Functional renormalization group study of the two-flavor linear sigma model in the presence of the axial anomaly

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The $SU(2)_A \times U(2)_V$-symmetric chiral linear sigma model in the presence of the axial anomaly is studied in the local-potential approximation of the functional renormalization group. The renormalization group flow is investigated in a truncation which reproduces recent results for the $U(2)_A \times U(2)_V$-symmetric model in the limit of vanishing axial anomaly strength. We search for the conjectured $O(4)$ fixed point in the presence of the $U(1)_A$ anomaly and analyze its stability properties.

I. INTRODUCTION

Pisarski and Wilczek [1] investigated the most general renormalizable Lagrangian which is invariant under the chiral $U(N_f)_L \times U(N_f)_R$ symmetry of quantum chromodynamics (QCD), where N_f denotes the number of quark flavors. Choosing the $[\bar{N}_f, N_f] + [N_f, \bar{N}_f]$ representation of $SU(N_f)_L \times SU(N_f)_R$ [2], in Euclidean space this Lagrangian reads

$$\mathcal{L}_\Phi = \frac{1}{2}\,\mathrm{Tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + \frac{1}{2} m_\Phi^2 \,\mathrm{Tr}\Phi^\dagger \Phi \\ + \frac{\pi^2}{3} g_1 (\mathrm{Tr}\Phi^\dagger\Phi)^2 + \frac{\pi^2}{3} g_2 \,\mathrm{Tr}(\Phi^\dagger\Phi)^2, \quad (1)$$

where Φ is a complex-valued $N_f \times N_f$ matrix. The anomalous breaking of the $U(1)_A$ symmetry contained in $U(N_f)_L \times U(N_f)_R$ is due to instantons [3] (see also Ref. [4]) and is commonly referred to as $U(1)_A$ anomaly. The authors of Ref. [1] conjectured that, for $N_f = 2$, the chiral phase transition of QCD can be of second order in the presence of the $U(1)_A$ anomaly. In this case, it would fall into the $O(4)$ universality class. In the following, we shall refer to this statement as $O(4)$ conjecture.

The term commonly introduced into Eq. (1) in order to explicitly break the $U(1)_A$ symmetry is

$$\det \Phi^\dagger + \det \Phi. \quad (2)$$

In Appendix A we show that, for $N_f = 2$, the most general form of the anomaly including terms up to naive scaling dimension four is [5]

$$\mathcal{L}_A = c(\det \Phi^\dagger + \det \Phi) + y(\det \Phi^\dagger + \det \Phi)\mathrm{Tr}\Phi^\dagger\Phi \\ + z[(\det \Phi^\dagger)^2 + (\det \Phi)^2]. \quad (3)$$

These terms must be added to Eq. (1),

$$\mathcal{L} = \mathcal{L}_\Phi + \mathcal{L}_A, \quad (4)$$

if one wants to study the impact of the $U(1)_A$ anomaly on the chiral phase transition. For $N_f = 2$ and including terms up to naive scaling dimension four, the Lagrangian (4) is the most general Lagrangian invariant under

$SU(2)_A \times U(2)_V$ and respecting parity symmetry. We note that the terms $\sim y, z$ are always induced by the renormalization group (RG) flow if $c \neq 0$. Therefore, in the following we shall use the notion "in the presence of the anomaly," whenever $c \neq 0$. Note also that

$$(\det \Phi^\dagger + \det \Phi)^2 \\ = -\mathrm{Tr}(\Phi^\dagger\Phi)^2 + (\mathrm{Tr}\Phi^\dagger\Phi)^2 + [(\det \Phi^\dagger)^2 + (\det \Phi)^2], \quad (5)$$

so that the square of the term (2) is not linearly independent from the other invariants contained in Eq. (3). Finally note that

$$i(\det \Phi^\dagger - \det \Phi) \quad (6)$$

is not invariant under CP transformations [1].

In this work, we consider the case $N_f = 2$. Denoting

$$\Phi = (\sigma + i\eta)t_0 + \vec{t} \cdot (\vec{a} + i\vec{\pi}), \quad (7)$$

with $t_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $t_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $t_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $t_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we rewrite the Lagrangian (4) into the form [6]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma \partial_\mu \sigma + \partial_\mu \vec{\pi} \cdot \partial_\mu \vec{\pi} + \partial_\mu \eta \partial_\mu \eta \\ + \partial_\mu \vec{a} \cdot \partial_\mu \vec{a}) + U, \quad (8)$$

$$U = \frac{1}{2}\mu^2(\sigma^2 + \vec{\pi}^2 + \eta^2 + \vec{a}^2) + \frac{\lambda_1}{4!}(\sigma^2 + \vec{\pi}^2 + \eta^2 + \vec{a}^2)^2 \\ + \lambda_2[(\sigma^2 + \vec{\pi}^2)(\eta^2 + \vec{a}^2) - (\sigma\eta - \vec{\pi}\cdot\vec{a})^2] \\ + c(\sigma^2 - \eta^2 + \vec{\pi}^2 - \vec{a}^2) + y(\sigma^2 + \vec{\pi}^2 + \eta^2 + \vec{a}^2) \\ \times (\sigma^2 - \eta^2 + \vec{\pi}^2 - \vec{a}^2) + z\frac{1}{2}(\eta^2 + \vec{a}^2 - \sigma^2 - \vec{\pi}^2 \\ - 2\vec{a}\cdot\vec{\pi} + 2\eta\sigma)(\eta^2 + \vec{a}^2 - \sigma^2 - \vec{\pi}^2 + 2\vec{a}\cdot\vec{\pi} - 2\eta\sigma), \quad (9)$$

where $\lambda_1 \equiv 4! \frac{\pi^2}{3}(g_1 + \frac{1}{2}g_2)$, $\lambda_2 \equiv 2 \frac{\pi^2}{3} g_2$, $\mu^2 \equiv m_\Phi^2$. For $c = 0$, $y = 0$, and $z = 0$ Eq. (8) reduces to the $U(2)_L \times U(2)_R$-symmetric Lagrangian (1).

The RG flow for the Lagrangian (1) was analyzed for different values of N_f. The results from the ϵ expansion [5,7] prove that for $N_f = 2$ the $O(8)$-symmetric infrared (IR) fixed point is unstable, which is confirmed from functional renormalization group (FRG) studies [6] as well as from lattice calculations [8]. The absence of an IR stable fixed point is a sufficient criterion for the phase transition to be of first order. In the presence of the anomaly ($c \neq 0$), however, to our knowledge the RG flow for the Lagrangian (8) has not yet been calculated explicitly, neither in the ϵ expansion nor in the FRG framework. The FRG study presented in Refs. [9,10] neglects the fields η and $\vec{a}$ from the beginning. Other RG results in the presence of the anomaly can be found in the literature only for cases where the anomaly term acts as a coupling of order higher than two (see for example Refs. [11–14]). Also, studying how c approaches ∞ has not yet been investigated explicitly on the level of RG flow equations. In this work we want to fill these gaps by appropriately extending the study presented in Ref. [6].

For the remainder of this introductory section, we would like to make a couple of remarks. The first one concerns the *universality hypothesis*. The RG approach towards critical phenomena defines (universality) classes of microscopically very different models, which lie in the basin of attraction of a certain IR stable fixed point and hence share the same critical exponents. Each universality class can be uniquely defined by the universal eigenvalues of the stability matrix associated with the IR stable fixed point (we neglect cases where the critical exponents depend on the couplings, as in Baxter's famous two-dimensional eight-vertex model [15]). Also the fixed-point potential is characteristic for a given universality class and determines the symmetry of the fixed point.

Consider a certain symmetry group G_0 and the most general G_0-invariant Landau-Wilson potential for a certain representation $\Gamma(G_0)$. For the sake of simplicity let us assume that the Landau-Wilson potential has a single IR stable fixed point FP_0 in coupling space, and thus falls into the universality class of FP_0.

