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Accurate bottom-quark mass from Borel QCD sum rules for f_B and f_{B_s}

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We prove that Borel QCD sum rules for heavy–light currents yield very strong correlations between the b-quark mass m_b and the B-meson decay constant f_B, namely, $\delta f_B/f_B \approx -8\delta m_b/m_b$. This fact opens the possibility of an accurate sum-rule extraction of m_b by using f_B as input. Combining precise lattice QCD determinations of f_B with our sum-rule analysis based on the three-loop $O(\alpha_s^2)$ heavy–light correlation function leads to $\bar{m}_b(\bar{m}_b) = (4.247 \pm 0.034)$ GeV.

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I. INTRODUCTION

The b-quark mass—for instance, the $\overline{\mathrm{MS}}$ running mass at renormalization scale ν, $\bar{m}_b(\nu)$, or $m_b \equiv \bar{m}_b(\bar{m}_b)$—is one of the fundamental parameters of the standard model and therefore its precise knowledge is highly desirable. The latest edition of the Review of Particle Physics reports $m_b = (4.18 \pm 0.03)$ GeV [1].

A direct way to determine m_b is by means of lattice QCD simulations; however, since the physical b-quark is too heavy for current lattice setups, the determination of m_b from pure lattice QCD requires either the extrapolation of the lattice results from lighter simulated masses or the use of the heavy-quark effective theory (HQET) formulated on the lattice. Using the former approach, the values $m_b = (4.29 \pm 0.14)$ GeV [2] and $m_b = (4.35 \pm 0.12)$ GeV [3] have recently been deduced, while the results $m_b = (4.26 \pm 0.09)$ GeV [4], $m_b = (4.25 \pm 0.11)$ GeV [5], and $m_b = (4.22 \pm 0.11)$ GeV [6] have been determined adopting the HQET-based approach. All above findings have been obtained using unquenched gauge configurations with $N_f = 2$ dynamical flavors in the sea.

Recently, more accurate determinations of the b-quark mass have been performed by exploiting moment sum rules for two-point functions of heavy–heavy currents: low-n moment sum rules based on three-loop $O(\alpha_s^2)$ [7] and four-loop $O(\alpha_s^3)$ [8] fixed-order pQCD calculations combined with the experimental data yield $m_b = (4.209 \pm 0.050)$ GeV [7] and $m_b = (4.163 \pm 0.016)$ GeV [8], respectively. The latter finding has been confirmed by a study based on a combination of perturbative QCD and lattice QCD simulations with $N_f = 2 + 1$ dynamical flavors in the sea [9]. Combining large-n moments obtained within renormalization-group-improved next-to-next-to-leading logarithmic (NNLL) order Υ sum rules with the experimental data yields $m_b = (4.235 \pm 0.055_{(\mathrm{pert})} \pm 0.03_{(\mathrm{exp})})$ GeV [10].

In this paper, we show that the Borel QCD sum rules for heavy–light correlators provide the possibility to extract the bottom-quark mass with comparable accuracy if a precise value for the B-meson decay constant f_B is adopted as input.

Let us first explore what degree of sensitivity of f_B to the precise value of the b-quark mass can be expected on the basis of simple quantum-mechanical considerations. Nonrelativistic potential models predict the following relationship between the ground-state wave function at the origin, $\psi(r = 0)$, and the ground-state binding energy ε:

$$|\psi(r = 0)| \propto \varepsilon^{3/2}. \tag{1.1}$$

Equation (1.1) is exact for any ground state in a purely Coulomb or purely harmonic-oscillator potential (or, to be more precise, in any model where the potential involves only one coupling constant). Moreover, this relation proves to be a good approximation for ground states in potentials given by the sum of confining and Coulomb interactions [11].

Now, taking into account that the decay constant is the analogue of the wave function at the origin and incorporating the known scaling behavior of the decay constant of a heavy meson in the heavy-quark limit [12]—which should work well for the beauty mesons—one obtains an approximate relation between the B-meson mass M_B and the heavy-quark pole mass m_Q:

$$f_B\sqrt{M_B} = \kappa(M_B - m_Q)^{3/2}. \tag{1.2}$$

Keeping the ground-state mass fixed and equal to its experimental value $M_B = 5.27$ GeV, we can easily pin down the dependence of f_B on small variations δm_Q of the heavy-quark pole mass near some average value of m_Q. Taking into account that $f_B \approx 200$ MeV for $m_Q \approx$ 4.6–4.7 GeV we end up with $\kappa \approx$ 0.9–1.0 and $\delta f_B \approx -0.5\delta m_Q$ or, equivalently,

$$\frac{\delta f_B}{f_B} \approx -(11\text{–}12)\frac{\delta m_Q}{m_Q}. \tag{1.3}$$

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TABLE I. Some recent QCD sum-rule predictions for f_B from heavy–light two-point functions.

	Reference [15]	Reference [16]	Reference [17]	Reference [18]
m_b (GeV)	4.05 ± 0.06	4.21 ± 0.05	4.245 ± 0.025	4.236 ± 0.069
f_B (MeV)	203 ± 23	210 ± 19	193 ± 15	206 ± 7

TABLE II. Some recent lattice-QCD evaluations of f_B and f_{B_s}.

Collaboration	N_f	f_B (MeV)	f_{B_s} (MeV)	f_{B_s}/f_B
ETMC I [2]	2	195 ± 12	232 ± 10	1.19 ± 0.05
ETMC II [3]	2	197 ± 10	234 ± 6	1.19 ± 0.05
ALPHA [6]	2	193 ± 10	219 ± 12	1.13 ± 0.09
HPQCD I [20]	2 + 1	191 ± 9	228 ± 10	1.188 ± 0.018
HPQCD II [21]	2 + 1	189 ± 4	225 ± 4	—
FNAL/MILC [22]	2 + 1	196.9 ± 9.1	242 ± 10	1.229 ± 0.026
Our average		191.5 ± 7.3	228.8 ± 6.9	1.198 ± 0.030

Thus, the sensitivity of f_B to the precise value of the heavy-quark mass should be rather high: a variation of the quark mass by $+100$ MeV entails $\delta f_B \approx -50$ MeV. Clearly, a similar effect should be observable in the outcomes of QCD sum rules [13,14].

