Exact blocking formulas for spin and gauge models

Yuzhi Liu,[1,2] Y. Meurice,[1] M. P. Qin,[3] J. Unmuth-Yockey,[1] T. Xiang,[3] Z. Y. Xie,[3] J. F. Yu,[3] and Haiyuan Zou[1]

[1]*Department of Physics and Astronomy, The University of Iowa, Iowa City, Iowa 52242, USA*
[2]*Theoretical Physics Department, Fermi National Accelerator Laboratory, Batavia, Illinois 60510, USA*
[3]*Institute of Physics, Chinese Academy of Sciences, P.O. Box 603, Beijing 100190, China*

Using the example of the two-dimensional (2D) Ising model, we show that in contrast to what can be done in configuration space, the tensor renormalization group formulation allows one to write exact, compact, and manifestly local blocking formulas and exact coarse-grained expressions for the partition function. We argue that similar results should hold for most models studied by lattice gauge theorists. We provide exact blocking formulas for several 2D spin models [the $O(2)$ and $O(3)$ sigma models and the SU(2) principal chiral model] and for the three-dimensional gauge theories with groups Z_2, U(1) and SU(2). We briefly discuss generalizations to other groups, higher dimensions and practical implementations.

I. INTRODUCTION

Lattice field theory is a well-developed numerical method which allows us to study the nonperturbative behavior of asymptotically free theories. The continuum limit of these theories is reached in the limit of zero bare coupling. If a mass gap remains present in this limit, the lattice spacing becomes exponentially small compared to the physical scale associated with the mass gap. Keeping the physical volume reasonably large requires an exponentially large volume or clever extrapolations. For gauge theories with enough massless fermions, a nontrivial infrared fixed point may appear (for a review see Ref. [1]). Evidence for quasiconformal behavior requires small masses and large volume. The practical demands of these two limits are sometimes too taxing to lift controversies regarding the existence of nontrivial fixed points. A well-known example is SU(3) with 12 fundamental fermions [2–6].

The above considerations make clear that reaching exponentially large volumes is a highly desirable outcome for lattice field theorists. This goal could be reached if a reasonably accurate blocking procedure could be designed. By blocking (or block-spinning or coarse-graining) we mean a partial integration procedure used in the renormalization group (RG) approach [7–9] to replace the initial degrees of freedom on sites or links corresponding to a lattice spacing a by some new ones assigned to the sites or links corresponding to the a lattice spacing ba while keeping the macroscopic observables and extensive quantities unchanged. We call b the scaling factor (typically $b = 2$). By blocking n times, linear lattices of order b^n can be reached, which is the aforementioned goal.

It is often believed that blocking in configuration space, for instance by summing over the spins in a block while keeping their sum constant, is tedious but straightforward. The example of the two-dimensional (2D) Ising model with two-by-two blocks (briefly discussed in Sec. II) can

be used to show that the procedure is far from straightforward because it generates an arbitrarily large number of new interactions of arbitrarily large range that are difficult to enumerate and control. It is possible to invent approximations where no new interactions are generated by the blocking process. Examples are the Migdal-Kadanoff approximation [10,11], the approximate recursion formula [12] or other hierarchical approximations [13,14]. However, the lack of reference to an exact procedure makes the systematic improvement of these approximations difficult.

In this paper, we show that, in contrast to the difficult situation encountered in configuration space, the tensor renormalization group (TRG) formulation allows us to write *exact* blocking formulas for several classes of spin and gauge models. For these models, the partition function can be written as a product of tensors attached to sites (or links, or plaquettes) with their indices suitably contracted (traced). After blocking, the partition function has exactly the same form as before except for the fact that the lattice spacing is twice as big and that the sum over the indices has more terms. The recursion formulas for the tensors are manifestly local and do not generate new types of tensors. However, the ability to reinterpret these results in terms of blocked configurations and interactions is probably lost.

The TRG approach of classical lattice models was introduced in Refs. [15–17] and was motivated by tensor states developed in RG studies of quantum models [18]. For this reason, we often refer to sums over tensor indices as sums over states. Improved methods to take the environment into account were proposed in Refs. [19,20]. One important purpose of our article is to show that TRG methods can be applied to many models studied by lattice gauge theorists and that detailed comparisons with standard Monte Carlo simulations should be performed.

In Sec. II, we start with the well-understood case of the 2D Ising model on a square lattice for which very accurate TRG-based numerical calculations [21] were performed.

The construction of the initial tensor can be performed using singular-value decomposition (SVD). This task can be simplified by using the character expansion techniques used to reformulate lattice models in terms of dual variables [22]. As the recursion formula is iterated, the number of states increases rapidly and truncation methods are necessary. Optimal methods with apparent convergence when going to a sufficiently large number of states were discussed in Ref. [21]. It is interesting to notice that two-state approximations provide much better estimates of the critical exponents [23] than the Migdal-Kadanoff approximation. However, there is an intermediate region for the number of states retained where oscillations appear in individual tensor components and new techniques need to be developed if we want to work within this "intermediate-number-of-states zone." This situation has been recently documented and analyzed in Ref. [24] which also provides a nice introduction to the TRG method.

The rest of the paper is organized as follows. We provide the exact tensor recursion formulas for the 2D $O(2)$ and $O(3)$ sigma models (Sec. III), the 2D SU(2) principal chiral model (Sec. IV) and the three-dimensional (3D) Z_2, U(1) and SU(2) pure gauge theories (Sec. V). Again, the calculation of the initial tensor can be done from expansions used in dual formulations [22,25–28] even though we do not deal with the dual variables here. The generalizations of these results and ongoing practical applications are discussed in the conclusions.

II. BLOCKING THE 2D ISING MODEL

In order to appreciate the importance of having exact blocking formulas, we discuss the 2D Ising model on a square lattice with nearest-neighbor ferromagnetic interactions and an inverse temperature β. We first work in configuration space (with spin and blocked spin variables) and then with the TRG approach. Some of the features observed generalize to other Abelian spin models.

A. Configuration-space blocking

We first describe an algorithm to block spin *once* in configuration space and try to generate a new energy function. The main purpose of the discussion is to show that unless approximations are made, this new energy function is very different from the original one. We consider square blocks with a checkerboard configuration partitioning the blocks into A and B blocks, as illustrated in Figs. 1 and 2.