Let us now add another coupling term to the potential, which breaks G_0 to a subgroup G of G_0. One obtains a different model which is only invariant under this subgroup $G \subset G_0$. The presence of an additional coupling term could induce another IR stable fixed point FP associated with G. The existence of the fixed point FP_0 need not be compromised, but it does not need to be IR stable anymore. However, if FP_0 remains IR stable, and if FP does not exist, and if no separatrix exists in the RG flow, the new G-invariant model necessarily falls into the same universality class as the previous G_0-invariant model.

In the literature there exist different versions of the universality hypothesis with slightly varying scope and content [15–17]. We state the universality hypothesis as follows: two Lagrangians for two different order parameters lie in the same universality class, if (i) the spatial dimension is the same for both systems, (ii) the order parameters have the same number of components, (iii) the symmetries of the Lagrangians are isomorphic, and (iv) there are no long-range interactions in both Lagrangians. [Usually, long-range interactions yield mean-field values for the critical exponents and one does not have nontrivial universal behavior. In the presence of "middle-range" interactions critical exponents can be different for two Lagrangians fulfilling the criteria (i)–(iii).] Whereas conditions (i), (ii), and (iv) are necessary conditions, criterion (iii) is sufficient but not necessary since, according to the above discussion, the fixed point for the full symmetry group G_0 can remain IR stable even in the presence of terms which break the symmetry to $G \subset G_0$, and therefore the two Lagrangians are both in the universality class associated with FP_0. It is an open question how to turn (iii) into a necessary condition, and if further conditions are necessary in order to exclude exceptions [17].

On the other hand, there exists a plethora of (more or less reliable) criteria which can serve to rule out the existence of an IR stable fixed point (see, for example, Refs. [18–21] and references therein). The best-known ones were already given by Landau and Lifshitz [19,22], namely the case where the representation $\Gamma(G)$ of the group associated with the G-invariant Landau-Wilson potential is not irreducible (such that there is a linear invariant), or the case where the third power of the representation, $\Gamma^3(G)$, contains the trivial representation (such that there is a third-order invariant which drives the transition first order).

Furthermore, unless one is interested in multicritical behavior (for a related work, see Ref. [23]), it is commonly assumed [24] that in case of two quadratic invariants, one can restrict the discussion of critical behavior near second-order phase transitions to simpler models, one for each of the competing order parameters (the quadratic invariants). This is because in general the couplings associated to the two quadratic invariants vanish at different critical temperatures, each corresponding to a different phase transition. One may naively think that it should therefore be possible to ignore one of the order parameters, when discussing the phase transition for the other one. However, the second invariant introduces another relevant direction in coupling space which may render an IR stable fixed point corresponding to a second-order phase transition associated with one of the order parameters unstable.

Our second remark concerns the role of baryon number conservation in the chiral phase transition. The $U(1)_A$ anomaly explicitly breaks the $U(1)_A$ symmetry contained in $G \equiv U(N_f)_V \times U(N_f)_A \simeq U(1)_V \times U(1)_A \times [SU(N_f)/Z(N_f)]_L \times [SU(N_f)/Z(N_f)]_R$ down to $Z(N_f)_A$, where $\simeq$ symbolizes group isomorphy. The group $U(1)_V$ is associated with baryon number conservation and should not be

broken (spontaneously) during the phase transition. Thus one usually argues that one can neglect it when studying the chiral phase transition, leaving $[SU(N_f)_L \times SU(N_f)_R]/Z(N_f)_V \rightarrow SU(N_f)_V/Z(N_f)_V$ for the symmetry breaking pattern relevant for the chiral phase transition in the presence of the anomaly [5]. The spontaneous breaking of a discrete symmetry does not yield Goldstone modes, such that it is sufficient to consider the breaking of the continuous group $G' \equiv SU(N_f)_L \times SU(N_f)_R$ in the chiral phase transition in the presence of the anomaly. We nevertheless consider an effective theory for the order parameter invariant under $U(1)_V \times G'$ in the search for the IR fixed point associated to spontaneous breaking of $SU(N_f)_L \times SU(N_f)_R$.

Our final remark concerns the $O(4)$ conjecture. Aside from criterion (iv) which we do not discuss here, we conclude that if the chiral phase transition of two-flavor QCD in the presence of the anomaly is of second order, then the Lagrangian (8) should fall into the same universality class as QCD. The Lagrangian (4) however has 8 degrees of freedom, whereas the $O(4)$ model has only four, which at first glance would mean that criterion (ii) of the universality hypothesis is not fulfilled. It is therefore *a priori* not clear that the IR stable fixed point for the Lagrangian (4) is an $O(4)$ fixed point. It might as well correspond to another universality class, characterized by $SU(2)_A \times U(2)_V$ critical exponents. To justify the $O(4)$ conjecture, first note that the choice of the representation depends on the physical degrees of freedom one intends to study. In the presence of the anomaly, one can make use of the isomorphism

$$SU(2) \times SU(2)/Z(2) \simeq SO(4), \qquad (10)$$

which means that $SU(2) \times SU(2)$ is locally isomorphic to $O(4)$. Accordingly, $SU(2) \times SU(2)$ has an $O(4)$ representation. For $N_f = 2$, the representation of the Lagrangian (8), or (4), respectively, is reducible. It consists of the sum of two equivalent $O(4)$ representations [2,5,25], $\Phi_1 = \sigma t_0 + i \vec{t} \cdot \vec{\pi}$ and $\Phi_2 = i \eta t_0 + \vec{t} \cdot \vec{a}$, which are both irreducible, but not faithful, representations of $SU(2) \times SU(2)$. Therefore, the symmetry of QCD allows for an $O(4)$ representation, if only the sigma and pion are light particles. At mean-field level this can be confirmed. The analysis in Refs. [26,27] shows that if we identify $\vec{\pi}$ with the Goldstone modes (the pions), the fields η and $\vec{a}$ are massive at the critical point, whereas the field σ is as light as the pions (and can be interpreted as the chiral partner of the pion). Since at the critical point only the modes with smallest mass are relevant (i.e., which count as components of the order parameter), we conclude that, if the mean-field approximation were justified, the IR fixed point would indeed be the stable Wilson-Fisher fixed point of the $O(4)$ model.

Of course, the mean-field approximation neglects quantum fluctuations (such as instantons), which might change the universality class or might lead to the instability of the fixed point. For this reason we study the FRG flow for the Lagrangian (8) in this paper. One could argue that for very large anomaly strength, $c \rightarrow -\infty$, η- and $\vec{a}$-loop diagrams should be suppressed according to the Appelquist-Carazzone decoupling theorem [28] due to the very high tree-level mass for the corresponding fields. Since the ϵ expansion deals only with loop diagrams, one can indeed expect to find the $O(4)$ fixed point [29]. However, this argument says nothing about (a) the stability of the $O(4)$ fixed point and (b) the cases of small and intermediate anomaly strength.

Let us note that in consistency with Refs. [1,6] we work with the dimensionally reduced theory, which is justified because the diverging correlation length at a second-order phase transition leads to dimensional reduction [30]. Again in agreement with the aforementioned references, we restrict ourselves to the case $T \gtrsim T_c$ where the system is driven towards the critical point from the side of the restored phase. This allows us to assume vanishing vacuum expectation values for all fields.

As a final remark, we want to point the reader to several FRG studies related to our work in a larger context, the list of which is, however, incomplete. Here we want to mention the recent work on applying the FRG method to QCD [31–35], a strategy how to combine first-principle QCD flows with effective models [36], and some investigations of effective models for QCD [37,38].

This paper is organized as follows. At the beginning of Sec. II we explain the method we use. In Secs. II A and II B, respectively, we consider two equivalent parametrizations of the potential. This not only serves as a check of our results, but also illustrates our general remarks given in Appendix C on how to obtain the correct flow equations when working with a parametrization in terms of the original field components instead of invariants. We derive the flow equations and analyze the stability properties of the fixed points. In Sec. II C we explain why the $O(4)$ fixed point becomes stable in the special case of infinite anomaly strength. Our arguments are supported by a discussion of the analogous situation in a simpler model in Sec. III. Section IV concludes this work with a summary of our results.

