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Weyl-invariant Kaluza-Klein theory and the teleparallel equivalent of Weyl-invariant general relativity

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A simple method is shown to demonstrate that the teleparallel equivalent of general relativity can be generalized to the Weyl-invariant models. We will also show explicitly that Weyl symmetry is preserved step by step throughout the 5D Kaluza-Klein dimensional-reduction process. As a result, the dimensional reduced model will be shown to be a theory with two scalar fields. When a symmetry-breaking potential is introduced, a strong constraint will effectively turn off one of the scalar fields. For heuristic reasons, the stability properties of the power-law solution associated with the resulting one-scalar-field model will be presented explicitly. In particular, all stable modes can be solved explicitly as functions of the free parameter associated with the symmetry-breaking potential.

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I. INTRODUCTION

Scale-invariant theory has been proposed to be a successful model as the effective theory of our physical Universe for a number of reasons [1–3]. Accumulating evidence also indicates that the scale symmetry could be very important in many physical applications of interest such as scale-invariant QCD [4]. In addition, it is also claimed that Weyl symmetry could be related to the Higgs problem in the electroweak theory [5]. Moreover, the Weyl gauge field is also known to be a possible dark-matter candidate. Many applications of Weyl symmetry can also be found in the study of the evolution of the early Universe [6–10].

In addition, the Kaluza-Klein (KK) theory is one of the successful approaches to unify the gravitational and electromagnetic interactions. In this paper, we will try to reveal the enriched symmetries of the Weyl-invariant Kaluza-Klein theory by incorporating the symmetric properties of the U_1 gauge field A_μ and the Weyl connection field S_μ in a harmonic way.

On the other hand, the teleparallel equivalent of general relativity (TEGR) has been known to provide a new way to look at the geometrical structure of Weyl-Cartan-Weitzenböck gravity. Indeed, the equivalence relation is given by the identity

$$R = -T - 2D^\mu T_\mu$$

with $T_\mu = T^\nu{}_{\mu\nu}$ defined as the trace of the torsion tensor $T^\alpha{}_{\mu\nu}$. The applications of the generalized theory with $f(T)$ have also been a focus of research interests lately. Note that this identity remains valid in any arbitrary dimension. There have thus been a number of interesting progresses in both the TEGR and Kaluza-Klein approaches to the gravitational theories lately [11–19].

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In this paper, we will show that this equivalence relation can be generalized to incorporate Weyl symmetry in a consistent way. For a simple demonstration, we will also analyze the effect of a simple 5D Weyl-invariant Kaluza-Klein model in this paper. The dimensional reduced action will be presented as an apparently 4D Weyl-invariant model by redefining all scalar fields according to their proper conformal dimensions. As a result, the resulting 4D model will have two scalar fields actively coupled to the system. One of them is the original Weyl scalar field ψ responsible for the scale symmetry in 5D. The other one is the scalar field ϕ that is associated with the dynamics of the 5D metric.

Note that there is a very special property associated with the Weyl-invariant theory: once the scale symmetry is broken by the introduction of a symmetry-breaking potential $V(\psi, \chi)$, the consistent vacuum configuration of the system will be quite different as compared to most other conventional theories. To be more specific, many field theories admit a vacuum of the form $\partial_\psi V(\psi_0, \chi_0) = \partial_\chi V(\psi_0, \chi_0) = 0$ that acts effectively as a cosmological constant. We will show that the vacuum of the Weyl-invariant model takes the form $\psi_0 \partial_\psi V(\psi_0, \chi_0) + \chi_0 \partial_\chi V(\psi_0, \chi_0) = 4V(\psi_0, \chi_0)$ for any symmetry-breaking potential $V(\psi, \chi)$ coupled to the Weyl-invariant system [20–29].

In fact, this constraint is valid for all on-shell scalar fields, not only for the vacuum configuration. Therefore, the physical scalar fields will be frozen to one of the solutions to the constraint equation [1]. If $V = V(\chi)$ is a function of the scalar field χ only, the scalar field has to be a constant once a symmetry-breaking potential is introduced. If there is another scalar field ψ coupled to the symmetry-breaking potential, e.g., $V = V(\psi, \chi)$, this constraint equation simply implies that the dynamics of one of the two scalar fields can be absorbed into the dynamics of the other scalar field. The resulting theory will then be effectively equivalent to a one-scalar-field model with a conventional effective potential free from any further constraint derived from Weyl symmetry.

Motivated by the 4D Weyl-invariant gravity derived from the 5D Weyl-invariant Kaluza-Klein model, we will hence study the possible cosmological implications of a one-scalar-field model with an effective symmetry-breaking potential. In particular, we will discuss the effect of this model in the evolution of a Bianchi type I (BI) space-time [30–43].

This paper will be organized as follows: (i) a brief introduction is presented in Sec. I; (ii) in Sec. II, we will briefly review Weyl symmetry in an n-dimensional space-time; (iii) the Weyl-invariant generalization of the TEGR will be shown explicitly in Sec. III; (iv) the Kaluza-Klein approach to the Weyl-invariant 5D model will be shown in Sec. IV; (v) in Sec. V, an effective 4D Weyl-invariant gravity will be presented in detail. We will also show that a strong constraint on the symmetry-breaking potential is present in this section. (vi) We will set a gauge choice to remove the dynamics of one of the scalar fields in Sec. VI. A natural and compatible choice of the Weyl vector meson will also be imposed in the BI metric space. (vii) A set of power-law solutions will be shown in detail in Sec. VII in the BI space. (viii) A stability analysis of the power-law solution will be presented in Sec. VIII. (ix) Finally, conclusions and discussions will be summarized in Sec. IX.

II. WEYL-INVARIANT GRAVITY IN n DIMENSIONS

Note that the Weyl transformation is a gauge transformation that relates physical fields in different length scale according to their conformal dimensions. For example, dimension-one scalar field ψ and the metric field $g_{\mu\nu}$ will transform, respectively, as

$$\psi \to \psi^\Omega = \Omega^{-1}\psi, \tag{2.1}$$

$$g_{\mu\nu} \to g^\Omega_{\mu\nu} = \Omega^2 g_{\mu\nu}. \tag{2.2}$$

In order to preserve Weyl symmetry, the ordinary derivative ∂_μ will have to be replaced by a Weyl covariant derivative ∇_μ such that the transformation properties of $\nabla_\mu \mathcal{T}$ can remain the same as any tensor $\mathcal{T}$ throughout the formulation. For example, the Weyl covariant derivative of a scalar field ψ can be defined as

$$\nabla_\mu \psi = (\partial_\mu - S_\mu)\psi, \tag{2.3}$$

with the help of a Weyl gauge field S_μ. Note that S_μ is also referred to as the Weyl connection or the Weyl vector meson. As a result, the scale transformation of $\nabla_\mu\psi$ will be similar to the scale transformation of the scalar field ψ:

$$\nabla_\mu\psi \to (\nabla_\mu\psi)^\Omega = \Omega^{-1}\nabla_\mu\psi \tag{2.4}$$

provided that the Weyl gauge field transforms as

$$S_\mu \to S^\Omega_\mu = S_\mu - \partial_\mu \ln\Omega. \tag{2.5}$$

Similarly, the Weyl covariant derivative of any tensor field $\mathcal{T}$ should take the following form:

$$\nabla_\mu \mathcal{T} = (\partial_\mu + nS_\mu)\mathcal{T}, \tag{2.6}$$

if the tensor field $\mathcal{T}$ transforms as

$$\mathcal{T} \to \mathcal{T}^\Omega = \Omega^n \mathcal{T} \tag{2.7}$$

under the Weyl transformation. For example, the Weyl covariant derivative of the metric field $g_{\mu\nu}$ should take the following form:

$$\tilde{\partial}_\alpha g_{\mu\nu} \equiv (\partial_\alpha + 2S_\alpha)g_{\mu\nu}. \tag{2.8}$$