We treat the spins in the B blocks as fixed background and proceed to calculate unnormalized probabilities for the total spin in the A blocks. For a given A block, there are eight background spins belonging to the four nearest-neighbor B blocks, as shown in Fig. 1. The total spin ϕ_A in the A block takes values ± 4, ± 2 and 0. For each value of ϕ_A and for each of the background configurations, we can sum over the known Boltzmann weights and obtain 5×2^8
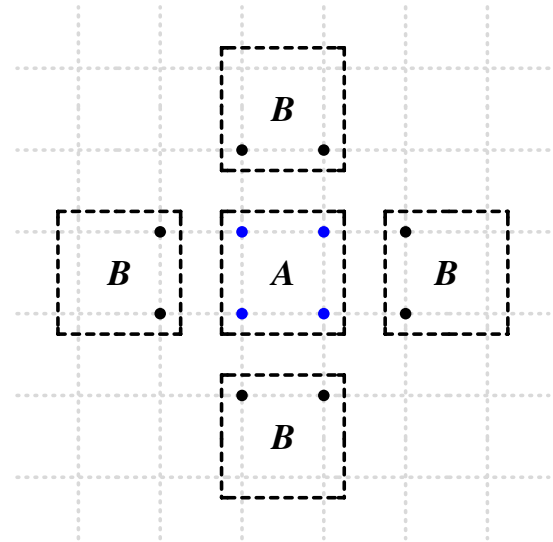

FIG. 1 (color online).   First blocking on AB checkerboard.

unnormalized probabilities. The next step is to try to block spin in the B blocks. We consider a given B block such as the one at the center of Fig. 2. This can be done by combining our previous results for the four nearest-neighbor A blocks. We can now sum over the Boltzmann weights corresponding to each of the values ± 4, ± 2 and 0 in the B block. This clearly generates probabilities for the 5^5 configurations of the four ϕ_As and the central ϕ_B and consequently generates nearest-neighbor interactions among these blocked variables.

The crucial point is that the results depend on the remaining background spins attached to the four A blocks and so the sums over the remaining spins in the B blocks cannot be done independently of one another. There are twelve such spins in the four B blocks. We denote these
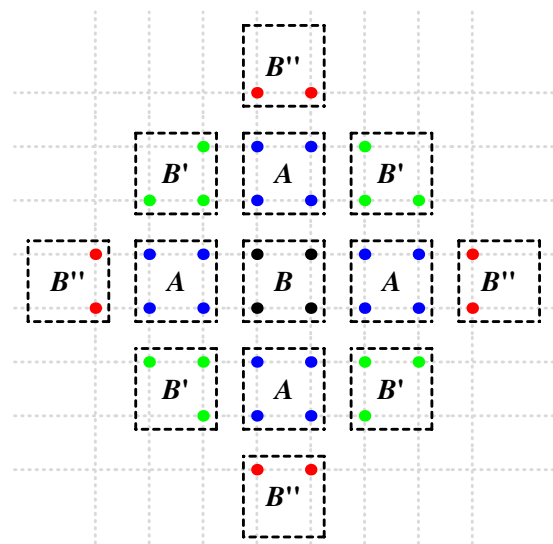

FIG. 2 (color online).   Second blocking on AB checkerboard.

four blocks as B', which are located diagonally from the central B block under consideration. There are eight spins in the next-to-nearest-neighbor B blocks, denoted as B''. If we try to construct the new energy function numerically, this requires $5^5 2^{20}$ (about 3 billion) memory entries. It is clear that these correlations will generate more than nearest-neighbor interactions.

We should now pause and discuss what has been accomplished so far and what remains to be done. For this purpose, we can tile the original lattice with the "diamonds" of Fig. 2. The unshared spins inside the diamonds have been blocked. This is 5/8 of the total number of spins. The remaining spins (3/8 of the total number of spins) are at the shared boundaries of the diamonds and form diagonals. The B' blocks are shared by two diamonds and the B'' blocks are shared by four. One can in principle combine four such diamonds into a new diamond with twice the linear size. This discussion makes clear that arbitrary range interactions are generated and finding a new energy function in terms of polynomials of the ϕ_A and ϕ_B seems to be a Herculean task at least as difficult as calculating the exact partition function in a comparable volume.

B. TRG blocking

In contrast, blocking is amazingly simple in the TRG formulation. For each link we can write

$$\exp(\beta \sigma_1 \sigma_2) = \cosh(\beta)\Big(1 + \sqrt{\tanh(\beta)}\,\sigma_1 \sqrt{\tanh(\beta)}\,\sigma_2\Big)$$
$$= \cosh(\beta) \sum_{n_{12}=0,1} \Big(\sqrt{\tanh(\beta)}\,\sigma_1 \sqrt{\tanh(\beta)}\,\sigma_2\Big)^{n_{12}}.$$
(2.1)

Using this identity for each link in the partition function, we can then regroup the four terms involving a given spin σ_i and sum over its two values ± 1. The results can be expressed in terms of a tensor $T^{(i)}_{xx'yy'}$ which can be visualized as a cross attached to the site i with the four legs covering half of the four links attached to i. The horizontal indices x, x' and vertical indices y, y' take the values 0 and 1 as the index n_{12} in Eq. (2.1). The tensor is zero for an odd number of 1's. For an even number of 1's, a factor $\sqrt{\tanh(\beta)}$ appears for each 1 irrespective of the direction. This can be summarized as follows:

$$T^{(i)}_{xx'yy'} = f_x f_{x'} f_y f_{y'} \delta(\mathrm{mod}[x + x' + y + y', 2]), \quad (2.2)$$

where $f_0 = 1$ and $f_1 = \sqrt{\tanh(\beta)}$. The delta symbol is 1 if $x + x' + y + y'$ is zero modulo 2 and zero otherwise.

The partition function of the model can now be written as

$$Z = \mathrm{Tr} \prod_i T^{(i)}_{xx'yy'}. \tag{2.3}$$

Tr is a short notation for contractions (sums over 0 and 1) over the links joining nearest neighbors on the lattice. This expression reproduces the proper closed paths of the high-temperature expansion.

We now use this reformulation to block spin [21,23]. We consider an isotropic procedure with a square block enclosing four sites as in the previous subsection and sum over the states inside the block associated with the nearest-neighbor links joining these four points. This defines a new rank-4 tensor $T'_{XX'YY'}$ where each index now takes four values,

$$T'_{X(x_1,x_2)X'(x'_1,x'_2)Y(y_1,y_2)Y'(y'_1,y'_2)}$$
$$= \sum_{x_U, x_D, x_R, x_L} T_{x_1 x_U y_1 y_L} T_{x_U x'_1 y_2 y_R} T_{x_D x'_2 y_R y'_2} T_{x_2 x_D y_L y'_1}, \quad (2.4)$$

where $X(x_2, x_2)$ is a notation for the product states. In Ref. [23] we used the convention $X(0, 0) = 1$, $X(1, 1) = 2$, $X(1, 0) = 3$, $X(0, 1) = 4$. This is represented graphically in Fig. 3. The partition function can be written as

$$Z = \mathrm{Tr} \prod_{2i} T'^{(2i)}_{XX'YY'}, \tag{2.5}$$

where $2i$ denotes the sites of the coarser lattice with twice the lattice spacing of the original lattice. As pointed out in Ref. [23] the TRG blocking is exact and can be written compactly because the procedure separates, unambiguously, the degrees of freedom inside the block which are integrated over from those kept to communicate with the neighboring blocks.

C. Abelian factorization and connection to configuration space

Equation (2.2) shows that the initial tensor factors nicely. This property can be extended to spin models with an Abelian group. For an explicit example see $O(2)$ in Sec. III A. The general reasoning goes as follows. The Boltzmann weight associated with a link can be expanded in characters (Fourier modes). For an Abelian group, each character of the expansion is a product of two characters involving each of the site variables. Similarly the coefficient of the expansion can be written as the product of two square roots of itself, each one being associated with a given site. One can then regroup all the characters and square roots associated with the variable of a given site. The initial tensor is obtained after integrating over the site variable. This gives a Kronecker delta times a product similar to what is seen in Eq. (2.2). The reasoning immediately extends to arbitrary dimension.