Recently, several QCD sum-rule analyses [15–18] of beauty-meson decay constants relying on three-loop heavy–light correlators [19] have been published; see Table I (note that all results collected in Table I are obtained by applying the QCD sum-rule method to essentially the same analytical expression for the correlator).

At first glance, the QCD sum-rule results for f_B seem to be very stable and practically independent of the input value of m_b. This, however, may not be regarded as an argument in favor of the published predictions: obviously, the results in Table I do not follow the general pattern discussed above; for instance, the central values of m_b reported in [15,18] differ by some 200 MeV, whereas the corresponding decay constants remain almost unchanged. Therefore, we are forced to conclude that not all the results in Table I are equally trustable.

Recall that the values of the ground-state parameters in Table I are strongly influenced by (i) the way one reorganizes the three-loop perturbative result in terms of the pole or the running mass of the heavy quark, and (ii) by one's way of fixing the auxiliary parameters of the sum-rule approach, particularly the effective continuum threshold.

The goal of this paper is to present a critical detailed analysis of the sum-rule extraction of f_B. Our main conclusion is that if the appropriate expression for the correlator in terms of the running heavy-quark mass is used and consistent procedures for extracting the bound-state parameters are applied, the QCD sum-rule results are in excellent agreement with the behavior expected from quantum mechanics: the decay constant f_B obtained from QCD sum rules is strongly correlated with the input value of the heavy-quark mass m_b. For all other input parameters of the correlator (quark condensate, α_s, renormalization scale, etc.) fixed, we find

$$f_B(m_b) = \left(192.0 - 37\frac{m_b - 4.247\ \text{GeV}}{0.1\ \text{GeV}} \pm 3_{(\text{syst})}\right)\ \text{MeV}. \tag{1.4}$$

Evidently, the dependence of f_B on m_b agrees very well with the semiqualitative quantum-mechanical expression (1.3). The strong correlation between f_B and m_b opens the possibility to deduce an accurate value of m_b using f_B as input. Combining our sum-rule analysis based on the heavy–light correlator known to order α_s^2 with the average of the most recent determinations of f_B from lattice QCD, $f_B^{\text{LQCD}} = (191.5 \pm 7.3)$ MeV (see Table II), leads to the accurate estimate

$$m_b = (4.247 \pm 0.034)\ \text{GeV}. \tag{1.5}$$

This paper is organized as follows: in the next section, we discuss the convergence of the operator product expansion (OPE) series for the correlator expressed in terms of either pole or running quark mass. Section III presents the details of the extraction procedure with particular emphasis on the related uncertainties of the extracted parameters. Section IV is devoted to the extraction of the b-quark mass. Section V summarizes our conclusions.

II. CORRELATION FUNCTION, OPERATOR PRODUCT EXPANSION, AND HEAVY-QUARK MASS

The basic object for our study of the decay constants of heavy pseudoscalar B (or B_s) mesons is the correlator [13,14]

$$\Pi(p^2) = i\int d^4x e^{ipx}\langle 0|T(j_5(x)j_5^\dagger(0))|0\rangle \tag{2.1}$$

of two pseudoscalar heavy–light currents

$$j_5(x) = (m_b + m)\bar{q}(x)i\gamma_5 b(x), \tag{2.2}$$

where $q(x)$ denotes the field of the light quark of mass m, that is, $q(x) \equiv d(x)$ for B and $q(x) \equiv s(x)$ for B_s. The OPE for this correlator may be calculated by using perturbative QCD and adding nonperturbative power corrections given in terms of vacuum condensates. The QCD sum rule for this correlator is obtained by equating the Borelized OPE for the correlator (2.1), $\Pi(p^2) \to \Pi(\tau)$, and the Borelized correlator calculated by insertion of intermediate hadron states:

$$\begin{aligned}\Pi(\tau) &= f_B^2 M_B^4 e^{-M_B^2\tau} + \int_{s_{\rm phys}}^{\infty} ds e^{-s\tau}\rho_{\rm hadr}(s)\\ &= \int_{(m_b+m)^2}^{\infty} ds e^{-s\tau}\rho_{\rm pert}(s,\mu) + \Pi_{\rm power}(\tau,\mu),\end{aligned} \tag{2.3}$$

where M_B is the mass of the B (or B_s) meson and f_B is the decay constant of the B (or B_s) meson, defined by

$$(m_b + m)\langle 0|\bar{q}i\gamma_5 b|B\rangle = f_B M_B^2. \tag{2.4}$$

In (2.3), $s_{\rm phys} = (M_{B^*} + M_P)^2$ is the physical continuum threshold, fixed by the mass M_{B^*} of the beauty vector meson and the mass M_P of the lightest pseudoscalar with appropriate quantum numbers (the pion or the kaon, respectively).

For large values of τ, the contributions of excited states to the Borelized correlator (2.3) decrease faster than the ground-state contribution and thus $\Pi(\tau)$ is saturated by the ground state. Therefore, knowing the correlator at large τ provides direct access to the ground-state parameters. However, analytic results for the correlator are obtained from a truncated OPE, which yields a good approximation to the correlator only at not too large τ, where excited states still contribute sizably to $\Pi(\tau)$.

To exclude the excited-state contributions from the sum rule (2.3), one adopts the duality *Ansatz*: all contributions of excited states are counterbalanced by the perturbative contribution above an *effective continuum threshold*, $s_{\rm eff}(\tau)$, which differs from the physical continuum threshold. While the physical continuum threshold is a constant determined by the masses of the lightest hadrons that may be produced from the vacuum by the interpolating current, the effective continuum threshold is a parameter of the sum-rule approach. The effective continuum threshold has interesting and nontrivial properties which have been discussed in great detail in [23]. In particular, it has been demonstrated that the "true" effective threshold which correctly reproduces the true ground-state parameters is a τ-dependent function [24].

Applying the duality assumption entails the following relation between the ground-state contribution and the OPE:

$$\begin{aligned}f_B^2 M_B^4 e^{-M_B^2\tau} &= \int_{(m_b+m)^2}^{s_{\rm eff}(\tau)} ds e^{-s\tau}\rho_{\rm pert}(s,\mu) + \Pi_{\rm power}(\tau,\mu)\\ &\equiv \Pi_{\rm dual}(\tau, s_{\rm eff}(\tau)).\end{aligned} \tag{2.5}$$

We refer to the right-hand side of this equation as the *dual correlator* $\Pi_{\rm dual}(\tau, s_{\rm eff}(\tau))$.

