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Anomalous magnetohydrodynamics

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Anomalous symmetries induce currents which can be parallel rather than orthogonal to the hypermagnetic field. Building on the analogy of charged liquids at high magnetic Reynolds numbers, the persistence of anomalous currents is scrutinized for parametrically large conductivities when the plasma approximation is accurate. Different examples in globally neutral systems suggest that the magnetic configurations minimizing the energy density with the constraint that the helicity be conserved coincide, in the perfectly conducting limit, with the ones obtainable in ideal magnetohydrodynamics where the anomalous currents are neglected. It is argued that this is the rationale for the ability to extend to anomalous magnetohydrodynamics the hydromagnetic solutions characterized by finite gyrotropy. The generally covariant aspects of the problem are addressed with particular attention to conformally flat geometries which are potentially relevant for the description of the electroweak plasma prior to the phase transition.

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I. INTRODUCTION

There are physical situations were electric currents are directed along the magnetic field itself. For instance, in the analysis of ordinary hydromagnetic nonlinearities it is customary to study the evolution of the magnetic field averaged over the turbulent flow, be it compressible (as in acoustic turbulence) or incompressible [1,2]. In the latter case the effective current density is proportional to the magnetic field as established in different situations and extensively reviewed in a number of textbooks [3–5]. For this to happen a necessary condition is the parity breaking associated with the turbulent velocity field which has to be globally non-mirror-symmetric for sufficiently large kinetic Reynolds numbers. In this situation the averaged scalar product of the bulk velocity of the plasma with the bulk vorticity (sometimes called kinetic gyrotropy [1,5]) does not vanish [i.e. $\langle \vec{v}\cdot(\vec{\nabla}\times\vec{v})\rangle \neq 0$] and the kinetic energy of the plasma can be in principle transferred to the magnetic field.

Provided pseudoscalar species exist in the plasma, the effective Ohmic currents can be oriented along the magnetic field even if turbulent flows are absent. For instance, in the case of axions [6–8] the standard model is supplemented by a (global) $U_{PQ}(1)$. This symmetry is broken at the Peccei-Quinn scale F_a and leads to a dynamical pseudo-Goldstone boson (the axion) presumably acquiring a small mass because of soft instanton effects at the QCD phase transition. If an axionic density was present in the early Universe, bounds can be obtained for the Peccei-Quinn symmetry-breaking scale. These bounds together with other constraints leave a window of opportunity of $F_a \simeq \mathcal{O}(10^{10})$ GeV with many uncertainties concerning the axion mass [7]. Pseudoscalar species can also arise in the low-energy limit of superstring models, but in spite of its specific physical origin the pseudoscalar field (say, ψ) can couple to the Abelian gauge field strength $Y_{\mu\nu}$ as $(\psi/M)Y_{\mu\nu}\tilde{Y}^{\mu\nu}$, where $\tilde{Y}^{\mu\nu}$ is the dual field strength and M is related, in the axion case, to the Peccei-Quinn scale.

In the symmetric phase of the electroweak theory the hypercharge current can flow along the hypermagnetic field. Both the current and the magnetic field are usual vector fields and the proportionality factor is related to the chemical potential of the anomalous charges. This effect arises in gauge theories at finite density where it can happen that cold fermionic matter with nonzero anomalous Abelian charges is unstable against the creation of the Abelian gauge field [9,10]. The existence of currents parallel to the Abelian gauge field strength has been also analyzed in the electroweak plasma [11] with the aim of understanding how hypercharge fields may be converted into fermions in a hot environment. A magnetic field intensity parallel (or antiparallel) to the current density is also thought to be one of the potential consequences of the existence of the quark-gluon plasma and it has been more recently studied in the context of heavy-ion collisions [12] as well as in holographic approaches [13] (see also Ref. [14]). Chiral anomalies alter the evolution of the corresponding current but also the evolution of the gauge fields and of the corresponding Ohmic currents. This point is common to all the themes mentioned in this paragraph.

In the present investigation the persistence of anomalous currents will be scrutinized in globally neutral and conducting plasmas at high temperatures, such as the ones occurring prior to matter radiation equality or in the symmetric phase of the electroweak theory. The terminology anomalous magnetohydrodynamics (AMHD in what follows) refers to the evolution of hydromagnetic nonlinearities in the presence of anomalous symmetries

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both in cold and hot environments. While in standard magnetohydrodynamics the particle currents are all conserved, in AMHD there are charges which are anomalous (i.e. not covariantly conserved because of the macroscopic effects related to the triangle anomaly). The specific point addressed in this paper concerns the highly conducting limit of the anomalous plasma. In particular the idea is to address various situations where there are (at least) two currents: one anomalous and the other covariantly conserved. The latter current has to be identified, in the present approach, with the induced Ohmic current. In Sec. II the case of ordinary hydromagnetic nonlinearities shall be reviewed and some basic terminology will be introduced. Section III is devoted to the birefringence induced by the axial couplings with the aim of deriving the evolution of the slow modes of the globally neutral and conducting plasma in the simplified situation where the total energy-momentum tensor of the system is covariantly conserved. In Sec. IV we shall move to the case where there are two currents—one anomalous and the other conducting—and the global neutrality of the plasma is always assumed. It will be shown that the second law of thermodynamics constrains the conduction current which must contain both magnetic and vortical components. In Sec. V the ideal and the resistive limits of AMHD will be studied in the case of a conformally flat background geometry. Section VI contains the concluding remarks. In the Appendix various results shall be swiftly derived with the aim of easing the derivations presented in the bulk of the paper. Unlike previous analyses (bounded to a special relativistic treatment) we shall privilege here the generally covariant approach which is more suitable for the applications to curved space-times and, more specifically, to conformally flat background geometries.

II. HYDROMAGNETIC NONLINEARITIES

Magnetohydrodynamics (MHD in what follows) can be investigated within two different but in some sense complementary approaches: the ideal (or perfectly conducting) limit where the conductivity goes to infinity (i.e. the $\sigma_c \to \infty$ limit) and the real (or resistive) limit where the conductivity is finite (see, for instance, Refs. [15–17]). The ordinary magnetic diffusivity equation in ideal MHD can be simply written as

$$\frac{\partial \vec{B}}{\partial \tau} = \vec{\nabla} \times (\vec{v} \times \vec{B}) + \mathcal{O}(\sigma_c^{-1}), \tag{2.1}$$

where $\vec{v}$ denotes the bulk velocity of the plasma and $\vec{B}$ is the magnetic field intensity. Batchelor [18] pioneered the general picture of the interaction between the magnetic field and a conducting liquid by exploiting the analogy with a bulk velocity vortex in an incompressible liquid. Assuming—as is often done in statistical fluid mechanics—that the bulk velocity of the charged fluid is stationary and isotropic, the correlation function of the velocity field can be written as [1,2]

$$\langle v_i(\vec{k}, \tau) v_j(\vec{p}, \tau') \rangle = [\mathcal{A}_1(k) P_{ij}(\hat{k}) + \mathcal{A}_2(k) \epsilon_{ijk} \hat{k}^k] \times \delta^{(3)}(\vec{k} + \vec{p}) f(\tau, \tau'), \tag{2.2}$$

where $P_{ij}(\hat{k}) = (\delta_{ij} - \hat{k}_i \hat{k}_j)$ and $\hat{k}_i = k_i/k$. In the Markovian approximation $f(\tau, \tau')$ is proportional to $\delta(\tau - \tau')$ and the power spectra can have different forms which are not immediately relevant for the present considerations. Using Eqs. (2.2) and (2.1) the effective evolution equation for the magnetic field averaged over the bulk velocity field is

$$\frac{\partial \vec{H}}{\partial \tau} = \alpha \vec{\nabla} \times \vec{H}, \qquad \alpha = -\frac{\tau_c}{3} \langle \vec{v} \cdot (\vec{\nabla} \times \vec{v}) \rangle. \tag{2.3}$$

Since the ideal hydromagnetic limit is a slow description valid for large distances, the displacement current can be neglected so that $\vec{\nabla} \times \vec{H}$ is proportional to the current density $\vec{j}$. But then Eq. (2.3) implies that there is an effective Ohmic current proportional to $\vec{H}$ [5]. In the Zeldovich interpretation [5,19], Eq. (2.3) suggests that an ensemble of screw-like vortices with zero mean helicity is able to generate loops in the magnetic flux. Equations (2.2) and (2.3) have been analyzed for a number of astrophysical applications and describe the physical situation where kinetic energy is transferred to magnetic energy.

The plasma description following from MHD can also be phrased in terms of the conservation of two interesting quantities, i.e. the magnetic flux and the magnetic helicity [15–17],

$$\frac{d}{d\tau} \int_{\Sigma} \vec{B} \cdot d\vec{\Sigma} = -\nu_{\rm mag} \int_{\Sigma} \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) \cdot d\vec{\Sigma}, \tag{2.4}$$

$$\frac{d}{d\tau} \int_V d^3x \vec{A} \cdot \vec{B} = -2\nu_{\rm mag} \int_V d^3x \vec{B} \cdot (\vec{\nabla} \times \vec{B}), \tag{2.5}$$

where V is a fiducial volume comoving with the conducting fluid and Σ is the corresponding boundary surface; we defined $\nu_{\rm mag} = 1/(4\pi\sigma_c)$. Up to a gauge coupling constant the magnetic helicity coincides with the Chern-Simons number. The quantity $\vec{B} \cdot (\vec{\nabla} \times \vec{B})$ is sometimes called magnetic gyrotropy in full analogy with the kinetic gyrotropy already mentioned in the Introduction.

In a conducting plasma the kinetic and magnetic Reynolds numbers are defined as $R_{\rm kin} = v_{\rm rms} L_v / \nu_{\rm kin}$ and $R_{\rm mag} = v_{\rm rms} L_B / \nu_{\rm mag}$, where $v_{\rm rms}$ estimates the bulk velocity of the plasma while $\nu_{\rm kin}$ denotes the coefficient of thermal diffusivity; L_v and L_B are, respectively, the correlation scales of the velocity field and of the magnetic field. In the ideal hydromagnetic limit (i.e. $\sigma_c \to \infty$, $\nu_{\rm mag} \to 0$ and $R_{\rm mag} \to \infty$) the flux is exactly conserved and the number of links and twists in the magnetic flux lines are also preserved by the time evolution. If $R_{\rm kin} \gg 1$ and $R_{\rm mag} \leq \mathcal{O}(1)$ the system is still turbulent; however, since the total time derivative of the magnetic flux and of the magnetic helicity are both $\mathcal{O}(\nu_{\rm mag})$ the terms on the right-hand side of Eqs. (2.4) and (2.5) cannot be neglected.

