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Simple "invariance" of two-body decay kinematics

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We study the two-body decay of a mother particle into a massless daughter. We further assume that the mother particle is *unpolarized* and has a *generic* boost distribution in the laboratory frame. In this case, we show analytically that the laboratory frame energy distribution of the massless decay product has a *peak*, whose location is identical to the (fixed) energy of that particle in the *rest* frame of the corresponding mother particle. Given its simplicity and "invariance" under changes in the boost distribution of the mother particle, our finding should be useful for the determination of masses of mother particles. In particular, we anticipate that such a procedure will then *not* require a full reconstruction of this two-body decay chain (or, for that matter, information about the rest of the event). With this eventual goal in mind, we make a proposal for extracting the peak position by fitting the data to a well-motivated analytic function describing the shape of such an energy distribution. This fitting function is then tested on the theoretical prediction for top quark pair production and its decay, and it is found to be quite successful in this regard. As a proof of principle of the usefulness of our observation, we apply it for measuring the mass of the top quark at the LHC, using simulated data and including experimental effects.

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It is very well known that in the *rest* frame of a mother particle undergoing a two-body decay, the energy of each of the daughter particles is fixed in terms of mother and the daughter particle masses. Turning this fact around, we can determine the mass of the mother particle if we can measure these rest-frame energies of the daughter particles.

However, often the mother particle is produced in the laboratory with a boost, that too with a magnitude and direction which is (*a priori*) not known. Moreover, the boost of mother particles produced at hadron colliders is different in each event. Such a boost *distribution* depends on the production mechanism of the particle and on the structure functions of the hadrons in the initial state of the collision and is thus a complicated function. In turn, the fact that the mother has a different boost in each event implies that when we consider the *observed* energy of the two-body decay product in the laboratory frame, we get a distribution in it. Thus it seems like the information that was encoded in the rest frame energy is lost, and we are prevented from extracting (at least at an easily tractable level) the mass of the mother particle along the lines described above.

We show that, remarkably, if one of the daughter particles from the two-body decay is massless and the mother is unpolarized, then such is *not* the case. Specifically, in this case, we demonstrate that the distribution of the daughter particle's energy in the laboratory frame has a *peak* precisely at its corresponding rest-frame energy.

This result is interesting *per se*. Furthermore, we expect that it will lead to formulation of new methods for mass measurements. Obviously, for this purpose, we need to be able to determine the location of this peak accurately from the observed energy distribution of the massless daughter. To this end, we propose and motivate an analytic function that can be used to fit the data on the energy distribution and thus extract the peak position. We show that this function is a suitable one using the top quark decay, $t \to W^- b$, as a test case; namely, it fits very well the *theory* prediction for energy spectrum of the resulting b jets. Simulating a realistic experimental situation, we then show that we can extract the value of the top mass from the position of the peak in the b-jet energy distribution along with the well-measured mass of the W boson.

Let us consider the decay of a heavy particle B of mass m_B, i.e., $B \to Aa$, where a is a *massless* visible particle. For the subsequent arguments, the properties of the particle A (other than its mass denoted by m_A) are irrelevant. In the rest frame of particle B, the energy of the particle a is simply given by

$$E^* = \frac{m_B^2 - m_A^2}{2m_B}. \tag{1}$$

Here and henceforth the starred quantity denotes that it is measured in the rest frame of particle B, i.e., the mother particle. If the mother particle (originally at rest) is boosted by a Lorentz factor γ in going to the laboratory frame, then the energy of particle a seen in the laboratory frame is

$$E = E^* \gamma (1 + \beta \cos\theta^*), \tag{2}$$

where θ^* defines the direction of emission of particle a in the rest frame of B with respect to the boost direction $\vec{\beta}$ of the mother B in the laboratory frame. Note that both $\cos\theta^*$ and γ can vary event by event. Therefore we get a *probability distribution* for the observed energy, which is the focus of our paper. Because of our assumption of the mother being not polarized, the probability distribution of $\cos\theta^*$ is flat. This implies that, for a fixed γ, the distribution of E is flat as well. More precisely, since $\cos\theta^* \in [-1, 1]$, for any fixed γ the shape of the distribution of E is a simple ''rectangle'' spanning the range[1]