Here the notation $\tilde{A}$ will be used to denote Weyl covariant generalization of an operator or a field A. This is designed to specify the difference between the introduction of the spin connection $\Gamma^\alpha_{\mu\nu}$ and the Weyl connection S_α. Indeed, in order to preserve the covariant properties under the general coordinate transformation $x^\mu \to x'^\mu(x)$, we also need to introduce the spin connection $\Gamma^\alpha_{\mu\nu}$. For example, $D_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\alpha_{\mu\nu}A_\alpha$ for any type $T(0,1)$ tensor field A_μ. For clarity, the notation D_μ will be used to denote the covariant derivative that preserves the covariant properties under the general coordinate transformations, while the notation $\tilde{\partial}_\mu$ will denote the Weyl covariant derivative involving only the Weyl connection S_μ. Finally, the notation $\nabla_\mu \equiv \tilde{D}_\mu$ will thus be defined as the fully covariant derivative under both the Weyl and the general coordinate transformation. For example, we can show that

$$\begin{aligned} \nabla_\mu A^\mu &= (\partial_\mu + \tilde{\Gamma}_\mu - 2S_\mu)A^\mu \\ &= [D_\mu + (n-2)S_\mu]A^\mu \\ &= \nabla^\mu A_\mu = (\partial^\mu - \tilde{\Gamma}^\mu)A_\mu \\ &= [D^\mu + (n-2)S^\mu]A_\mu, \end{aligned} \tag{2.9}$$

assuming that A_μ does not change under the Weyl transformation. Here we have taken into account the fact that $A^\nu = A_\alpha g^{\alpha\nu}$ transforms effectively as $g^{\alpha\nu}$ under the Weyl transformation. In addition, we have also used the following identities of the spin connections: $\tilde{\Gamma}_\mu \equiv \tilde{\Gamma}^\nu_{\mu\nu} = \Gamma_\mu + nS_\mu$ and $\tilde{\Gamma}^\mu \equiv g^{\alpha\nu}\tilde{\Gamma}^\mu_{\alpha\nu} = \Gamma^\mu + (2-n)S^\mu$.

Indeed, the Weyl covariant generalization of the spin connection can be shown to be

$$\begin{aligned} \tilde{\Gamma}^\sigma{}_{\mu\nu} &= \frac{1}{2}g^{\sigma\rho}(\tilde{\partial}_\mu g_{\nu\rho} + \tilde{\partial}_\nu g_{\rho\mu} - \tilde{\partial}_\rho g_{\mu\nu}) \\ &= \Gamma^\sigma{}_{\mu\nu} + (S_\mu g_\nu{}^\sigma + S_\nu g_\mu{}^\sigma - S^\sigma g_{\mu\nu}). \end{aligned} \tag{2.10}$$

Note that the Weyl covariant generalization of the spin connection $\tilde{\Gamma}^\sigma{}_{\mu\nu}$ is scale invariant by itself.

As a result, we can show that the action of a minimal Weyl-invariant theory in n dimensions is given by

$$\int d^n x \sqrt{g} \mathcal{L} = \int d^n x \sqrt{g} \left[\psi^{n-2} \tilde{R} + \psi^{n-4} \left(-\frac{1}{2} \nabla_\mu \psi \nabla^\mu \psi - \frac{1}{4} H_{\mu\nu} H^{\mu\nu} \right) \right]. \tag{2.11}$$

Here the Weyl-invariant field strength of the gauge field is defined as $H_{\mu\nu} \equiv \partial_\mu S_\nu - \partial_\nu S_\mu$. In addition, the Weyl covariant scalar curvature $\tilde{R}$ is defined as

$$\begin{aligned}\tilde{R} &= \partial_\rho \tilde{\Gamma}^\rho{}_{\nu\mu} - \partial_\nu \tilde{\Gamma}_\mu + \tilde{\Gamma}_\lambda \tilde{\Gamma}^\lambda{}_{\nu\mu} - \tilde{\Gamma}^\rho{}_{\nu\lambda} \tilde{\Gamma}^\lambda{}_{\rho\mu} \\ &= R - 2(n-1) D_\mu S^\mu - (n-1)(n-2) S_\mu S^\mu. \end{aligned} \tag{2.12}$$

Note that the Riemann tensor is a gauge-invariant tensor:

$$\tilde{R}^{\Omega}{}_{\mu\alpha\nu}^{\rho} = \tilde{R}^\rho{}_{\mu\alpha\nu}, \tag{2.13}$$

such that the scalar curvature $\tilde{R}$ transforms as

$$\tilde{R}^\Omega = \Omega^{-2} \tilde{R} \tag{2.14}$$

under the Weyl transformation. As a result, we can show that the action (2.11) given above is invariant under the Weyl transformation.

III. THE WEYL-INVARIANT TELEPARALLEL EQUIVALENT OF WEYL-INVARIANT GENERAL RELATIVITY

In order to obtain the teleparallel equivalent of Weyl-invariant general relativity in n dimensions, we need to write the metric as a combination of the vielbein:

$$g_{\mu\nu} = e_\mu{}^a e_{\nu a} \tag{3.1}$$

with the flat index a raised and lowered by the flat metric η_{ab}. Note that the torsion tensor is defined as

$$T^\lambda{}_{\mu\nu} \equiv e^\lambda{}_b (D_\mu e_\nu{}^b - D_\nu e_\mu{}^b) = e^\lambda{}_b (\partial_\mu e_\nu{}^b - \partial_\nu e_\mu{}^b) \tag{3.2}$$

with $D_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a - \Gamma^\alpha_{\mu\nu} e_\alpha{}^a$ the covariant derivative of the vielbein $e_\nu{}^a$. In addition, we can define the contorsion tensor as

$$K^\lambda{}_{\nu\mu} \equiv e^\lambda{}_b D_\mu e_\nu{}^b \tag{3.3}$$

with the explicit relation to the torsion tensor given by

$$K^\rho{}_{\mu\nu} = \frac{1}{2} (T_\mu{}^\rho{}_\nu + T_\nu{}^\rho{}_\mu - T^\rho{}_{\mu\nu}). \tag{3.4}$$

We will also define the contracted torsion and contorsion tensors as $T_\mu \equiv T^\nu{}_{\mu\nu}$ and $K_\mu \equiv K^\nu{}_{\mu\nu}$, respectively, for convenience. They are related by the simple relation

$$K_\mu = -T_\mu. \tag{3.5}$$

In addition, we can also show that $K^\rho{}_{\mu\nu} g^{\mu\nu} = T^\rho = g^{\rho\mu} T_\mu$. In order to prove the equivalence relation between general relativity and teleparallel gravity given, respectively, by the action R and $-T$, we need to define the torsion scalar as

$$T = \frac{1}{4} T^{\mu\nu\alpha} T_{\mu\nu\alpha} + \frac{1}{2} T^{\mu\nu\alpha} T_{\alpha\nu\mu} - T^\mu T_\mu. \tag{3.6}$$

Before we can demonstrate the proof of the TEGR relation, we need to show that the Ricci curvature tensor is related to the contorsion tensor by the following relation:

$$R_{\mu\nu} = D_\nu K_\mu - D_\lambda K^\lambda{}_{\mu\nu} + K^\rho{}_{\alpha\nu} K^\alpha{}_{\mu\rho} - K_\lambda K^\lambda{}_{\mu\nu}. \tag{3.7}$$