Finally, if $R_{\rm mag} \gg 1$ and $R_{\rm kin} \ll 1$ the fluid is not kinetically turbulent but the magnetic flux is conserved. This occurs, incidentally, after matter-radiation equality but before decoupling [20]. The considerations developed here are bound to the analysis of a number of toy models but they are potentially relevant in more realistic situations as long as the plasma can be considered globally neutral and perfectly conducting. First-order phase transitions, if they occurred in the early Universe, can provide a source of kinetic turbulence and, hopefully, the possibility of inverse cascades which could lead to an enhancement of the correlation scale of a putative large-scale magnetic field [21]. Will AMHD help in connection with the possible onset of inverse cascades in astrophysical environments? In the past it has been argued that field configurations carrying magnetic helicity may be effective in obtaining inverse cascades [22]. These speculations are as complicated as a precise modelling of turbulence itself. On top of that at the electroweak scale the ratio between the magnetic and the kinetic Reynolds numbers (the so called Prandtl number) is very large. This is a further complication in comparison with laboratory plasmas where magnetic and kinetic diffusivities are typically of the same order (see also, in this regard, Ref. [20]). The extension of the viewpoint conveyed in the present analysis to a kinetically turbulent environment is not implausible but shall not be attempted here. For the present ends what matters are the physical analogies of the forthcoming discussions with the physics of charged liquids at high magnetic Reynolds number.

III. DYNAMICAL PSEUDOSCALAR FIELDS

Turbulence at high Reynolds numbers is sufficient for the existence of Ohmic currents flowing, on average, along the magnetic field direction. Such a requirement is, however, not necessary since similar phenomena can arise thanks to pseudoscalar species. Denoting with S_{ψ} the pseudoscalar contribution and with S_{Y} the gauge part, the corresponding actions can be written as[1]

$$S_{\psi} + S_{Y} = \int d^4x\sqrt{-g}\biggl[\frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\psi\partial_{\beta}\psi - W(\psi) - j^{\alpha}Y_{\alpha}\biggr] - \frac{1}{16\pi}\int d^4x\sqrt{-g}\biggl[Y_{\alpha\beta}Y^{\alpha\beta} + \frac{\psi}{M}Y_{\alpha\beta}\tilde{Y}^{\alpha\beta}\biggr], \tag{3.1}$$

where $j^{\alpha} = j^{\alpha}_{(+)} - j^{\alpha}_{(-)}$ and $j^{\alpha}_{\pm} = \tilde{n}_{\pm}u^{\alpha}_{\pm}$; the velocities $u^{\alpha}_{(\pm)}$ satisfy $g_{\alpha\beta}u^{\alpha}_{(\pm)}u^{\beta}_{(\pm)} = 1$ and $Y_{\alpha\beta}$ is the gauge field strength. The equations for ψ and $Y^{\mu\nu}$ are obtained by minimizing the variation of the action (3.1) and they are

$$g^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta}\psi + \frac{\partial W}{\partial\psi} = -\frac{1}{16\pi M}Y_{\alpha\beta}\tilde{Y}^{\alpha\beta}, \tag{3.2}$$

$$\nabla_{\alpha}Y^{\alpha\beta} = 4\pi j^{\beta} - \frac{\partial_{\alpha}\psi}{M}\tilde{Y}^{\alpha\beta}, \tag{3.3}$$

where ∇_{α} denotes the covariant derivative. The exchange of energy and momentum between the charged species is responsible for the existence of a finite conductivity. The presence of an energy-momentum transfer Γ implies

$$\nabla_{\mu}T^{\mu\nu}_{(+)} + \Gamma g^{\alpha\nu}(p_{-} + \rho_{-})u_{\alpha} = Y^{\nu\alpha}j^{(+)}_{\alpha}, \tag{3.4}$$

$$\nabla_{\mu}T^{\mu\nu}_{(-)} - \Gamma g^{\alpha\nu}(p_{-} + \rho_{-})u_{\alpha} = -Y^{\nu\alpha}j^{(-)}_{\alpha}, \tag{3.5}$$

where u_{α} denotes the total velocity field and $T^{\mu\nu}_{(\pm)}$ is

$$T^{\mu\nu}_{(\pm)} = (p_{\pm} + \rho_{\pm})u^{\mu}_{(\pm)}u^{\nu}_{(\pm)} - p_{\pm}g^{\mu\nu}. \tag{3.6}$$

The relation between u^{μ} and $u^{\nu}_{(\pm)}$ is

$$u^{\mu}u^{\nu} = (1+\gamma_{+})\Omega_{+}u^{\mu}_{(+)}u^{\nu}_{(+)} + (1+\gamma_{-})\Omega_{-}u^{\mu}_{(-)}u^{\nu}_{(-)}, \tag{3.7}$$

where $\gamma_{\pm} = p_{\pm}/\rho_{\pm}$ and $\Omega_{\pm} = \rho_{\pm}/(\rho_{+} + \rho_{-})$; note also that $g_{\alpha\beta}u^{\alpha}u^{\beta} = 1$. Equations (3.4) and (3.5) can be summed and subtracted. From the sum we get the equation for the total energy-momentum tensor of the charges, i.e.

$$\nabla_{\mu}T^{\mu\nu}(\rho, p) = Y^{\nu\alpha}j_{\alpha}, \qquad j_{\alpha} = j^{(+)}_{\alpha} - j^{(-)}_{\alpha}. \tag{3.8}$$

From the difference of Eqs. (3.4) and (3.5) (multiplied by the corresponding charge concentrations) an evolution equation for the total current can be obtained. In the limit where the rate of interaction dominates against the rate of variation of the geometry this combination leads to a relation between the current and the gauge field strength, i.e. Ohm's law, which will be introduced in a moment.

For the subsequent applications it is useful to rephrase the evolution of the system in terms of the evolution of the energy-momentum tensors for $T^{\nu}_{\mu}(\psi)$ and $T^{\nu}_{\mu}(Y)$,

$$\nabla_{\mu}T^{\mu}_{\nu}(\psi) = -\frac{\partial_{\nu}\psi}{16\pi M}Y_{\alpha\beta}\tilde{Y}^{\alpha\beta}, \tag{3.9}$$

$$\nabla_{\mu}T^{\mu}_{\nu}(Y) = -Y_{\nu\alpha}j^{\alpha} + \frac{\partial_{\nu}\psi}{16\pi M}Y_{\alpha\beta}\tilde{Y}^{\alpha\beta}, \tag{3.10}$$

where

$$T^{\nu}_{\mu}(\psi) = \partial_{\mu}\psi\partial^{\nu}\psi - \delta^{\nu}_{\mu}\biggl[\frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\psi\partial_{\beta}\psi - W(\psi)\biggr], \tag{3.11}$$

[1] The conventions will be the following. Greek indices run over the four-dimensional space-time. Latin (lowercase) indices run over three-dimensional spatial geometry. The signature of the metric is mostly minus, i.e. $(+,-,-,-)$.

$$T_{\mu}^{\nu}(Y) = \frac{1}{4\pi}\left[-Y_{\mu\alpha}Y^{\nu\alpha} + \frac{\psi}{M}Y_{\mu\alpha}\tilde{Y}^{\nu\alpha} + \frac{1}{4}\delta_{\mu}^{\nu}(Y_{\alpha\beta}Y^{\alpha\beta} + Y_{\alpha\beta}\tilde{Y}^{\alpha\beta})\right]. \tag{3.12}$$

Consider now a conformally flat geometry of the type $g_{\mu\nu} = a^2(x)\eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric and the scale factor can be a function of a generic space-time point. The gauge field strengths can be written as $Y_{i0} = -a^2(x)e_i$ and $Y_{ij} = -a^2(x)b^k\epsilon_{ijk}$ and the equations for the hyperelectric and hypermagnetic fields are given, in this case, by

$$\vec{\nabla}\cdot\vec{E} = 4\pi(n_+ - n_-) - \frac{1}{M}\vec{\nabla}\psi\cdot\vec{B}, \tag{3.13}$$

$$\vec{\nabla}\cdot\vec{B} = 0, \qquad \vec{\nabla}\times\vec{E} = -\partial_\tau\vec{B}, \tag{3.14}$$

$$\vec{\nabla}\times\vec{B} = \partial_\tau\vec{E} + \frac{1}{M}[\partial_\tau\psi\vec{B} + \vec{\nabla}\psi\times\vec{E}] + 4\pi[n_+\vec{v}_+ - n_-\vec{v}_-], \tag{3.15}$$

where $\vec{v}_\pm = a\vec{u}_\pm$; furthermore, $\vec{E} = a^2\vec{e}$ and $\vec{B} = a^2\vec{b}$. The expressions of the total charge and current density appearing in Eqs. (3.13) and (3.15) come from the definition of the comoving concentrations of charged species [i.e. $n_\pm(x) = a^3(x)\tilde{n}_\pm$] and from the comoving velocity field (i.e. $\vec{v}_\pm = a\vec{u}_\pm$). The covariant conservations of the currents leads to the evolution equations of the comoving concentrations, i.e. $\partial_\tau n_\pm + \vec{\nabla}\cdot(n_\pm\vec{v}_\pm) = 0$.

For the present ends the interesting situation concerns a globally neutral plasma. From Eq. (3.13) $\vec{\nabla}\cdot\vec{E} = 0$ provided $n_+ = n_- = n_0$ and, at the same time, ψ is spatially homogenous, i.e. $\vec{\nabla}\psi = 0$. In the limit where the rate of interaction between the charged species is larger than the rate of variation of the geometry the total Ohmic current can be expressed in covariant language as [23] $j^\alpha = \sigma_c Y^{\nu\alpha}u_\alpha$, where σ_c is the conductivity of the system. From the expression of the Ohmic current $j^\alpha u_\alpha = 0$ and this is why we can also write $j^\mu h_\mu^\nu = \sigma_c Y^{\nu\alpha}u_\alpha$. If we now project the latter expression along u_ν we obtain an identity. Conversely, if we project $j^\mu h_\mu^\nu$ along $h_\nu^\beta = (\delta_\nu^\beta - u_\nu u^\beta)$ we shall obtain, again, $j^\alpha = \sigma_c Y^{\nu\alpha}u_\alpha$ since, by definition, $h_\mu^\nu h_\nu^\beta = h_\mu^\beta$.