$$x \equiv \frac{E}{E^*} \in \left[\left(\gamma - \sqrt{\gamma^2 - 1}\right), \left(\gamma + \sqrt{\gamma^2 - 1}\right)\right], \quad (3)$$

where x defines the dimensionless energy variable of the visible particle in the laboratory frame normalized by its rest-frame energy. A few crucial observations are in order. First, the lower (upper) bound of Eq. (3) is smaller (larger) than 1 for an arbitrary γ, which implies that *every* rectangle contains E^*. Remarkably, E^* is the only value of the energy to enjoy such a property as long as the distribution of mother particle boost is nonvanishing in a small region around $\gamma = 1$. Furthermore, the energy distribution being flat for every γ, there is no other value of the energy which gets a larger contribution than E^*. Thus, upon ''stacking up'' the rectangles of different widths, corresponding to a *range* of γ's, we see that the peak of the energy distribution of the particle a is unambiguously located at $E = E^*$. In fact, this argument goes through even for a *massive* daughter, provided we restrict boosts of the mother particle to $\gamma < (2\gamma^{*2} - 1)$, where γ^* denotes the boost of the daughter in the rest frame of the mother. Secondly, such rectangles are *asymmetric* with respect to the point $E = E^*$; i.e., the upper bound is farther from it than is the lower bound. Thus, the energy distribution of the particle a has a longer tail toward high energy with respect to such a peak.

More formally, the normalized differential decay width in x for a *fixed* γ is given by

$$\frac{1}{\Gamma}\frac{d\Gamma}{dx}\bigg|_{\mathrm{fixed}\,\gamma} = \frac{\Theta\left(x - \gamma + \sqrt{\gamma^2 - 1}\right)\Theta\left(-x + \gamma + \sqrt{\gamma^2 - 1}\right)}{2\sqrt{\gamma^2 - 1}}, \quad (4)$$

where $\Theta(x)$ is the usual Heaviside step function and the two step functions here merely define the allowed range of x. Next, consider a probability distribution of boosts of the mother given by $g(\gamma)$. A given energy of daughter in laboratory frame (x) can actually result from a specific range of values of the mother boost (γ), as per Eq. (4). So, we have to superpose these contributions weighted by the boost distribution, giving

$$f(x) \equiv \frac{1}{\Gamma}\frac{d\Gamma}{dx} = \int_{\frac{1}{2}(x+\frac{1}{x})}^{\infty} d\gamma \frac{g(\gamma)}{2\sqrt{\gamma^2 - 1}}. \quad (5)$$

The lower end in the integral here was derived from the solution to the equation, $x = \gamma \pm \sqrt{\gamma^2 - 1}$, for γ where the positive (negative) signature is relevant for $x \geq 1$ ($x < 1$). From Eq. (5) we can also compute the first derivative of $f(x)$, that is,

$$f'(x) = \frac{\mathrm{sgn}(1 - x)}{2x} g\left(\frac{1}{2}\left(x + \frac{1}{x}\right)\right). \quad (6)$$

We assume that $g(\gamma)$ has no zeros in the range of γ strictly between 1 and the kinematical limit of the collider at hand. In what follows we show that this is sufficient to guarantee that there is a peak at E^*. We consider the two possibilities for $g(1)$. Namely, if it vanishes, then $f'(x = 1) \propto g(1) = 0$, and the distribution has its unique extremum at $E = E^*$, following from our assumption on $g(\gamma)$. If $g(1) \neq 0$, then $f'(x)$ flips its sign at $x = 1$ so that the energy distribution has a cusp at $E = E^*$. Also, the function $f(x)$ is positive and vanishes for both $x \to 0$ and $x \to \infty$ since those two limits lead to a trivial definite integral in Eq. (5). Combining all these features, we see that the point $E = E^*$ is necessarily the peak of the distribution for both values of $g(1)$. This completes the formal proof of the peak location for a generic boost distribution of the mother particle.