The proof is quite straightforward. Indeed, we can show explicitly that

$$R_{\mu\nu} = e^\lambda{}_a [D_\mu, D_\lambda] e_\nu{}^a \tag{3.8}$$

holds by directly appealing to the definition of the contorsion tensor given by Eq. (3.3). As a result, we can easily derive the following relation:

$$R = -2 D_\mu T^\mu + g^{\mu\nu} K^\rho{}_{\alpha\nu} K^\alpha{}_{\mu\rho} + T_\mu T^\mu \tag{3.9}$$

by taking the trace of the Ricci tensor $R = g^{\mu\nu} R_{\mu\nu}$. Finally, we can show that the following identity:

$$g^{\mu\nu} K^\rho{}_{\alpha\nu} K^\alpha{}_{\mu\rho} = -\frac{1}{4} T^{\mu\nu\alpha} T_{\mu\nu\alpha} - \frac{1}{2} T^{\mu\nu\alpha} T_{\alpha\nu\mu} \tag{3.10}$$

holds in any arbitrary dimension. With this result, we can finally make the conclusion that R and T are related by the well-known TEGR relation

$$R + T + 2 D_\mu T^\mu = 0. \tag{3.11}$$

Note that this proof has nothing to do with the dimension of space-time. It hence holds in any arbitrary dimension. Therefore, we can fairy say that the TEGR action can be written as

$$-\int dx^n \sqrt{g} T = \int dx^n \sqrt{g} R \tag{3.12}$$

by ignoring the total derivative term $\sqrt{g} D_\mu T^\mu$.

A. The Weyl-invariant teleparallel equivalent of Weyl-invariant general relativity

In addition to the TEGR relation shown above, we can also show that this relation remains valid in its Weyl covariant generalization. The proof is in fact quite straightforward. We can simply add an appropriate Weyl connection to both sides of the above equation in order to make them manifestly Weyl covariant term by term. More specifically, we want to show that Eq. (3.11) can be generalized as

$$\tilde{R} = -\tilde{T} - 2\nabla^\mu(\tilde{T}_\mu). \quad (3.13)$$

If we can show explicitly that the S_μ-dependent terms on the left-hand side of the above equation equal exactly to the S_μ-dependent terms on the right-hand side of the above equation, then we can show that Eq. (3.13) will reduce to Eq. (3.11). This then proves that the equivalence relation, Eq. (3.11), can be generalized to its Weyl-invariant version, Eq. (3.13).

Note that the partial derivative of the vielbein is given by

$$\partial_\alpha e_\mu{}^a \to \tilde{\partial}_\alpha e_\mu{}^a \equiv (\partial_\alpha + S_\alpha) e_\mu{}^a, \quad (3.14)$$

such that the Weyl transformation properties of $\tilde{\partial}_\alpha e_\mu{}^a$ and $e_\mu{}^a$ are similar to each other:

$$[\tilde{\partial}_\alpha e_\mu{}^a]^\Omega = \Omega \tilde{\partial}_\alpha e_\mu{}^a. \quad (3.15)$$

As a result, we can show that the functional form of the Weyl-invariant teleparallel Lagrangian $\tilde{T}$ and the original torsion Lagrangian T are related by the following equation:

$$\tilde{T} = T - 2(n-2)S^\mu T_\mu - (n-1)(n-2)S_\mu S^\mu. \quad (3.16)$$

Note that the Weyl transformation of $\tilde{T}$ is given by

$$\tilde{T}^\Omega = \Omega^{-2}\tilde{T}. \quad (3.17)$$

In addition, we can also show that the Weyl gauge field dependence of $\nabla_\mu \tilde{T}^\mu$ is given by

$$\nabla_\mu \tilde{T}^\mu = D_\mu T^\mu + (n-2)S_\mu T^\mu + (n-1)D_\mu S^\mu + (n-1)(n-2)S_\mu S^\mu. \quad (3.18)$$

Together with the relation between $\tilde{R}$ and R shown earlier in Eq. (2.12),

$$\tilde{R} = R - 2(n-1)D_\mu S^\mu - (n-1)(n-2)S_\mu S^\mu, \quad (3.19)$$

we can formally show that

$$\tilde{R} + \tilde{T} + 2\nabla_\mu \tilde{T}^\mu = R + T + 2D_\mu T^\mu = 0. \quad (3.20)$$

Note again that this equivalence relation is also known to remain valid in any arbitrary dimension n.

IV. KALUZA-KLEIN APPROACH TO WEYL-INVARIANT GRAVITY

The Kaluza-Klein approach is a successful method to unify the Yang-Mills field and gravitational fields. It is also known that Yang-Mills symmetry is responsible for the phase transformation (θ), while Weyl symmetry is responsible for the scale transformation (ω), described by the following transformation:

$$\psi \to \exp[\omega + i\theta]\psi \quad (4.1)$$

with $\Omega(x) \equiv \exp[-\omega(x)]$ and $\theta(x)$ the local scale and phase transformation parameters, respectively. Therefore, the 5D Weyl-invariant theory can merge these two symmetries in a harmonic way. Hence, for simplicity, we would like to study the impact of a 5D Weyl-invariant theory with a U_1 gauge field interaction derived directly from the dimensional reduction process. Following the n-D Weyl-invariant action given in (2.11), the 5D Weyl-invariant action takes the following form:

$$\int d^5x\sqrt{\mathbf{g}}\left[\psi^3\tilde{\mathbf{R}} + \psi\left(-\frac{1}{2}\nabla_A\psi\nabla^A\psi - \frac{1}{4}H_{AB}H^{AB}\right) - V(\psi)\right]. \quad (4.2)$$

Note that we have included the potential term $V(\psi)$ explicitly here. For example, a scale-invariant potential term will take the form $V = \lambda\psi^5$. It could also be some symmetry-breaking potential that will break Weyl symmetry explicitly. In order to treat the conformal dimension of all the fields involved appropriately, we will write the standard 5D metric in the following form:

$$\mathbf{g}_{AB} = \begin{pmatrix} g_{\mu\nu} + \phi^{-2}A_\mu A_\nu & \phi^{-2}A_\mu \\ \phi^{-2}A_\nu & \phi^{-2} \end{pmatrix}. \quad (4.3)$$

Here the 5D metric will transform as

$$\mathbf{g}_{AB} \to \mathbf{g}^\Omega_{AB} = \Omega^2\mathbf{g}_{AB} \quad (4.4)$$

consistent with the Weyl transformation of its 4D counterparts:

$$g_{\mu\nu} \to g^\Omega_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad (4.5)$$

$$\phi \to \phi^\Omega = \Omega^{-1}\phi. \quad (4.6)$$

Note that we have used the following notations to distinguish the difference of the field variables and indices in 5D and 4D: (i) boldface (normal) letters, e.g., $\mathbf{g}_{AB}(g_{\mu\nu})$, will denote 5(4)D fields and variables, and (ii) capital roman (lowercase greek) indices $A = 0, 1, 2, 3, 4$ ($\mu = 0, 1, 2, 3$) will denote 5(4)D space-time indices.