The value of the conductivity depends on the specific properties of the plasma. In particular, defining m as the mass of the lightest charge carrier we have that $\sigma_c \simeq T/\alpha_{em}$ for $T \gg m$ and $\sigma_c \simeq (T/\alpha_{em})(T/m)^{1/2}$ in the opposite limit. The total Ohmic current is then given by

$$\vec{J} = \sigma\left(\vec{E} + \vec{v}\times\vec{B} + \frac{\vec{\nabla}p}{n_0} - \frac{\vec{J}\times\vec{B}}{n_0}\right), \tag{3.16}$$

where n_0 denotes the rescaled charge concentration while $\sigma = \sigma_c a$ and $\vec{J} = a^3\vec{j}$. Dropping the third and fourth terms on the right-hand side of Eq. (3.16) (i.e. the thermoelectric and Hall terms which are of higher order in the spatial gradients) the (hyper)electric field can be expressed in terms of the total current.

In the resistive approximation, the hyperelectric field is not exactly orthogonal to the hypermagnetic one. The source of this mismatch depends on the specific dynamical situation and, in the present case, the induced hyperelectric field is

$$\vec{E} \simeq \frac{\vec{\nabla}\times\vec{B}}{4\pi\sigma} - \frac{\partial_\tau\psi\vec{B} - \vec{\nabla}\psi\times(\vec{v}\times\vec{B})}{4\pi\sigma} - \vec{v}\times\vec{B} + \mathcal{O}(\sigma^{-2}). \tag{3.17}$$

The first term on the right-hand side of Eq. (3.17) is the analog of the MHD contribution. The second and third contributions contain both temporal and spatial derivatives of ψ and describe the energy-momentum transfer from the pseudoscalar field to the hypermagnetic field. Depending on the initial topology of the hypermagnetic flux lines this process can even produce hypermagnetic knots (see the third and fourth papers of Ref. [11]). In Eq. (3.17) there are various other terms proportional to the gradients of ψ and carrying terms $\mathcal{O}(\sigma^{-2})$, $\mathcal{O}(\sigma^{-3})$ and so on and so forth. Using Eq. (3.17) into Eq. (3.14) we obtain the wanted form of the magnetic diffusivity equation,

$$\frac{\partial\vec{B}}{\partial\tau} = \frac{\vec{\nabla}\times(\partial_\tau\psi\vec{B})}{4\pi M\sigma} - \frac{\vec{\nabla}\times[(\vec{\nabla}\psi)\times(\vec{v}\times\vec{B})]}{4\pi\sigma M} + \vec{\nabla}\times(\vec{v}\times\vec{B}) + \frac{\nabla^2\vec{B}}{4\pi\sigma}. \tag{3.18}$$

Equation (3.18) generalizes Eq. (2.2) and the first term on the right-hand side can be interpreted as a current density flowing along the magnetic field. From Eq. (3.18) it can be immediately argued that whenever the conductivity is high (and the ideal limit can be enforced) the magnetic current is suppressed by the value of the conductivity.

The total energy-momentum tensor $T_{\rm tot}^{\mu\nu}$ is covariantly conserved, i.e. $\nabla_\mu T_{\rm tot}^{\mu\nu} = 0$ as it can be easily argued by combining Eqs. (3.4), (3.5), (3.9), and (3.10). Also the total entropy of the system is covariantly conserved. However, recalling the first principle of thermodynamics and the fundamental thermodynamic identity, it can be easily shown that the entropy of the global fluid of charged species obeying Eq. (3.8) is not conserved and the corresponding evolution equation of the entropy four-vector is

$$\nabla_\mu \varsigma_f^\mu = \frac{\sigma}{T}Y_{\alpha\beta}Y^{\nu\alpha}u_\nu u^\beta, \qquad \varsigma_f = \frac{\rho + p}{\bar{T}}, \tag{3.19}$$

where $\varsigma_f^\mu = \varsigma_f u^\mu$ and the term appearing on the right-hand side of the conservation equation is nothing but the relativistic generalization of the heating due to the Joule effect; $\bar{T} = aT$ denotes the comoving temperature. When the conductivity vanishes gauge fields can be amplified thanks to the coupling to the pseudoscalar field and this phenomenon has been studied in various space-times and, in particular, during a quasi–de Sitter stage of expansion [24–26] (see also the third and fourth papers in Ref. [11]). If we start with a field configuration carrying zero magnetic helicity the pseudoscalar coupling discussed here can produce configurations characterized by nonvanishing magnetic helicity which have been dubbed hypermagnetic knots.

The presence of anomalous symmetries has some relation to the so-called magnetogenesis problem (see Ref. [27] for an explanation of this terminology). In short the punch line of the discussion is that anomalous vertices cannot alone produce sizeable magnetic fields. The reason for this statement can be already appreciated from Eq. (3.15). The pumping action of the axionic coupling carries a time derivative of the axion, a wave number and one inverse power of M. This term must overwhelm the contribution of the Laplacian of the magnetic field if amplification is to take place. In short this never happens except from a tiny slice of Fourier modes concentrated around the event horizon (during the de Sitter phase) or the particle horizon (during a decelerated phase) [24–26]. In the case of a single chemical potential the situation is even worse, in some sense, because any possible breaking of conformal invariance is completely overwhelmed by the contribution of the conductivity, as we shall see in the forthcoming sections.

IV. ANOMALOUS SYMMETRIES AT FINITE DENSITY

The derivation of the previous section assumed the global neutrality of the plasma and the covariant conservation of the total energy-momentum tensor. The latter approach shall now be reversed by dealing directly with the currents rather than with a specific form of the action. In this respect the simplified model discussed in the present section is instructive insofar as it contemplates the simultaneous presence of two currents, one anomalous and the other nonanomalous (to be identified, in the language of the previous section, with the hyperelectric current). The logic is, in short, the following. The anomalous current j_R^μ is not covariantly conserved because of the anomaly contribution,

$$\nabla_\mu j_R^\mu = \mathcal{A}_R Y_{\mu\nu}\tilde{Y}^{\mu\nu}. \tag{4.1}$$

The equations of the energy-momentum tensor of the fluid and the covariant conservation of the nonanomalous four-current are instead

$$\nabla_\mu T^{\mu\nu} = Y_{\nu\alpha} j^\alpha, \qquad \nabla_\mu j^\mu = 0. \tag{4.2}$$

The equation for $T^{\mu\nu}$ appearing in Eq. (4.2) can be split in terms of the two projections along u^ν and along $h_\alpha^\nu = \delta_\alpha^\nu - u^\nu u_\alpha$ [see Eqs. (A1) and (A2) of Appendix A]. The system of equations (4.1) and (4.2) approximately describes different physical situations ranging from the anomalous plasma in the symmetric phase of the electroweak theory [11], to the models of chiral liquids [12] which are proposed as a simplified framework for the discussion of the quark-gluon plasma. Indeed, above the critical temperature of the corresponding phase transition the electroweak symmetry is restored, and the nonscreened gauge field strength $Y_{\mu\nu}$ corresponds to the $\mathrm{U}(1)_Y$ hypercharge group. The system of equations (4.1) and (4.2) extends the hydrodynamic approach described in Ref. [28] to the extent that j^μ does not coincide with j_R^μ and the ambient plasma is globally neutral. As already mentioned, unlike previous analyses bounded to a special relativistic treatment, the generally covariant discussion is more suitable for the class of problems addressed here.

A. Useful thermodynamic relations

Denoting with μ_R the chemical potential associated with the anomalous species, the first principle of thermodynamics implies

$$dE = TdS - pdV + \mu_R dN_R. \tag{4.3}$$

Dividing the fundamental thermodynamics identity (i.e. $E = TS - pV + \mu_R N_R$) by a fiducial volume we obtain the well-known relation $\rho + p = T\varsigma + \mu_R \tilde{n}_R$. Differentiating the fundamental thermodynamic identity and subtracting the obtained result from Eq. (4.3) a known relation between the ordinary derivatives of the temperature, of the chemical potential and of the pressure can be obtained and it is, in the present case,

$$\varsigma\partial_\alpha T + \tilde{n}_R \partial_\alpha \mu_R = \partial_\alpha p. \tag{4.4}$$

The anomalous current of Ref. [11] was associated with the slowest perturbative processes related to the $\mathrm{U}(1)_Y$ anomaly, namely the processes flipping the chirality of the right electron which are in thermal equilibrium until sufficiently late because of the smallness of their Yukawa coupling. The origin of the anomalous current is not essential for the present ends but what matters is the physical and mathematical distinction between anomalous and conduction (possibly Ohmic) currents. According to this approach, the general expression of the anomalous current must contemplate an inviscid contribution supplemented by a viscous term, i.e. $j_R^\mu = \tilde{n}_R u^\mu + \nu_R^\mu$, where ν_R^μ denotes the dissipative coefficient. The four-velocity of the anomalous species coincides with the bulk velocity of the plasma and, therefore, $u_R^\mu \simeq u^\mu$. This assumption simplifies the discussion a bit and corresponds to the logic followed in this investigation (see also Ref. [11]) where the single-fluid approach is privileged. As in the analysis of nonanomalous plasmas (see Sec. II), in AMHD it is also possible to discuss a multifluid approach entailing different velocities for the different species.

Let us now pause for a moment and recall the main features of the dissipative description adopted hereunder. Whenever dissipative effects are included in both the energy-momentum tensor and in the particle current the physical meaning of the four-velocity u^μ must be specified. In the Eckart approach u^{μ} coincides with the velocity of particle transport [29]. Conversely, in the Landau approach [30] the velocity u^{μ} coincides with the velocity of the energy transport defined by the $(0i)$ component of the energy-momentum tensor giving the energy flux. The Landau approach shall be privileged with the important caveat that in a perfect conductor Lorentz invariance is broken and a preferred frame (i.e. the plasma frame) arises naturally; in this frame hyperelectric fields are exactly vanishing when the conductivity goes to infinity. In the Landau approach we shall have that the global charge neutrality of the plasma is enforced by requiring that $j^{\mu}u_{\mu} = 0$. If the plasma *is not* globally neutral, i.e. $j^{\mu}u_{\mu} = \tilde{n}$, then a second chemical potential must be introduced so that Eq. (4.4) will be

$$\varsigma\partial_{\alpha}T + \tilde{n}_{R}\partial_{\alpha}\mu_{R} + \tilde{n}\partial_{\alpha}\mu = \partial_{\alpha}p, \tag{4.5}$$

and $w = T\varsigma + \tilde{n}\mu + \tilde{n}_{R}\mu_{R}$. The case of a plasma which is not neutral will be treated in more detail in Appendix C with the purpose of showing that the coefficients of the magnetic and vortical currents are subjected to a higher degree of arbitrariness, as we shall discuss more precisely at the end of this section.