II. LINEAR SIGMA MODEL FROM FRG

In this section we investigate the FRG flow of the Lagrangian (8) in different parametrizations proceeding in analogy to Ref. [6]. We reproduce the result of Ref. [6] in the limit $c, y, z \rightarrow 0$.

We use the FRG equation in the local-potential approximation:

$$\frac{\partial U_k}{\partial k} = K_d k^{d+1} \sum_i \frac{1}{E_i^2}, \qquad E_i^2 \equiv k^2 + M_i^2,$$

$$K_d \equiv \frac{2\pi^{d/2}}{d\Gamma(d/2)(2\pi)^d}, \qquad (11)$$

where $\mathcal{L}_k = \frac{1}{2}\text{Tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + U_k$, with $\mathcal{L}_{k=\Lambda} = \mathcal{L}$ being the bare Lagrangian, and d is the spatial dimension. In the following, all our numerical results are for $d = 3$. M_i^2 denote the eigenvalues of the mass matrix

$$M_{ij} \equiv \frac{\partial^2 U_k}{\partial \phi_i \partial \phi_j}, \quad i, j = 1, \ldots, 8, \quad (12)$$

where the fields ϕ_i are given by σ, $\vec{\pi}$, η, and $\vec{a}$.

With the invariants

$$\varphi \equiv \sigma^2 + \vec{\pi}^2 + \eta^2 + \vec{a}^2,$$

$$\xi = (\sigma^2 + \vec{\pi}^2)(\eta^2 + \vec{a}^2) - (\sigma\eta - \vec{\pi} \cdot \vec{a})^2, \quad (13)$$

$$\alpha \equiv \sigma^2 - \eta^2 + \vec{\pi}^2 - \vec{a}^2,$$

and the abbreviation

$$\beta \equiv \alpha^2 - \frac{\varphi^2}{2} + 2\xi$$

$$= \frac{1}{2}(\eta^2 + \vec{a}^2 - \sigma^2 - \vec{\pi}^2 - 2\vec{a} \cdot \vec{\pi} + 2\eta\sigma)$$

$$\times (\eta^2 + \vec{a}^2 - \sigma^2 - \vec{\pi}^2 + 2\vec{a} \cdot \vec{\pi} - 2\eta\sigma),$$

the bare potential (9) reads

$$U(\varphi, \xi, \alpha) = \frac{1}{2}\mu^2\varphi + \frac{1}{4!}\lambda_1\varphi^2 + \lambda_2\xi + c\alpha + y\alpha\varphi + z\beta. \quad (14)$$

Using relation (5) and a different notation,

$$\varphi_1 = \sigma^2 + \vec{\pi}^2, \quad \varphi_2 = \eta^2 + \vec{a}^2, \quad \gamma = (\sigma\eta - \vec{\pi} \cdot \vec{a})^2, \quad (15)$$

we obtain

$$U(\varphi_1, \varphi_2, \gamma) = m_1^2\varphi_1 + m_2^2\varphi_2 + l_1\varphi_1^2 + l_2\varphi_2^2$$

$$+ l_{12}\varphi_1\varphi_2 + l_3\gamma, \quad (16)$$

where we introduced new couplings,

$$m_1^2 = \frac{1}{2}\mu^2 + c, \quad m_2^2 = \frac{1}{2}\mu^2 - c, \quad (17)$$

$$l_1 = y + \frac{\lambda_1}{4!} + \frac{z}{2}, \quad l_2 = -y + \frac{\lambda_1}{4!} + \frac{z}{2}, \quad (18)$$

$$l_{12} = \frac{\lambda_1}{12} + \lambda_2 - z, \quad l_3 = -(\lambda_2 + 2z).$$

Note that the number of linearly independent invariants is the same in expressions (14) and (16), respectively. When calculating the mass eigenvalues M_i, we have to simplify the computation by setting the values of several fields to

zero after having performed the second derivatives in Eq. (12). Keeping all fields nonzero, we obtain complicated expressions for the eigenvalues because an 8×8 matrix has to be diagonalized. One can circumvent the diagonalization using the relation

$$\sum_i \frac{1}{k^2 + M_i^2} = \text{Tr}\,\mathcal{M}^{-1}, \quad \mathcal{M}_{ij} \equiv M_{ij} + k^2\delta_{ij}. \quad (19)$$

However, it still would take a symbolic computation program a long time to expand the rhs of the FRG equation (11) in powers of the fields. Fortunately, the ϵ-expansion results from Ref. [1] can be reproduced by keeping nonzero values only for σ and one of the components of $\vec{a}$, say a_1 [6]. Note that this is not possible if we choose another field than a component of $\vec{a}$, since then $\xi = 0$ and we do not obtain a flow equation for λ_2. We further comment on the validity of this procedure from a more general perspective in Appendix C.

A. Parametrization in terms of invariants

In this section we use the parametrization (16) for the potential. It is nontrivial to rewrite all fields ϕ_i in terms of the above invariants. Since we have three invariants, the rewriting can be performed unambiguously only if we keep at least three fields nonzero. Keeping η, σ, and a_1 nonzero, we obtain the unambiguous mapping

$$\sigma = \sqrt{\varphi_1}, \quad a_1 = \sqrt{\frac{\varphi_1\varphi_2 - \gamma}{\varphi_1}}, \quad \eta = \sqrt{\frac{\gamma}{\varphi_1}}. \quad (20)$$

We also repeated our analysis using π_1 instead of η and found identical results.

We express the mass eigenvalues M_i in terms of φ_1, φ_2, and γ and expand the rhs of Eq. (11) in powers of these invariants. Then, inserting Eq. (16) on the lhs, we read off flow equations for the couplings by comparing coefficients. In order to calculate critical exponents we rescale quantities to obtain flow equations for dimensionless parameters. With

$$m_{i,k}^2 = k^2\bar{m}_{i,k}^2, \quad l_{i,k} = k^{4-d}\bar{l}_{i,k}, \quad (21)$$

we obtain

$$k\frac{\partial\bar{m}_1^2}{\partial k} = -2\bar{m}_1^2 - \frac{1}{3\pi^2}\left(\frac{12\bar{l}_1}{\bar{\epsilon}_1^2} + \frac{\bar{l}_3 + 4\bar{l}_{12}}{\bar{\epsilon}_2^2}\right), \quad (22)$$

$$k\frac{\partial\bar{m}_2^2}{\partial k} = -2\bar{m}_2^2 - \frac{1}{3\pi^2}\left(\frac{\bar{l}_3 + 4\bar{l}_{12}}{\bar{\epsilon}_1^2} + \frac{12\bar{l}_2}{\bar{\epsilon}_2^2}\right), \quad (23)$$

$$k\frac{\partial\bar{l}_{12}}{\partial k} = -\bar{l}_{12} + \frac{2[4(\bar{l}_1\bar{\epsilon}_2^3 + \bar{l}_2\bar{\epsilon}_1^3)(\bar{l}_3 + 6\bar{l}_{12}) + (\bar{l}_3^2 + 4\bar{l}_{12}^2)(\bar{\epsilon}_1 + \bar{\epsilon}_2)\bar{\epsilon}_1\bar{\epsilon}_2]}{3\pi^2\bar{\epsilon}_1^3\bar{\epsilon}_2^3}, \quad (24)$$

$$k\frac{\partial \bar{l}_1}{\partial k} = -\bar{l}_1 + \frac{2}{3\pi^2}\left(\frac{48\bar{l}_1^2}{\bar{\epsilon}_1^3} + \frac{\bar{l}_3^2 + 2\bar{l}_3\bar{l}_{12} + 4\bar{l}_{12}^2}{\bar{\epsilon}_2^3}\right), \quad (25)$$

$$k\frac{\partial \bar{l}_2}{\partial k} = -\bar{l}_2 + \frac{2}{3\pi^2}\left(\frac{48\bar{l}_2^2}{\bar{\epsilon}_2^3} + \frac{\bar{l}_3^2 + 2\bar{l}_3\bar{l}_{12} + 4\bar{l}_{12}^2}{\bar{\epsilon}_1^3}\right), \quad (26)$$