Note that the inverse of $\mathbf{g}_{AB}$ can be shown to be

$$\mathbf{g}^{AB} = \begin{pmatrix} g^{\mu\nu} & -A^\mu \\ -A^\nu & \phi^2 + A^2 \end{pmatrix}. \quad (4.7)$$

We can also show that the determinant of the 5D metric is

$$\mathbf{g} = \phi^{-2}g. \quad (4.8)$$

In order to bring the dimensional reduced action into a standard four-dimensional form, we will assume that, throughout the dimensional reduction process,

(i) all field variables will be independent of the fifth dimension coordinate x^4,
(ii) $\int dx^4$ will be set as 1 for convenience,
(iii) the Weyl vector meson takes the following form:

$$\mathbf{S}_A = (S_\mu, \lambda_4). \quad (4.9)$$

In addition, we can also derive the following identities:

$$\tilde{\mathbf{R}} = \mathbf{R} - 8\mathbf{D}_A\mathbf{S}^A - 12\mathbf{S}_A\mathbf{S}^A, \quad (4.10)$$

$$\boldsymbol{R} = R + 2\frac{1}{\phi}D^2\phi - \frac{4}{\phi^2}(D_\mu\phi)^2 - \frac{1}{4}\phi^{-2}F^2, \quad (4.11)$$

$$\boldsymbol{D}_A\boldsymbol{S}^A = D_\mu\boldsymbol{S}^\mu - \frac{\partial_\mu\phi}{\phi}\boldsymbol{S}^\mu. \quad (4.12)$$

Therefore, the dimensional reduction action of the Weyl-invariant gravity can be shown to be

$$\begin{aligned} S = \int d^4x\sqrt{g}\bigg\{\frac{\psi^3}{\phi}\bigg[&\tilde{R} + \frac{2}{\phi}\nabla_\mu\nabla^\mu\phi - \frac{4}{\phi^2}(\nabla_\mu\phi)^2 \\ &- \frac{1}{4}\phi^{-2}F^2 + 8\nabla_\mu(\lambda_4A^\mu) - 8\frac{\nabla_\mu\phi}{\phi}\lambda_4A^\mu \\ &- 12\lambda_4^2(\phi^2 + A^2)\bigg] + \frac{\psi}{\phi}\bigg[-\frac{1}{2}((\nabla_\mu\psi + \lambda_4\psi A_\mu)^2 \\ &+ (\lambda_4\psi)^2\phi^2) - \frac{1}{4}((H_{\mu\nu} - 2(\partial_\mu\lambda_4)A_\nu)^2 \\ &- (2(\partial_{(\mu}\lambda_4)A_{\nu)})^2 + 2(\partial_\mu\lambda_4)^2\phi^2)\bigg]\bigg\}, \end{aligned} \quad (4.13)$$

with $\tilde{R} = R - 6D_\mu S^\mu - 6S_\mu S^\mu$.

V. EFFECTIVE ACTION AND THE FIELD EQUATIONS

For simplicity, we will assume that $\lambda_4 = 0$. As a result, we have the following effective action:

$$\begin{aligned} S = \int d^4x\sqrt{g}\bigg\{&\frac{\psi^3}{\phi}\bigg[\tilde{R} - 6\frac{\nabla_\mu\psi}{\psi}\frac{\nabla^\mu\phi}{\phi} - \frac{1}{2}\Big(\frac{\nabla_\mu\psi}{\psi}\Big)^2\bigg] \\ &- \frac{\psi^3}{4\phi^3}F^2 - \frac{\psi}{4\phi}H^2\bigg\} \end{aligned} \quad (5.1)$$

once the integration by part is performed on the $\nabla_\mu\nabla^\mu\phi$ term. In addition, by parametrizing $\psi^3/\phi = \chi^2$, the action takes the familiar form with two independent scalar fields χ and ψ:

$$\begin{aligned} S = \int d^4x\sqrt{g}\bigg\{&\chi^2\bigg[\tilde{R} - \frac{37}{2}\frac{\nabla_\mu\psi\nabla^\mu\psi}{\psi^2} + 12\frac{\nabla_\mu\chi}{\chi}\frac{\nabla^\mu\psi}{\psi}\bigg] \\ &- \frac{\chi^6}{4\psi^6}F^2 - \frac{\chi^2}{4\psi^2}H^2 - V\bigg\}. \end{aligned}$$

Here we have added a symmetry-breaking potential $V = V(\chi, \psi)$ as a reference potential.

A. Constraint on the symmetry-breaking potential

Note that we will prove that the following constraint [29]:

$$(\psi\partial_\psi + \chi\partial_\chi)V = 4V \quad (5.2)$$

holds for any scalar field potential coupled to the system. This constraint can be derived from the identity $\delta(\sqrt{g}V) = 0$ under the scale transformation of all the fields involved in the effective Lagrangian. To be more specific, we have

$$\delta(\mathcal{L}_g) = -\delta(V_g) = 0 \quad (5.3)$$

for the deviation derived from the scale transformation. Here we have defined $\mathcal{L}_g = \sqrt{g}\mathcal{L}$ and $V_g = \sqrt{g}V$ for convenience.

In order to look closely at the physics of the constraint, we will derive it in a rigorous way. Indeed, we can show that

$$\begin{aligned} \delta\mathcal{L}_g = &\frac{\delta\mathcal{L}_g}{\delta g_{\mu\nu}}\delta g_{\mu\nu} + \frac{\delta\mathcal{L}_g}{\delta\nabla_\alpha g_{\mu\nu}}\delta\nabla_\alpha g_{\mu\nu} + \frac{\delta\mathcal{L}_g}{\delta\psi}\delta\psi \\ &+ \frac{\delta\mathcal{L}_g}{\delta\nabla_\alpha\psi}\delta\nabla_\alpha\psi + \frac{\delta\mathcal{L}_g}{\delta\chi}\delta\chi + \frac{\delta\mathcal{L}_g}{\delta\nabla_\alpha\chi}\delta\nabla_\alpha\chi \\ &+ \frac{\delta\mathcal{L}_g}{\delta\partial_\mu S_\nu}\delta\partial_\mu S_\nu. \end{aligned} \quad (5.4)$$

Note that all δS_α components have been rearranged into the Weyl covariant derivative terms in combinations of $\delta\nabla_\alpha g_{\mu\nu}$, $\delta\nabla_\alpha\psi$, and $\delta\nabla_\alpha\chi$. This follows directly from the built-in structures associated with the prescribed scale symmetry. Note again that the deviation or variation of any function F, δF, shown above can be all kinds of variations not necessarily restricted to the scale transformation.

We can now restrict δF as the deviation derived from the scale transformations, i.e.,

$$\psi^\Omega = \Omega^{-1}\chi, \qquad \psi^\Omega = \Omega^{-1}\psi, \qquad g^\Omega_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad (5.5)$$

such that

$$\Omega\frac{\delta\chi^\Omega}{\delta\Omega} = -\chi^\Omega, \qquad \Omega\frac{\delta\psi^\Omega}{\delta\Omega} = -\psi^\Omega, \qquad \Omega\frac{\delta g^\Omega_{\mu\nu}}{\delta\Omega} = 2g^\Omega_{\mu\nu}, \quad (5.6)$$

for infinitesimal variation derived from $g^\Omega_{\mu\nu}$. First of all, the last term in Eq. (5.4) vanishes since (i) the deviation of $\delta\partial_\mu S_\nu \propto \partial_\mu\partial_\nu\Omega$ is symmetric with respect to μ, ν, and (ii) $\frac{\delta\mathcal{L}_g}{\delta\partial_\mu S_\nu}$ is always antisymmetric with respect to μ, ν. Property (ii) follows from the fact that the $\partial_\mu S_\nu$ dependence in $\mathcal{L}_g$ is derived from the H^2 term that is always antisymmetric with respect to μ, ν by definition. Therefore, we have

$$\begin{aligned} \Omega\frac{\delta\mathcal{L}_g}{\delta\Omega} = &2\frac{\delta\mathcal{L}_g}{\delta g_{\mu\nu}}g_{\mu\nu} + 2\frac{\delta\mathcal{L}_g}{\delta\nabla_\alpha g_{\mu\nu}}\nabla_\alpha g_{\mu\nu} - \frac{\delta\mathcal{L}_g}{\delta\psi}\psi \\ &- \frac{\delta\mathcal{L}_g}{\delta\nabla_\alpha\psi}\nabla_\alpha\psi - \frac{\delta\mathcal{L}_g}{\delta\chi}\chi - \frac{\delta\mathcal{L}_g}{\delta\nabla_\alpha\chi}\nabla_\alpha\chi \end{aligned} \quad (5.7)$$

for the scale transformation shown above. Note that we have just removed the superscript Ω from all scale-transformed field variables for convenience. With the Euler-Lagrange equations of all the fields $g_{\mu\nu}$, χ, ψ enforced, we finally end up with the expression

$$\Omega\frac{\delta\mathcal{L}_g}{\delta\Omega} = 2\nabla_\alpha\left[\frac{\delta\mathcal{L}_g}{\delta\nabla_\alpha g_{\mu\nu}}g_{\mu\nu}\right] - \nabla_\alpha\left[\frac{\delta\mathcal{L}_g}{\delta\nabla_\alpha\psi}\psi\right] - \nabla_\alpha\left[\frac{\delta\mathcal{L}_g}{\delta\nabla_\alpha\chi}\chi\right]. \tag{5.8}$$