B. Joule heating

Equations (4.1), (4.2), and (4.3) can be rephrased in terms of the entropy density. The projection of the first expression reported in Eq. (4.2) along the four-velocity u^{ν} implies—according to the results of Appendix A—the relation

$$\nabla_{\mu}[(p+\rho)u^{\mu}] - u^{\nu}\partial_{\nu}p - u^{\nu}Y_{\nu\alpha}j^{\alpha} = 0, \tag{4.6}$$

which can be further modified by using Eq. (4.3) together with the fundamental thermodynamic identity; the result of this manipulation is

$$\nabla_{\mu}[\varsigma u^{\mu} - \bar{\mu}_{R}\nu_{R}^{\mu}] + \nu_{R}^{\mu}\partial_{\mu}\bar{\mu}_{R} + \mathcal{A}_{R}\bar{\mu}_{R}Y_{\mu\nu}\tilde{Y}^{\mu\nu} = \frac{u^{\nu}}{T}Y_{\nu\alpha}j^{\alpha}, \tag{4.7}$$

where $\bar{\mu}_{R} = \mu_{R}/T$ denotes the rescaled chemical potential. Equation (4.7) can be manipulated by inserting the explicit expressions of the Ohmic [23] and anomalous currents i.e. $j^{\alpha} = \sigma_{c}Y^{\alpha\nu}u_{\nu} + \nu^{\alpha}$ and $j^{\alpha} = n_{R}u^{\alpha} + \nu_{R}^{\alpha}$,

$$\nabla_{\mu}[\varsigma u^{\mu} - \bar{\mu}_{R}\nu_{R}^{\mu}] + \nu_{R}^{\mu}\partial_{\mu}\bar{\mu}_{R} + \mathcal{A}_{R}\bar{\mu}_{R}Y_{\mu\nu}\tilde{Y}^{\mu\nu} = \left(\frac{\sigma_{c}}{T}\right)Y^{\alpha\beta}Y_{\nu\alpha}u^{\nu}u_{\beta} - \frac{\nu^{\alpha}}{T}u^{\beta}Y_{\beta\alpha}. \tag{4.8}$$

The second law of thermodynamics implies that the covariant divergence of the entropy four-vector ς^{μ} must be positive semidefinite, i.e. $\nabla_{\mu}\varsigma^{\mu} \geq 0$. Absent any anomalous current, the entropy of the fluid obeys $\nabla_{\mu}\varsigma^{\mu} = (\sigma_{c}/T)Y_{\alpha\beta}Y^{\nu\alpha}u_{\nu}u^{\beta}$, where the term on the right-hand side is the relativistic generalization of the Joule effect. This is indeed the same kind of relation already obtained in Eq. (3.19) of the preceding section.

The specific definition of the entropy four-vector depends on the chemical potential of the system. However, since the coefficient $\mathcal{A}_{R}$ does not have a definite sign, the anomalous currents may even lead to violation of the second principle of thermodynamics (e.g. $\nabla_{\mu}\varsigma^{\mu} < 0$). Starting from a covariantly conserved total energy-momentum tensor without dissipative effects, the entropy four-vector is covariantly conserved. The increase of the entropy signals the presence of dissipative effects, as in the case of Joule heating. Conversely the decrease of the entropy is the result of an incomplete definition of the entropy four-vector which is not sufficiently general, as argued in Ref. [28]. Two further kinetic coefficients $\mathcal{S}_{\omega}$ and $\mathcal{S}_{B}$ will then be introduced so that the generalized entropy four-vector ς^{μ} will become

$$\varsigma^{\mu} = \varsigma u^{\mu} - \bar{\mu}_{R}\nu_{R}^{\mu} + \mathcal{S}_{\omega}\omega^{\mu} + \mathcal{S}_{B}\mathcal{B}^{\mu}, \tag{4.9}$$

where $\mathcal{S}_{\omega}$ and $\mathcal{S}_{B}$ depend on the chemical potential and the pressure, but the arguments of these functions shall not be explicitly written to avoid tedious expressions.

The vorticity four-vector ω^{μ} appearing in Eq. (4.9) is defined as

$$\omega^{\mu} = \tilde{f}^{\mu\alpha}u_{\alpha}, \qquad f_{\beta\gamma} = \nabla_{\beta}u_{\gamma} - \nabla_{\gamma}u_{\beta}, \tag{4.10}$$

where $\tilde{f}^{\mu\alpha} = E^{\mu\alpha\beta\gamma}f_{\beta\gamma}/2$ is the dual tensor. In Appendices A and B a collection of technical results on the general relativistic treatment of the magnetic and vortical currents has been included. The results reported in the appendices are by no means exhaustive and are only instrumental in easing the derivation of some expressions appearing hereunder. In connection with Eq. (4.9) it is interesting to notice that the appearance of the vortical current in the relativistic treatment can be physically motivated from the observation that the sum of the vorticity and of the magnetic field is conserved by the time evolution in flat space-time and in the nonrelativistic limit. More specifically in an electron-ion plasma, introducing the ion mass M, the sum $[(M/e)\vec{\omega} + \vec{B}]$ is conserved [31,32] and this is essentially the Einstein–de Haas effect [31]. This conservation law can be generalized in curved space-time geometries [32]. Finally, by inserting the entropy four-vector defined in Eq. (4.9) into Eq. (4.8) it is straightforward to obtain the following result:

$$\nabla_{\mu}\varsigma^{\mu} - \frac{\sigma_{c}}{T}Y^{\alpha\beta}Y_{\nu\alpha}u^{\nu}u_{\beta} = \nabla_{\mu}(\mathcal{S}_{\omega}\omega^{\mu} + \mathcal{S}_{B}\mathcal{B}^{\mu}) + \frac{\nu^{\alpha}u^{\beta}}{T}Y_{\alpha\beta} - \partial_{\beta}\bar{\mu}_{R}\nu_{R}^{\beta} - \mathcal{A}_{R}\bar{\mu}_{R}Y_{\alpha\beta}\tilde{Y}^{\alpha\beta}. \tag{4.11}$$

C. Hypermagnetic and vortical currents

The coefficients ν^{α} and ν_{R}^{α} appearing in Eqs. (4.8) and (4.11) must also be expressible as a combination of the vortical current and of the hypermagnetic current. Four different coefficients parametrize the relation between $(\nu^\alpha, \nu_R^\alpha)$ and $(\omega^\alpha, \mathcal{B}^\alpha)$,

$$\nu^\alpha = \Lambda_\omega \omega^\alpha + \Lambda_B \mathcal{B}^\alpha, \qquad \nu_R^\alpha = \Lambda_{R\omega}\omega^\alpha + \Lambda_{RB}\mathcal{B}^\alpha. \tag{4.12}$$

Provided that the coefficients introduced in Eq. (4.12) are specifically related to $\mathcal{S}_\omega$ and $\mathcal{S}_B$, the whole expression on the right-hand side of Eq. (4.11) vanishes and the left-hand side of Eq. (4.11) reproduces the standard result due to Joule heating in a conducting plasma. The relation stemming from Eq. (4.11) can be obtained with simple manipulations and it is given by

$$\omega^\alpha \partial_\alpha \mathcal{S}_\omega + \mathcal{B}^\alpha \partial_\alpha \mathcal{S}_B + \mathcal{S}_\omega \nabla_\alpha \omega^\alpha + \mathcal{S}_B \nabla_\alpha \mathcal{B}^\alpha + 4\bar{\mu}_R \mathcal{A}_R \mathcal{E}^\alpha \mathcal{B}_\alpha = \frac{\mathcal{E}_\alpha \nu^\alpha}{T} + \nu_R^\alpha \partial_\alpha \bar{\mu}_R. \tag{4.13}$$

Exploiting the general results of Eqs. (A7) and (A8) in the case of the Ohmic current supplemented by the dissipative coefficient, the generally covariant four-divergences of ω^α and $\mathcal{B}^\alpha$ are[2]

$$\nabla_\alpha \omega^\alpha = -\frac{2}{w}\omega^\alpha \partial_\alpha p - \frac{2}{w}\nu^\beta \omega^\alpha Y_{\alpha\beta} - \frac{2\sigma_c}{w} Y^{\beta\gamma} Y_{\alpha\beta} u_\gamma \omega^\alpha, \tag{4.14}$$

$$\nabla_\alpha \mathcal{B}^\alpha = 2Y_{\rho\sigma}\omega^\rho u^\sigma + \frac{1}{w}\tilde{Y}^{\mu\alpha} u_\mu \partial_\alpha p + \frac{\sigma_c}{w}\tilde{Y}^{\mu\alpha} Y^{\beta\gamma} Y_{\alpha\beta} u_\gamma u_\mu + \frac{1}{w}\tilde{Y}^{\mu\alpha} Y_{\alpha\beta}\nu^\beta u_\mu. \tag{4.15}$$

Introducing the fields $\mathcal{E}^\mu = Y^{\mu\alpha} u_\alpha$ and $\mathcal{B}^\mu = \tilde{Y}^{\mu\alpha} u_\alpha$, Eqs. (4.14) and (4.15) can be further modified,

$$\nabla_\alpha \omega^\alpha = -\frac{2}{w}\omega^\alpha \partial_\alpha p - \frac{2}{w}\omega^\alpha \mathcal{E}_\alpha \nu^\beta u_\beta - \frac{2}{w} u^\rho \mathcal{B}^\sigma \omega^\alpha [\nu^\beta + \sigma_c \mathcal{E}^\beta] E_{\alpha\beta\rho\sigma}, \tag{4.16}$$

$$\nabla_\alpha \mathcal{B}^\alpha = 2\omega^\alpha \mathcal{E}_\alpha - \frac{1}{w}\partial_\alpha p \mathcal{B}^\alpha - \frac{1}{w} u^\beta \nu_\beta \mathcal{E}_\alpha \mathcal{B}^\alpha. \tag{4.17}$$