$$k\frac{\partial \bar{l}_3}{\partial k} = -\bar{l}_3 + \frac{4\bar{l}_3[4\bar{l}_1\bar{\epsilon}_2^3 + 4\bar{l}_2\bar{\epsilon}_1^3 + (3\bar{l}_3 + 4\bar{l}_{12})(\bar{\epsilon}_1 + \bar{\epsilon}_2)\bar{\epsilon}_1\bar{\epsilon}_2]}{3\pi^2\bar{\epsilon}_1^3\bar{\epsilon}_2^3},$$

$$(27)$$

where we omitted the index k and used the abbreviation

$$\bar{\epsilon}_i = 1 + 2\bar{m}_i^2. \quad (28)$$

In order to find the fixed points we have to set the left-hand sides to zero and solve the resulting system of equations. Since the equations are nonlinear, this has to be done numerically, using starting values for which a standard root-finding algorithm converges towards a solution. We applied an algorithm with randomized starting values in a reasonably large domain of parameter space, each of the starting values lying in the interval $[-10^4, 10^4]$. We found the nontrivial solutions given in Table I where they are listed together with the corresponding eigenvalues of the stability matrix. Using 10^6 different starting values, we

checked that these are the only solutions for given starting values in the above domain of parameter space.

The method how to calculate the eigenvalues of the stability matrix is described in Appendix B. From comparison with the corresponding eigenvalues for the $O(N)$ models in the same approximation scheme (local-potential approximation, fourth-order truncation in the fields), see Appendix C, we can unambiguously identify those fixed points in Table I with $O(N)$ critical exponents. Let us start our discussion with the fixed points FP_6 and FP_7. From the vanishing of the couplings in the upper part of Table I we see that for each of these fixed points the fixed-point potential is that of an $O(4)$ model. Fixed point FP_6 is that for the $O(4)$ representation $\Phi_1 = \sigma t_0 + i\vec{t}\cdot\vec{\pi}$, while FP_7 that for $\Phi_2 = i\eta t_0 + \vec{t}\cdot\vec{a}$. From the eigenvalues of the stability matrix in the lower part of Table I one observes that both fixed points have more than one negative eigenvalue, which means that they are unstable. Comparison of the second and third eigenvalue with the last two columns in Table II of Appendix C also tells us that they have one relevant $O(4)$ scaling direction.

For fixed point FP_5, the two masses $\bar{m}_i^2$ and the two coupling constants $\bar{l}_i$ are identical, while $\bar{l}_{12} = 0$. This means that the fixed-point potential is that of two independent, identical $O(4)$ models. From the lower part of Table I we see that this fixed point is a multicritical fixed point

TABLE I. Fixed points in the presence of nonzero anomaly strength, in $d = 3$ dimension, in the FRG analysis in the local-potential approximation, with couplings up to quartic order. The bar denotes rescaled quantities.

FP	$\bar{m}_1^2$	$\bar{m}_2^2$	$\bar{l}_1$	$\bar{l}_2$	$\bar{l}_{12}$	$\bar{l}_3$
FP_1	-3.80278	-0.197224	-355.58	0.273944	29.6088	0
FP_2	-0.197224	-3.80278	0.273944	-355.58	29.6088	0
FP_3	-1.34694	-0.333929	-17.0334	0.128417	5.64724	-5.93079
FP_4	-0.333929	-1.34694	0.128417	-17.0334	5.64724	-5.93079
FP_5	-0.0555556	-0.0555556	0.216617	0.216617	0	0
FP_6	-0.0555556	0	0.216617	0	0	0
FP_7	0	-0.0555556	0	0.216617	0	0
FP_8	-0.0555556	-0.0555556	0.108308	0.108308	0.216617	0.433234
FP_9	-0.0675676	-0.0675676	0.149643	0.149643	0.299286	0
FP_{10}	0.609013	-1.18037	-34.5716	7.9826	-9.98504	129.304
FP_{11}	-1.18037	0.609013	7.9826	-34.5716	-9.98504	129.304

FP	Stability-matrix eigenvalues
FP_1	$\{9.64793, -5.66667, -0.585909 + 4.07239\,i, -0.585909 - 4.07239\,i, 3.83241, -1.30852\}$
FP_2	$\{9.64793, -5.66667, -0.585909 + 4.07239\,i, -0.585909 - 4.07239\,i, 3.83241, -1.30852\}$
FP_3	$\{29.6235, 14.0524 + 4.23653\,i, 14.0524 - 4.23653\,i, 0.917927 + 9.64911\,i, 0.917927 - 9.64911\,i, -1.17232\}$
FP_4	$\{29.6235, 14.0524 + 4.23653\,i, 14.0524 - 4.23653\,i, 0.917927 + 9.64911\,i, 0.917927 - 9.64911\,i, -1.17232\}$
FP_5	$\{-1.77069, -1.77069, 1.27069, 1.27069, -0.666667, 0\}$
FP_6	$\{-2., -1.77069, 1.27069, -1., -0.833333, -0.5\}$
FP_7	$\{-2., -1.77069, 1.27069, -1., -0.833333, -0.5\}$
FP_8	$\{-2., -1.77069, 1.27069, -0.666667, 0, 0\}$
FP_9	$\{-1.98804, -1.71971, 1.34471, 0.613041, -0.25, -0.25\}$
FP_{10}	$\{-28.9145, -16.865, -10.9156, -3.11604 + 5.87462\,i, -3.11604 - 5.87462\,i, -1.28288\}$
FP_{11}	$\{-28.9145, -16.865, -10.9156, -3.11604 + 5.87462\,i, -3.11604 - 5.87462\,i, -1.28288\}$

TABLE II. Stability-matrix eigenvalues, y_i, for the Wilson-Fisher fixed point of the $O(N)$ model, $d = 3$, FRG (in local-potential approximation), up to quartic coupling. The bar denotes rescaled quantities.

N	$\frac{1}{2}\bar{\mu}_*^2$	$\bar{\lambda}_{1*}$	$\nu = -1/y_1$	y_2
1	-0.03846	7.76271	$0.54272 = -1/ -1.84256$	1.1759
2	-0.04545	6.67366	$0.55149 = -1/ -1.81327$	1.21327
4	-0.05556	5.1988	$0.564751 = -1/ -1.77069$	1.27069
8	-0.06757	3.59143	$0.581495 = -1/ -1.71971$	1.34471

with two relevant $O(4)$ scaling directions. The third negative eigenvalue of the stability matrix renders this an unstable fixed point. Fixed point FP_8 is another unstable multicritical fixed point with a single $O(4)$ scaling direction.

From the vanishing of $\bar{l}_3$ and the fact that $\bar{l}_1 = \bar{l}_2 = \bar{l}_{12}/2$, the fixed-point potential for FP_9 is that of an $O(8)$ model. The stability matrix indicates that this fixed point is unstable. Comparison of the eigenvalues of the stability matrix with Table II shows that it has one relevant $O(8)$ scaling direction. Since all eigenvalues of the stability matrix are negative, fixed points FP_{10} and FP_{11} are ultraviolet (UV) stable fixed points. Fixed points FP_1 and FP_2 are unstable fixed points, none of them belonging to one of the $O(N)$ universality classes.

Finally, FP_3 and FP_4 are IR stable fixed points. While for the other fixed points the (rescaled) eigenvalues of the (squared) mass matrix are always positive semidefinite, for FP_3 and FP_4 we find one negative eigenvalue in all minima of the (rescaled) fixed-point potential $\bar{U}(\bar{\sigma}, \bar{\eta}, \bar{a}_1)$, which corresponds to an unphysical situation. However, this could be an artifact of our fourth-order truncation of the potential (16), and the masses could be real-valued in higher order [39]. In that case, these IR stable fixed points are in the $SU(2) \times U(2)$ universality class. Nevertheless, at our truncation order we have to reject them.

