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Rapid growth of superradiant instabilities for charged black holes in a cavity

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Scalar fields confined either by a mass term or by a mirrorlike boundary condition have unstable modes in the background of a Kerr black hole. Assuming a time dependence as $e^{-i\omega t}$, the growth time scale of these unstable modes is set by the inverse of the (positive) imaginary part of the frequency, $\mathrm{Im}(\omega)$, which reaches a maximum value of the order of $\mathrm{Im}(\omega)M \sim 10^{-5}$, attained for a mirrorlike boundary condition, where M is the black hole mass. In this paper we study the minimally coupled Klein-Gordon equation for a charged scalar field in the background of a Reissner-Nordström black hole and show that the unstable modes, due to a mirrorlike boundary condition, can grow several orders of magnitude faster than in the rotating case: we have obtained modes with up to $\mathrm{Im}(\omega)M \sim 0.07$. We provide an understanding, based on an analytic approximation, of why the instability in the charged case has a shorter time scale than in the rotating case. This faster growth, together with the spherical symmetry, makes the charged case a promising model for studies of the fully nonlinear development of superradiant instabilities.

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I. INTRODUCTION

In classical relativistic gravity, black holes are observer independent space-time regions unable to communicate with their exterior [1]. Thus, within this description, information captured by black holes is trapped therein forever and cannot be recovered by exterior observers.

Given this picture it is intriguing, at first, to realize that there is a classical process through which energy can be extracted from a black hole: superradiant scattering. In one form, this process amounts to the amplification of waves impinging on a Kerr black hole, provided the frequency ω and azimuthal quantum number m of the wave modes obey the condition $\omega < m\Omega_+$, where Ω_+ is the angular velocity of the outer Kerr horizon [2–4]. The extraction of energy and consequent decrease of the black hole mass M is, however, necessarily accompanied by the extraction of angular momentum and consequent decrease of the black hole spin J. In fact, it was shown by Christodoulou [5] that the particle analogue of this process—the Penrose process [6]—is irreversible, subsequently realized to mean that the black hole area never decreases [7]. Finally, the identification between black hole area and entropy [8,9] made clear that it is only rotational energy that is being extracted from the black hole, not information.

In another form, superradiant scattering amounts to the amplification of charged waves impinging on a Reissner-Nordström (RN) black hole, provided the frequency ω and the charge q of the wave modes obey the condition $\omega < q\Phi_+$, where Φ_+ is the electric potential of the outer Reissner-Nordström horizon [10]. The extraction of (Coulomb) energy and consequent decrease of the black hole mass M is, in this case, necessarily accompanied by the extraction of charge and consequent decrease of the black hole charge Q, such that, again, the area and entropy of the RN black hole does not decrease.

The existence of superradiant modes can be converted into an instability of the background if a mechanism to trap these modes in a vicinity of the black hole is provided: heuristically, these modes are then recurrently scattered off the black hole and amplified, eventually producing a non-negligible backreaction on the background. This possibility, anticipated by Zel'dovich [11], was named *the black hole bomb* by Press and Teukolsky [4] and has been studied extensively in the Kerr case within the linear analysis (see e.g., [12–18]). One of the outcomes of these studies is that the maximum growth rate for the instabilities is associated with modes with an imaginary part of their frequency, $\mathrm{Im}(\omega)$, of $\mathrm{Im}(\omega)M \sim 1.74 \times 10^{-7}$ for massive fields and $\mathrm{Im}(\omega)M \sim 6 \times 10^{-5}$ for mirrorlike boundary conditions [12,18]. The growth time scale is set by the inverse of $\mathrm{Im}(\omega)$.

The unstable states found in the Kerr case are localized in a potential well found outside the potential barrier of the effective potential. The growth of such states can be seen at linear level, but a fully nonlinear study is required to address the endpoint of this instability. It has been suggested that such endpoint is attained after an explosive event called a bosenova [19]. Progress in understanding this endpoint has been making use, and will certainly require further use, of fully nonlinear numerical simulations [20,21].