Concerning Eqs. (4.16) and (4.17) a simple comment is in order. In the Landau approach the terms $u_\beta \nu^\beta$ and $u_\alpha \nu_R^\alpha$ vanish exactly. This is of course also true when the dissipative coefficients are defined as in Eq. (4.12) as can be explicitly verified since, by definition, $u_\beta \omega^\beta$ and $u_\beta \mathcal{B}^\beta$ vanish exactly. Equations (4.16) and (4.17) can be finally inserted into Eq. (4.13) with the result that

$$\omega^\alpha \mathcal{P}_\alpha + \mathcal{B}^\alpha \mathcal{Q}_\alpha + \left[2\mathcal{S}_B - \left(\frac{\Lambda_\omega}{T}\right)\right]\omega^\alpha \mathcal{B}_\alpha + \left[4\bar{\mu}_R \mathcal{A}_R - \left(\frac{\Lambda_B}{T}\right)\right](\mathcal{E}^\alpha \mathcal{B}_\alpha) - \frac{2}{w}\sigma_c \omega^\alpha \mathcal{E}^\beta u^\mu \mathcal{B}^\nu E_{\alpha\beta\mu\nu} \mathcal{S}_\omega = 0, \tag{4.18}$$

where $\mathcal{P}_\alpha$ and $\mathcal{Q}_\alpha$ are defined, respectively, as

$$\mathcal{P}_\alpha = \partial_\alpha \mathcal{S}_\omega - \frac{2}{w}\mathcal{S}_\omega \partial p - \partial_\alpha \bar{\mu}_R \Lambda_{R\omega}, \tag{4.19}$$

$$\mathcal{Q}_\alpha = \partial_\alpha \mathcal{S}_B - \frac{\mathcal{S}_B}{w}\mathcal{S}_\omega \partial_\alpha p - \partial_\alpha \bar{\mu}_R \Lambda_{RB}. \tag{4.20}$$

The last term in Eq. (4.18) contains the explicit dependence on the conductivity. All the other terms of similar origin vanish because of the symmetry properties of the various currents. The results of Eqs. (4.18), (4.19), and (4.20) follow easily if we recall that, by definition, $u^\alpha \omega_\alpha$, $u^\beta \mathcal{E}_\beta$ and $u^\gamma \mathcal{B}_\gamma$ are all vanishing.

In Eq. (4.18) there should also be a term containing $\mathcal{S}_B$ and corresponding to the one including the explicit dependence on S_ω and on the conductivity (i.e. the term proportional to $\omega^\alpha \mathcal{E}^\beta u^\mu \mathcal{B}^\nu E_{\alpha\beta\mu\nu}$). This term vanishes, as expected, since it would have the same form as the last term of Eq. (4.18) but with ω^α replaced by $\mathcal{B}^\alpha$: the overall coefficient will therefore contain the contraction of $\mathcal{B}^\alpha \mathcal{B}^\nu$ with $E_{\nu\alpha\beta\mu}$ (which is totally antisymmetric) so that the final contribution of this term will vanish exactly. It is relevant to stress here that the possibility of a consistent analysis of the conducting case rests on the inclusion of the electric degrees of freedom. It would therefore be incorrect to set $\mathcal{E}^\alpha = 0$ from the beginning since this would forbid a precise analysis of the perfectly conducting limit which is one of the purposes of the present investigation.

D. Consistency relations

To satisfy Eq. (4.18) the four-vectors multiplying ω^α and $\mathcal{B}^\alpha$ must vanish together with the coefficients of the terms multiplied by $\omega^\alpha \mathcal{B}_\alpha$ and $\mathcal{E}^\alpha \mathcal{B}_\alpha$. Moreover the supplementary term proportional to $\omega^\alpha \mathcal{E}^\beta u^\mu \mathcal{B}^\nu E_{\alpha\beta\mu\nu}$ must also vanish. To preserve the second principle of thermodynamics in a globally neutral plasma with anomalous currents and Joule heating we must have that

$$\mathcal{P}_\alpha = 0, \qquad \mathcal{Q}_\alpha = 0, \qquad \Lambda_B = 4\mu_R \mathcal{A}_R, \qquad \Lambda_\omega = 2T\mathcal{S}_B, \qquad \mathcal{S}_\omega = 0. \tag{4.21}$$

If, as established, $\mathcal{S}_\omega = 0$ then Eq. (4.18) also implies that $\Lambda_{R\omega} = 0$. All the coefficients we ought to determine depend on $\bar{\mu}_R$ and on the pressure. Thus the conditions of Eq. (4.21) are equivalent to the following system of equations:

[2]Recall that w denotes, in the present paper, the enthalpy density.

$$\left(\frac{\partial \mathcal{S}_B}{\partial p} - \frac{S_B}{w}\right)\partial_\alpha p + \left(\frac{\partial \mathcal{S}_B}{\partial \bar{\mu}_R} - \Lambda_{RB}\right)\partial_\alpha \bar{\mu}_R = 0, \quad (4.22)$$

$$\Lambda_\omega = 2T\mathcal{S}_B, \qquad \Lambda_B = 4\mathcal{A}_R\bar{\mu}_R T. \quad (4.23)$$

The standard thermodynamic relations giving the partial derivatives of the pressure and of the rescaled chemical potential with respect to the temperature are

$$\left(\frac{\partial p}{\partial T}\right) = \frac{w}{T} + \tilde{n}_R\left(\frac{\partial \bar{\mu}_R}{\partial T}\right),$$
$$\left(\frac{\partial \bar{\mu}_R}{\partial T}\right) = -\frac{w}{\tilde{n}_R T^2} + \frac{1}{\tilde{n}_R T}\left(\frac{\partial p}{\partial T}\right), \quad (4.24)$$

implying that the partial derivatives of each variable with respect to the temperature (when the other variable is held fixed) are

$$\left(\frac{\partial p}{\partial T}\right)_{\bar{\mu}_R} = \frac{w}{T}, \qquad \left(\frac{\partial \bar{\mu}_R}{\partial T}\right)_p = -\frac{w}{\tilde{n}_R T^2}. \quad (4.25)$$

With the results of Eqs. (4.24) and (4.25), Eqs. (4.22) and (4.23) can be explicitly solved,

$$\mathcal{S}_B(\bar{\mu}_R, T) = T a_B(\bar{\mu}_R), \qquad \Lambda_{RB} = \frac{\partial}{\partial \bar{\mu}_R}[T a_B(\bar{\mu}_R)], \quad (4.26)$$

$$\Lambda_\omega(\bar{\mu}_R, T) = 2T^2 a_B(\bar{\mu}_R), \qquad \Lambda_B(\bar{\mu}_R, T) = 4\mathcal{A}_R\bar{\mu}_R T, \quad (4.27)$$

where $a_B(\bar{\mu}_R)$ is an arbitrary function of the rescaled chemical potential. Note also that Λ_B is fully determined in terms of the coefficient of the anomaly and it is, in practice, only a function of the chemical potential itself since, by definition, $\bar{\mu}_R T = \mu_R$. These consistency relations will also be discussed in Sec. V.

All in all, the presence of an anomalous current induces—thanks to the second principle of thermodynamics—two further terms in the Ohmic current. Starting with a globally neutral plasma with an anomalous current, the second principle of thermodynamics implies that the nonanomalous current must contain magnetic and vortical contributions resembling the magnetic currents induced by pseudoscalar fields. The induced current can be compared with the effective action for the hypercharge fields at finite fermionic density. In the case of right electrons $\mathcal{A}_R = -g'^2 y_R^2/(64\pi^2)$, where g' denotes the gauge coupling and $y_R = -2$ is the hypercharge assignment of the right electrons. In the comoving frame (see Appendix B) the interaction induced by the computed term is

$$-4\sqrt{-g}\mu_R\mathcal{A}_R Y_\mu \tilde{Y}^{\mu\nu}\frac{g_{\nu 0}}{g_{00}} = \frac{g'^2}{4\pi^2}\mu_R\epsilon^{ijk}Y_{ij}Y_k. \quad (4.28)$$

The results discussed so far refer to the globally neutral case where the current is Ohmic. If the plasma is not globally neutral the degree of arbitrariness in the determination of the consistency relations increases since the coefficients $\mathcal{S}_\omega$ and $\mathcal{S}_B$ will also depend on the chemical potential of the global charge of the plasma. This analysis is reported, for completeness, in Appendix C and has also been discussed, from a different perspective, in Ref. [33].

V. IDEAL AND RESISTIVE LIMITS IN AMHD

The generally covariant discussion of the magnetic and Ohmic currents will now serve as a starting point for the analysis of conformally flat background geometries of the Friedmann-Robertson-Walker type, $g_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}$, which are just slightly more restrictive than the ones discussed in Sec. III. The evolution equations of the system become particularly simple in terms of the rescaled electric and magnetic fields already introduced in Eqs. (3.13), (3.14), and (3.15),

$$\vec{\nabla}\cdot\vec{E} = 0, \qquad \vec{\nabla}\cdot\vec{B} = 0, \quad (5.1)$$

$$\vec{\nabla}\times\vec{E} + \partial_\tau\vec{B} = 0,$$
$$\vec{\nabla}\times\vec{B} - \partial_\tau\vec{E} = 4\pi\vec{J} + \bar{\Lambda}_\omega\vec{\omega} - \bar{\Lambda}_B\vec{B}, \quad (5.2)$$

where the two quantities $\bar{\Lambda}_\omega$ and $\bar{\Lambda}_B$ are defined as $\bar{\Lambda}_\omega = 4\pi a^2\Lambda_\omega$ and $\bar{\Lambda}_B = 4\pi a\Lambda_B$. Using Eqs. (4.26) and (4.27) their explicit form is

$$\bar{\Lambda}_\omega = 8\pi a^2 T^2 a_B(\bar{\mu}_R), \qquad \bar{\Lambda}_B = 16\pi T a\mathcal{A}_R\bar{\mu}_R. \quad (5.3)$$

From the projection of Eq. (4.2) in the direction orthogonal to u_ν [as discussed in Eq. (A2) of Appendix A] the evolution equations of the bulk velocity of the plasma are given by

$$\partial_\tau[W\vec{v}] + (\vec{v}\cdot\vec{\nabla})[W\vec{v}] + \vec{v}\,\vec{\nabla}\cdot[W\vec{v}]$$
$$= -\vec{\nabla}P + \vec{J}\times\vec{B} + \eta\left[\nabla^2\vec{v} + \frac{1}{3}\vec{\nabla}(\vec{\nabla}\cdot\vec{v})\right], \quad (5.4)$$

$$\partial_\tau\epsilon + \vec{\nabla}\cdot[W\vec{v}] - \vec{E}\cdot\vec{J} = 0, \quad (5.5)$$

where W denotes the rescaled enthalpy density and (ϵ, P) are the rescaled energy density and pressure,

$$W = a^4 w = a^4(p+\rho) = \epsilon + P,$$
$$P = a^4 p, \qquad \epsilon = a^4\rho. \quad (5.6)$$