Clearly, even if the QCD inputs $\rho_{\rm pert}(s,\mu)$ and $\Pi_{\rm power}(\tau,\mu)$ are known, the extraction of the decay constant requires, in addition, a criterion for determining $s_{\rm eff}(\tau)$. As first step, however, we need a reasonably convergent OPE for both correlator and dual correlator. For heavy–light systems, the relative sizes of the lowest-order terms, which contain powers of the heavy-quark mass, turn out to depend strongly on one's choice of the renormalization scheme and scale.

The best-known three-loop calculations of the perturbative spectral density [19] have been performed in form of an expansion in terms of the $\overline{\rm MS}$ strong coupling $\alpha_s(\mu)$ and the pole mass of the heavy-quark M_b:

$$\begin{aligned}\rho_{\rm pert}(s,\mu) &= \rho^{(0)}(s, M_b^2) + \frac{\alpha_s(\mu)}{\pi}\rho^{(1)}(s, M_b^2)\\ &\quad + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \rho^{(2)}(s, M_b^2, \mu) + \cdots.\end{aligned} \tag{2.6}$$

The correlator (2.1) and its Borel image (2.3) do not depend on the renormalization scale μ. Unfortunately, this nice property is lost if one works with truncated expansions: both the perturbative expansion truncated at fixed order in α_s and the lowest-order power corrections $\Pi_{\rm power}(\tau,\mu)$ given in terms of condensates and the radiative corrections to the latter depend on μ.

The pole mass has been used in most sum-rule studies since the pioneering work [14]. It turns out, however, that the OPE for the dual correlator expressed in terms of the pole mass M_b exhibits a bad convergence, as illustrated in Fig. 1.

An alternative option [16] is to reorganize the perturbative expansion in terms of the running $\overline{\rm MS}$ mass, $\bar{m}_b(\nu)$, by substituting M_b in the spectral densities $\rho^{(i)}(s, M_b^2)$ via its perturbative expansion in terms of the running mass $\bar{m}_b(\nu)$ (while keeping the integration variable s fixed)[1]

$$M_b = \bar{m}_b(\nu)\left(1 + \frac{\alpha_s(\nu)}{\pi} r_1 + \left(\frac{\alpha_s(\nu)}{\pi}\right)^2 r_2 + \cdots\right). \tag{2.7}$$

Since the original correlator is known to order α_s^2, it suffices to use the relation between M_b and $\bar{m}_b(\mu)$ to α_s^2

[1]Note that the scale μ of $\alpha_s(\mu)$ in the expansion (2.6) is not necessarily equal to the scale ν in the relation (2.7) between M_b and $\bar{m}_b(\nu)$; the two scales are, in principle, independent. As noticed in [10], setting these scales equal to each other leads to a reduced dependence of the truncated correlator on the then common scale; however, more realistic error estimates are obtained if one studies the sensitivity of the truncated correlator to independent variations of the scales μ and ν.

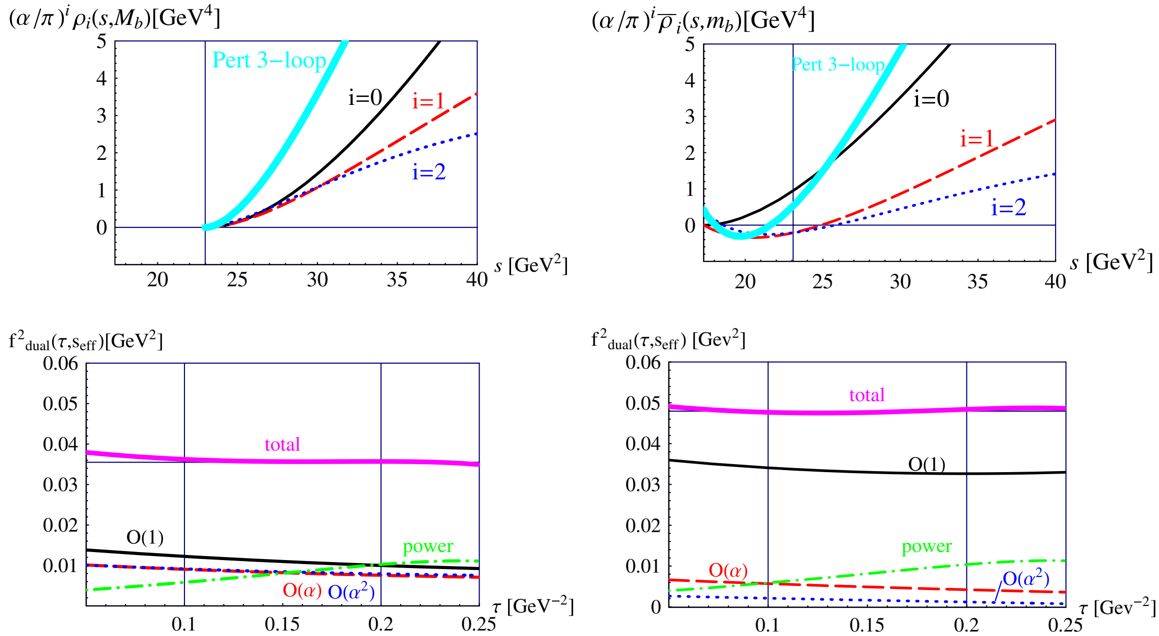

FIG. 1 (color online). OPE computed in terms of pole mass (left) and $\overline{\rm MS}$ mass (right) of the b-quark. First row: spectral densities; second row: corresponding sum-rule findings for f_B. In both cases, a typical value of the effective continuum threshold is used: $s_0 = 35$ GeV^2. Bold solid lines labelled "Pert 3-loop" and "total," respectively: total result; black solid lines: $O(1)$ contribution; red dashed lines: $O(\alpha_s)$ contribution; blue dotted lines: $O(\alpha_s^2)$ contribution; green dash-dotted lines: power contributions.

accuracy (although the α_s^3 term is also known [25]). Moreover, one should omit all terms of order α_s^3 and higher, induced by the substitution $M_b \to \bar{m}_b(\nu)$ in the results of [19]. Explicit expressions for the perturbative spectral densities and power corrections may be found in [16,19] and are not given here. Notice that two different scales, μ and ν, naturally emerge when reorganizing the perturbative expansion from the pole b-quark mass to the running b-quark mass. One may set these scales equal to each other; we, however, leave the scales independent from each other and investigate the impact of particular choices of the scales μ and ν on the extracted values of m_b and the decay constants f_B and f_{B_s}.