Considerably less attention has been devoted to the charged case, perhaps due to the lack of astrophysical motivation. Moreover, the studies found in the literature [22–24] discard the possibility of an instability in the asymptotically flat case, since an analysis of the effective potential shows no potential well for quasibound states

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compatible with the superradiance condition. Similar conclusions follow for quasi-normal modes boundary conditions [25,26]. This picture can be altered, however, if the black hole is enclosed in a cavity. The purpose of this work is to show that when imposing a mirrorlike boundary condition at some radial coordinate r_m, the superradiant instability occurs and can have a much shorter time scale than in the rotating case. Very slow instabilities prove challenging to follow numerically since the very small growth rate may be masked by numerical errors or other physical effects; an example of the latter is discussed in [17]. Thus, our result suggests that RN black holes inside a cavity provide a setup that may facilitate the numerical study of the nonlinear development of superradiant instabilities, not only because of the shorter time scale but also due to the spherical symmetry. Such nonlinear study will certainly yield valuable lessons for the more relevant, but harder, rotating case.

This paper is organized as follows. In Sec. II we describe the basic setup for a charged massive scalar field with a mirrorlike boundary condition in the background of a RN black hole. In Sec. III we discuss the results for the imaginary part of the frequencies for various values of the background and field parameters. In Sec. IV we provide an understanding of why the growth rate of instabilities in the charged case can become larger than in the rotating case and discuss our results.

Throughout the paper we will use Planck units such that $G = c = \hbar = 1$, and we use the black hole mass M as a scale. Then, for instance, conventional physical units for the charge of the scalar field and the charge of the black hole can be recovered by making the substitutions $q \to q \frac{1}{\sqrt{\hbar c}} \frac{M}{M_{\rm pl}}$ and $Q \to \frac{Q}{\sqrt{\hbar c}} \frac{M_{\rm pl}}{M}$.

II. MIRRORED QUASIBOUND STATES

We shall consider a massive, charged scalar field, Ψ, with mass μ and charge q, propagating in the background of a Reissner-Nordström black hole. As explained in the Introduction, in order to have superradiant scattering, the field needs to be charged, thus making it natural to be massive as well, in view of the known fundamental particles. Written in Boyer-Lindquist–type coordinates, the line element of the background is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where

$$f(r) = \frac{(r-r_+)(r-r_-)}{r^2}, \qquad r_\pm \equiv M \pm \sqrt{M^2 - Q^2}. \quad (2)$$

In the linear regime, the dynamics of the scalar field is described by the wave equation

$$[(D^\nu - iqA^\nu)(D_\nu - iqA_\nu) - \mu^2]\Psi = 0, \quad (3)$$

where the electric potential satisfies $A_\nu dx^\nu = -Q/rdt$.

Setting $\Psi(t, r, \theta, \phi) = e^{-i\omega t}\sum_{\ell,m} Y_\ell^m(\theta, \phi)R_\ell(r)/r$, with $Y_\ell^m(\theta, \phi)$ the spherical harmonics, the radial equation for each mode can be written as

$$f(r)\frac{d^2}{dr^2}R_\ell(r) + f'(r)\left(\frac{d}{dr}R_\ell(r) - \frac{1}{r}R_\ell(r)\right) + \left(\frac{1}{f(r)}\left(\omega - \frac{qQ}{r}\right)^2 - \frac{\ell(\ell+1)}{r^2} - \mu^2\right)R_\ell(r) = 0, \quad (4)$$

where $f'(r) = df(r)/dr$, and $-\ell(\ell+1)$ is the eigenvalue of the angular operator. This equation can be written in a Schrödinger-like form by applying a transformation of coordinates $r = r(r^*)$,

$$\left[-\frac{d^2}{dr^{*2}} + V(r)\right]R_\ell(r) = \omega^2 R_\ell(r), \quad (5)$$

where r^* is the Regge-Wheeler tortoise coordinate defined by $dr^* = dr/f(r)$. The effective potential is given by

$$V(r) = \frac{2qQ\omega}{r} - \frac{q^2Q^2}{r^2} + f(r)\left(\frac{l(l+1)}{r^2} + \mu^2 + \frac{f'(r)}{r}\right). \quad (6)$$

The fact that the effective potential depends on the unknown ω makes it unorthodox as compared to standard potentials in Schrödinger-like problems. Some information, nevertheless, can be obtained by studying this unorthodox potential. In particular, it has been shown in [23] that quasibound states of a massive charged scalar field in an extreme RN geometry do not contain superradiant states. It has been proven that the conditions for (i) the effective potential to have a well and (ii) the frequency to obey $\omega < \omega_c \equiv q\Phi_+$ cannot be satisfied simultaneously. For the nonextremal RN geometry, on the other hand, it was shown in [22], using a matching technique, that the condition $qQ < M\mu$ is necessary for the field to satisfy the regular boundary conditions at infinity. But, as pointed out in the same reference, this condition is not satisfied by superradiant states. The previous inequality is the Newton-Coulomb requirement for the gravitational force to exceed the electrostatic force, which is naturally associated as a condition for bound states.