Finally, we can show that the variation of $\delta\nabla_\alpha g_{\mu\nu}$ is equivalent to the variation of $2\delta S_\alpha g_{\mu\nu}$. This equivalence also applies to the variations of $\delta\nabla_\alpha\psi$ ($\sim -\delta S_\alpha\psi$) and $\delta\nabla_\alpha\chi$ ($\sim -\delta S_\alpha\chi$). As a result, we have

$$\Omega\frac{\delta\mathcal{L}_g}{\delta\Omega} = \nabla_\alpha\left[\frac{\delta\mathcal{L}_g}{\delta S_\alpha}\right]. \tag{5.9}$$

With the Euler-Lagrange equation of S_α, we can thus derive the following result:

$$\Omega\frac{\delta\mathcal{L}_g}{\delta\Omega} = \nabla_\mu\nabla_\alpha\left[\frac{\delta\mathcal{L}_g}{\delta\nabla_\mu S_\alpha}\right] = 0. \tag{5.10}$$

The vanishing of the above equation again follows from the antisymmetric property of $\delta\mathcal{L}_g/\delta(\nabla_\mu S_\alpha)$, since all terms in $\mathcal{L}_g$ are scale invariant except the symmetry-breaking potential $-V_g$. Hence we derive the final constraint equation $\delta V_g/\delta\Omega = 0$. From the scale transformation of all fields involved, we can then show that

$$\delta V_g = \frac{\delta V_g}{\delta g_{\mu\nu}}\delta g_{\mu\nu} + \frac{\delta V_g}{\delta\psi}\delta\psi + \frac{\delta V_g}{\delta\chi}\delta\chi = \Omega\left[2\frac{\delta V_g}{\delta g_{\mu\nu}}g_{\mu\nu} - \frac{\delta V_g}{\delta\psi}\psi - \frac{\delta V_g}{\delta\chi}\chi\right] = 0. \tag{5.11}$$

As a result, we prove the constraint equation

$$(\psi\partial_\psi + \chi\partial_\chi)V = 4V. \tag{5.12}$$

Note that this constraint is actually derived from the strong constraint hidden in the field equations. The field equations tend to freeze part of the dynamics of the fields involved in the symmetry-breaking potential.

Indeed, we can show that the metric equations take the following form:

$$\frac{1}{2}\mathcal{L}g_{\mu\nu} - \chi^2\tilde{R}_{\mu\nu} + \frac{1}{2}(\nabla_\mu\nabla_\nu\chi^2 + \nabla_\nu\nabla_\mu\chi^2) - g_{\mu\nu}\nabla^2\chi^2 + \frac{37}{2}\frac{\chi^2}{\psi^2}\nabla_\mu\psi\nabla_\nu\psi - 6\frac{\chi}{\psi}(\nabla_\mu\chi\nabla_\nu\psi + \nabla_\nu\chi\nabla_\mu\psi) + \frac{\chi^6}{2\psi^6}F_{\mu\alpha}F_\nu{}^\alpha + \frac{\chi^2}{2\psi^2}H_{\mu\alpha}H_\nu{}^\alpha = 0. \tag{5.13}$$

In addition, the variational equations of χ and ψ can be shown to be

$$2\chi\left[\tilde{R} - \frac{37}{2}\frac{\nabla_\mu\psi\nabla^\mu\psi}{\psi^2} + 12\frac{\nabla_\mu\chi}{\chi}\frac{\nabla^\mu\psi}{\psi}\right] - 12\frac{\nabla_\mu\chi\nabla^\mu\psi}{\psi} - 12\nabla_\mu\left(\frac{\chi\nabla^\mu\psi}{\psi}\right) - \frac{3\chi^5}{2\psi^6}F^2 - \frac{\chi}{2\psi^2}H^2 - \partial_\chi V = 0, \tag{5.14}$$

$$37\nabla_\mu\left(\frac{\chi^2\nabla^\mu\psi}{\psi^2}\right) + 37\frac{\chi^2}{\psi^3}\nabla_\mu\psi\nabla^\mu\psi - 12\frac{\chi}{\psi^2}\nabla_\mu\chi\nabla^\mu\psi - 12\nabla^\mu\left(\frac{\chi}{\psi}\nabla_\mu\chi\right) + \frac{3\chi^6}{2\psi^7}F^2 + \frac{\chi^2}{2\psi^3}H^2 - \partial_\psi V = 0. \tag{5.15}$$

As a result, we can obtain the following equation from the metric equation and scalar field equations:

$$(\psi\partial_\psi + \chi\partial_\chi)V - 4V = 25\nabla_\mu\left(\frac{\chi^2\nabla^\mu\psi}{\psi}\right). \tag{5.16}$$

We can also derive the variational equation of S_μ as

$$\nabla_\nu\left(\frac{\chi^2}{\psi^2}H^{\nu\mu}\right) + 25\left(\frac{\chi^2\nabla^\mu\psi}{\psi}\right) = 0. \tag{5.17}$$

Hence a further covariant derivative ∇_μ on the above equation lead to the constraint equation

$$\nabla_\mu\left(\frac{\chi^2\nabla^\mu\psi}{\psi}\right) = 0. \tag{5.18}$$

This also concludes our proof of the constraint equation of the scalar field potential

$$(\psi\partial_\psi + \chi\partial_\chi)V = 4V. \tag{5.19}$$

Note that we can choose a gauge with $\Omega = \chi$ such that $\chi^\omega = \chi^{-1}\chi = 1$ and turns off the χ field effectively for convenience. Once we do that, the only effect of the scalar field will be transformed to the constraint on V. In other words, once a gauge is chosen for $\chi =$ const, the dynamics of χ field will be turned off completely and thus push forward a constraint on the symmetry-breaking potential V. We will be back with this point shortly near the end of Sec. VI.