The sum over the internal states in Fig. 3 and Eq. (2.4) is very similar to a sum over momenta in Feynman diagrams. Three of the sums are absorbed using the Kronecker delta associated with three vertices, but there is a global condition on the external legs that, if satisfied, guarantees that the fourth condition is satisfied. Consequently there is, in
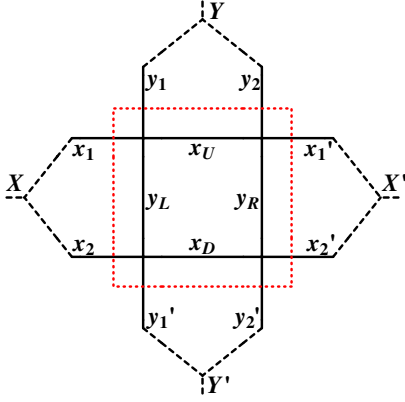
FIG. 3 (color online).   Graphical representation of $T'_{XX'YY'}$.

general, a sum over "states circulating in the loop" which is enclosed in the block. This sum of factorizable terms is apparently not factorizable and it seems impossible to rewrite the blocked tensor as coming from a blocked energy function with an Abelian symmetry. This argument indicates that a direct connection between blocked tensors and blocked energy functions might be difficult or impossible to find.

III. TRG FORMULATION OF $O(2)$ AND $O(3)$ SIGMA MODELS

The Hamiltonian for the $O(N)$ nonlinear sigma models can be written as

$$H = -\sum_{\langle ij \rangle} \mathbf{S_i} \cdot \mathbf{S_j}, \tag{3.1}$$

with $\mathbf{S_i}$ a unit vector in $\mathbb{R}^N$, or equivalently a point on an $(N-1)$-dimensional unit sphere. We will discuss explicitly the Abelian case $N = 2$ and the non-Abelian case $N = 3$ in two dimensions. In both cases, the TRG expression of the partition function has the same form as Eq. (2.3) for the Ising model, with the only differences being the range of the indices and the initial values. Similarly, the blocking of the tensor has the same form as Eq. (2.4) and will not be written explicitly.

A. $O(2)$ model

For $N = 2$, $\mathbf{S_i}$ is a unit vector staying at each site i: $(\cos(\theta_i), \sin(\theta_i))$. The partition function reads

$$Z = \int \prod_i \frac{d\theta_i}{2\pi} \, e^{\beta \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)}. \tag{3.2}$$

Using

$$e^{\beta \cos(\theta_i - \theta_j)} = \sum_{n_{ij} = -\infty}^{+\infty} e^{i n_{ij}(\theta_i - \theta_j)} I_{n_{ij}}(\beta), \tag{3.3}$$

where the I_n are the modified Bessel functions of the first kind. From the basic property of the exponential, it is

possible to collect all the factors involving a given θ_i and integrate over this variable. This results in a tensor attached to the site i. In two dimensions,

$$T^i_{n_{ix}, n_{ix'}, n_{iy}, n_{iy'}}$$
$$= \sqrt{I_{n_{ix}}(\beta)} \sqrt{I_{n_{iy}}(\beta)} \sqrt{I_{n_{ix'}}(\beta)} \sqrt{I_{n_{iy'}}(\beta)} \delta_{n_{ix}+n_{iy}, n_{ix'}+n_{iy'}}. \tag{3.4}$$

The sign convention is that we have positive signs for the left and top indices and negative signs for the right and bottom indices. This allows us to write the partition function and the blocking of the tensor similarly to the Ising model. The only difference is that the sums run over the integers. As the $I_n(\beta)$ decay rapidly for large n and fixed β (namely like $1/n!$) there is no convergence issue. The generalization to higher dimensions is straightforward ($2D$ indices in D dimensions).

B. $O(3)$ model

For $N = 3$, $\mathbf{S_i}$ is a unit vector at site i: $(\sin(\theta_i) \times \cos(\phi_i), \sin(\theta_i) \sin(\phi_i), \cos(\theta_i))$. In terms of these variables, the energy function can then be written as

$$H = -\sum_{\langle ij \rangle} \cos \gamma_{ij}, \tag{3.5}$$

where γ_{ij} is the angle between $\mathbf{S_i}$ and $\mathbf{S_j}$ and $\cos \gamma_{ij}$ can be expressed in terms of the angles as

$$\cos \gamma_{ij} = \cos \theta_i \cos \theta_j + \sin \theta_i \sin \theta_j \cos(\phi_i - \phi_j). \tag{3.6}$$

Expanding as for $O(2)$ and using $I_n(\beta) = I_{-n}(\beta)$,

$$e^{\beta \cos \gamma} = I_0(\beta) + 2 \sum_{n=1}^{\infty} I_n(\beta) \cos(n\gamma), \tag{3.7}$$

we can then use the Chebyshev polynomials of the first kind to re-express

$$\cos(n\gamma) = T_n(\cos \gamma). \tag{3.8}$$

Using the Legendre polynomials $P_n(\cos \gamma)$, we can write

$$e^{\beta \cos \gamma_{ij}} = I_0(\beta) + 2 \sum_{n=1}^{\infty} I_n(\beta) \sum_{l=0}^{n} a_{nl} P_l(\cos \gamma_{ij}), \tag{3.9}$$

with

$$a_{nl} = \frac{2l+1}{2} \int_{-1}^{1} T_n(x) P_l(x) dx. \tag{3.10}$$

By using the addition theorem for spherical harmonics

$$P_l(\cos \gamma_{ij}) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y^*_{lm}(\theta_j, \phi_j) Y_{lm}(\theta_i, \phi_i), \tag{3.11}$$

Eq. (3.9) can be written as

$$e^{\beta \cos \gamma_{ij}} = \sum_{l=0}^{\infty} A_l(\beta) \sum_{m=-l}^{l} Y_{lm}^*(\theta_j, \phi_j) Y_{lm}(\theta_i, \phi_i), \quad (3.12)$$

where

$$A_0(\beta) = I_0(\beta) + 2 \sum_{n=1}^{\infty} I_n(\beta) a_{n0} 4\pi,$$

$$A_1(\beta) = 2 \sum_{n=1}^{\infty} I_n(\beta) a_{n1} \frac{4\pi}{3},$$

$$\vdots \qquad\qquad\qquad (3.13)$$

$$A_l(\beta) = 2 \sum_{n=l}^{\infty} I_n(\beta) a_{nl} \frac{4\pi}{2l+1}.$$

Again, the Bessel functions control the decay for large l. The elements of the T tensor can then be written as

$$T_{(l_1,m_1),(l_2,m_2),(l_3,m_3),(l_4,m_4)}$$
$$= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta Y_{l_1 m_1}(\theta, \phi) Y_{l_2 m_2}^{\star}(\theta, \phi)$$
$$\times Y_{l_3 m_3}(\theta, \phi) Y_{l_4 m_4}^*(\theta, \phi) \sqrt{A_{l_1} A_{l_2} A_{l_3} A_{l_4}}. \quad (3.14)$$