The evidence presented in the previous paragraph points out that quasibound states for a massive charged scalar field in a RN background cannot be in the superradiant regime. There is, however, a way to obtain superradiant quasibound states in this background. The point is that the frequencies of the quasibound states are determined by both the parameters of the system and the boundary conditions; thus, changing the latter may allow bound states to obey the superradiant condition.

The standard boundary behavior for the quasibound states of a massive scalar field that can extend to asymptotic infinity is to decay exponentially; this follows from the fact that the effective potential tends asymptotically to the mass, generating a well. If a mirror is placed at some radial coordinate r_m outside the black hole, on the other hand, the outer boundary condition is modified so that the field vanishes exactly at r_m and its proper frequencies become determined by the position of the mirror [4,12,27]. Since one can place the mirror at arbitrary positions, the scalar field might have frequencies that are in the superradiant regime. As we show in the next section, this is indeed the case, and, most interestingly, the value of the time scale of the instability for the charged black hole can become considerably shorter than in the rotating counterpart of this problem.

In order to compute the spectrum of bound states, we found it more convenient to numerically integrate the radial equation (4), imposing the appropriate boundary conditions. In the vicinity of the horizon r_+, we impose an ingoing wavelike condition [4,12]

$$R_\ell(r) \sim e^{-i(\omega-\omega_c)r^*}. \tag{7}$$

The outer boundary condition is determined by the position of the mirror r_m. At this radius the mirrored states must vanish and hence $R_\ell(r_m) = 0$. The algorithm to find the frequencies is then the following: we start integrating the radial equation with the behavior given by (7) outward from $r = r_+(1+\varepsilon)$—in the calculations presented in the following section we used typically $\varepsilon \sim 10^{-8}$—with an arbitrary value of ω and stop the integration at the radius of the mirror. This procedure gives us a value for the wave function at r_m, as a function of the frequency. The integration is repeated varying the frequency until $R_\ell(r_m) = 0$ is reached with the desired precision, thus obtaining the frequency of the mirrored state.

III. RESULTS

We shall now exhibit the behavior of the imaginary part of the frequency (also the real part in Fig. 2) as a function of the mirror radius r_m, for various values of q, μ, and Q. All the modes displayed correspond to $\ell = 1$, since these are the modes for which the instability is expected to be stronger [22,28].

In Fig. 1 we show the imaginary part of the frequency as a function of the mirror radius for different values of the ratio q/μ and Q.

We see that when the ratio is unity, $\mathrm{Im}(\omega) < 0$ for any value of the black hole charge; that is, there are no superradiant modes. As the ratio increases, however, mirror radii greater than some minimum value will have positive