Equations (5.4) and (5.5) can be simplified in the case of an incompressible closure where $\vec{\nabla}\cdot\vec{v} = 0$ even if this is probably not the most physically justified closure prior to matter-radiation equality (see e.g. Ref. [20]). For the slow modes of the plasma the displacement current can be dropped in Eq. (5.2) so that the generalized magnetic diffusivity equation is

$$\partial_\tau\vec{B} = \vec{\nabla}\times(\vec{v}\times\vec{B}) + \frac{\nabla^2\vec{B}}{4\pi\sigma} + \frac{\vec{\nabla}\times(\bar{\Lambda}_\omega\vec{\omega})}{4\pi\sigma} - \frac{\vec{\nabla}\times(\bar{\Lambda}_B\vec{B})}{4\pi\sigma}. \quad (5.7)$$

Equation (5.7) should be compared with Eq. (3.18) in the pseudoscalar case. Focusing our attention on the terms containing the conductivity, we have

$$\frac{\bar{\Lambda}_{\omega}}{4\pi\sigma} = \frac{\bar{T}^2}{\sigma} a_B(\bar{\mu}_R), \qquad \frac{\bar{\Lambda}_B}{4\pi\sigma} = \frac{\bar{T}}{4\pi\sigma} \mathcal{A}_R \bar{\mu}_R, \tag{5.8}$$

where $\bar{T} = aT$ denotes the comoving temperature and $\sigma = \sigma_c a$ is the comoving conductivity.

The rescaled chemical potential enters the infinitely conducting limit of Eq. (5.8) since it is generally plausible that $\bar{\mu}_R \ll 1$ while $\bar{T}$ and σ are approximately constant in time. The smallness of the particle asymmetries is the rationale for the minuteness of the rescaled chemical potentials in approximate thermal equilibrium. Positing, for simplicity, that all the species can be treated as being ultrarelativistic at temperatures larger than a certain reference temperature (e.g. the temperature of the electroweak phase transition) and assuming the minimal standard model of electroweak interactions with three families and massless neutrinos, there are three conserved global charges supplemented by the hypercharge and by the third component of the weak isospin. If the plasma is hypercharge neutral the value of the chemical potential can be estimated from the asymmetry in the case where all the standard model charges are in complete thermal equilibrium [11]. If all the asymmetry is attributed to the right electrons (which is, in some sense, the most favorable situation) then $\bar{\mu}_R = (87\pi^2/220) N_{\rm eff}(n_R/\varsigma)$, where $N_{\rm eff} = 106.75$. This means that, indeed, $\bar{\mu}_R \ll 1$.

Denoting with m the mass of the lightest charge carrier, $\sigma \propto \sigma_0 \bar{T}(1 + ma/\bar{T})^{-1/2}$ and σ_0 can be estimated in explicit models like the ones of Ref. [34]. In the case of an electromagnetic plasma $\sigma_0 \propto \alpha_{\rm em}^{-1}$. The balance between the two terms in Eq. (5.8) depends on the value of $a_B(\bar{\mu}_R)$ (which is not fixed), but in the limit of infinite conductivity Eq. (5.7) leads to

$$\partial_\tau \vec{B} = \vec{\nabla} \times (\vec{v} \times \vec{B}) + \mathcal{O}\left(\frac{\bar{\mu}_R}{\sigma}\right), \tag{5.9}$$

which is qualitatively similar to the result of Eq. (3.18). Defining the vector potential in the Coulomb gauge, Eq. (5.9) becomes, up to small corrections, $\partial_\tau \vec{A} = \vec{v} \times (\vec{\nabla} \times \vec{A})$. The classic analysis of Woltjer and Chandrasekhar [35] (see also Refs. [36,37]) can then be exploited. The magnetic energy density shall then be minimized in a finite volume under the assumption of constant magnetic helicity by introducing the Lagrange multiplier ζ. By taking the functional variation of[3]

$$\mathcal{G} = \int_V d^3x \{|\vec{\nabla} \times \vec{A}|^2 - \zeta \vec{A} \cdot (\vec{\nabla} \times \vec{A})\} \tag{5.10}$$

with respect to $\vec{A}$ and by requiring $\delta\mathcal{G} = 0$, the configurations minimizing $\mathcal{G}$ are such that $\vec{\nabla} \times \vec{B} = \zeta \vec{B}$. These configurations have been used to describe hypermagnetic knots (see the third and fourth papers of Ref. [11]); in this case ζ has the dimension of inverse length and gives the scale of the hypermagnetic knot, which is related to Chern-Simons waves. Configurations with finite energy and finite helicity can also be constructed [11,38]. The configurations with constant ζ also represent the lowest state of magnetic energy which a closed system may attain in the case where anomalous currents are present, provided the ambient plasma is perfectly conducting.

The limit $\sigma \to \infty$ can be corroborated by explicit solutions that are valid in the presence of anomalous symmetries in conformally flat space-time geometries and by minimizing, asymptotically, the functional of Eq. (5.10). Let us now use the configurations (5.10) and try to find solutions of our system. For the sake of simplicity we shall assume the constancy of the rescaled enthalpy W both in space and time. This means that the rescaled energy density and pressure are also constant in time provided that the plasma is dominated by radiation and $P = \epsilon/3$. For consistency the fluid should be incompressible in the absence of the relativistic fluctuations of the geometry (see, however, Ref. [20]). Under these simplifying (but realistic) assumptions Eqs. (5.4) and (5.7) can be rewritten as

$$\partial_\tau \vec{v} = \vec{v} \times \vec{\omega} - \vec{\nabla}\left(\frac{P}{W} + \frac{v^2}{2}\right) + \frac{\vec{J} \times \vec{B}}{W} + \nu_{\rm kin} \nabla^2 \vec{v}, \tag{5.11}$$

$$\partial_\tau \vec{B} = \vec{\nabla} \times (\vec{v} \times \vec{B}) + \lambda_\omega \vec{\nabla} \times \vec{\omega} - \lambda_B \vec{\nabla} \times \vec{B} + \nu_{\rm mag} \nabla^2 \vec{B}, \tag{5.12}$$

where $\nu_{\rm kin} = (\eta/W)$ and $\nu_{\rm mag} 1/(4\pi\sigma)$. Equations (5.11) and (5.12) are symmetric for the generalized self-similarity transformations

$$\vec{x} \to \ell \vec{x}, \qquad \tau \to \ell^{1-\delta} \tau, \qquad \vec{v} \to \ell^{\delta} \vec{v}, \qquad \vec{B} \to \ell^{\delta} \vec{B} \tag{5.13}$$

in the so-called inertial range [i.e. when the magnetic forcing is absent from the right-hand side of Eq. (5.11)] and provided that $(\nu_{\rm kin}, \nu_{\rm mag})$ transform as $(\nu_{\rm kin}, \nu_{\rm mag}) \to (\nu_{\rm kin} \nu_{\rm mag}) \ell^{1+\delta}$. The similarity transformation of Eq. (5.13) holds true if $(\lambda_\omega, \lambda_B)$ transform as $\lambda_\omega \to \lambda_\omega \ell^{\delta}$ and $\lambda_B \to \lambda_B \ell^{\delta}$. Recalling that $\lambda_\omega \propto f_\omega(\bar{\mu}_R) \nu_{\rm mag}$ and $\lambda_B \propto f_B(\bar{\mu}_R) \nu_{\rm mag}$, then it also follows that $f_\omega(\bar{\mu}_R)$ and $f_B(\bar{\mu}_R)$ must scale as ℓ^{-1} if the symmetry holds true. The latter considerations generalize the similarity symmetry used by Olesen (see e.g. the third paper of Ref. [21]) to analyze the conditions for inverse cascades in the standard hydromagnetic situation.

[3]Following the treatment of Ref. [35] (see also Refs. [36,37]) we assume that V is the fiducial volume of a closed system. In the present case it could be identified, for instance, with the volume of the particle horizon at a given epoch after the end of inflation.

Two solutions shall now be discussed. In the first case the magnetic field is given by $\vec{B} = \vec{H}_0 + \vec{H}$, where $\vec{H}_0$ is a space-time constant while $\vec{H}$ and $\vec{v}$ are inhomogeneous and depend both on space and time. In the second case both $\vec{B}$ and $\vec{v}$ will be taken to be fully inhomogeneous. Defining the auxiliary fields $\vec{h} = \vec{H}/\sqrt{4\pi W}$ and $\vec{h}_0 = \vec{H}_0/\sqrt{4\pi W}$, Eqs. (5.11) and (5.12) are expressible as

$$\partial_\tau \vec{v} = \vec{v} \times (\vec{\nabla} \times \vec{v}) + (\vec{\nabla} \times \vec{h}) \times \vec{h} + (\vec{h}_0 \cdot \vec{\nabla})\vec{h} - \sqrt{\frac{4\pi}{W}} \lambda_\omega \sigma(\vec{\omega} \times \vec{h}_0 + \vec{\omega} \times \vec{h}) + \nu_{\rm kin} \nabla^2 \vec{v}, \quad (5.14)$$

$$\partial_\tau \vec{h} = \vec{\nabla} \times (\vec{v} \times \vec{h}) + (\vec{h}_0 \cdot \vec{\nabla})\vec{v} + \frac{\lambda_\omega}{\sqrt{4\pi W}} \vec{\nabla} \times \vec{\omega} - \lambda_B \vec{\nabla} \times \vec{h} + \nu_{\rm mag} \nabla^2 \vec{h}. \quad (5.15)$$

After careful inspection of Eqs. (5.14) and (5.15) there are two possibilities for a consistent solution. If $\lambda_\omega = 0$ and $h_0 \neq 0$, Eqs. (5.14) and (5.15) are solved provided that the functional of Eq. (5.10) is minimized and, consequently, $\vec{\nabla} \times \vec{h} = k\vec{h}$ and $\vec{\nabla} \times \vec{v} = k\vec{v}$. The full solution can be expressed in a specific Cartesian coordinate system as

$$\vec{v}(k, z, \tau) = v(\tau)[\cos(kz)\hat{e}_x - \sin(kz)\hat{e}_y], \quad \vec{h}(k, z, \tau) = h(\tau)[\sin(kz)\hat{e}_x + \cos(kz)\hat{e}_y]. \quad (5.16)$$