The direction convention is shown in Fig. 4. Equation (3.14) can be further simplified by expanding the product of two spherical harmonics in terms of spherical harmonics themselves,

$$Y_{l_1 m_1}(\theta, \phi) Y_{l_3 m_3}(\theta, \phi) = \sum_{L=l_{\min}}^{l_{\max}} G_L^{(m_1, m_3, l_1, l_3)} Y_L^{m_1+m_3}(\theta, \phi). \quad (3.15)$$

Explicit formulas for $G_L^{(m_1, m_3, l_1, l_3)}$ and a discussion of the Gaunt coefficients can be found in Ref. [29]. The summation bounds are
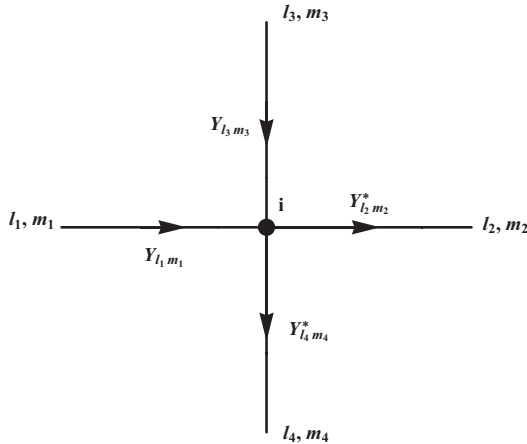

FIG. 4.   2D $O(3)$.

$$l_{\max} = l_1 + l_3,$$

$$l_{\min} = \begin{cases} \lambda_{\min} & \text{if } l_{\max} + \lambda_{\min} \text{ is even,} \\ \lambda_{\min} + 1 & \text{if } l_{\max} + \lambda_{\min} \text{ is odd,} \end{cases} \quad (3.16)$$

$$\lambda_{\min} = \max(|l_1 - l_3|, |m_1 + m_3|).$$

The angular integration in Eq. (3.14) can be performed using the orthonormal property of the spherical harmonics with the result

$$T_{(l_1,m_1),(l_2,m_2),(l_3,m_3),(l_4,m_4)}$$
$$= \delta_{m_1+m_3,m_2+m_4} \sum_L G_L^{(m_1,m_3,l_1,l_3)} G_L^{*(m_2,m_4,l_2,l_4)} \sqrt{A_{l_1} A_{l_2} A_{l_3} A_{l_4}}. \quad (3.17)$$

In contrast to $O(2)$, there are now two indices associated with each leg of the tensor and the factorization of the initial tensor is lost.

IV. TRG FOR SU(2) PRINCIPAL CHIRAL MODELS

The partition function for the principal chiral model reads

$$Z = \prod_n \int dU(n) \prod_{ni} \exp\left\{\frac{\beta}{2} \text{Re}[\text{tr}[U(n)U^\dagger(n+i)]]\right\}, \quad (4.1)$$

with i a unit vector in one of the spatial directions and n a spatial location. Then, since the action only depends on the trace of the matrix representation of the group elements, we can write it in terms of a character expansion,

$$\exp\left\{\frac{\beta}{2} \text{Re}[\text{tr}[U(n)U^\dagger(n+i)]]\right\} = \sum_r F_r(\beta) \chi^r(U(ni)), \quad (4.2)$$

with the sum over the representations of the group and $U(ni)$ a short notation for $U(n)U^\dagger(n+i)$. χ_r is the trace in the irreducible representation r of SU(2). The explicit form of the measure, $\chi^r(U)$ and $F_r(\beta)$ (in terms of Bessel functions), for SU(2) and other groups can be found in Ref. [30]. Some of the results below will be written using the conventions from Ref. [29]. We can rewrite the partition function as

$$Z = \prod_n \int dU(n) \prod_{ni} \sum_{r(ni)} F_{r(ni)}(\beta) \chi^{r(ni)}(U(ni)). \quad (4.3)$$

Let $A \in SU(2)$ and let $D_{mn}^r(A)$ be the matrix elements in the rth irreducible representation (the "Wigner D-functions"). Then to extract the angle dependence inside of the χ's we note that the χ's are the traces of these representations and thus

$$\chi^r(AB) = D_{mn}^r(A) D_{nm}^r(B). \quad (4.4)$$

For a 2D lattice, there are four times when a product of pairs of sites contains the same site, so if we multiply out all of the nearest-neighbor pairs and collect the single same site together, we obtain

$$Z = \prod_n \int dU(n) \prod_{ni} \sum_{r(ni)} F_{r(ni)}(\beta)$$
$$\times \sum_{m,k} D_{mk}^{r(ni)}(U(n)) D_{km}^{r(ni)}(U(n+i)) \quad (4.5)$$

$$= \sum_{\{r's\}} \sum_{\{m's\}} \sum_{\{n's\}} \prod_l (F_{r_{1,l}}(\beta) F_{r_{2,l}}(\beta)$$
$$\times F_{r_{3,l}}(\beta) F_{r_{4,l}}(\beta))^{\frac{1}{2}} \prod_n \int dU(n) D_{m_1 n_1}^{r_{1,l}}(U)$$
$$\times D_{m_2 n_2}^{r_{2,l}}(U) D_{m_3 n_3}^{r_{3,l}}(U) D_{m_4 n_4}^{r_{4,l}}(U), \quad (4.6)$$

with l a product over the sites of the lattice, and r_1, r_2, r_3, r_4 the four links incoming and outgoing from the site. Each F_r is shared by two sites on the lattice since they are located on the links. Then from the integrals we get a constraint at each site on the lattice, and a product over constraints and link variables, F_r.

To preform the integration over the site variables we can use the Clebsch-Gordan series to rewrite two D-functions as a single D-function with accompanying Clebsch-Gordan symbols,

$$D_{m_1 n_1}^{r_1}(U) D_{m_2 n_2}^{r_2}(U) = \sum_{r=|r_1 - r_2|}^{r_1 + r_2} \sum_{m,n} C_{m_1 m_2 n}^{r_1 r_2 r} D_{mn}^r(U) C_{n_1 n_2 n}^{r_1 r_2 r}. \quad (4.7)$$

Then for the integrals above we can change out the four Ds for two, and using $D_{mn}^r = (-1)^{n-m} D_{-m-n}^{r*}$ and their orthogonality [28]

$$\int dU D_{m_1 n_1}^{r_1}(U) D_{m_2 n_2}^{*r_2}(U) = \frac{1}{2r_1+1} \delta_{r_1 r_2} \delta_{m_1 m_2} \delta_{n_1 n_2}, \quad (4.8)$$

we can write down the integral exactly,

$$\int dU D_{m_1 n_1}^{r_1}(U) D_{m_2 n_2}^{r_2}(U) D_{m_3 n_3}^{r_3}(U) D_{m_4 n_4}^{r_4}(U)$$
$$= \sum_{r',m',n'} \sum_{r'',m'',n''} C_{m_1 m_2 m'}^{r_1 r_2 r'} C_{n_1 n_2 n'}^{r_1 r_2 r'} C_{m_3 m_4 m''}^{r_3 r_4 r''} C_{n_3 n_4 n''}^{r_3 r_4 r''}$$
$$\times \int dU D_{m'n'}^{r'} D_{-m''-n''}^{*r''} (-1)^{n''-m''} \quad (4.9)$$

$$= \sum_{r',m',n'} \sum_{r'',m'',n''} C_{m_1 m_2 m'}^{r_1 r_2 r'} C_{n_1 n_2 n'}^{r_1 r_2 r'} C_{m_3 m_4 m''}^{r_3 r_4 r''} d_{r'}^{-1}$$
$$\times (-1)^{n''-m''} C_{n_3 n_4 n''}^{r_3 r_4 r''} \delta_{m',-m''} \delta_{n',-n''} \delta_{r',r''} \quad (4.10)$$

$$= \sum_{r',m',n'} d_{r'}^{-1} (-1)^{m'-n'} C_{m_1 m_2 m'}^{r_1 r_2 r'} C_{n_1 n_2 n'}^{r_1 r_2 r'} C_{m_3 m_4 -m'}^{r_3 r_4 r'} C_{n_3 n_4 -n'}^{r_3 r_4 r'}. \quad (4.11)$$