One caveat is in order here: the spectral density (2.6) involves an implicit θ function restricting the integration region in the correlator: for instance, for a massless light quark, it reads $\theta(s - M_b^2)$. Switching from the pole to the running mass, $M_b \to \bar{m}_b(\nu)$, this θ function has to be expanded in powers of α_s, a step which induces "surface" terms $\delta(s - M_b^2)$ and their derivatives. The spectral densities $\rho^{(i)}(s, M_b^2)$, however, have zeros of second order at this threshold $s = M_b^2$; consequently, to the $O(\alpha_s^2)$ accuracy considered, the surface terms do not contribute and one merely has to perform the replacement $\theta(s - M_b^2) \to \theta(s - \bar{m}_b^2(\nu))$. The surface terms enter the game at order α_s^3 and higher.

In order to appreciate the amount of improvement achieved by reorganizing the perturbative expansion in terms of the running mass, Fig. 1 shows the perturbative spectral densities and the estimates for f_B arising from the sum rule (2.5) for two choices of the b-quark mass: the pole mass M_b and the running $\overline{\rm MS}$ mass $\bar{m}_b(\nu)$. All results are given for $m_b = 4.163$ GeV, corresponding to two-loop and three-loop pole masses $M_b^{\text{2-loop}} = 4.75$ GeV and $M_b^{\text{3-loop}} = 4.89$ GeV [25]. Since we work at $O(\alpha_s^2)$ accuracy, we use for consistency the two-loop value of M_b to obtain the results depicted in Fig. 1. For the other relevant OPE parameters, we adopt the following values [1,26]:

$$
\begin{aligned}
m_d(2\,\text{GeV}) &= (3.5 \pm 0.5)\,\text{MeV},\\
m_s(2\,\text{GeV}) &= (95 \pm 5)\,\text{MeV},\\
\alpha_s(M_Z) &= 0.1184 \pm 0.0007,\\
\langle \bar{q}q\rangle(2\,\text{GeV}) &= -((269 \pm 17)\,\text{MeV})^3,\\
\langle \bar{s}s\rangle(2\,\text{GeV})/\langle \bar{q}q\rangle(2\,\text{GeV}) &= 0.8 \pm 0.3,\\
\left\langle \frac{\alpha_s}{\pi} GG \right\rangle &= (0.024 \pm 0.012)\,\text{GeV}^4.
\end{aligned}
\tag{2.8}
$$

The sum-rule estimates shown in Fig. 1 are obtained for $\mu = \nu = m_b$ and for a τ-independent effective threshold $s_{\rm eff}$. Clearly, the choice of the heavy-quark mass (that is, pole or running) used in the OPE makes a great difference for the numerical values of the truncated heavy–light correlators and of the resulting decay constants.

The above observations may be summarized as follows:

(1) When the dual correlator is calculated in terms of the heavy-quark pole mass, its perturbative expansion exhibits no sign of convergence; the contributions of the $O(1)$, $O(\alpha_s)$, and $O(\alpha_s^2)$ terms are of nearly the same magnitude. Therefore, in this

scheme one cannot expect higher orders to give smaller contributions.

(2) Formulating the perturbative series in terms of the heavy-quark $\overline{\rm MS}$ mass yields a clear hierarchy of contributions.

(3) The decay constant extracted from the pole-mass truncated OPE ($f_B = 188$ MeV) is substantially smaller than that from the $\overline{\rm MS}$-mass OPE truncated at the same order ($f_B = 220$ MeV). Nevertheless, both decay constants exhibit a satisfactory degree of stability over a wide range of the Borel parameter. We therefore stress again that mere Borel stability is by far not sufficient to guarantee the reliability of any sum-rule extraction of bound-state features. We have illustrated these findings before in some exactly solvable quantum-mechanical examples [11,23].

Because of the evident lack of convergence of the truncated pole-mass OPE for the correlator, we employ the $\overline{\rm MS}$-mass OPE in our subsequent sum-rule analysis.

III. EXTRACTION OF THE DECAY CONSTANT

According to the standard procedures of the QCD sum-rule approach, its application requires the following steps.

1. The Borel window

The working τ window is chosen such that the OPE gives a sufficiently accurate description of the exact correlator (i.e., all higher-order radiative and power corrections are small) and at the same time the ground state gives a "sizable" contribution to the correlator. Hence, we require [11,24,27] that the power corrections do not exceed 30% of the dual correlator (to fix the maximal τ) and that the ground-state contribution does not fall below 10% (to fix the minimal τ).

In practice, our τ window for the $B_{(s)}$ mesons is $0.05 \lesssim \tau\ (\text{GeV}^{-2}) \lesssim 0.175$. Such a window is much more extended than the τ range usually adopted in the literature, e.g., $0.17 \lesssim \tau\,(\text{GeV}^{-2}) \lesssim 0.25$ [16] or $0.20 \lesssim \tau(\text{GeV}^{-2}) \lesssim 0.26$ [18]. We observe (i) that our upper bound in τ is much safer with respect to the convergence properties of both perturbative and power-correction series, and (ii) that our lower bound in τ produces a dual correlator (2.5) which represents the ground-state contribution in a much wider range of values of τ. The latter property corresponds to the fact that the quantity $[\Pi_{\rm dual}(\tau, s_{\rm eff}(\tau)) \cdot e^{M_B^2\tau}]$ should exhibit a plateau in a wide range of values of τ, which makes the extraction of the decay constant f_B much more reliable.

Finally, we notice that it would be extremely unreasonable to assume a τ-independent effective threshold $s_{\rm eff}$ in a τ window where the impact of the contamination of excited states in the full correlator changes quite significantly, as explicitly shown in Ref. [24].