B. Parametrization in terms of original fields

In this section, in contrast to the previous one, we keep the potential parametrized in terms of the original fields ϕ_i. This avoids the use of the chain rule together with tedious rewriting procedures and serves as a check of our results. As in the previous section, we expand the rhs of Eq. (11) and read off flow equations for the couplings by comparison of coefficients, but now the expansion is in powers of the original fields ϕ_i instead of the invariants φ_i, γ. Again, in order to obtain the correct flow equations, accounting for all three anomaly terms, we have to keep at least three fields nonzero after having performed the second derivatives in Eq. (12). For a general rule which and how many fields one has to keep at a minimum, in a case where the invariants are not known, we refer to Appendix C.

For checking purposes we keep an additional field nonzero, say π_1, and set π_2, π_3, a_2, and a_3 to zero after having

computed the second derivatives. This means that the comparison of coefficients is carried out using the potential (9) for $\pi_2 = \pi_3 = a_2 = a_3 = 0$ on the lhs of the flow equation (11). In this case the (scale-dependent) potential (9) reads

$$
\begin{aligned}
U_k = {} & a_1^2 m_{2,k}^2 + \eta^2 m_{2,k}^2 + \sigma^2 m_{1,k}^2 + \pi_1^2 m_{1,k}^2 \\
& + \lambda_{a\eta}(a_1^4 + \eta^4) + \lambda_{\sigma\pi}(\sigma^4 + \pi_1^4) \\
& + \delta_1(\pi_1^2 a_1^2 + \eta^2 \sigma^2) + \delta_2 a_1^2 \eta^2 + \delta_0(a_1^2 \sigma^2 + \pi_1^2 \eta^2) \\
& + \kappa \pi_1 a_1 \eta \sigma + \delta_3 \pi_1^2 \sigma^2,
\end{aligned} \tag{29}
$$

with

$$
\lambda_{a\eta} \equiv \frac{\lambda_1}{24} - y + \frac{z}{2}, \qquad \lambda_{\sigma\pi} \equiv \frac{\lambda_1}{24} + y + \frac{z}{2},
$$
$$
\delta_0 \equiv \frac{\lambda_1}{12} + \lambda_2 - z, \qquad \delta_1 \equiv \frac{\lambda_1}{12} - 3z,
$$
$$
\delta_2 \equiv \frac{\lambda_1}{12} + z - 2y, \qquad \delta_3 \equiv \frac{\lambda_1}{12} + z + 2y,
$$
$$
\kappa \equiv 4z + 2\lambda_2.
$$

Note that

$$
\delta_3 = 2\lambda_{\sigma\pi}, \qquad \delta_2 = 2\lambda_{a\eta}, \qquad \delta_0 = \delta_1 + \frac{\kappa}{2},
$$
$$
y = \frac{\lambda_{\sigma\pi}}{2} - \frac{\lambda_{a\eta}}{2}, \qquad z = -\frac{\delta_1}{4} + \frac{\lambda_{\sigma\pi}}{4} + \frac{\lambda_{a\eta}}{4},
$$
$$
\lambda_1 = 3\delta_1 + 9\lambda_{a\eta} + 9\lambda_{\sigma\pi}, \qquad \lambda_2 = \frac{\delta_1}{2} + \frac{\kappa}{2} - \frac{\lambda_{\sigma\pi}}{2} - \frac{\lambda_{a\eta}}{2}.
$$

We verified that we obtain unambiguous flow equations for $m_{1,k}^2$, $m_{2,k}^2$, $\lambda_{1,k}$, $\lambda_{2,k}$, y_k, and z_k, no matter from which of the coefficients in Eq. (29) we extract them (which is a freedom we have due to the additional field we have kept nonzero). We do not state the flow equations and fixed points again; we have checked that they are equivalent to those found in Sec. II A.

C. Physical anomaly strength

So far we have considered only a finite anomaly strength. According to Ref. [25], however, the limit $c \to -\infty$ should be closer to reality: in order to reproduce the correct vacuum mass of the eta meson in the two-flavor quark-meson model at tree level, one has to choose a value for the anomaly strength $[|c| \sim (958 \text{ MeV})^2]$ which exceeds a physically reasonable UV cutoff scale for the RG flow ($k \sim 600$ MeV). Therefore, on all scales relevant for the RG flow, effectively $c \to -\infty$. More precisely, instead of this limit, we should rather consider the limit

$$
m_{2,k}^2 = \frac{1}{2}\mu_k^2 - c_k \to \infty, \tag{30}
$$

otherwise $m_{1,k}^2 \equiv \frac{1}{2}\mu_k^2 + c_k \to -\infty$ would impose severe constraints on the RG flow in order to finally obtain positive-definite masses for σ and $\vec{\pi}$.

In the limit $m_{2,k}^2 \to \infty$ the flow equations (23)–(27) simplify to

$$k\frac{\partial \bar{m}_1^2}{\partial k} = -2\bar{m}_1^2 - \frac{4\bar{l}_1}{\pi^2 \bar{\epsilon}_1^2}, \tag{31}$$

$$k\frac{\partial \bar{l}_{12}}{\partial k} = -\bar{l}_{12} + \frac{8\bar{l}_1(\bar{l}_3 + 6\bar{l}_{12})}{3\pi^2 \bar{\epsilon}_1^3}, \tag{32}$$

$$k\frac{\partial \bar{l}_1}{\partial k} = -\bar{l}_1 + \frac{32\bar{l}_1^2}{\pi^2 \bar{\epsilon}_1^3}, \tag{33}$$

$$k\frac{\partial \bar{l}_2}{\partial k} = -\bar{l}_2 + \frac{2}{3\pi^2}\frac{\bar{l}_3^2 + 2\bar{l}_3\bar{l}_{12} + 4\bar{l}_{12}^2}{\bar{\epsilon}_1^3}, \tag{34}$$

$$k\frac{\partial \bar{l}_3}{\partial k} = -\bar{l}_3 + \frac{16\bar{l}_3\bar{l}_1}{3\pi^2 \bar{\epsilon}_1^3}. \tag{35}$$

The above flow equations have only one nontrivial fixed point, namely the $O(4)$ fixed point,

$$(\bar{m}_1^2 = -0.0555556, \bar{l}_1 = 0.216617, \bar{l}_2 = 0, \bar{l}_{12} = 0, \bar{l}_3 = 0). \tag{36}$$

Calculating its stability-matrix eigenvalues,

$$\{-1.77069, 1.27069, -1, -0.83334, -0.5\}, \tag{37}$$

we find that it is IR unstable. According to standard rules one would, erroneously, conclude that the phase transition cannot be of second order. According to common sense, however, this cannot be true since the fields η and $\vec{a}$ are infinitely heavy, so that fluctuations of these fields are completely suppressed and cannot affect the critical behavior. In Sec. III we explain, using a simpler model as an example, why we have to neglect the spurious negative eigenvalues when inferring the order of the phase transition. From the discussion in Sec. III, we conclude that couplings occurring only in front of terms involving infinitely heavy fields have to be neglected in the stability analysis of fixed points. This can be also understood from the fact that the fluctuations represented by infinitely heavy fields are zero.

Inserting the fixed-point solution (36) into the rescaled potential (16), we obtain

$$\bar{U}_{k=0} = -0.0555556(\bar{\sigma}^2 + \vec{\bar{\pi}}^2) + 0.216617(\bar{\sigma}^2 + \vec{\bar{\pi}}^2)^2. \tag{38}$$

Since the fixed-point potential is $O(4)$ symmetric, we can choose $\vec{\bar{\pi}}_0 = 0$ in the vacuum state. Then, the rescaled vacuum is given by

$$(\bar{\sigma}_0 = 0.358099, \vec{\bar{\pi}}_0 = 0). \tag{39}$$

Using these vacuum expectation values we calculate the rescaled mass eigenvalues (i.e., the rescaled physical masses):

$$\bar{M}_\sigma^2 = 2/9, \qquad \bar{M}_{\pi_i}^2 = 0, \qquad \bar{M}_\eta^2 \to \infty, \qquad \bar{M}_{a_i}^2 \to \infty. \tag{40}$$

We see that, as expected, we have three Goldstone bosons, the three pions $\vec{\pi}$, whereas η and $\vec{a}$ are infinitely heavy and thus decouple. Considering Eq. (16), we conclude that the couplings $\bar{l}_2$, $\bar{l}_{12}$, and $\bar{l}_3$ appear only in front of terms involving infinitely heavy fields and must not be included in the stability analysis. Including only $\bar{m}_1^2$ and $\bar{l}_1$, we find the stability-matrix eigenvalues

$$\{-1.77069, 1.27069\}, \tag{41}$$

from which we finally conclude that there exists a stable $O(4)$ fixed point in case of infinite anomaly strength. We also note that we verified that above the critical dimension, $d \geq 4$, the Gaussian fixed point becomes IR stable with mean-field critical exponent $\nu = 1/2$, as expected.