This allows us to write the partition function directly as

$$Z = \sum_{\{r's\}} \sum_{\{m's\}} \sum_{\{n's\}} \prod_l (F_{r_{1,l}}(\beta) F_{r_{2,l}}(\beta) F_{r_{3,l}}(\beta) F_{r_{4,l}}(\beta))^{\frac{1}{2}}$$
$$\times \sum_{r',m',n'} d_{r'}^{-1} (-1)^{m'-n'} C_{m_1 m_2 m'}^{r_1 r_2 r'} C_{n_1 n_2 n'}^{r_1 r_2 r'} C_{m_3 m_4 -m'}^{r_3 r_4 r'} C_{n_3 n_4 -n'}^{r_3 r_4 r'}, \quad (4.12)$$

and gives us a T tensor of the form

$$T_{(r_1,m_1,n_1)(r_2,m_2,n_2)(r_3,m_3,n_3)(r_4,m_4,n_4)}$$
$$= (F_{r_1}(\beta) F_{r_2}(\beta) F_{r_3}(\beta) F_{r_4}(\beta))^{\frac{1}{2}} \sum_{r',m',n'} d_{r'}^{-1} (-1)^{m'-n'}$$
$$\times C_{m_1 m_2 m'}^{r_1 r_2 r'} C_{n_1 n_2 n'}^{r_1 r_2 r'} C_{m_3 m_4 -m'}^{r_3 r_4 r'} C_{n_3 n_4 -n'}^{r_3 r_4 r'}. \quad (4.13)$$

This T tensor can be used just as a typical spin-model tensor with four (grouped) indices. The typical contraction between tensor legs can be carried out with the help of a grouped set of Kronecker deltas,

$$\tilde{\delta}_{(r,i,i')(r',j,j')} = \delta_{r,r'} \delta_{i,j} \delta_{i',j'}. \quad (4.14)$$

This tensor ensures the same representation per link, and circulates the trace of the matrix indices along the link between sites.

V. TRG FORMULATIONS OF LATTICE GAUGE MODELS

In this section, the tensor-network forms for the partition function of Abelian and non-Abelian gauge models, including 3D Z_2 gauge theory, D-dimensional U(1) gauge models ($D = 2, 3, 4$), and SU(2) gauge models are shown. Two formulations of the TRG method are constructed: one is left-right asymmetric and the other symmetric.

A. Three-dimensional Z_2 gauge theory

We first consider a simple gauge theory on a lattice—the three-dimensional compact Z_2 gauge theory, with the partition function

$$Z = \sum_{\{\sigma\}} \exp\left(\beta \sum_P \sigma_{12} \sigma_{23} \sigma_{34} \sigma_{41}\right), \quad (5.1)$$

where the action is a sum over all the plaquettes and the fields $\sigma_{ij} = \pm 1$ are attached to each link of the lattice. We can now proceed as in Eq. (2.1) and write a single plaquette contribution using a sum with $n = 0$ or 1 of

$$\left(\sqrt[4]{\tanh(\beta)} \sigma_{12} \sqrt[4]{\tanh(\beta)} \sigma_{23} \sqrt[4]{\tanh(\beta)} \sigma_{34} \sqrt[4]{\tanh(\beta)} \sigma_{41}\right)^n.$$

Regrouping the factors with a given σ_l and summing over ± 1 we obtain a tensor attached to this link,

$$A^{(l)}_{n_1 n_2 n_3 n_4} = (\sqrt[4]{\tanh \beta})^{n_1 + n_2 + n_3 + n_4}$$
$$\times \, \delta(\text{mod}[n_1 + n_2 + n_3 + n_4, 2]). \quad (5.2)$$

The four links attached to a given plaquette p must carry the same index 0 or 1. For this purpose we introduce a new tensor,

$$B^{(p)}_{m_1 m_2 m_3 m_4} = \delta(m_1, m_2, m_3, m_4) = \begin{cases} 1, & \text{all } n_i \text{ are the same,} \\ 0, & \text{otherwise.} \end{cases}$$
$$(5.3)$$

The partition function can now be written as

$$Z = (2 \cosh \beta)^{3V} \text{Tr} \prod_l A^{(l)}_{n_1 n_2 n_3 n_4} \prod_p B^{(p)}_{m_1 m_2 m_3 m_4}, \quad (5.4)$$

where V is the volume of the system and Tr is a notation for a sum over all the shared plaquettes. A graphical representation of the tensors is provided in Fig. 5. One can check that the new expression for the partition function reproduces the strong-coupling expansion.

1. Asymmetric formulation

By using three A tensors and three B tensors as shown Fig. 6, a basic cell can be constructed. There are twelve external legs. We can recombine the indices attached to the legs pointing in the same directions using product states (labeled by capital letters). For instance, $X = x_1 \otimes x_2$, and similarly with the other directions. Proceeding this way, we obtain a new tensor $T_{6XX'YY'ZZ'}$ which can be treated as in the case of a 3D spin model. However, in the positive (X, Y, Z) and negative (X', Y', Z') directions, the opposite legs are associated with different tensors. For instance, X is associated with A and X' with B.
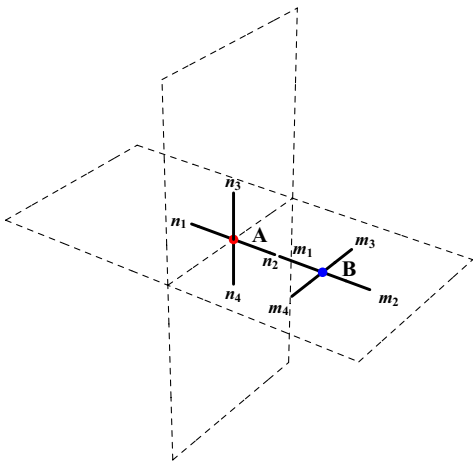
The partition function can be rewritten as the tensor-network state of the new T_6 tensor at each cube c,

$$Z = (2 \cosh \beta)^{3V} \text{Tr} \prod_c T^{(c)}_{6XX'YY'ZZ'}. \quad (5.5)$$

To block spin, we can use anisotropic steps by contracting the lattice alternatively in the x-axis, y-axis, and z-axis directions. In each step, the lattice size is reduced by a factor of 2 in the appropriate direction and a new T_6' tensor is generated as

$$T'_{6XX''\tilde{Y}(Y_1,Y_2)\tilde{Y}'(Y_1',Y_2')\tilde{Z}(Z_1,Z_2)\tilde{Z}'(Z_1',Z_2')}$$
$$= \sum_{X'} T_{6XX'Y_1Y_1'Z_1Z_1'} T_{6X'X''Y_2Y_2'Z_2Z_2'}, \quad (5.6)$$

where $\tilde{Y}(Y_1, Y_2)$ is the notation for the product states $\tilde{Y} = Y_1 \otimes Y_2$ and similarly with the other directions. The partition function can then be rewritten as the trace of the product of T_6' tensors as before blocking.