In addition to the field equations of $g_{\mu\nu}$, ψ, S_μ, we also have the variational equation of A_μ:

$$\nabla_\nu\left(\frac{\chi^6}{\psi^6}F^{\nu\mu}\right) = 0. \tag{5.20}$$

VI. EFFECTIVE ACTION WITH $\nabla_\mu\psi = 0$

We will now choose a gauge such that $\chi = 1$ or a constant scale factor that can be absorbed into the redefinition of the Newtonian constant. We will also write $\psi = \exp[\hat{\varphi}]$ for convenience. As a result, we need to solve the following field equations:

$$T_{\mu\nu} = \frac{1}{2} g_{\mu\nu} \mathcal{L}_m + 6 S_\mu S_\nu + \frac{37}{2} \nabla_\mu \hat{\varphi} \nabla_\nu \hat{\varphi} + 6(S_\mu \nabla_\nu \hat{\varphi} + S_\nu \nabla_\mu \hat{\varphi}) + \frac{1}{2} \exp[-6\hat{\varphi}] F_{\mu\alpha} F_\nu{}^\alpha + \frac{1}{2} \exp[-2\hat{\varphi}] H_{\mu\alpha} H_\nu{}^\alpha, \tag{6.1}$$

$$24 D_\mu S^\mu + 3 \exp[-6\hat{\varphi}] F^2 + \exp[-2\hat{\varphi}] H^2 = 2\partial_{\hat{\varphi}} V, \tag{6.2}$$

$$\nabla_\mu \left(\frac{\chi^2 \nabla^\mu \psi}{\psi} \right) = D_\mu \left(\frac{\chi^2 \nabla^\mu \psi}{\psi} \right) \underset{\chi=1}{\rightarrow} D_\mu \nabla^\mu \hat{\varphi} = 0, \tag{6.3}$$

$$\nabla_\nu \left(\frac{\chi^6}{\psi^6} F^{\nu\mu} \right) = D_\nu \left(\frac{\chi^6}{\psi^6} F^{\nu\mu} \right) \underset{\chi=1}{\rightarrow} D_\nu (\exp[-6\hat{\varphi}] F^{\nu\mu}) = 0, \tag{6.4}$$

$$\nabla_\nu \left(\frac{\chi^2}{\psi^2} H^{\nu\mu} \right) + 25 \left(\frac{\chi^2 \nabla^\mu \psi}{\psi} \right) \underset{\chi=1}{\rightarrow} D_\nu (\exp[-2\hat{\varphi}] H^{\nu\mu}) + 25 \nabla^\mu \hat{\varphi} = 0 \tag{6.5}$$

with $G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu} R/2$ the Einstein tensor and $\mathcal{L}_m$ the matter part of the full Lagrangian defined by

$$\mathcal{L}_m = -6 S_\mu S^\mu - \frac{37}{2} \nabla_\mu \hat{\varphi} \nabla^\mu \hat{\varphi} - 12 S_\mu \nabla^\mu \hat{\varphi} - \frac{1}{4} \exp[-6\hat{\varphi}] F^2 - \frac{1}{4} \exp[-2\hat{\varphi}] H^2 - V. \tag{6.6}$$

Now we will try to solve the equations of motion for a solution with the scalar field $\hat{\varphi}$ satisfying the equation $\nabla_\mu \hat{\varphi} = 0$. This implies immediately that $S_\mu = \partial_\mu \hat{\varphi}$. In other words, the Weyl vector meson field S_μ is in a so-called pure gauge condition such that the field strength vanishes identically, i.e., $H_{\mu\nu} = 0$. As the result, the solution with $\nabla_\mu \hat{\varphi} = 0$ is an exact solution to the S_μ equation (6.5).

We will make a brief remark about the choice of the special solution $\nabla_\mu \hat{\varphi} = 0$. This paper will focus on the application of the Weyl-invariant KK theory in the cosmological evolution during our early Universe. For a simple demonstration, we will also assume that the space belongs to the class of homogeneous spaces, or the Bianchi-type metric spaces. To be more specific, we will focus on the study the application of this theory in the BI space throughout the rest of this paper. The metric of the BI space can be read off directly from the definition given by

$$ds^2 = -dt^2 + \exp[2\alpha(t) - 4\sigma(t)] dx^2 + \exp[2\alpha(t) + 2\sigma(t)](dy^2 + dz^2). \tag{6.7}$$

We will hence assume that $\hat{\varphi} = \hat{\varphi}(t)$ and $S_\mu = S_\mu(t)$ are compatible with the BI metric space.

Note also that, with the special solution under the condition $\nabla_\mu \hat{\varphi} = 0$, a scale transformation looks exactly like a translational transformation on the $\hat{\varphi}$ field. Indeed, $\hat{\varphi}^\omega = \hat{\varphi} - \omega$ if $S_\mu^\omega = S_\mu - \partial_\mu \omega$ under the scale transformation with $\Omega = \exp[\omega]$. Note also that the field tensor $H_{\mu\nu}$ will remain null for any scale transformation if it starts out as a null field tensor $H_{\mu\nu} = 0$. This is exactly the reason why the null field tensor condition is referred to as the pure gauge condition; namely, the Weyl vector tensor can be gauged away by a scale transformation.

Once the condition $\nabla_\mu \hat{\varphi} = 0$ is enforced on the field equations, the field equations turn into the following form:

$$T_{\mu\nu} = \frac{1}{2} g_{\mu\nu} \mathcal{L}_m + 6 \partial_\mu \hat{\varphi} \partial_\nu \hat{\varphi} + \frac{1}{2} \exp[-6\hat{\varphi}] F_{\mu\alpha} F_\nu{}^\alpha, \tag{6.8}$$

$$24 D_\mu \partial^\mu \hat{\varphi} + 3 \exp[-6\hat{\varphi}] F^2 = 2\partial_{\hat{\varphi}} V, \tag{6.9}$$

$$D_\nu (\exp[-6\hat{\varphi}] F^{\nu\mu}) = 0, \tag{6.10}$$

with

$$\mathcal{L}_m = -6 \partial_\mu \hat{\varphi} \partial^\mu \hat{\varphi} - \frac{1}{4} \exp[-6\hat{\varphi}] F^2 - V. \tag{6.11}$$

This set of field equations is equivalent to the field equations derived from the effective Lagrangian given by

$$\mathcal{L}_1 = R - 6 \partial_\mu \hat{\varphi} \partial^\mu \hat{\varphi} - \frac{1}{4} \exp[-6\hat{\varphi}] F^2 - V. \tag{6.12}$$

Note that the constraint equation $\partial_\kappa V + \partial_{\hat{\varphi}} V = 4V$ can remain valid if we choose the scalar potential as $V = \exp[\lambda(\hat{\varphi} - \kappa) + 4\kappa]$. Here we have defined $\chi = \exp[\kappa]$ for convenience, too.

Hence by assuming $V = \exp[\lambda(\hat{\varphi} - \kappa) + 4\kappa]$, we will have a system with an effective Lagrangian given by $\mathcal{L}_1$ with no further constraint to be enforced on the scalar field $\hat{\varphi}$. As a result, the Weyl field strength tensor $H_{\mu\nu}$ will effectively be decoupled from the system. It will only be related to the Weyl vector meson through the relation $S_\mu = \partial_\mu \hat{\varphi}$.

VII. ANISOTROPICALLY EXPANDING UNIVERSE

The system with the effective Lagrangian $\mathcal{L}_1$ is known to admit a set of stable power-law solutions on the Bianchi type I metric space. Indeed, we can define $\varphi = \sqrt{12}\hat{\varphi}$ and write the Lagrangian $\mathcal{L}_1$ as [44]

$$\mathcal{L}_1 = R - \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{4} f^2(\varphi) F_{\mu\nu} F^{\mu\nu} - V(\varphi), \tag{7.1}$$

with $f^2(\varphi, \tau) = \exp[2a\varphi]$, and $a = -\sqrt{3}/2$.

The only difference with the model in Ref. [44] is that some of the parameters involved are no longer free parameters. Therefore, there are strong constraints to be imposed on the choice of the parameters. Under these conditions, stable solutions may not persist as a result of the hidden Weyl KK theory. Therefore, for heuristic reason, we will briefly review the process of solving the field equations for a set of power-law solutions. Because of the fact that many of the parameters are fixed by the Weyl-invariant KK theory, we can in fact solve the perturbation equation and derive all the perturbation modes explicitly.