2. The effective continuum threshold

To find $s_{\rm eff}(\tau)$, we employ a previously developed algorithm [11,24], which has proven to provide a reliable extraction of the ground-state parameters in quantum-mechanical models and of the charmed-meson decay constants in QCD [27]. We introduce the *dual invariant mass* $M_{\rm dual}$ and the *dual decay constant* $f_{\rm dual}$ by the definitions

$$
\begin{aligned}
M_{\rm dual}^2(\tau) &\equiv -\frac{d}{d\tau}\log \Pi_{\rm dual}(\tau, s_{\rm eff}(\tau)),\\
f_{\rm dual}^2(\tau) &\equiv M_B^{-4} e^{M_B^2\tau}\Pi_{\rm dual}(\tau, s_{\rm eff}(\tau)).
\end{aligned}
\tag{3.1}
$$

By construction, the dual mass should reproduce the true ground-state mass M_B. So, the deviation of $M_{\rm dual}$ from M_B measures the contamination of the dual correlator by excited states. Starting from an *Ansatz* for $s_{\rm eff}(\tau)$ and requiring a minimum deviation of $M_{\rm dual}$ from M_B in the τ window generates a variational solution for $s_{\rm eff}(\tau)$. With the latter at our disposal, $f_{\rm dual}(\tau)$ yields the desired decay-constant estimate. Since we deal with a limited τ window, it suffices to consider polynomials in τ, including also the standard assumption for the effective threshold, a τ-independent constant:

$$
s_{\rm eff}^{(n)}(\tau) = \sum_{j=0}^{n} s_j^{(n)}\tau^j. \tag{3.2}
$$

We obtain the expansion coefficients $s_j^{(n)}$ by minimizing the squared difference between $M_{\rm dual}^2$ and M_B^2 in the τ window:

$$
\chi^2 \equiv \frac{1}{N}\sum_{i=1}^{N}[M_{\rm dual}^2(\tau_i) - M_B^2]^2. \tag{3.3}
$$

3. Uncertainties in the extracted decay constant

The resulting value of the decay constant $f_{B_{(s)}}$ is, beyond doubt, sensitive to the input values of the OPE parameters, which determine what we call the *OPE-related error*, and to the details of the adopted prescription for fixing the behavior of the effective continuum threshold $s_{\rm eff}(\tau)$, which we will refer to as the *systematic error.*

OPE-related error: We estimate the size of the OPE-related error by performing a bootstrap analysis [28], allowing the OPE parameters to vary over the ranges indicated in (2.8) and using 1000 bootstrap events. Gaussian distributions for all OPE parameters but the scales μ and ν are employed. For the latter, we assume uniform distributions in the range $3 \le \mu,\ \nu\ (\text{GeV}) \le 6$. The resulting distribution of the decay constant turns out to be close to Gaussian shape. Hence, the quoted OPE-related error is a Gaussian error.

Systematic error: The systematic error, encoding the limited intrinsic accuracy of the sum-rule method, constitutes a rather subtle point. In quantum mechanics, we observed, for polynomial parametrizations of the effective continuum threshold $s_{\rm eff}(\tau)$, that the band of results obtained from linear, quadratic, and cubic *Ansätze* for the effective threshold encompasses the true value of the decay constant [24]. Moreover, the extraction procedures in quantum mechanics and in QCD proved to be strikingly similar [11]. Thus, the half-width of this band may be regarded as a realistic estimate for the systematic uncertainty of the prediction. The ultimate efficiency and reliability of this algorithm have already been established for the decay constants of D and D_s mesons [27]. Here, we apply this technique to the B and B_s mesons.

A. Decay constant of the B meson

Recall that the τ window for the $B_{(s)}$ mesons is fixed by the above criteria to be equal to $\tau = (0.05\text{–}0.175)\ \text{GeV}^{-2}$. Figure 2 shows the corresponding results for the effective continuum threshold $s_{\text{eff}}(\tau)$ and the extracted f_B. Obviously, in this window the τ-dependent effective thresholds reproduce the meson mass M_B much better than the constant one [Fig. 2(a)]. This signals that those dual correlators that correspond to such τ-dependent thresholds are less contaminated by the excited states.

According to Fig. 2(d), the dependence of our QCD sum-rule prediction for the B-meson decay constant f_B on m_b and the quark condensate $\langle\bar{q}q\rangle \equiv \langle\bar{q}q(2\ \text{GeV})\rangle$, for fixed values of the other OPE parameters, may be well parametrized by

$$\begin{aligned} f_B^{\text{dual}}&(m_b, \mu=\nu=m_b, \langle\bar{q}q\rangle) \\ &= \left[192.0 - 37\left(\frac{m_b - 4.247\ \text{GeV}}{0.1\ \text{GeV}}\right)\right. \\ &\quad \left. + 4\left(\frac{|\langle\bar{q}q\rangle|^{1/3} - 0.269\ \text{GeV}}{0.01\ \text{GeV}}\right) \pm 3_{(\text{syst})}\right]\text{MeV}, \qquad (3.4) \end{aligned}$$

representing the range of results obtained for $n = 1, 2, 3$ in the *Ansatz* (3.3) within the two short-dashed lines in Fig. 2(d).

Note that our algorithm, relying on polynomial functions, provides a clear and unambiguous prescription for fixing the effective continuum thresholds. The τ dependence of the latter is crucial for deriving the dual mass, the definition of which involves a derivative with respect to τ. On the other hand, our decay-constant prediction may be reproduced by the constant effective continuum threshold $s_{\text{eff}} = (33.1 \pm 0.5)$ GeV. However, in order to obtain this very range of values, one has to apply our algorithm, which takes advantage of the freedom provided by the τ dependence of the thresholds.

Performing the bootstrap analysis of the OPE uncertainties and adding the half-width of the band deduced from our τ-dependent *Ansätze* for the effective continuum threshold of degree $n = 1, 2, 3$ as (intrinsic) systematic error, we find

$$f_B = (192.0 \pm 14.3_{(\text{OPE})} \pm 3.0_{(\text{syst})})\ \text{MeV}. \qquad (3.5)$$

The main contributions to the OPE uncertainty in the extracted f_B arise from the renormalization-scale dependence and the errors in m_b and the quark condensate. Let us emphasize that for $m_b = 4.05$ GeV one gets $f_B = 265$ MeV which is very far from the result reported in [15]; cf. Table I.