As advertised at the beginning, our finding can be utilized to measure a combination of m_A and m_B given by E^*, which means that m_B can be determined if m_A is known, or vice versa. For this purpose, we are required to extract the location of the peak accurately from data. Clearly, having a theoretical prediction from first principles for the shape of $f(x)$ is very hard because the boost distribution $g(\gamma)$ is inherently process dependent. Nevertheless, we know some properties of $f(x)$ which are listed below: (i) the value of $f(x)$ remains the same under $x \leftrightarrow \frac{1}{x}$, (ii) f is maximized at $x = 1$, (iii) f vanishes as x approaches 0 or ∞, (iv) f becomes a δ function in some limit of its parameters. The first property follows from the x dependence of f arising only from the lower limit of the integral in Eq. (5), and the second from Eq. (6) and the argument thereafter. The third one is also manifest from Eq. (5) as mentioned above. Finally, the last one reflects the fact that when the mother particle is not boosted, we get a fixed value of energy given in Eq. (1), i.e., a delta function.

Being aware of the constraints given above, we propose the following ''simple'' function as an ansatz for $f(x)$:

$$f(x) = K_1^{-1}(p)\exp\left[-\frac{p}{2}\left(x + \frac{1}{x}\right)\right], \quad (7)$$

where p is a parameter which encodes the width of the peak and the normalization factor $K_1(p)$ is a modified Bessel function of the second kind of order 1. One can easily prove that the proposed ansatz can be reduced to a δ function for any sufficiently large p using the asymptotic behavior of $K_1(p)$ such that

$$K_1(p) \xrightarrow{p\to\infty} \sim \frac{e^{-p}}{\sqrt{p}}\left(1 + \mathcal{O}\left(\frac{1}{p}\right)\right). \quad (8)$$

[1]This result is quite well known and was used for a measurement of the W mass at lepton colliders [1].

Finally, we can show that the above ansatz does not have a cusp (at E^*) so that it is more suitable for the case of $g(1) = 0$ such as pair production of mothers.

In order to test the goodness of the ansatz given in Eq. (7) we use it to fit a *theoretical* prediction for the distribution of b-jet energy in top decay. The bottom quark is not massless; it is nonetheless highly boosted in the rest frame of top quark, namely, $\gamma^* \approx 15$. Based on our earlier discussion of the massive case, our argument for the peak in b-jet energy being at E^* is invalidated for boosts of the top quark which are so large ($\gamma \gtrsim 500$) as to have a negligible probability. Hence, we expect the peak to be very close to E^*. Similarly, we expect that the first of the functional properties of the energy spectrum Eq. (5) will be only negligibly violated by the nonzero mass of the bottom quark. This justifies the use of the ansatz Eq. (7) to fit the b-jet energy spectrum.

Specifically, we study a sample of fully leptonic top decays from the process, $pp \to t\bar{t} \to b\bar{b}\mu^- e^+ \nu_e \bar{\nu}_\mu$, at the Large Hadron Collider (LHC) with 7 TeV center-of-mass energy. To compute the theory prediction for the given process we employ MADGRAPH5 1.4.2 [2] taking $m_{\rm top}$ of 173 GeV and the patron distribution functions (PDFs) CTEQ6L1 [3] with default choice of the renormalization and factorization scales.

The result of the associated fit is exhibited in Fig. 1 which shows a very good agreement between the theory prediction from MADGRAPH5 and the fitting function. To quantify the goodness of the ansatz with an objective measure we compute both the Kolmogorov-Smirnov (KS) [4] and the χ^2 value. The latter is computed taking bin counts for a luminosity of 5 fb^{-1} at LHC with $\sqrt{s} =$ 7 TeV assuming that the error on each bin count is Gaussian. The result is $\chi^2 = 39.3$ for 198 degrees of freedom while the KS test statistic is 0.012, which, rather than being taken in any statistical sense, should be taken as an indication that our ansatz gives a very good fit to the theory