The functions $v(\tau)$ and $h(\tau)$ appearing in Eq. (5.16) must then obey:

$$\partial_\tau v = k h_0 h - \nu_{\rm kin} k^2 v, \quad \partial_\tau h = -k h_0 v - \lambda_B k h - \nu_{\rm mag} k^2 h. \quad (5.17)$$

As anticipated there is also a second solution of Eqs. (5.14) and (5.15) which can be obtained by setting $h_0 = 0$ and by demanding that the velocity and the rescaled hypermagnetic field are parallel, i.e. $\vec{v} \times \vec{h} = 0$ (i.e. $\vec{v} \| \vec{h}$). In the latter case, defining $\vec{v} = v(\tau)\hat{n}$ and $\vec{h} = h(\tau)\hat{n}$, we have that

$$\partial_\tau v + k^2 \nu_{\rm kin} v = 0, \quad \partial_\tau h + k^2 \nu_{\rm mag} h = -\frac{k^2 \lambda_\omega}{\sqrt{4\pi W}} v - k\lambda_B h, \quad (5.18)$$

where, as before, $\vec{\nabla} \times \vec{h} = k\vec{h}$ and Eq. (5.10) is minimized. Equations (5.17) and (5.18) can be used to investigate the limit of the solutions for infinite conductivity and check that it coincides with the solution of the limit. For instance, Eq. (5.17) in the infinite conductivity limit (i.e. $\nu_{\rm mag} \to 0$ and $\lambda_B \to 0$) for an inviscid fluid (i.e. $\nu_{\rm kin} \to 0$) can be solved with the result that $v(\tau) = v_* \cos(kh_0\tau + \varphi_*)$ and $h(\tau) = -v_* \sin(kh_0\tau + \varphi_*)$, which is exactly the solution expected in the absence of anomalous currents (see e.g. the last two papers in Ref. [39]).

VI. CONCLUDING REMARKS

Hydromagnetic nonlinearities in charged liquids at high magnetic Reynolds numbers lead to large-scale magnetic fields which are parallel rather than orthogonal to the current. Anomalous symmetries produce a similar effect that may even interfere with standard hydromagnetic results in a turbulent environment. Two distinct but equally plausible situations have been specifically scrutinized in a globally neutral system at finite conductivity: a plasma containing pseudoscalar species and the anomalous currents induced by finite-density effects.

The analysis of pseudoscalar species is simplified by the covariant conservation of the total energy-momentum tensor of the system. The slow modes (i.e. the modes for which the propagation of electromagnetic disturbances is negligible) obey a generalized magnetic diffusivity equation where the anomalous effects are suppressed as long as the plasma is globally neutral, the pseudoscalar field is quasihomogeneous, and the conductivity is parametrically large. Instead of positing a specific action it is possible to consider the currents themselves as the building blocks of the physical description of the plasma. The simplest case in the framework of anomalous magnetohydrodynamics contains two currents—one anomalous and the other nonanomalous—that are both constrained by the canonical form of the Joule heating and by the second principle of thermodynamics. Supplementary terms have been shown to arise in the Ohmic current. While this treatment resembles the hydrodynamic approach to anomalous symmetries, in the present analysis, the hyperelectric current is not anomalous. The generalized magnetic diffusivity equation has been shown to also include terms proportional to the vorticity four-vector, which is intuitively plausible by thinking of the Einstein–de Haas effect in a globally neutral plasma. The anomalous currents contribute to the evolution of the bulk velocity of the plasma and to the generalized magnetic diffusivity equation. The perfectly conducting limit suppresses the anomalous contributions and the configurations minimizing the energy density with the constraint that the magnetic helicity be conserved, which coincides with the ones obtainable in ideal magnetohydrodynamics where anomalous currents are absent. This observation has been used to derive hypermagnetic knot solutions in a hot plasma from their magnetic counterpart.

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APPENDIX A: SOME USEFUL GENERALLY COVARIANT RELATIONS

Consider a generally relativistic plasma characterized by a gauge field strength $Y_{\alpha\beta}$, current j_α and four-velocity u_α.

Using the equations of the gauge fields (i.e. $\nabla_\mu Y^{\mu\nu} = 4\pi j^\nu$ and $\nabla_\mu \tilde{Y}^{\mu\nu} = 0$) the conservation of the energy-momentum tensor implies that $\nabla_\mu T^{\mu\nu} = Y_{\nu\alpha} j^\alpha$. The latter relation can be projected along two orthogonal directions, i.e. u^ν and $h^\alpha_\nu = \delta^\alpha_\nu - u^\alpha u_\nu$, with the result

$$\nabla_\mu [w u^\mu] - u^\nu \partial_\nu p = Y_{\nu\alpha} u^\nu j^\alpha, \tag{A1}$$

$$w u^\mu \nabla_\mu u_\nu - \partial_\nu p + u_\nu u^\mu \partial_\mu p = Y_{\nu\beta} j^\beta - Y_{\alpha\beta} u^\alpha j^\beta u_\nu, \tag{A2}$$

where $w = (\rho + p)$ denotes the enthalpy density of the fluid. The electric and magnetic fields are nonrelativistic concepts, while in relativistic terms the correct quantity to employ is the Maxwell field strength and its dual. It is sometimes useful to decompose the gauge field strength in terms of $\mathcal{E}^\mu$ and $\mathcal{B}^\mu$,

$$\begin{aligned} Y_{\alpha\beta} &= \mathcal{E}_{[\alpha} u_{\beta]} + \frac{1}{2} E_{\alpha\beta\rho\sigma} u^{[\rho} \mathcal{B}^{\sigma]}, \\ E_{\alpha\beta\rho\sigma} &= \sqrt{-g}\,\epsilon_{\alpha\beta\rho\sigma}, \end{aligned} \tag{A3}$$

where $\epsilon_{\alpha\beta\rho\sigma}$ is the Levi-Civita symbol in four dimensions and $\mathcal{E}_{[\alpha} u_{\beta]} = \mathcal{E}_\alpha u_\beta - \mathcal{E}_\beta u_\alpha$. From the definition of dual field strength in a four-dimensional curved space-time, i.e. $\tilde{Y}^{\mu\nu} = E^{\mu\nu\rho\sigma} Y_{\rho\sigma}/2$, we shall have, in terms of $\mathcal{E}_\alpha$ and $\mathcal{B}_\beta$,

$$\tilde{Y}^{\alpha\beta} = \mathcal{B}^{[\alpha} u^{\beta]} + \frac{1}{2} E^{\alpha\beta\rho\sigma} \mathcal{E}_{[\rho} u_{\sigma]}. \tag{A4}$$

In full analogy with the gauge field strength we can also define the vorticity four-vector,

$$\begin{aligned} \omega^\mu &= \tilde{f}^{\mu\alpha} u_\alpha \equiv \frac{1}{2} E^{\mu\alpha\beta\gamma} u_\alpha f_{\beta\gamma}, \\ f_{\beta\gamma} &= \nabla_\beta u_\gamma - \nabla_\gamma u_\beta. \end{aligned} \tag{A5}$$

Equation (A5) can be inverted in terms of $f_{\gamma\beta}$ and the result is

$$f_{\gamma\beta} = -E_{\gamma\beta\lambda\sigma} \omega^\lambda u^\sigma + u_{[\gamma} u^\sigma \nabla_\sigma u_{\beta]}. \tag{A6}$$

Recalling Eqs. (A1), (A2), and (A6) the covariant derivative of ω^μ can therefore be expressed as

$$\nabla_\mu \omega^\mu = -\frac{2\omega^\alpha}{w}\left(\partial_\alpha p + Y_{\alpha\sigma} j^\sigma\right). \tag{A7}$$

In analogy with Eq. (A7) the covariant divergences of $\mathcal{B}^\mu$ and $\mathcal{E}^\mu$ become

$$\nabla_\mu \mathcal{B}^\mu = 2 Y_{\rho\sigma} \omega^\rho u^\sigma + \frac{u_\mu \partial_\alpha p}{w} \tilde{Y}^{\mu\alpha} + \frac{u_\mu Y_{\alpha\beta}}{w} j^\beta \tilde{Y}^{\mu\alpha}, \tag{A8}$$

$$\nabla_\mu \mathcal{E}^\mu = 4\pi j^\alpha u_\alpha - \tilde{Y}^{\mu\rho} \omega_\mu u_\rho + Y^{\beta\gamma} \frac{u_\beta \partial_\gamma p}{w} + \frac{Y^{\beta\gamma} u_\beta Y_{\gamma\alpha} j^\alpha}{w}. \tag{A9}$$

In the special case where the plasma is not globally neutral and the electric current is $j^\alpha = \tilde{n} u^\alpha$, Eqs. (A7) and (A8) become, respectively,

$$\nabla_\mu \omega^\mu = -2\frac{\omega^\alpha \partial_\alpha p}{w} - 2\tilde{n}\frac{\mathcal{E}^\alpha \omega_\alpha}{w}, \tag{A10}$$

$$\nabla_\mu \mathcal{B}^\mu = 2\mathcal{E}_\alpha \omega^\alpha - \frac{\mathcal{B}^\alpha \partial_\alpha p}{w} - \frac{\tilde{n}}{w} \mathcal{E}_\alpha \mathcal{B}^\alpha, \tag{A11}$$

$$\nabla_\mu \mathcal{E}^\mu = 4\pi\tilde{n} - \omega_\alpha \mathcal{B}^\alpha - \frac{\mathcal{E}^\alpha \partial_\alpha p}{w} - \frac{\tilde{n}\mathcal{E}_\alpha \mathcal{E}^\alpha}{w}. \tag{A12}$$

In the absence of gauge fields, the relativistic generalization of the Helmholtz equation can be written as

$$u^\alpha \nabla_\alpha \omega^\mu + \nabla_\alpha u^\alpha \omega^\mu - \omega^\alpha \nabla_\alpha u^\mu + (u^\alpha \omega^\mu + u^\mu \omega^\alpha) \times \frac{\partial_\alpha p}{w} = 0. \tag{A13}$$

APPENDIX B: COMOVING FRAME AND PHYSICAL FIELDS

In comoving coordinates $u_\mu = g_{0\mu}/\sqrt{g_{00}}$ and $u^\mu = \delta^\mu_0/\sqrt{g_{00}}$. In the comoving frame the auxiliary fields defined in Eq. (A4) are $\mathcal{E}^\mu = (0, \mathcal{E}^i)$ and $\mathcal{B}^\mu = (0, \mathcal{B}^i)$, where

$$\mathcal{E}^i = \frac{Y^{i0}}{\sqrt{g_{00}}}, \qquad \mathcal{B}^i = \frac{\tilde{Y}^{i0}}{\sqrt{g_{00}}}. \tag{B1}$$

Since $\mathcal{E}^i$ and $\mathcal{B}^i$ are not three-dimensional fields but rather the spatial components of a contravariant four-vector, the corresponding covariant components will be obviously given by $\mathcal{E}_m = g_{mi}\sqrt{g_{00}} Y^{i0}$ and $\mathcal{B}_m = g_{mi}\sqrt{g_{00}} \tilde{Y}^{i0}$.