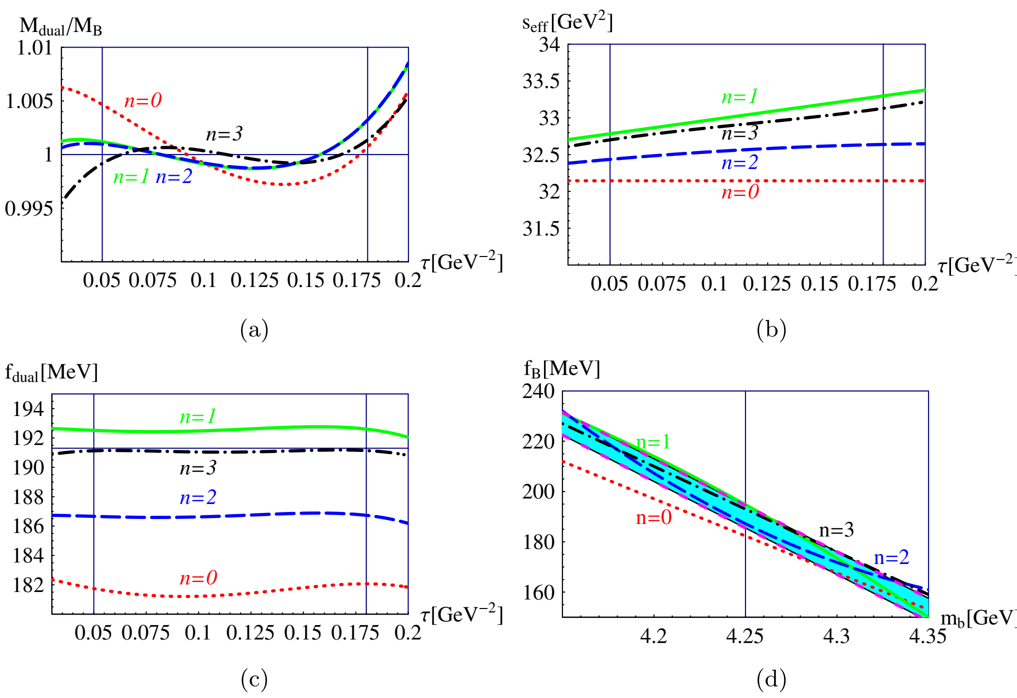

FIG. 2 (color online). Dual mass $M_{\text{dual}}(\tau)$ (a), corresponding to τ-dependent effective continuum threshold $s_{\text{eff}}(\tau)$ according to our *Ansatz* (3.2), determined by minimizing the expression (3.3) (b), and dual decay constant $f_{\text{dual}}(\tau)$ (c). Results for $m_b \equiv \bar{m}_b(\bar{m}_b) = 4.25$ GeV, $\mu = \nu = m_b$, and central values of the other relevant parameters are shown. (d) Dual decay constant of the B meson vs m_b for $\mu = \nu = m_b$ and central values of all the other OPE parameters. The integer $n = 0, 1, 2, 3$ is the degree of the $s_{\text{eff}}(\tau)$ polynomial in the *Ansatz* (3.2). Red dotted line: $n = 0$; green solid line: $n = 1$; blue dashed line: $n = 2$; black dot-dashed line: $n = 3$; red short-dashed lines: delimiters of the range of results.

B. Decay constant of the B_s meson

A similar procedure yields for the B_s meson

$$f_{B_s}^{\mathrm{dual}}(m_b, \mu = \nu = m_b, \langle \bar{s}s \rangle) = \left[228.0 - 43\left(\frac{m_b - 4.247\ \mathrm{GeV}}{0.1\ \mathrm{GeV}} \right) + 3.5\left(\frac{|\langle \bar{s}s \rangle|^{1/3} - 0.248\ \mathrm{GeV}}{0.01\ \mathrm{GeV}} \right) \pm 4_{(\mathrm{syst})} \right] \mathrm{MeV}. \quad (3.6)$$

Performing the bootstrap analysis of the OPE uncertainties, we obtain

$$f_{B_s} = (228.0 \pm 19.4_{(\mathrm{OPE})} \pm 4_{(\mathrm{syst})})\ \mathrm{MeV}. \quad (3.7)$$

C. f_{B_s}/f_B

The resulting ratio of the B and B_s decay constants reads

$$f_{B_s}/f_B = 1.184 \pm 0.023_{(\mathrm{OPE})} \pm 0.007_{(\mathrm{syst})}, \quad (3.8)$$

in excellent agreement with the recent lattice results summarized in Table II. The error in the ratio (3.8) arises mainly from the uncertainties in the quark condensates $\langle \bar{s}s \rangle / \langle \bar{q}q \rangle = 0.8 \pm 0.3$.

IV. EXTRACTION OF THE BOTTOM-QUARK MASS

The results of the previous section reveal a strong sensitivity of the sum-rule predictions for f_B and f_{B_s} on the