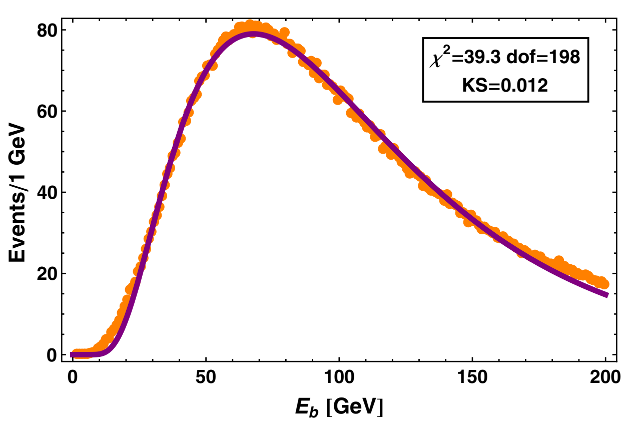

FIG. 1 (color online). The orange dots are the theory prediction for $d\sigma/dE_b$ in the process $pp \to t\bar{t} \to b\bar{b}\mu^- e^+ \nu_e \bar{\nu}_\mu$ computed with MADGRAPH5 at LHC with $\sqrt{s} = 7$ TeV. The purple line is the best fit of our ansatz Eq. (7).

curve. We have investigated the sensitivity of this result to the choice of the PDFs by repeating the same fit for the theory prediction obtained using the MRST2002NLO PDFs set of Ref. [5]. We observe negligible differences from the result obtained with CTEQ6L1.

So far, we have found that the ansatz in Eq. (7) is very good at reproducing the theory prediction. In fact, this success suggests that the ansatz may be used to measure the combination of masses in Eq. (1) from *experimental* data. In order to investigate this possibility, we go back to the example of the top quark; namely, we would like to use the fitting function in order to extract the peak of the observed energy distribution of the b jet and measure the top quark mass by plugging this value and the well-known mass of the W boson into Eq. (1).

Before getting into details, we would like to mention that we do not necessarily aim at getting a result for the value of $m_{\rm top}$ that is competitive with the current measurements. Rather, we aim at finding what is the sensitivity of our method for measuring top quark mass in a realistic setup. In fact, for a fair comparison it should be remarked that the current measurements of $m_{\rm top}$ rely on rather complicated tools and often advocate templates for the distributions that require a detailed knowledge of the underlying dynamics of the top quark decay. On the contrary, our method is extremely simple: it is based on pure kinematics and does not rely at all on detailed knowledge of the above-mentioned dynamics (as long as the top quark is produced unpolarized). As such we can regard our study of the mass measurement of the top as a proof of principle that our method can be used to measure the mass of heavy particles, in particular, new physics particles.

In lieu of actual experimental data we use a sample of Monte Carlo (MC)-*simulated* collision events. Namely, we further process the previous parton-level event sample to include the effects of showering and hadronization as described in PYTHIA 6.4 [6] with detector response simulated by DELPHES 1.9 [7] and jets made with FASTJET [8,9] using the *anti*-k_T algorithm [10] with the parameter choice $R = 0.4$. Furthermore, we impose cuts on the final state following the selections of Ref. [11] for the $e\mu$ final state. We consider an ensemble of 100 pseudoexperiments, each of which is equivalent to 5 fb^{-1} of data from the LHC at $\sqrt{s} = 7$ TeV. For each pseudoexperiment we perform a fit with our ansatz Eq. (7). From the extracted value of the peak of the distribution we get a measurement of $m_{\rm top}$. The distribution over the 100 pseudoexperiments of $m_{\rm top}$ and its 1σ error are symmetric around the central values and do not show special features. For a bin size of 4 GeV the average best-fit $m_{\rm top}$ and the 1σ error resulting from the fit are

$$\langle m_{\rm top} \rangle = 173.1 \pm 2.5 \text{ GeV} \tag{9}$$

with a median of the reduced χ^2 of our fit equal to 1.1. We have checked that the result of the fit is stable under