It is straightforward but tedious to write an isotropic blocking formula involving the product of eight T_6 tensors. It is also possible to find tensors associated with the partition function in the temporal gauge. The A tensors on the temporal links disappear while those on the spatial links have a space-time asymmetry. This will be important for numerical applications.
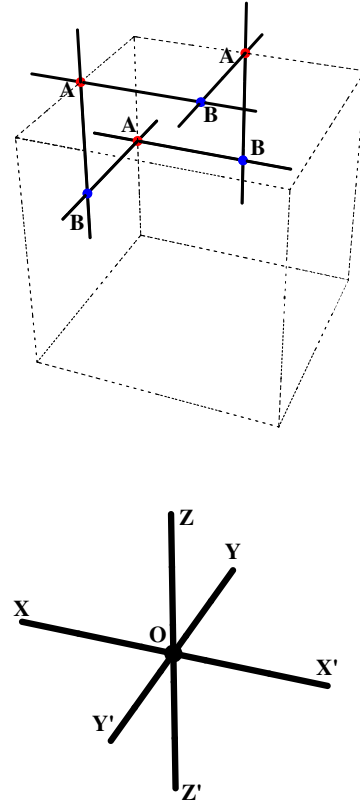

FIG. 6 (color online). (Top) A new basic cell in an original cube. (Bottom) The equivalent T_6 tensor with center (1/4, 3/4, 3/4) in the original cube.

FIG. 5 (color online). A tensor and B tensors.

2. Symmetric formulation

The difference between the positive and negative directions in the previous formulation can be avoided by introducing new tensors. First, we notice that the A and B tensors do not suffer from this asymmetry. However they do not close under blocking. To see this we can try to combine the B tensors of two adjacent plaquettes in the same plane into a new one. This does not work because the A tensor on the common link induces two new legs orthogonal to the plane and pointing in opposite directions. This is the effect that is eliminated in the Migdal-Kadanoff approximation by bond-sliding. Here, we want an exact formula so we modify the B tensor to form a $\tilde{B}$ tensor with six indices (see Fig. 7) with the initial value

$$\tilde{B}_{n_1 n_2 n_3 n_4 z z'} = B_{n_1 n_2 n_3 n_4} \delta_{z z'} \quad (5.7)$$

for a plaquette in the x-y plane and with similar expressions for the two other planes. The new legs piercing the plaquettes can be traced by introducing a new tensor $C_{x x' y y' z z'}$ at the center of the cubes, as shown in Fig. 8, with the initial value

$$C_{x x' y y' z z'} = \delta_{x x'} \delta_{y y'} \delta_{z z'}, \quad (5.8)$$

where the δ_{ij} is the Kronecker delta function. In general, the C tensor has the same indices as the T_6 tensor shown at the bottom of Fig. 6, but its center is $(1/2, 1/2, 1/2)$ in the original cube.

We can now rewrite the partition function as

$$Z = K(2 \cosh \beta)^{3V} \mathrm{Tr} \prod_l A^{(l)} \prod_p \tilde{B}^{(p)} \prod_c C^{(c)}, \quad (5.9)$$

where the indices are implicit to keep the formula short. The Kronecker delta in the initial values can be summed along open or closed lines (depending on the boundary conditions) and give rise to a power of 2 that can be eliminated by adjusting the constant K. The other traces are as in the original expression of the partition function.

A blocking procedure can be constructed by sequentially combining two cubes into one in each of the directions. This is illustrated in one direction in Fig. 8.

Now we can write explicit blocking formulas. On the link of the new lattice formed by two cubes, two parallel A
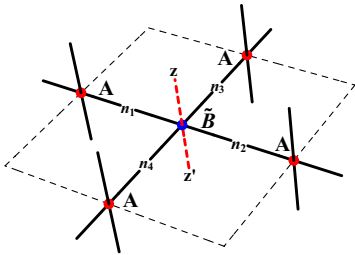

FIG. 7 (color online). The A tensors on the links and the $\tilde{B}$ tensor at the center of the plaquette.
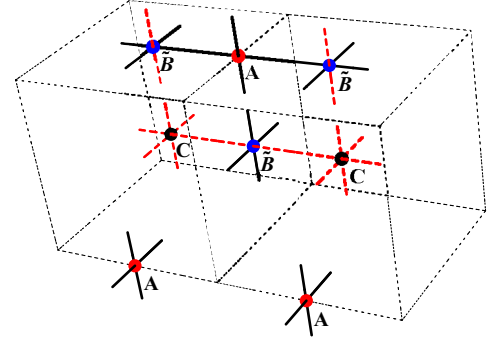

FIG. 8 (color online). Blocking procedure.

tensors form the new A' tensor with product states (capital letters). Each tensor element is

$$A'_{X(x_1,x_2) X'(x'_1,x'_2) Y(y_1,y_2) Y'(y'_1,y'_2)} = A_{x_1 x'_1 y_1 y'_1} \times A_{x_2 x'_2 y_2 y'_2}. \quad (5.10)$$

On the new face, two $\tilde{B}$ tensors and one A tensor form a new $\tilde{B}'$ tensor,

$$\begin{aligned} \tilde{B}'_{x x' Y(y_1,y_2) Y'(y'_1,y'_2) Z(z_1,z_2,z_3) Z'(z'_1,z'_2,z'_3)} \\ = \sum_{x_3, x'_3} \tilde{B}_{x x_3 y_1 y'_1 z_1 z'_1} A_{x_3 x'_3 z_3 z'_3} \tilde{B}_{x'_3 x' y_2 y'_2 z_2 z'_2}. \end{aligned} \quad (5.11)$$

At the center, two C tensors and one $\tilde{B}$ tensor form a new C' tensor,

$$\begin{aligned} C'_{x x' Y(y_1,y_2,y_3) Y'(y'_1,y'_2,y'_3) Z(z_1,z_2,z_3) Z'(z'_1,z'_2,z'_3)} \\ = \sum_{x_2, x'_2} C_{x x_2 y_1 y'_1 z_1 z'_1} \tilde{B}_{x_2 x'_2 y_2 y'_2 z_2 z'_2} C_{x'_2 x' y_3 y'_3 z_3 z'_3}. \end{aligned} \quad (5.12)$$

B. U(1) gauge models

In this section, we formulate the U(1) gauge models in D dimensions in terms of tensor-network states. The partition function of these models can be written as

$$Z = \prod_{\langle ij \rangle} \int_{-\pi}^{\pi} \frac{d\theta_{ij}}{2\pi} \exp\left(\beta \sum_P \cos\left(\theta_{12} + \theta_{23} - \theta_{43} - \theta_{14}\right)\right),$$

$$(5.13)$$

where the product is running through all the links of the lattice and the sum is over all the plaquettes.