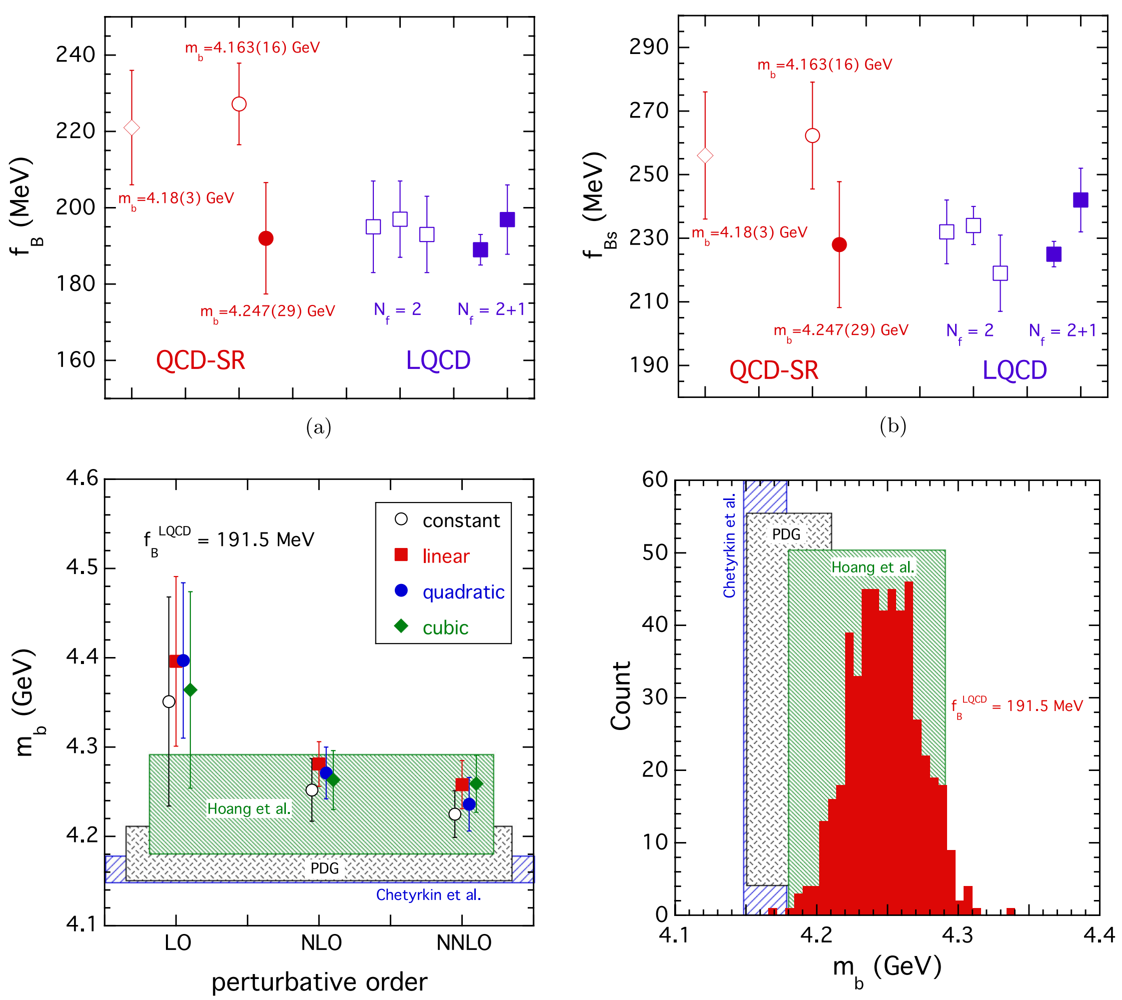

FIG. 3 (color online). (a) Summary of our results for f_B. Lattice (LQCD) outcomes are from [2,3,6] for two dynamical light flavors ($N_f = 2$) and from [21,22] for three dynamical flavors ($N_f = 2 + 1$). For the τ-dependent QCD sum-rule (QCD-SR) result, the error shown is the sum of the OPE and systematic uncertainties in (3.5), added in quadrature. (b) Similar findings for f_{B_s}. (c) The value of m_b extracted from the sum rule (2.5) by a bootstrap analysis of the OPE uncertainties making use of the central value $f_B = 191.5$ MeV and the other relevant parameters collected in (2.8). The dependence of m_b on the number of terms in the perturbative expansion of the correlator is indicated by LO, NLO, and NNLO. The shaded areas correspond to $\pm 1\sigma$ intervals of the results by PDG [1], Chetyrkin *et al.* [8] and Hoang *et al.* [10]. (d) Distribution of m_b as obtained by the bootstrap analysis described in the text. Gaussian distributions for all the OPE parameters (apart from the scales μ and ν) with the associated uncertainties collected in (2.8) are employed. For the independent parameters μ and ν, uniform distributions in the range 3 GeV $< \mu$, $\nu <$ 6 GeV are assumed.

precise value of m_b, in accordance with our simple quantum-mechanical analysis. This feature opens the promising possibility to extract an accurate value of the b-quark mass $m_b \equiv \bar{m}_b(\bar{m}_b)$ by exploiting the accurate lattice results for f_B and f_{B_s}.

The latest lattice-QCD findings for these decay constants are recalled in Table II and Figs. 3(a) and 3(b) (see also [29]). Using these results and applying the algorithms described above, the sum rule (2.5) yields the results for m_b shown in Fig. 3. Figure 3(c) presents the extracted values of m_b depending on the number of terms kept in the perturbative part of the correlator. Moving from $O(1)$ leading order (LO) to $O(\alpha_s)$ next-to-leading order (NLO) accuracy of the perturbative expansion has two effects: first, the central m_b value decreases sizeably from $m_b^{\rm LO} = (4.38 \pm 0.1_{\rm (OPE)} \pm 0.020_{\rm (syst)})$ GeV to $m_b^{\rm NLO} = (4.27 \pm 0.04_{\rm (OPE)} \pm 0.015_{\rm (syst)})$ GeV, and, second, the OPE error also reduces considerably. Adding the $O(\alpha_s^2)$ (NNLO) correction does not, however, entail a sizeable change of the predictions: $m_b^{\rm NNLO} = (4.247 \pm 0.027_{\rm (OPE)} \pm 0.011_{\rm (syst)})$ GeV. Obviously, the extracted values of m_b exhibit a nice ‘‘convergence’’ depending on the accuracy of the perturbative correlation function.

The OPE error in the extracted $m_b^{\rm NNLO}$ is related to the variations of the OPE parameters in the ranges given in (2.8) and the independent variations of the scales μ and ν in the range 3 GeV $\leq \mu, \nu \leq$ 6 GeV. The individual contributions to the OPE error read 14 MeV (μ, ν), 20 MeV (quark condensate), 7 MeV (gluon condensate), 8 MeV (α_s), and 4 MeV (light-quark mass). Adding these values in the quadrature gives 27 MeV. The systematic uncertainty in the extracted value of m_b is found as the spread of the results for different *Ansätze* for the effective continuum threshold and amounts to 11 MeV. To obtain the final estimate for m_b one should further add the (Gaussian) error 18 MeV, related to the uncertainty in the lattice value of $f_B = (191.5 \pm 7.3)$ MeV.