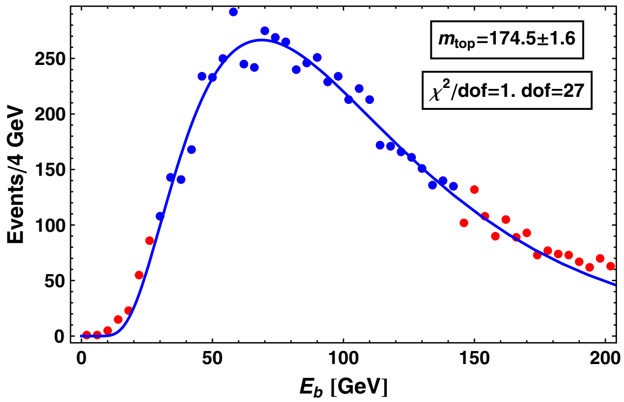

FIG. 2 (color online). An instance of the result of the fit on the energy distribution of the b jets in a pseudoexperiment. For the fit we binned data in bins of 4 GeV. Only the blue (dark) data points are used in the fit, which correspond to using only the part of the spectrum from 30 to 150 GeV.

changes of the range of energies taken in the fit and changes of the bin size. For illustration purpose we show the outcome of one of the pseudoexperiments in Fig. 2.

The obtained value of m_{top} is rather good: the error is small and the central value is compatible with the input value within the 1σ error. Furthermore, the obtained χ^2 is good. All this tells that the usefulness of our function Eq. (7) is not spoiled by the selection criteria for top quark decay events nor by detector effects. Even though our assessment of the power of the technique does not take the background and the entire realm of detector effects into consideration, we regard it as rather encouraging for the determination of the masses of heavy particles at colliders based only on the kinematics of the decay.

Note that our analysis does not include resolvable—and thus necessarily hard—radiation from the bottom quark in the above process. This extra radiation will turn the decay into a genuine three-body one so that our formalism will not apply to it. Therefore, if one wants to interpret our result in Eq. (9) as a realistic mass measurement, then the corrections from this process should be taken into account. Nevertheless, we have estimated that the cross section for it is smaller than the leading order one by one order of magnitude or more. Also, it might be possible to veto the extra radiation to further suppress such a contribution.

In conclusion, we have shown that for the two-body decay of an unpolarized boosted mother particle, the energy spectrum of a massless daughter in the laboratory frame encodes in a rather simple manner information about the masses involved in the decay. We showed as well how this can be used for mass measurements at hadronic colliders, which represents a remarkable twist in this paradigm. Indeed, instead of using longitudinally or fully Lorentz-invariant quantities for this purpose, we extracted the mass of the top quark from a Lorentz-variant observable, i.e., the energy of the b jet. The crucial point is that even though the distribution of this quantity depends on the possible boosts of the mother particle, the peak position in it is "invariant." Another merit of our method is that it does not rely on any measurement of the other particle of the two-body decay so that we can extract some information about masses even if the latter is invisible: a case that would not be tractable with, for example, measurement of invariant mass only. It is also clear that our method and the traditional techniques for mass measurement are sensitive to different kind of detector effects, In general, we thus expect that there will be a large degree of complementarity of our method with more traditional ones.

Finally, we emphasize that the proposed technique, despite being based on a fitting function, relies only on the minimal assumptions of absence of polarization and the presence of a nontrivial boost distribution of the mother particle; i.e., it does not require any other prior knowledge about the underlying physics model governing the decay of the particle whose mass we want to measure. This suggests that our method will be especially suitable for the mass measurement of new particles which might be discovered at the LHC, where we (*a priori*) would not know such details.

We envisage a number of applications of our finding here about the distributions of the energy of a daughter particle from a two-body decay. In particular, in forthcoming work we shall present results on the measurement of the masses of *specific* new physics particles [12]. Meanwhile, there has already been a suggestion to use the energy peak in distinguishing signal from background in searching for superpartners of the top quark: see Ref. [13]. And, an application of this result in distinguishing decays of bottom partners into bottom quark accompanied by one vs two (massive) invisible particles (as in dark matter models) is worked out in detail in Ref. [14]. This latter analysis can be generalized to the case of other three- and two-body decays.

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