Using the Fourier expansion with the Bessel functions as in Eq. (3.2) and collecting the factors for each link, we obtain the tensor

$$A_{n_1 \dots n_{2(D-1)}} = \prod_{i=1}^{2(D-1)} \sqrt[4]{I_{n_i}(\beta)} \delta\left(\sum_{i=1}^{2(D-1)} (-1)^{i+1} n_i\right), \quad (5.14)$$

where the I_ns are the modified Bessel functions. For any D, we can use a B tensor that ensures that the four indices attached to a plaquette are identical just like for the Z_2 case. The partition function can be written as

$$Z = \mathrm{Tr}\prod_l A^{(l)}_{n_1\ldots n_{2(D-1)}}\prod_p B^{(p)}_{m_1 m_2 m_3 m_4}. \qquad (5.15)$$

We can construct the blocking procedure in both the asymmetric and symmetric ways by following what has been done for Z_2. From a geometric viewpoint, a basic cell in a D-dimensional lattice contains D A tensors with $2(D-1)$ legs each, and $\frac{D(D-1)}{2}$ B tensors, each always with four legs. We now will consider $D = 2$, 3 and 4 separately.

1. $D = 2$

For $D = 2$, the A tensor is just a diagonal matrix. This allows us to block two adjacent B tensors and get another B tensor. We can also construct an asymmetric tensor T_4 from a basic cell as illustrated in Fig. 9. If one leg is fixed, all the other three are also fixed because of the constraint of the B tensor. Thus,

$$T_4 \equiv T_{xx'yy'} = I_x(\beta)\delta(x, x', y, y'). \qquad (5.16)$$

This tensor can be blocked isotropically with the result $(I_x(\beta))^4\delta(x, x', y, y')$. For periodic boundary conditions we can pursue this process, and we get the known answer

$$Z = \sum_{n=-\infty}^{\infty} I_n(\beta)^{L_x \times L_y}, \qquad (5.17)$$

where $L_x \times L_y$ is the area of the system. For open boundary conditions, we can represent the three-indices tensor at the boundary as a four-indices tensor with an index 0 for the leg going outside the boundary. With this, only the $n = 0$ term survives from the sum obtained with periodic boundary conditions. This is very similar to the one-dimensional $O(2)$ model.

2. $D = 3$

The treatment is almost identical to the 3D Z_2 gauge model. The geometric construction is the same but the
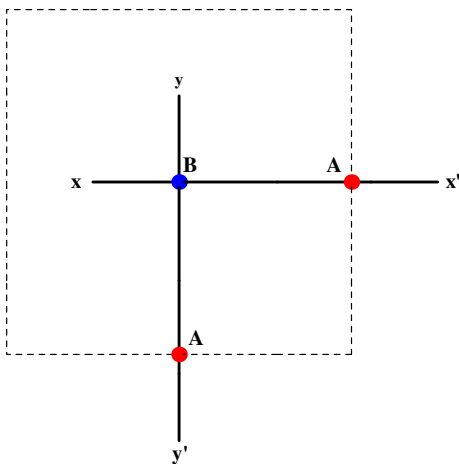
initial tensor is given by Eq. (5.14) and the initial sums run over the integers instead of 0 and 1 for Z_2.

3. $D = 4$

The basic cell of tensors in $D = 4$ is illustrated in Fig. 10. There are four A tensors with six legs and six B tensors in one basic cell of the hypercube. There are three legs pointing in each of the directions. Following the asymmetric procedure, we can combine each of these three legs into a single index, build a rank-8 tensor and block as in the spin model case. It seems possible to follow the symmetric procedure and build a modified B tensor with four additional legs in the directions orthogonal to the plaquette, a modified C tensor with two additional legs pointing in the direction orthogonal to the cubes and a new tensor with eight legs located at the center of the hypercubes. Blocking in one direction is then performed by contracting two similar tensors with a tensor associated to the object with one less dimension in between.

C. Abelian factorization

All the initial A tensors calculated have the factorization property shown by Abelian spin models and discussed in Sec. II C. The same reasoning can be used for Abelian gauge theories. Assuming the character expansion for the single plaquette weight

$$e^{-\beta S_p} = \sum_r F_r(\beta)\chi^r(U), \qquad (5.18)$$

we can rewrite $\chi^r(U)$ as the product of the four characters for each of the four links $\chi^r(U_1)\chi^r(U_2)\chi^r(U_3)\chi^r(U_4)$, factorize the contributions associated with each link and then integrate over the link variables. The initial tensor reads
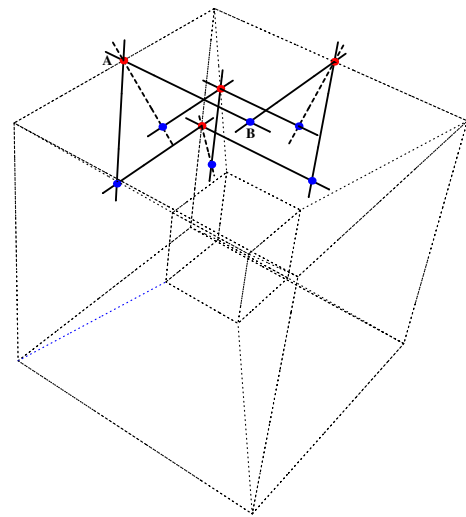

FIG. 9 (color online). T_4 tensor which contains two A tensors and one B tensor.

FIG. 10 (color online).   Four-dimensional U(1) tensor in a hypercube. The A tensors are on the links, and the B tensors are at the center of the plaquettes.

$$A_{ijkl} = (F_i(\beta)F_j(\beta)F_k(\beta)F_l(\beta))^{\frac{1}{4}}\delta^g_{i+k,j+l}. \quad (5.19)$$

The main difference with the spin model is the appearance of the fourth root instead of the square root.

D. TRG formulation of 3D SU(2) gauge

Using the conventions from Ref. [29], and following a procedure described in Ref. [28], we start with a partition function for the 3D SU(2) gauge model,

$$Z = \prod_{ni} \int dU(ni) \prod_{nij} \exp\left\{\frac{\beta}{4} \text{Re}[\text{tr}[U(nij)]]\right\}, \quad (5.20)$$

with $U(nij)$ the product of group elements around a plaquette and ni the links of the lattice. Since the action only depends on the trace of matrix representations of SU(2), we can rewrite the action as a character expansion,

$$e^{-\beta S_p} = \sum_r F_r(\beta)\chi^r(U(nij)). \quad (5.21)$$

Again, see Ref. [30] for explicit forms. Then the partition function can be written as

$$Z = \prod_{ni} \int dU(ni) \prod_{nij} \sum_{r(nij)} F_{r(nij)}(\beta)\chi^{r(nij)}(U(nij)). \quad (5.22)$$