III. COUPLED VECTOR MODEL

In order to justify why we can neglect the spurious negative eigenvalues occurring in Sec. II C, we discuss a simpler model where the reasons become transparent. We consider the case of the most simple coupled vector model, which involves two scalar fields ϕ_1 and ϕ_2:

$$U = m_1^2\phi_1^2 + m_2^2\phi_2^2 + \frac{\lambda_{11}}{24}\phi_1^4 + \frac{\lambda_{12}}{12}\phi_1^2\phi_2^2 + \frac{\lambda_{22}}{24}\phi_2^4. \tag{42}$$

For a mean-field analysis of the model we refer to Ref. [40], for a leading-order ϵ expansion to Ref. [41].

Using the method of Taylor expansion and comparison of coefficients, we find the following flow equations:

$$k\frac{\partial \bar{m}_1^2}{\partial k} = -2\bar{m}_1^2 - \frac{1}{36\pi^2}\left(\frac{3\bar{\lambda}_{11}}{\bar{\epsilon}_1^2} + \frac{\bar{\lambda}_{12}}{\bar{\epsilon}_2^2}\right),$$
$$k\frac{\partial \bar{m}_2^2}{\partial k} = -2\bar{m}_2^2 - \frac{1}{36\pi^2}\left(\frac{3\bar{\lambda}_{22}}{\bar{\epsilon}_2^2} + \frac{\bar{\lambda}_{12}}{\bar{\epsilon}_1^2}\right), \tag{43}$$

$$k\frac{\partial \bar{\lambda}_{11}}{\partial k} = -\bar{\lambda}_{11} + \frac{1}{\pi^2}\left(\frac{\bar{\lambda}_{11}^2}{\bar{\epsilon}_1^3} + \frac{\bar{\lambda}_{12}^2}{9\bar{\epsilon}_2^3}\right),$$
$$k\frac{\partial \bar{\lambda}_{22}}{\partial k} = -\bar{\lambda}_{22} + \frac{1}{\pi^2}\left(\frac{\bar{\lambda}_{22}^2}{\bar{\epsilon}_2^3} + \frac{\bar{\lambda}_{12}^2}{9\bar{\epsilon}_1^3}\right), \tag{44}$$

$$k\frac{\partial \bar{\lambda}_{12}}{\partial k} = -\bar{\lambda}_{12} + \frac{\bar{\lambda}_{12}}{9\pi^2 \bar{\epsilon}_1^3 \bar{\epsilon}_2^3}[2\bar{\lambda}_{12}(\bar{\epsilon}_1 + \bar{\epsilon}_2)\bar{\epsilon}_2\bar{\epsilon}_1 + 3\bar{\lambda}_{22}\bar{\epsilon}_1^3 + 3\bar{\lambda}_{11}\bar{\epsilon}_2^3], \tag{45}$$

where we again used the abbreviation (28).

In this work we are only interested in the Ising fixed point,

$$(\bar{m}_1^2 = -0.03846, \bar{m}_2^2 = 0,$$
$$\bar{\lambda}_{11} = 7.76271, \bar{\lambda}_{12} = 0, \bar{\lambda}_{22} = 0), \tag{46}$$

the stability-matrix eigenvalues of which,

$$\{-2, -1.84256, 1.1759, -1, -0.666667\}, \quad (47)$$

indicate that it appears to be unstable in the $\bar{m}_2^2$, $\bar{\lambda}_{12}$, and $\bar{\lambda}_{22}$ directions.

Examining the above flow equations in the limit $\bar{m}_2^2 \to \infty$,

$$k\frac{\partial \bar{m}_1^2}{\partial k} = -2\bar{m}_1^2 - \frac{1}{12\pi^2}\frac{\bar{\lambda}_{11}}{\bar{\epsilon}_1^2}, \qquad k\frac{\partial \bar{\lambda}_{11}}{\partial k} = -\bar{\lambda}_{11} + \frac{1}{\pi^2}\frac{\bar{\lambda}_{11}^2}{\bar{\epsilon}_1^3},$$
$$(48)$$

$$k\frac{\partial \bar{\lambda}_{22}}{\partial k} = -\bar{\lambda}_{22} + \frac{1}{9\pi^2}\frac{\bar{\lambda}_{12}^2}{\bar{\epsilon}_1^3}, \qquad k\frac{\partial \bar{\lambda}_{12}}{\partial k} = -\bar{\lambda}_{12} + \frac{1}{3\pi^2}\frac{\bar{\lambda}_{12}\bar{\lambda}_{11}}{\bar{\epsilon}_1^3},$$
$$(49)$$

we still find negative eigenvalues corresponding to the unstable $\bar{\lambda}_{12}$ and $\bar{\lambda}_{22}$ directions, respectively. Obviously, we have the same situation as in Sec. II C. Formally, the negative eigenvalues would indicate that the Ising fixed point is IR unstable. In this particular case, however, one cannot conclude from this that the phase transition is fluctuation-induced first order. Fluctuations in $\bar{\lambda}_{12}$ and $\bar{\lambda}_{22}$ direction are completely suppressed due to the infinitely heavy ϕ_2 field and cannot affect the critical behavior. To prove this, we investigate in detail the scale evolution of the dimensionful

potential for different initial values for the parameters in the UV. Using the invariants

$$\varphi_1 = \phi_1^2, \qquad \varphi_2 = \phi_2^2, \quad (50)$$

we make the following ansatz for the potential running under the RG flow:

$$U_k = V_k(\varphi_1) + W_k(\varphi_1)\varphi_2 + X_k(\varphi_1)\varphi_2^2. \quad (51)$$

Having expressed the mass eigenvalues M_i^2 in terms of φ_1 and φ_2, we expand the rhs of Eq. (11) and read off flow equations for $V_k(\varphi_1)$, $W_k(\varphi_1)$, and $X_k(\varphi_1)$ by comparison of coefficients. We solve the resulting system of three partial differential equations together with the initial conditions

$$V_{k=\Lambda}(\varphi_1) = m_{1,\Lambda}^2\varphi_1 + \frac{\lambda_{11,\Lambda}}{24}\varphi_1^2,$$

$$W_{k=\Lambda}(\varphi_1) = m_{2,\Lambda}^2 + \frac{\lambda_{12,\Lambda}}{12}\varphi_1, \qquad X_{k=\Lambda}(\varphi_1) = \frac{\lambda_{22,\Lambda}}{24}.$$
$$(52)$$