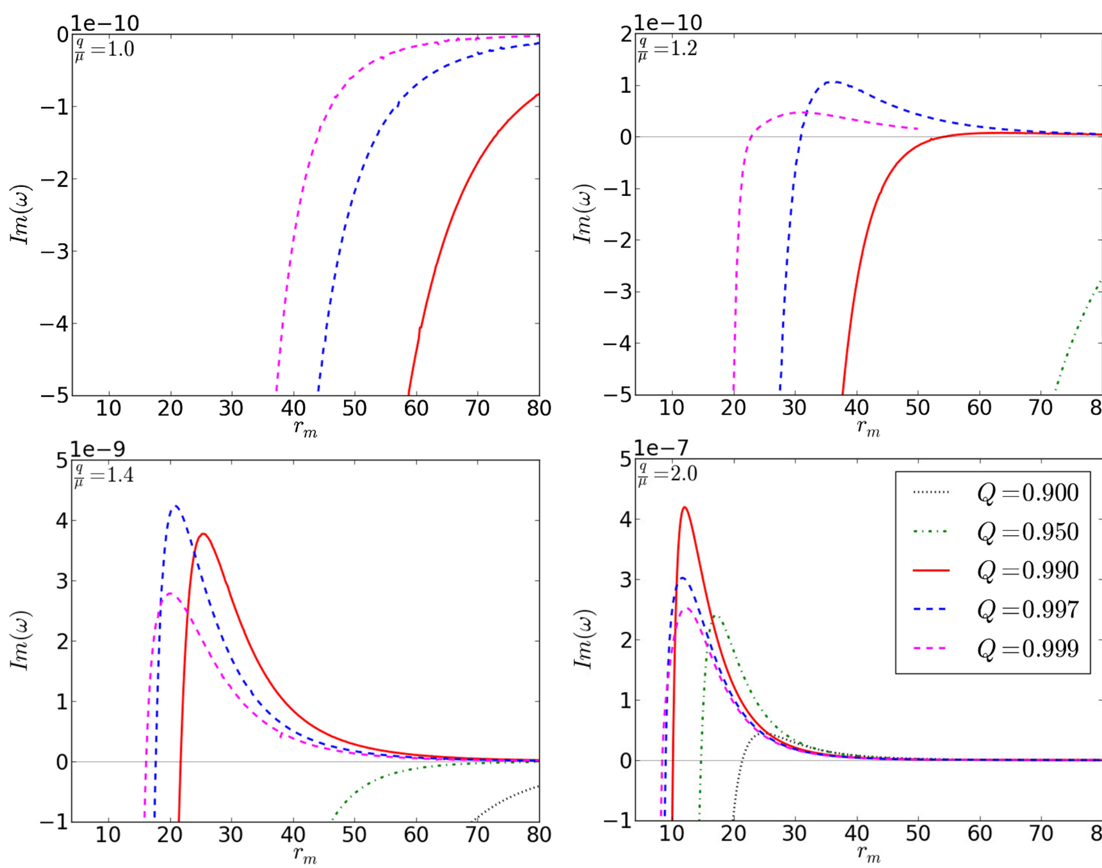

FIG. 1 (color online). The imaginary part of the frequency plotted versus the radius of the mirror for various ratios of the scalar charge, q to scalar mass μ. The scalar mass is $\mu = 0.3$. We included the line $\mathrm{Im}(\omega) = 0$ to help visualize where the curves have $\mathrm{Im}(\omega) > 0$.