Indeed, we will focus on the effect of the scale-invariant effective action to the evolution of the physical Universe in the BI metric space. We will also choose BI-compatible gauge fields $A_\mu = (0, A_x(t), 0, 0)$. First of all, the field equation for the vector field A_μ is

$$\partial_\mu(\sqrt{g} f^2(\varphi) F^{\mu\nu}) = 0, \tag{7.2}$$

that can be integrated to obtain the following solution:

$$\dot{A}_x(t) = p_A f^{-2} \exp[-\alpha - 4\sigma], \tag{7.3}$$

with p_A a constant of integration. In addition, the Friedman equation and the variational field equations of φ, α, σ can be shown to be

$$\dot{\alpha}^2 = \dot{\sigma}^2 + \frac{1}{12}\dot{\varphi}^2 + \frac{p_A^2}{12 f^2}\exp[-4\alpha - 4\sigma] + \frac{1}{6}V, \tag{7.4}$$

$$\ddot{\varphi} = -3\dot{\alpha}\,\dot{\varphi} + p_A^2 f^{-3} f_\varphi \exp[-4\alpha - 4\sigma] - \partial_\varphi V, \tag{7.5}$$

$$\ddot{\alpha} = -3\dot{\alpha}^2 + \frac{p_A^2}{12 f^2}\exp[-4\alpha - 4\sigma] + \frac{1}{2}V, \tag{7.6}$$

$$\ddot{\sigma} = -3\dot{\alpha}\,\dot{\sigma} + \frac{p_A^2}{6 f^2}\exp[-4\alpha - 4\sigma]. \tag{7.7}$$

A. Power-law solutions

In this subsection, we would like to find a set of power-law solution of the following form:

$$\alpha = \zeta \log(t); \quad \sigma = \eta \log(t); \quad \varphi = \xi \log(t) + \varphi_0. \tag{7.8}$$

For later convenience, we will assume that there is a scalar potential term of the form

$$V = v_0 \exp[\lambda\varphi] \tag{7.9}$$

with $u = v_0 \exp[\lambda\varphi_0]$ as the initial value. Note that λ is the only free parameter introduced to represent the scale of symmetry breaking. For simplicity, we will also introduce a new variable $l = p_A^2 \exp[-2a\varphi_0]$. As a result, we can show that the field equations (7.5)–(7.7) reduce to a set of algebraic equations as follows:

$$al = \xi(3\zeta - 1) + \lambda u, \tag{7.10}$$

$$\frac{l}{12} = \zeta^2 - \eta^2 - \frac{1}{12}\xi^2 - \frac{1}{6}u, \tag{7.11}$$

$$\frac{l}{12} = \zeta(3\zeta - 1) - \frac{1}{2}u, \tag{7.12}$$

$$\frac{l}{6} = \eta(3\zeta - 1). \tag{7.13}$$

This set of equations can be simplified by writing the following parameters in units of $3\zeta - 1$:

$$l = (3\zeta - 1)\tilde{l}, \tag{7.14}$$

$$u = (3\zeta - 1)\tilde{u}. \tag{7.15}$$

As a result, the field equations reduce to the following form:

$$a\tilde{l} = \xi + \lambda\tilde{u}, \tag{7.16}$$

$$\frac{\tilde{l}}{12} = \zeta - \frac{1}{2}\tilde{u}, \tag{7.17}$$

$$\frac{\tilde{l}}{6} = \eta. \tag{7.18}$$

There are also two extra constraint equations

$$2\zeta + 2\eta + a\xi = 2, \tag{7.19}$$

$$\lambda\xi = -2, \tag{7.20}$$

derived respectively from the power counting of the scalar-photon coupling and the scalar potential.

Eliminating $\tilde{u}$ from the field equations (7.16)–(7.18) and Friedmann equation (7.11), we can obtain an equation of l as a function of ζ. Then we can eliminate ζ from this equation with the help of the σ equation (7.18). The result is

$$\tilde{l} = \frac{2\lambda(\lambda + 2a) - 4}{\lambda(\lambda + 2a)} = 2\frac{\lambda^2 - \sqrt{3}\lambda - 2}{\lambda(\lambda - \sqrt{3})}. \tag{7.21}$$

Here we have assumed $\zeta \neq 1/3$. Hence we can solve the parameters ζ, $3\zeta - 1$, and η as

$$\zeta = \frac{4a(2\lambda + 3a) + 4 + \lambda^2}{6\lambda(\lambda + 2a)} = \frac{\lambda^2 - 4\sqrt{3}\lambda + 13}{6\lambda(\lambda - \sqrt{3})}, \tag{7.22}$$

$$3\zeta - 1 = \frac{4a(\lambda + 3a) - \lambda^2 + 4}{2\lambda(\lambda + 2a)} = \frac{13 - \lambda^2 - 2\sqrt{3}\lambda}{2\lambda(\lambda - \sqrt{3})}, \tag{7.23}$$

$$\eta = \frac{\tilde{l}}{6}. \tag{7.24}$$

We can also obtain the solution of $\tilde{u}$ as

$$\tilde{u} = \frac{2(\lambda a + 2a^2 + 1)}{\lambda(\lambda + 2a)} = \frac{5 - \sqrt{3}\lambda}{\lambda(\lambda - \sqrt{3})}. \tag{7.25}$$

In summary, the power-law solution exists only when the parameters ζ, η, u, l are functions of the parameter λ related by the following relations:

$$\zeta = \frac{\lambda^2 - 4\sqrt{3}\lambda + 13}{6\lambda(\lambda - \sqrt{3})}, \tag{7.26}$$

$$\eta = \frac{\lambda^2 - \sqrt{3}\lambda - 2}{3\lambda(\lambda - \sqrt{3})}, \tag{7.27}$$

$$u = \frac{(\sqrt{3}\lambda - 5)(\lambda^2 + 2\sqrt{3}\lambda - 13)}{2\lambda^2(\lambda - \sqrt{3})^2}, \tag{7.28}$$

$$l = -\frac{(\lambda^2 - \sqrt{3}\lambda - 2)(\lambda^2 + 2\sqrt{3}\lambda - 13)}{\lambda^2(\lambda - \sqrt{3})^2}. \tag{7.29}$$

We can also compute the expansion rate of the scale factor $\dot{a}_i$ that is related to $\zeta - 2\eta$ and $\zeta + \eta$ given by

$$\zeta - 2\eta = -\frac{(\lambda + \sqrt{7})(\lambda - \sqrt{7})}{2\lambda(\lambda - \sqrt{3})}, \tag{7.30}$$

$$\zeta + \eta = \frac{\lambda - \sqrt{3}}{2\lambda}. \tag{7.31}$$

In addition, the average slow-roll parameter $\varepsilon \equiv -\dot{H}/H^2 = 1/\zeta$ and the anisotropy $\Sigma/H \equiv \dot{\sigma}/\dot{\alpha} = \eta/\zeta$ can also be derived accordingly [44].