Figure 1 illustrates the potential for various values of the RG flow parameter k for various values of $m_{2,\Lambda}^2$ and $\lambda_{12,\Lambda}$ for fixed values of $m_{1,\Lambda}^2$, $\lambda_{11,\Lambda}$, and $\lambda_{22,\Lambda} = 0$. We observe that the influence of the coupling $\bar{\lambda}_{12}$ on the shape of the
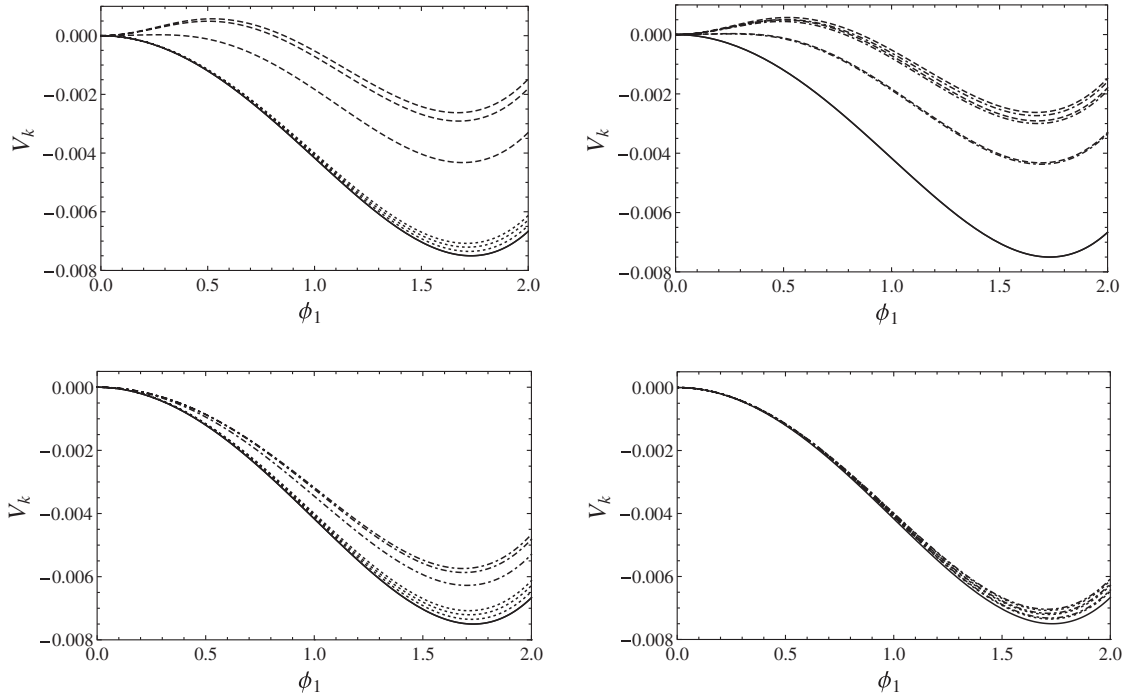

FIG. 1. Scale evolution of the potential V_k. In each panel, the solid line is the same and corresponds to the start of the evolution in the UV ($k/\Lambda = 1$). Furthermore, in each panel there are two sets of three curves (drawn with identical line mode). These three curves correspond to the RG potentials at the scales $k/\Lambda = 0.7$, $k/\Lambda = 0.4$, and $k/\Lambda = 0.12$ respectively. For all panels, $m_{1,\Lambda}^2 = -0.005\Lambda^2$, $\lambda_{11,\Lambda} = 0.02\Lambda$, $\lambda_{22,\Lambda} = 0$. In the upper left, the lower left, and the lower right panels, the dotted curves correspond to $m_{2,\Lambda}^2 = 0$, $\lambda_{12,\Lambda} = 0$ (and therefore coincide with solutions for the Ising model). In the upper left and upper right panels, the dashed curves are for $m_{2,\Lambda}^2 = 0$, $\lambda_{12,\Lambda} = 8\Lambda$. In the lower left panel, the dot-dashed curves are for $m_{2,\Lambda}^2 = 0.5\Lambda^2$ and $\lambda_{12,\Lambda} = 8\Lambda$. In the lower right panel, the dot-dashed curves are for $m_{2,\Lambda}^2 = 8\Lambda^2$ and $\lambda_{12,\Lambda} = 8\Lambda$.

potential becomes smaller for larger values of m_2^2. We have checked that the same is true for nonzero values of the coupling $\bar{\lambda}_{22}$. We also observe that the RG-evolved potential exhibits the typical shape for a (fluctuation-induced) first-order phase transition in the case of a light φ_2 field (upper panels), while the transition remains of second order for a heavy φ_2 field (lower panels).

IV. CONCLUSIONS

We investigated the conjecture that the two-flavor chiral phase transition of QCD can be of second order in the presence of the axial anomaly. We studied the most general renormalizable Lagrangian invariant under $SU(2)_A \times U(2)_V$, using the FRG method in the local-potential approximation. We took into account all possible 't Hooft determinant-like terms, the couplings of which we denoted as c, y, and z, respectively. We distinguished between the case of finite and the limit of divergent anomaly strength c.

Our conclusions are as follows. An $O(4)$ IR fixed point indeed exists for the two-flavor linear sigma model in the presence of the axial anomaly. However, it is only IR stable in the case of infinite anomaly strength. This case is reasonable if the IR value of the anomaly strength exceeds the cutoff scale of the linear sigma model, which is true at the mean-field level but not beyond this admittedly very crude approximation. For finite anomaly strength, however, we found that the $O(4)$ IR fixed point is unstable. Nevertheless, we find other IR stable fixed points which are in the $SU(2) \times U(2)$ universality class. These have unphysical mass-matrix eigenvalues in our fourth-order truncation of the potential and were thus neglected in our considerations. However, in a scheme which accounts for higher orders, they might become physical [39], indicating that the two-flavor chiral phase transition of QCD could be of second order, but not with $O(4)$ critical exponents. We want to note that the possibility of another universality class $[U(2)_L \times U(2)_R / U(2)_V]$ has also been recently emphasized in Ref. [42]. On the other hand, if the $SU(2) \times U(2)$ fixed points should remain unphysical, the absence of other IR stable fixed points indicates that the phase transition should be fluctuation-induced first order. However, we note that the strength of the first-order phase transition depends on the initial values for the parameters in the UV and could be extremely weak, which would make it practically indistinguishable from a second-order phase transition in this case [39].

APPENDIX A: CONSTRUCTING INVARIANTS

In condensed-matter systems, finite groups G play an important role. For such groups, there exist practical methods how to construct the most general G-invariant Landau-Wilson polynomials for certain representations $\Gamma(G)$. These methods have been applied to study phase transitions in various condensed-matter systems [18,43–45]. For arbitrary continuous groups, however, such a program is, at the very least, not well documented. In the following we describe how to construct the $SU(2)_A \times SU(2)_V$ invariants for the $[\bar{2}, 2] + [2, \bar{2}]$ representation. We note that our method is not restricted to this special case, and we have checked that it can be successfully applied to other groups as well. However, one has to know the explicit form of the symmetry transformation for the representation of interest.

The $[\bar{2}, 2] + [2, \bar{2}]$ representation is 8 dimensional. Accordingly, the corresponding invariants of order N are polynomials in eight components which are in our notation the fields σ, $\vec{\pi}$, η, and $\vec{a}$, i.e., they are of the form

$$p = \sum_{m_i \in m} c_i m_i, \tag{A1}$$

where m denotes the set of all possible monomials of order N,

$$m = \{\sigma^{n_1} \pi_1^{n_2} \pi_2^{n_3} \pi_3^{n_4} \eta^{n_5} a_1^{n_6} a_2^{n_7} a_3^{n_8}\},$$
$$n_i \in \mathbb{N}, \quad \sum_i n_i = N, \tag{A2}$$

and the coefficients c_i are expected to be rational multiples of each other.

Infinitesimal $SU(2)_A$ transformations for the above representation are determined by [46]

$$\sigma' = \sigma + \vec{\alpha} \cdot \vec{\pi}, \qquad \pi_i' = \pi_i - \alpha_i \sigma,$$
$$\eta' = \eta - \vec{\alpha} \cdot \vec{a}, \qquad a_i' = a_i + \alpha_i \eta, \tag{A3}$$

where $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ consists of three infinitesimal angles. Infinitesimal $SU(2)_V$ transformations for the above representation are determined by

$$\sigma' = \sigma, \qquad \vec{\pi}' = \vec{\pi} + \vec{\beta} \times \vec{\pi},$$
$$\eta' = \eta, \qquad \vec{a}' = \vec{a} + \vec{\beta} \times \vec{a}, \tag{A4}$$

where $\vec{\beta} = (\beta_1, \beta_2, \beta_3)$ consists of three infinitesimal angles.