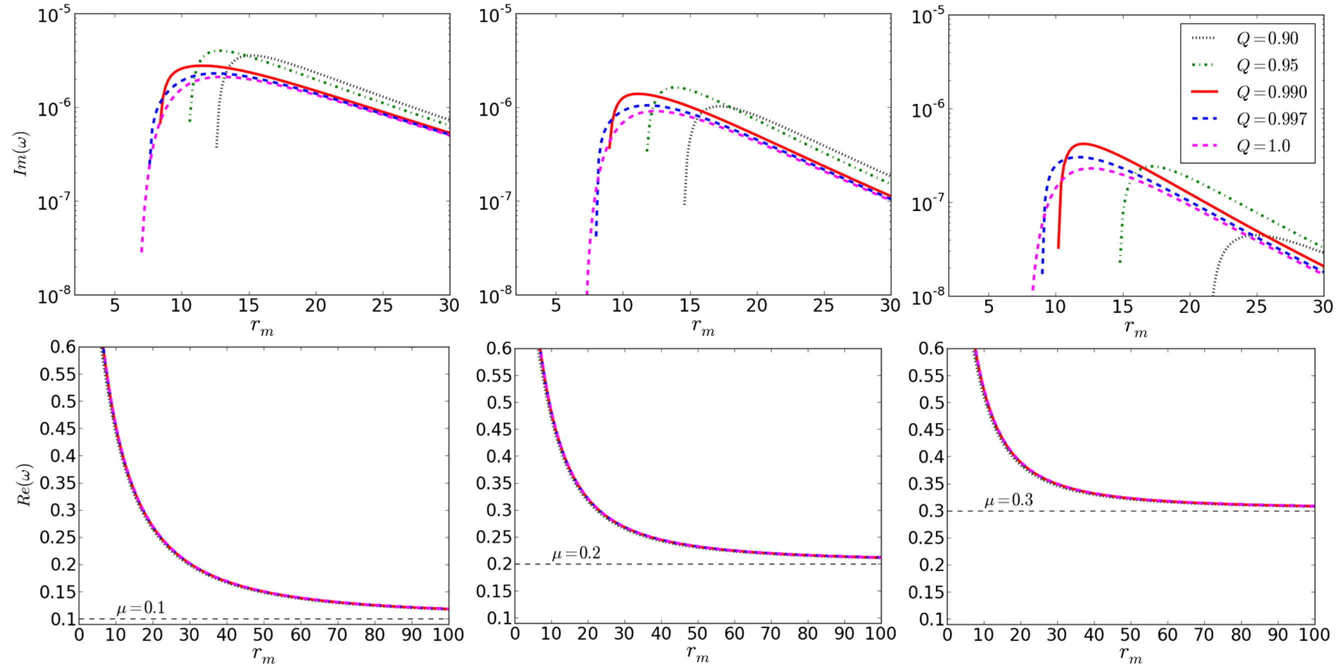

FIG. 2 (color online). The imaginary and real part of ω drawn as a function of the mirror radius r_m for various values of the black hole charge, Q, and the scalar mass, μ. $\mu = 0.1$, 0.2, 0.3 for the left, middle, and right column, respectively. We took $q = 0.6$.

imaginary parts for a given Q. These facts are compatible with the following interpretations. First, superradiant modes have a maximum frequency; thus, they will have a minimum wavelength and hence a minimum radius for the mirror is required for obtaining $\mathrm{Im}(\omega) > 0$. Second, an analysis of the area formula for RN black holes reveals that an increase of area requires the small quantities of charge dQ and mass dM extracted to be in a ratio greater than unity. Indeed, if the area $A = 4\pi r_+^2$ increases by a small quantity dA, then requiring $dA > 0$ yields

$$dA = \frac{\partial A}{\partial M} dM + \frac{\partial A}{\partial Q} dQ > 0 \Rightarrow \frac{dQ}{dM} > \frac{r_+}{Q} > 1. \quad (8)$$

The condition $dQ/dM > r_+/Q$ is actually surprisingly consistent with the numerical results in Fig. 1 to identify which values of Q can yield $\mathrm{Im}(\omega) > 0$, for each ratio q/μ. Since our goal is to analyze how large the instability may become, we shall focus in the following on values for the scalar charge and mass where their ratio is larger than unity.

In Figs. 2 and 3, we fix, respectively, the field charge q and the field mass μ. From these figures we observe that as the scalar mass (scalar charge) increases, the magnitude of the imaginary part of the frequency decreases (increases). A generic observation is that the real part of the frequencies approaches the numerical value of the field's mass and the imaginary part decreases monotonically as r_m increases. Moreover, from the latter figure we see that if the scalar charge is increased, the black hole charge which gives the maximum imaginary part of the frequency increases and eventually becomes the extremal case.

These results indicate that in order to get the maximum amplification of the scalar field, then (i) the black hole should be extremal, or at least close to extremal, and (ii) the scalar field should be as light as possible but with the highest possible charge. This latter expectation is confirmed in Fig. 4, where it is seen that fixing Q and r_m, the imaginary part of the frequency grows monotonically with q (as does the real part).

These data suggests that the growth of $\mathrm{Im}(\omega)$ with q will continue. As q increases, however, the integration of the radial equation becomes difficult. The first issue arises because the coefficients of (4) might differ by several orders of magnitude. Close to the horizon, for instance, the last term in parentheses of (4) can be up to five orders of magnitude larger than the terms that multiply the derivatives. In this sense, the equation becomes "stiff." In order to ameliorate this difficulty, we integrate the equation using different methods, and we report the values for the frequency at which both methods give the same value.[1] Second, for large values of the frequency, the leading terms of $R_\ell(r)$ and its derivative become very small close to the horizon; with such small values, the integrators find, very frequently, the trivial solution $R_\ell(r) \equiv 0$. For these reasons the largest imaginary part we can quote is $\mathrm{Im}(\omega) = 0.07099 \pm 0.0002$ for $q = 40$, $Q = 0.9$, $r_m = 5.0$.

[1]In most cases we used an explicit Runge-Kutta integrator of third and fourth order; however, for particular parameters we found it necessary to use the explicit modified midpoint method provided by MATHEMATICA [29].

