have verified that $\delta = 1/2$ is also nonintegrable, confirming the numerical evidence of Ref. [2] that for both $\delta > 1$ and $\delta < 1$, the general metric does not admit additional first integrals.

VI. CONCLUSIONS

Our main result can be stated as

Theorem 3.—There does not exist an additional, meromorphic first integral of the geodesic motion in the Zipoy-Voorhees metric [Eq. (1)]; i.e., the system is not Liouville integrable.

This confirms the previous considerations of Ref. [3] and goes much further than excluding first integrals polynomial in momenta up to a certain fixed small degree. Meromorphic functions include not only the analytic functions of both momenta and coordinates, but also rational and transcendental ones as long as their singularities are just poles. In particular, it follows that even if a conserved quantity exists, it cannot be expressed by an explicit formula of the above type. This result thus strongly reduces the possibility of using constants of motion expansion in solving the equations of geodesic motion or gravitational waves because the decomposition in terms of normal frequencies requires one to calculate their values directly from the initial conditions of the coordinates and momenta [9]. Of course, further techniques can be used to better understand and describe the motion, especially in the region where the dynamics is regular, but the fundamental physical property of this space-time is that no additional conservation law holds.

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