In a perfect conductor, i.e. when the conductivity is infinite, the electric fields are completely screened. Conversely, at finite conductivity electric fields are suppressed. In both cases Lorentz invariance is broken and it is convenient to introduce a frame (the so-called plasma frame) where the electric fields vanish. The spatial components of $\mathcal{E}^\mu$ and $\mathcal{B}^\mu$ do not coincide with the three-dimensional fields e^i and b^i. The three-dimensional fields can be defined as $Y^{i0} = g^{00} e^i$ and $Y^{ij} = -g^{00} \epsilon^{ijk} b_k$.

Since $\sqrt{-g} Y^{\mu\nu}$ and $\sqrt{-g}\tilde{Y}^{\mu\nu}$ are both invariant under Weyl rescaling, two Weyl-invariant combinations can be introduced, i.e. $\bar{\mathcal{E}}^\mu = \sqrt{-g} Y^{\mu\nu} \bar{u}_\nu$ and $\bar{\mathcal{B}}^\mu = \sqrt{-g} Y^{\mu\nu} \bar{u}_\nu$, where $\bar{u}_\nu$ satisfies $\eta_{\mu\nu} \bar{u}^\nu \bar{u}^\mu = 1$ and $\eta_{\mu\nu}$ is the Minkowski metric. The comoving electric and magnetic fields in three-dimensional notation are $\vec{E} = g^{00}\sqrt{-g}\vec{e}$ and $\vec{B} = g^{00}\sqrt{-g}\vec{b}$. Using the standard Arnowitt-Deser-Misner (ADM) decomposition the comoving fields are $\vec{E} = (\sqrt{\gamma}/N)\vec{e}$ and

$\vec{B} = (\sqrt{\gamma}/N)\vec{b}$ and coincide with the ones discussed, for instance, in Ref. [32]. Following the definitions spelled out in this appendix and consistently followed in the paper we have that

$$Y_{\mu\nu}\tilde{Y}^{\mu\nu} = 4\mathcal{B}_{\mu}\mathcal{E}^{\mu} = -4\frac{\vec{E}\cdot\vec{B}}{\sqrt{-g}}, \tag{B2}$$

where $\sqrt{-g} = N\sqrt{\gamma}$ in the framework of the ADM decomposition. Finally, in the case of a conformally flat geometry we can write that the metric is $g_{\mu\nu} = (-g)^{1/4}\eta_{\mu\nu}$ and the various definitions simplify so that, for instance, $\vec{E} = (-g)^{1/4}\vec{e}$, $\vec{B} = (-g)^{1/4}\vec{b}$, and so on and so forth.

APPENDIX C: THE CASE OF A NON-NEUTRAL PLASMA

The results of Eqs. (4.26) and (4.27) hold in the case of a globally neutral plasma where Ohmic and anomalous currents are simultaneously present. This situation will now be compared with the case where—instead of an Ohmic current—we have an ordinary particle current and the plasma is not globally neutral. In this case we shall have two chemical potentials: one related to the anomalous current and the other related to the particle current. The thermodynamical relations will therefore be modified and, for instance, the enthalpy density will be given by $w = T\varsigma + \tilde{n}\mu + n_R\mu_R$. Repeating the same steps discussed before, we shall have that

$$\nabla_{\alpha}[(\varsigma - \bar{\mu}_R - \bar{\mu})u^{\alpha}] + \nu_R^{\alpha}\partial_{\alpha}\bar{\mu}_R + \nu^{\alpha}\partial_{\alpha}\bar{\mu} - 4\mathcal{A}_R\bar{\mu}_R\mathcal{E}_{\alpha}\mathcal{B}^{\alpha} + \frac{\nu_{\alpha}\mathcal{E}^{\alpha}}{T} = 0. \tag{C1}$$

The same steps outlined above can then be repeated. By defining the entropy four-vector as in Eq. (4.9), the covariant four-divergence of ς^{μ} becomes

$$\nabla_{\mu}\varsigma^{\mu} = \nabla_{\mu}(\mathcal{S}_{\omega}\omega^{\mu} + \mathcal{S}_B\mathcal{B}^{\mu}) - \nu_R^{\alpha}\partial_{\alpha}\bar{\mu}_R - \nu^{\alpha}\partial_{\alpha}\bar{\mu} + 4\mathcal{A}_R\bar{\mu}_R\mathcal{E}_{\alpha}\mathcal{B}^{\alpha} - \frac{\nu_{\alpha}\mathcal{E}^{\alpha}}{T}. \tag{C2}$$

We can now recall, from the general expressions of Appendix A and B, that

$$\begin{aligned}\nabla_{\alpha}\omega^{\alpha} &= -\frac{2}{w}\omega^{\alpha}\partial_{\alpha}p - \frac{2}{w}\tilde{n}\mathcal{E}_{\alpha}\omega^{\alpha},\\ \nabla_{\alpha}\mathcal{B}^{\alpha} &= 2\omega^{\alpha}\mathcal{E}_{\alpha} - \frac{1}{w}\partial_{\alpha}p\mathcal{B}^{\alpha} - \frac{\tilde{n}}{w}\mathcal{E}_{\alpha}\mathcal{B}^{\alpha}.\end{aligned} \tag{C3}$$

In this case the expressions of $\mathcal{P}_{\alpha}$ and $\mathcal{Q}_{\alpha}$ of Eqs. (4.19) and (4.20) become

$$\mathcal{P}_{\alpha} = \partial_{\alpha}\mathcal{S}_{\omega} - \frac{2}{w}\mathcal{S}_{\omega}\partial p - \partial_{\alpha}\bar{\mu}_R\Lambda_{R\omega} - \partial_{\alpha}\bar{\mu}\Lambda_{\omega}, \tag{C4}$$

$$\mathcal{Q}_{\alpha} = \partial_{\alpha}\mathcal{S}_B - \frac{\mathcal{S}_B}{w}\mathcal{S}_{\omega}\partial p - \partial_{\alpha}\bar{\mu}_R\Lambda_{RB} - \partial_{\alpha}\bar{\mu}\Lambda_B. \tag{C5}$$

Two further conditions can be derived by requiring the coefficients of $\mathcal{E}_{\alpha}\omega^{\alpha}$ and $\mathcal{E}_{\alpha}\mathcal{B}^{\alpha}$ vanish. The two relations are

$$\begin{aligned}2\mathcal{S}_B - \frac{\Lambda_{\omega}}{T} - 2\frac{\tilde{n}}{w}\mathcal{S}_{\omega} &= 0,\\ 4\mathcal{A}_R\bar{\mu}_R - \frac{\Lambda_B}{T} - \frac{\tilde{n}}{w}\mathcal{S}_B &= 0.\end{aligned} \tag{C6}$$

In this case $\mathcal{S}_{\omega}$ is not bound to vanish but, conversely, the system depends on a number of arbitrary functions. More precisely, we have that

$$\begin{aligned}\mathcal{S}_{\omega}(T, \bar{\mu}, \bar{\mu}_R) &= T^2 a_{\omega}(\bar{\mu}, \bar{\mu}_R),\\ \mathcal{S}_B(T, \bar{\mu}, \bar{\mu}_R) &= T a_B(\bar{\mu}, \bar{\mu}_R),\end{aligned} \tag{C7}$$

$$\begin{aligned}\Lambda_{\omega}(T, \bar{\mu}, \bar{\mu}_R) &= \frac{\partial}{\partial\bar{\mu}}[T^2 a_{\omega}(\bar{\mu}, \bar{\mu}_R)],\\ \Lambda_B(T, \bar{\mu}, \bar{\mu}_R) &= \frac{\partial}{\partial\bar{\mu}}[T a_B(\bar{\mu}, \bar{\mu}_R)],\end{aligned} \tag{C8}$$

$$\begin{aligned}\Lambda_{\omega R}(T, \bar{\mu}, \bar{\mu}_R) &= \frac{\partial}{\partial\bar{\mu}_R}[T^2 a_{\omega}(\bar{\mu}, \bar{\mu}_R)],\\ \Lambda_{BR}(T, \bar{\mu}, \bar{\mu}_R) &= \frac{\partial}{\partial\bar{\mu}_R}[T a_B(\bar{\mu}, \bar{\mu}_R)].\end{aligned} \tag{C9}$$

From Eq. (C6) it follows that

$$\frac{\partial a_B}{\partial\bar{\mu}} = 4\mathcal{A}_R\bar{\mu}_R, \qquad \frac{\partial a_{\omega}}{\partial\bar{\mu}} = 2a_B. \tag{C10}$$

After integrating the two equations of Eq. (C10) we have that

$$\begin{aligned}a_B(\bar{\mu}, \bar{\mu}_R) &= 4\mathcal{A}_R\bar{\mu}_R\bar{\mu} + f(\bar{\mu}_R),\\ a_{\omega}(\bar{\mu}, \bar{\mu}_R) &= 4\mathcal{A}_R\bar{\mu}_R\bar{\mu}^2 + \bar{\mu}f(\bar{\mu}_R) + g(\bar{\mu}),\end{aligned} \tag{C11}$$

where $f(\bar{\mu}_R)$ and $g(\bar{\mu})$ are two arbitrary functions of the corresponding arguments. In the simplest situation we can set both arbitrary functions to zero and, therefore,

$$\begin{aligned}\Lambda_{\omega} &= 8\mathcal{A}_R\mu\mu_R\left(1 - \frac{2nT\bar{\mu}}{w}\right),\\ \Lambda_B &= 4\mathcal{A}_R\mu\mu_R\left(1 - \frac{nT\bar{\mu}}{w}\right).\end{aligned} \tag{C12}$$

In a relativistic plasma in thermal equilibrium, both corrections appearing in Eq. (C12) go as μ/T.

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