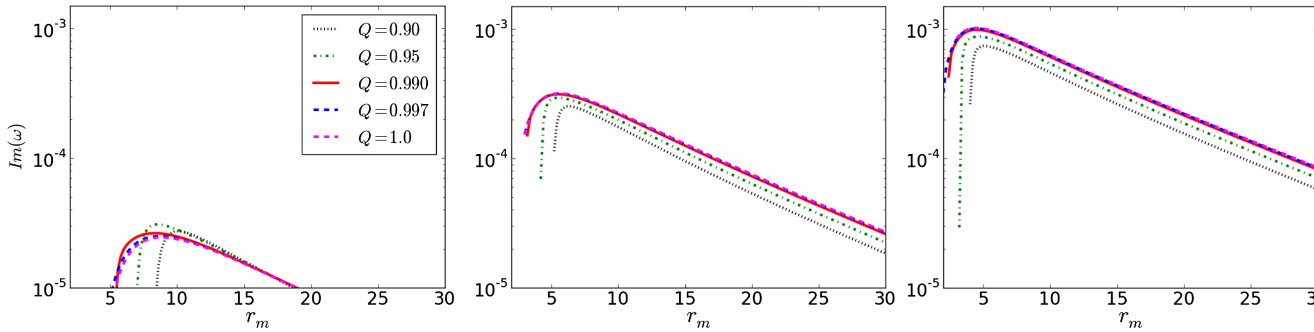

FIG. 3 (color online). The imaginary and real part of ω drawn as a function of the mirror radius r_m for various values of the black hole charge, Q, and the scalar charge, q. $q = 0.9$, 1.5, 2.0 for the left, middle, and right column, respectively. We took $\mu = 0.1$.

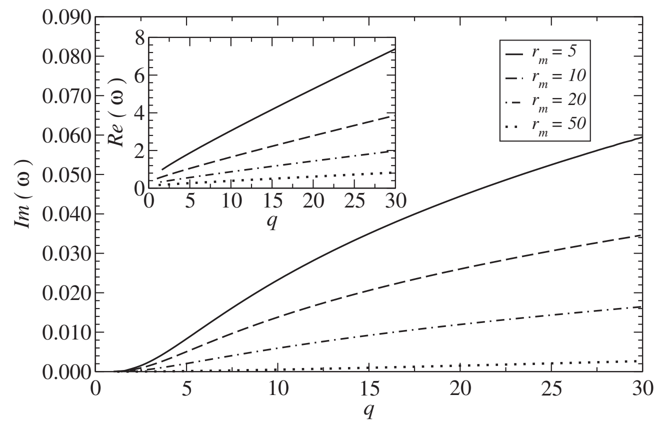

FIG. 4. The imaginary part (real part in the inset) of the frequency as a function of q for $Q = 0.9$, $\mu = 0.1$, and $r_m = 5$, 10, 20, 50.

In order to estimate the error in the frequencies, we use two integrators. With one integrator we get one frequency and one wave function, let us say $\omega_{(1)}$ and $R_{(1)}$, where $R_{(*)}$ stands for the radial part of the wave function. With the other integrator we get $\omega_{(2)}$ and $R_{(2)}$. When the value $\|R_{(1)} - R_{(2)}\|_\infty$ is less than 10^{-9}, we take the difference $|\omega_{(1)} - \omega_{(2)}|$ as the maximum error in the value of the frequency. The frequency reported is the linear interpolation between the two values.

IV. DISCUSSION AND CONCLUSIONS

The main message in this paper is that for RN black holes enclosed in a cavity, superradiant instabilities can be triggered by a charged scalar field; moreover, these instabilities can have a considerably shorter time scale than the analogous problem in the Kerr background. A hint to why there is such a difference between rotating and charged black holes comes from comparing the critical frequency for superradiance in both cases: $\omega_c = m\Omega_+$ for the rotating and $\omega_c = q\Phi_+$ for the charged black holes. It follows that in the charged case, q plays the same role that m plays in the rotating case. But whereas the former is bounded by ℓ, which should be taken to be $\ell = 1$ to maximize the instability [22,28], there is no bound on q. Thus, the critical frequency in a fixed RN background can be made as large as one wishes by increasing q, thus rendering plausible the existence of superradiant modes with very high frequency.

The fact that ω_c grows with q does not, however, lead to the conclusion that the imaginary part of the frequency should grow with q, but we can complement the above argument with another one to make this point, as follows.

As we have shown with our numerical results, the smaller the value of the mass, the greater the value of the imaginary component of the frequency. In the limit of a zero mass field, an estimate of the frequencies can be obtained analytically. The computation follows that in [12], where the Kerr black hole is considered and we provide only the main result.