Note that we are looking for expanding solutions. Hence the parameters u, l, $\zeta - 2\eta$, $\zeta + \eta$ have to be positive definite. These requirements then give rise to the following constraints on the choice of λ:

$$(\lambda^2 + 2\sqrt{3}\lambda - 13)\left(\lambda - \frac{5\sqrt{3}}{3}\right) > 0, \tag{7.32}$$

$$(\lambda^2 + 2\sqrt{3}\lambda - 13)(\lambda^2 - \sqrt{3}\lambda - 2) < 0, \tag{7.33}$$

$$(\lambda + \sqrt{7})(\lambda - \sqrt{7}) < 0, \tag{7.34}$$

$$\lambda(\lambda - \sqrt{3}) > 0. \tag{7.35}$$

Note that we can parametrize these inequalities with the help of the following expressions: $\lambda^2 + 2\sqrt{3}\lambda - 13 = (\lambda + \sqrt{3})^2 - 16$ and $\lambda^2 - \sqrt{3}\lambda - 2 = (\lambda - \sqrt{3}/2)^2 - 11/4$. Hence, we can write the roots in order as $A = -4 - \sqrt{3} < B = -\sqrt{7} < C = -(\sqrt{11} - \sqrt{3})/2 < O = 0 < D = \sqrt{3} < E = 4 - \sqrt{3} < F = (\sqrt{3} + \sqrt{11})/2 < G = \sqrt{7} < H = 5\sqrt{3}/3$. As a result, we can show that these inequalities can be written as

$$(\lambda - A)(\lambda - E)(\lambda - H) > 0, \tag{7.36}$$

$$(\lambda - A)(\lambda - C)(\lambda - E)(\lambda - F) < 0, \tag{7.37}$$

$$(\lambda - B)(\lambda - G) < 0, \tag{7.38}$$

$$\lambda(\lambda - D) > 0. \tag{7.39}$$

Hence it can be shown that the parameter λ has to fall into the region $-\sqrt{7} < \lambda < -(\sqrt{11} - \sqrt{3})/2$ for the existence of an expanding solution.

VIII. STABILITY ANALYSIS OF THE EXPANDING SOLUTIONS

We would like to check whether these solutions are stable against small perturbations of the fields when the parameter λ is in the region $-\sqrt{7} < \lambda < -(\sqrt{11} - \sqrt{3})/2$. Therefore, by perturbing the field equations (7.5)–(7.7), we can obtain the following set of perturbation equations:

$$\delta\ddot{\varphi} = -3\frac{\zeta}{t}\delta\dot{\varphi} - 3\frac{\xi}{t}\delta\dot{\alpha} - \frac{\lambda^2 u}{t^2}\delta\varphi - \frac{al}{t^2}[2a\delta\varphi + 4(\delta\alpha + \delta\sigma)], \tag{8.1}$$

$$\frac{2\zeta}{t}\delta\dot{\alpha} = 2\frac{\eta}{t}\delta\dot{\sigma} + \frac{\xi}{6t}\delta\dot{\varphi} + \frac{\lambda u}{6t^2}\delta\varphi - \frac{l}{12t^2}[2a\delta\varphi + 4(\delta\alpha + \delta\sigma)], \tag{8.2}$$

$$\delta\ddot{\alpha} = -6\frac{\zeta}{t}\delta\dot{\alpha} + \frac{\lambda u}{2t^2}\delta\varphi - \frac{l}{12t^2}[2a\delta\varphi + 4(\delta\alpha + \delta\sigma)], \tag{8.3}$$

$$\delta\ddot{\sigma} = -3\frac{\zeta}{t}\delta\dot{\sigma} - 3\frac{\eta}{t}\delta\dot{\alpha} - \frac{l}{6t^2}[2a\delta\varphi + 4(\delta\alpha + \delta\sigma)]. \tag{8.4}$$

Instead of taking the perturbation of fields as $\delta\alpha = \alpha_0 \exp[nt]$ that is incompatible with the power-law solutions shown above, we will assume the perturbation of fields as $\delta\alpha = A_0 t^n$, $\delta\sigma = B_0 t^n$, $\delta\varphi = C_0 t^n$ [44]. As a result, the above set of perturbation equations becomes a set of algebraic equations that can be written as a matrix equation:

$$\mathcal{D}\begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} \equiv \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} = 0, \tag{8.5}$$

with

$$\mathcal{D} = \begin{bmatrix} \frac{6n}{\lambda} - 4al, & -4al, & -[n(n-1) + 3\zeta n + \lambda^2 u + 2a^2 l] \\ 12\zeta n + 2l, & -12\eta n + 2l, & \frac{2n}{\lambda} + la - \lambda u \\ n(n-1) + 6\zeta n + \frac{l}{3}, & \frac{l}{3}, & \frac{la}{6} - \frac{1}{2}u\lambda \end{bmatrix}. \quad (8.6)$$

Nontrivial solutions of Eq. (8.5) are known to exist only when

$$\det \mathcal{D} = 0. \quad (8.7)$$

We can write Eq. (8.7) as a polynomial equation of n:

$$\det \mathcal{D} = 12\eta n(n^4 + a_3 n^3 + a_2 n^2 + a_1 n + a_0) = 0. \quad (8.8)$$

Therefore, we need to solve the following polynomial equation:

$$f(n) = (n^4 + a_3 n^3 + a_2 n^2 + a_1 n + a_0) = 0, \quad (8.9)$$

for nontrivial solutions. In fact, this polynomial equation can be solved as

$$f(n) = (n+1)(n-n_2)(n-n_+)(n-n_-) = 0, \quad (8.10)$$

with

$$n_2 = \frac{(\lambda + \sqrt{3} - 4)(\lambda + \sqrt{3} + 4)}{2\lambda(\lambda - \sqrt{3})} \quad (8.11)$$

and

$$n_\pm = \frac{\lambda(\lambda - \sqrt{3})(\lambda + \sqrt{3} - 4)(\lambda + \sqrt{3} + 4) \pm \kappa_1}{4\lambda(\lambda - \sqrt{3})}, \quad (8.12)$$

$$\kappa_1 = [(\lambda + \sqrt{3} - 4)(\lambda + \sqrt{3} + 4) \times (-8\sqrt{3}\lambda^3 + 65\lambda^2 - 22\sqrt{3}\lambda - 93)]^{1/2}. \quad (8.13)$$

Note that we have used the fact that $\lambda(\lambda - \sqrt{3}) > 0$ for the expanding solution region $B < \lambda < C$. It is also easy to show that $n_2 < 0$ in this expansion region. In addition, κ_1 is pure imaginary in this region. Therefore the perturbation admits totally three different nonpositive roots: $n = 0, -1, n_2$. Hence the expanding solution is stable against the perturbation shown here. In fact, we can also show numerically that the solution shown in this section remains an attractor fixed-point solution as shown in Ref. [44].

IX. CONCLUSION

We have briefly reviewed the Weyl symmetry in an n-dimensional space-time. In addition, the Weyl-invariant generalization of the TEGR relation has been shown explicitly in Sec. III. Moreover, the Kaluza-Klein approach to the Weyl-invariant 5D model was also shown in a complete and consistent approach. A very interesting constraint on the symmetry-breaking potential was also presented for heuristic reasons in this paper.

In addition, we also show that effective 4D Weyl-invariant gravity is equivalent to a one-scalar-field model if we assume that the Weyl vector field takes the form of a pure gauge field, $S_\mu = \partial_\mu \hat{\omega}$. This is achieved by assuming the Weyl covariant derivative of the scalar field ψ vanishes, namely, $\nabla_\mu \psi = 0$. A physical reason for this choice was given in Sec. VI. It was shown that this choice is a compatible choice in coherence with the BI metric space.

We hence discuss possible effects of this model in this paper. We found a set of power-law expanding solutions in the Bianchi type I universe. There is, however, a constraint on the parameter λ given by $-\sqrt{7} < \lambda < -(\sqrt{11} - \sqrt{3})/2$. A perturbation is also shown explicitly to obtain exact perturbation modes as functions of λ. The result shows that this set of power-law solutions are stable solutions in all the allowed region $-\sqrt{7} < \lambda < -(\sqrt{11} - \sqrt{3})/2$.

Evidence indicates that there is enriched information hidden in the Weyl-invariant theories and Kaluza-Klein theories as well as the TEGR approaches of gravity theories. They seem to work in a coherent way and present many interesting properties that deserve more attention.

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