The $O(\alpha_s^3)$ correction to the perturbative spectral density is, at present, not known. Nevertheless, on the basis of our findings we do not expect a sizeable shift of the central value of m_b due to the inclusion of the $O(\alpha_s^3)$ correction. One, however, might expect a reduction of the sensitivity of the extracted value of m_b to the precise values of the scales μ and ν and thus a further increase of the accuracy of the extracted value of the bottom-quark mass.

V. SUMMARY AND CONCLUSIONS

We performed a detailed QCD sum-rule analysis of the B- and B_s-meson decay constants, with particular emphasis on the study of the errors in the extracted decay-constant values: the OPE uncertainty due to the errors of the QCD parameters and the intrinsic error of the sum-rule approach due to the limited accuracy of the extraction procedure. Our main findings may be summarized by the following observations:

(1) The choice of the renormalization scheme used to define the heavy-quark mass is crucial for the convergence of the perturbative expansion of the two-point function: the latter exhibits in its pole-mass formulation no sign of convergence but develops in its running-mass formulation a clear hierarchy of the perturbative contributions. For the extracted decay constant, the pole-mass result is sizeably smaller than its running-mass counterpart, albeit both enjoy a perfect stability in the Borel parameter. Borel stability does not imply reliability of sum-rule results.

(2) The extraction of hadronic properties is significantly improved by allowing a Borel-parameter dependence for the effective continuum threshold, which then quite naturally increases the accuracy of the duality approximation. As shown already before in the charmed-meson sector [27], considering suitably optimized polynomial *Ansätze* for the effective continuum threshold provides an estimate of the intrinsic uncertainty of the method of QCD sum rules.

(3) For beauty mesons, a very strong correlation between the exact m_b value and the sum-rule result for f_B is found:

$$\frac{\delta f_B}{f_B} \approx -8\frac{\delta m_b}{m_b}. \tag{5.1}$$

This enables us to revert the problem and to make use of the precise lattice-QCD computations of f_B to extract the value of m_b. Combining our sum-rule analysis with the latest results for f_B and f_{B_s} from lattice QCD yields

$$m_b = (4.247 \pm 0.027_{\rm (OPE)} \pm 0.018_{\rm (exp)} \pm 0.011_{\rm (syst)})\ {\rm GeV}; \tag{5.2}$$

the OPE error is related to the uncertainties in the OPE input parameters, and the ‘‘exp’’ error is induced by the error in the lattice determination of f_B. Good news is that the systematic uncertainty of the sum-rule method, estimated from the spread of the results for different *Ansätze* of the effective continuum threshold, amounts to 11 MeV and remains under control. Adding all three errors in quadrature yields our final estimate

$$m_b = (4.247 \pm 0.034)\ {\rm GeV}. \tag{5.3}$$

With (5.3), the QCD sum rules for heavy–light correlators evaluated at $O(\alpha_s^2)$ accuracy yield, for the decay constants,

$$f_B = (192.0 \pm 14.3_{\rm (OPE)} \pm 3.0_{\rm (syst)})\ {\rm MeV}, \tag{5.4}$$

$$f_{B_s} = (228.0 \pm 19.4_{\rm (OPE)} \pm 4_{\rm (syst)})\ {\rm MeV}, \tag{5.5}$$

$$f_{B_s}/f_B = 1.184 \pm 0.023_{\rm (OPE)} \pm 0.007_{\rm (syst)}. \qquad (5.6)$$

Our algorithm enables us to provide both the OPE uncertainties and the intrinsic (systematic) uncertainty of the sum-rule method related to the limited accuracy of the extraction procedure. We observe an extreme sensitivity of the decay constant to the input value of the quark mass, but only for beauty mesons. It is not observed in the charm sector, where one finds $\delta f_D/f_D = -0.3\delta m_c/m_c$ [27]. Therefore, the extracted value of f_D is rather mildly sensitive to the precise value of m_c. In the charm sector, on the other hand, one observes a stronger sensitivity of the extracted value of f_D to the algorithm adopted for fixing the τ-dependent effective continuum threshold [27].

(4) Our value (5.2) of m_b is extracted from the Borel QCD sum rule for the heavy–light correlator known to $O(\alpha_s^2)$ accuracy. Taking into account that the value of m_b is changing only marginally when moving from the $O(\alpha_s)$ to $O(\alpha_s^2)$ accuracy of the correlator, we do not expect that the inclusion of the presently unknown $O(\alpha_s^3)$ correction will lead to a substantial change in the extracted value of m_b. Our result is compatible with the result [7]

$$m_b = (4.209 \pm 0.050)\ \text{GeV} \qquad (5.7)$$

found from moment sum rules for heavy–heavy correlators known to the same $O(\alpha_s^2)$ accuracy as in our analysis. We observe an excellent agreement with the prediction of the renormalization-group-improved NNLL analysis of the Υ sum rule [10],

$$m_b = (4.235 \pm 0.055_{\rm (pert)} \pm 0.003_{\rm (exp)})\ \text{GeV}. \qquad (5.8)$$

Our result agrees within 2σ with the PDG estimate

$$m_b = (4.18 \pm 0.03)\ \text{GeV}. \qquad (5.9)$$

We realize, however, a pronounced tension with the predictions of [8]

$$m_b = (4.163 \pm 0.016)\ \text{GeV}, \qquad (5.10)$$

and [30]

$$m_b = (4.171 \pm 0.009)\ \text{GeV}, \qquad (5.11)$$

based on sum rules for heavy–heavy correlators calculated to $O(\alpha_s^3)$ accuracy. As already noticed above, it seems unlikely that the $O(\alpha_s^3)$ correction may bring our result in agreement with the relatively low value of [8]; therefore, we expect that this tension will persist. The origin of this disagreement requires further considerations.

We conclude by emphasizing that the properly formulated Borel QCD sum rules for heavy–light correlators provide a competitive tool for the reliable calculation of heavy-meson properties and for the extraction of basic QCD parameters by making use of the results from lattice QCD and the experimental data. We point out that in the context of QCD sum rules based on correlation functions calculated at $O(\alpha_s^2)$ accuracy, Eq. (5.2) gives the appropriate value of the b-quark mass.

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