We shall now assume that the Compton wavelength of the scalar particle is larger than the typical size of the black hole, $1/\omega \gg M$ (we have restored the mass of the black hole for clarity). Within this approximation it is possible to divide the space-time outside the horizon in two regions—the near region, where $(r - r_+) \ll 1/\omega$ and the far region, where $(r - r_+) \gg M$. Then, one solves the wave equation in both regions separately, and where an overlap occurs—in the region—$M \ll (r - r_+) \ll 1/\omega$ the solutions are matched. Using this matching technique, an approximation for the real part of frequency of the $n_{\rm th}$ overtone ω_n in terms of the mirror radius can be obtained as $\mathrm{Re}(\omega_n) = j_{\ell+1/2,n}/r_m$, where $j_{\ell+1/2,n}$ is the $(n+1)$th root of the Bessel function of order ℓ, J_ℓ. The imaginary part can be approximated as

$$\mathrm{Im}(\omega_n) = -\gamma \frac{1}{r_m^{2(\ell+1)}}(\mathrm{Re}(\omega_n) - \omega_c), \tag{9}$$

where

$$\gamma = \frac{(-1)^\ell J_{-\ell-1/2}(j_{\ell+1/2,n})}{J'_{-\ell-1/2}(j_{\ell+1/2,n})}\left(\frac{\ell!}{(2\ell-1)!!}\right)^2 \times \frac{r_+^2(M^2-Q^2)^\ell}{(2\ell)!(2\ell+1)!}\frac{(2j_{\ell+1/2,n})^{2\ell+1}}{2\ell+1}\prod_{k=1}^{l}(k^2+4\tilde{\omega}^2), \tag{10}$$

TABLE I. Analytic and numerical frequencies for the mirrored states. The mirror is at $r_m = 100$ and $Q = 0.8$.

q	$\omega_{\text{Numerical}}$	$\omega_{\text{Analytical}}$
0.1	$0.0452 + 4.7542 \times 10^{-10} i$	$0.0449 + 9.8835 \times 10^{-10} i$
1.2	$0.0605 + 7.1405 \times 10^{-7} i$	$0.0449 + 7.1534 \times 10^{-7} i$
1.6	$0.0657 + 3.8595 \times 10^{-6} i$	$0.0449 + 1.6754 \times 10^{-6} i$

and $\tilde{\omega} = r_+^2 (j_{\ell+1/2,n}/r_m - \omega_c)/(r_+ - r_-)$. The salient feature we wish to emphasize is the dependence on q, which appears via the dependence on ω_c. By inspection, this suggests that $\mathrm{Im}(\omega)$ grows with ω_c, and hence with q. This is indeed the behavior observed in the numerical results previously presented.

In Table I we show some frequencies obtained by this analytic approximation and compare them with the values obtained with the numerical integration scheme previously presented. From the aforementioned approximations, one expects that lower frequencies yield a better analytical approximation. This is indeed observed for the real part. An empirical observation is that the imaginary part is better approximated by the analytical formula when the product qQ is of the order of unity. This is also seen for the frequencies displayed in the table. Thus, there are indeed regimes of applicability for which the analytic approximation is legitimate.

As already mentioned in the Introduction, the charged case we have studied herein does not seem to have astrophysical relevance, mainly because if $Q/M \gtrsim 10^{-13}(a/M)^{-1/2} \times (M/M_\odot)^{1/2}$, the (Kerr-Newman) black hole is expected to discharge very quickly [30]. The interest in our study lies on providing a setup wherein the nonlinear development of the superradiant instability might be more treatable.

Finally, what will be the end state, in this setup, of the nonlinear development of the superradiant instability? Since the scalar field, after being amplified by the instability, cannot leave the cavity, the end state is likely to be a scalar condensate around a charged black hole. Observe that such scalar hair is not precluded by the usual theorems [31], since these assume asymptotic flatness. This scenario parallels the fate of unstable charged black holes in asymptotically anti–de Sitter space-times, against the condensation of a scalar field, which have been of considerable interest over the last few years for studies of holographic superconductors (see, e.g., [32]). Therefore, we conjecture that in contrast to the asymptotically flat case, the Einstein-Maxwell scalar field theory in a cavity should have two families of spherically symmetric solutions, at least for some range of the physical parameters, given that the behavior displayed for the frequencies in the $\ell = 1$ case holds in a qualitatively similar way for $\ell = 0$ as well. And the existence, in the linear analysis, of a threshold mode with zero imaginary part (as seen from the plots in Fig. 1) is, as in other well-known cases such as the Gregory-Laflamme instability of cosmic strings [33], indicating such a branching in the solutions of this theory.

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