Under the transformation (A3), the polynomial p transforms as

$$p \rightarrow p' = \sum_{m_i \in m} c_i'(\vec{c}, \vec{\alpha}) m_i, \tag{A5}$$

where the new coefficients, c_i', depend on the coefficients $\vec{c}$ and the angles $\vec{\alpha}$, and where we only keep terms linear in α_i. Since invariants are defined by $p = p'$, we obtain a system of equations,

$$c_i = c_i'(\vec{c}, \vec{\alpha}), \tag{A6}$$

determining all invariants of order N.

For $N = 2$, the sum in Eq. (A1) runs from $i = 1$ to $i = 36$, since there are 36 different monomials of order $N = 2$. Using for example Mathematica's option SolveAlways [47], solutions for the coefficients c_i can be found, such that Eq. (A6) is fulfilled for arbitrary values of the angles α_i. Inserting the solution into the general ansatz (A1), we obtain

$$p = c_1(\sigma^2 + \vec{\pi}^2) + c_2(\eta^2 + \vec{a}^2) + c_3(\sigma\eta - \vec{\pi} \cdot \vec{a}). \tag{A7}$$

Since the coefficients c_i are independent from each other, there exist exactly three linearly independent invariants of order $N = 2$:

$$\varphi_1 = \sigma^2 + \vec{\pi}^2, \qquad \varphi_2 = \eta^2 + \vec{a}^2, \qquad \varphi_3 = \sigma\eta - \vec{\pi} \cdot \vec{a}. \tag{A8}$$

For $N = 4$, the sum in Eq. (A1) runs from $i = 1$ to $i = 330$, since there are 330 different monomials of order $N = 4$. Again, using Mathematica, we find solutions for the coefficients c_i, such that Eq. (A6) is fulfilled for arbitrary values of the angles α_i. Inserting the solution into the general ansatz (A1), we obtain

$$\begin{aligned} p = {} & c_1(\eta^2 + \vec{a}^2)^2 + c_2(\sigma^2 + \vec{\pi}^2)^2 + c_3(-\sigma\eta + \vec{\pi} \cdot \vec{a})^2 \\ & + c_4(-\sigma\eta + \vec{\pi} \cdot \vec{a})(\sigma^2 + \vec{\pi}^2) + c_5(\eta^2 + \vec{a}^2) \\ & \times (-\sigma\eta + \vec{\pi} \cdot \vec{a}) + c_6[(\eta^2 + \vec{a}^2)(\sigma^2 + \vec{\pi}^2) \\ & - (\sigma\eta - \vec{\pi} \cdot \vec{a})^2]. \end{aligned} \tag{A9}$$

Since the coefficients c_i are independent from each other, there exist exactly four linearly independent invariants of order $N = 4$:

$$\varphi_1^2, \varphi_2^2, \varphi_1\varphi_2, \gamma = \varphi_3^2. \tag{A10}$$

Note that the quadratic invariant φ_3 is not invariant under parity transformations

$$\sigma \to \sigma, \qquad \vec{\pi} \to \vec{\pi}, \qquad \eta \to -\eta, \qquad \vec{a} \to -\vec{a}, \tag{A11}$$

and therefore cannot appear in a theory without parity violation.

Note further that the invariants (A8) and (A10) are also invariant under $SU(2)_V$ transformations (A4). Proceeding along the same lines described above one can derive several additional invariants for this symmetry. Since these are not $SU(2)_A$ symmetric, and hence no $SU(2)_A \times SU(2)_V$ invariants, we do not list them here.

APPENDIX B: CRITICAL EXPONENTS FROM THE STABILITY MATRIX

In the following we describe how to calculate critical exponents proceeding in complete analogy to Ref. [48]. The method is appropriate as long as the anomalous dimension η is small, which is assumed to be the case in the

local-potential approximation. For given beta functions $\beta_i(\bar{p}) \equiv k\partial_k \bar{p}_i$ for the rescaled parameters $\bar{p} = \{\bar{p}_i\}$ (i.e., the rescaled mass terms and couplings) of the Lagrangian, the stability matrix for a fixed point is defined as

$$(S_{ij}) \equiv \left(\frac{\partial \beta_i}{\partial \bar{p}_j} \right) \bigg|_{\bar{p} = \bar{p}*}, \tag{B1}$$

where a fixed point $\{\bar{p}_i^*\}$ is determined by

$$\beta_i(\{\bar{p}_i^*\}) = 0. \tag{B2}$$

The stability properties of a fixed point can be determined from the eigenvalues of the stability matrix S. Eigenvalues with positive real part correspond to IR stable (UV unstable) directions, whereas eigenvalues with negative real part correspond to IR unstable (UV stable) directions. If a fixed point is IR (UV) stable in a direction in coupling space, the flow, for decreasing k, in the neighbourhood of the fixed point is directed towards it (away from it) in this direction in coupling space. For fixed points associated with second-order phase transitions, for every plane in coupling space an IR stable direction exists. However, since a phase transition always requires that a scaling variable (e.g., the temperature T) approaches a critical value, there has to exist at least one IR unstable direction. Tuning a system towards the critical point corresponds to tuning the parameters $\bar{p}_i$ to a point on the critical surface (a point which is attracted by the IR fixed point). Such a fixed point can be associated with a second-order phase transition and is simply called IR stable. In case of a single scaling variable, the eigenvalues of the stability matrix for an IR stable fixed point have positive real parts, except for one which is negative, say y_1. The critical exponent ν, determined by

$$T \to T_c: \xi \sim |T - T_c|^{-\nu}, \tag{B3}$$

is then given by

$$\nu = -\frac{1}{y_1}. \tag{B4}$$

APPENDIX C: $O(N)$ IR FIXED POINTS

In order to determine the universality class that a fixed point belongs to, one has to compare its stability-matrix eigenvalues with those of the fixed points defining certain universality classes. In Table II we list stability-matrix eigenvalues for $O(N)$ models,

$$U = \frac{1}{2}\mu^2 \sum_{n=1}^{N} \phi_i^2 + \frac{\lambda_1}{24} \left(\sum_{n=1}^{N} \phi_i^2 \right)^2, \tag{C1}$$

which are relevant for our discussion. We keep only terms with relevant canonical scaling dimension. Higher-order terms are irrelevant with respect to the Gaussian fixed point and should be negligible also with respect to nontrivial fixed points in a resummed ϵ expansion. This is different in

the FRG approach. At nontrivial fixed points, higher-order terms are expected to have a distinct effect on critical exponents [48]. However, in this work we are only interested in identifying the universality class of a fixed point, so we can neglect these higher orders, if we also do this in the model which we want to compare the $O(N)$ model with. For more evolved FRG studies of $O(N)$ models we refer for example to Refs. [9,48–52] and references therein.

As an example, we can now explicitly determine the universality class of the IR fixed point of the flow equations (69)–(71) in Ref. [6]. These are the flow equations we obtain from our results in the limit $c, y, z \rightarrow 0$. The IR fixed point is given by $(\bar{\mu}_*^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}) = (-0.135, 3.591, 0)$ and is unstable as one infers from the stability-matrix eigenvalues $\{-1.71971, 1.34471, -0.25\}$, which are in perfect agreement with the $O(8)$ values in Table II.

In the context of $O(N)$ models we can easily study the influence of setting fields to zero after having performed the second derivatives in Eq. (12). It does not affect the results at all, because the coefficients in the expansion of the flow equation (11) in terms of fields do not change by setting certain fields to zero. Due to the $O(N)$ symmetry we can read off the flow equation for λ_1 from any quartic term. This argument directly generalizes to all other potentials for which one obtains unambiguous flow equations for all couplings, keeping all fields nonzero. In such a case, instead of keeping all fields nonzero, one can set as many fields to zero [after having performed the second derivatives in Eq. (12)] as one likes, as long as one still obtains a flow equation for each coupling. The flow equations are the same in both cases.

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