We can rewrite the characters of the product of group elements as the trace over the product of the matrix representations of group elements,

$$\chi^r(U_1U_2U_3U_4) = D^r_{ij}(U_1)D^r_{jk}(U_2)D^r_{kl}(U_3)D^r_{li}(U_4), \quad (5.23)$$

and with these ''Wigner D-functions'' we can perform the product over plaquettes of the lattice, and gather together the four D-functions which all share the same link variable. In three dimensions there are four plaquettes for each link and so there are four D-functions per link variable. This situation is identical to the 2D principal chiral model, since in 2D there are four links impinging on a site. The only minor difference in this case is that each plaquette is bordered by four links, as opposed to 2D where each link is bordered by two sites. The consequence of this is that the character coefficients are shared more between the links. We can use the same analysis as before for the integration and write down the partition function directly,

$$Z = \prod_{ni} \sum_{r,m,n} \int dU(ni)(F_{r_1}(\beta)F_{r_2}(\beta)F_{r_3}(\beta)F_{r_4}(\beta))^{\frac{1}{4}}$$
$$\times D^{r_1}_{m_1n_1}(U)D^{r_2}_{m_2n_2}(U)D^{r_3}_{m_3n_3}(U)D^{r_4}_{m_4n_4}(U) \quad (5.24)$$

$$= \prod_{ni} \sum_{r's} \sum_{m's,n's} (F_{r_1}(\beta)F_{r_2}(\beta)F_{r_3}(\beta)F_{r_4}(\beta))^{\frac{1}{4}}$$
$$\times \sum_{r',m',n'} d^{-1}_{r'}(-1)^{m'-n'} C^{r_1r_2r'}_{m_1m_2m'} C^{r_1r_2r'}_{n_1n_2n'} C^{r_3r_4r'}_{m_3m_4-m'} C^{r_3r_4r'}_{n_3n_4-n'}.$$

$$(5.25)$$

This gives us an A tensor of the form

$$A_{(r_1,m_1,n_1)(r_2,m_2,n_2)(r_3,m_3,n_3)(r_4,m_4,n_4)}$$
$$= (F_{r_1}(\beta)F_{r_2}(\beta)F_{r_3}(\beta)F_{r_4}(\beta))^{\frac{1}{4}}$$
$$\times \sum_{r',m',n'} d^{-1}_{r'}(-1)^{m'-n'} C^{r_1r_2r'}_{m_1m_2m'} C^{r_1r_2r'}_{n_1n_2n'} C^{r_3r_4r'}_{m_3m_4-m'} C^{r_3r_4r'}_{n_3n_4-n'}.$$

$$(5.26)$$

Now, the model demands that there be a single representation assigned to each plaquette; we see this during the character decomposition when a single plaquette takes on a single representation. However, while the D-function matrix indices are traced out, they demand to be traced out in a specific way, namely, to close around a plaquette. This behavior must be obeyed during the tensor reconstruction. This can be handled by separate tensors. One tensor—the original B tensor from Abelian models—can remain the same and is responsible for keeping the representations the same on a plaquette. Next we need adjacent (link) matrix indices, ms and ns, to be contracted. This is achieved with four Kronecker deltas, each contracting a pair of adjacent indices. The initial value of the tensor is given by

$$\tilde{B}_{(r,i,i')(r',j,j')(r'',k,k')(r''',l,l')} = B_{rr'r''r'''}\delta_{i,j}\delta_{j',k}\delta_{k',l}\delta_{l',i'}. \quad (5.27)$$

We can now proceed as in the 3D Abelian case to write the partition function and perform blockings using A and B tensors. The only difference is that the single indices of the Abelian formulas need to be replaced by three indices.

VI. CONCLUSIONS

In conclusion, we have shown that the partition functions of the 2D $O(2)$ and $O(3)$ sigma models, the 2D SU(2) principal chiral model and the 3D gauge theories with the groups Z_2, U(1) and SU(2) can be written in terms of local tensors and that exact blocking formulas can be written for these models. The basic ingredient is the character expansion. This is available for any finite or compact group. For Abelian models, the factorization properties discussed for spin and gauge models should guarantee that the procedures described here can be extended for any compact Abelian group in any dimension. In the non-Abelian case, it is in addition necessary to re-express products of representations in terms of irreducible representations. Again, this is possible for $O(N)$ and SU(N) with larger N than the ones considered here.

Models with fermions have not been discussed. In Ref. [23], it was shown that standard SVD methods can be used to factorize exponentials of quadratic forms in Grassman numbers and then perform the local integrations. This yields tensors similar to the ones constructed for spin models. Combining this result with the ones for the gauge models presented here is an important goal. A first objective could be the 2D Schwinger model. It is interesting that the quantum treatment of this model in $1 + 1$ dimensions in terms of tensor network states has been proposed recently [31]. It would be very interesting to understand the standard quantum-classical correspondence in a unified tensor language.

Exact blocking formulas may be useful for analytical problems, such as the understanding of confinement for 4D SU(2) and the lack thereof for U(1) gauge theories [32]. There are many possible numerical applications of the blocking formulas presented here. The finite size of computer memory requires truncations and projections which are model dependent. The 2D $O(2)$ model can be treated as the 2D Ising model. The good agreement between TRG and Monte Carlo calculations of thermodynamics quantities and critical properties will be reported elsewhere [33]. Numerical implementations for 3D Ising gauge theory are under way.

In general, the computational demands are very different from those present in Monte Carlo simulations. For a given set of states in the external legs (which communicate with the other blocks as in Fig. 3), the sums over the internal states amount to solving a small lattice problem and takes little CPU time. However, the large number of combinations of external states requires many repetitions and can stretch the limit of computer memory.

The numerical treatment seems insensitive to sign problems. In conventional Monte Carlo simulations, calculations with complex β can only be achieved by a reweighing of configurations obtained with real β where the sign problem is absent. This only allows small imaginary values of β. In contrast, the TRG method allows larger imaginary parts. This allowed us to calculate the zeros of the partition function for the 2D Ising and $O(2)$ models in good agreement with existing results [33]. We plan to use the TRG method to study the Z_N clock and $O(2)$ models with a complex chemical potential and compare the results with those obtained with dual formulations [34] and world-line methods [35,36].

In summary, the TRG method is a very promising method to deal with models studied by lattice gauge theorists. We hope that the recent numerical success will extend to other models and that ultimately it will be useful to approach important problems, such as the phase diagram of QCD and the boundary of the conformal window for various multiflavor gauge theories.

ACKNOWLEDGMENTS

[1] T. DeGrand, Phil. Trans. R. Soc. A **369**, 2701 (2011).
[2] Z. Fodor, K. Holland, J. Kuti, D. Nogradi, and C. Schroeder, Phys. Lett. B **703**, 348 (2011).
[3] A. Deuzeman, M. P. Lombardo, T. Nunes Da Silva, and E. Pallante, Phys. Lett. B **720**, 358 (2013).
[4] A. Cheng, A. Hasenfratz, and D. Schaich, Phys. Rev. D **85**, 094509 (2012).
[5] Y. Aoki, T. Aoyama, M. Kurachi, T. Maskawa, K. Nagai, H. Ohki, A. Shibata, K. Yamawaki, and T. Yamazaki, Phys. Rev. D **86**, 054506 (2012).
[6] T. Appelquist, G. Fleming, M. Lin, E. Neil, and D. Schaich, Phys. Rev. D **84**, 054501 (2011).
[7] K. Wilson and J. Kogut, Phys. Rep. **12**, 75 (1974).
[8] J. L. Cardy, *Scaling and Renormalization in Statistical Physics* (Cambridge University Press, Cambridge, UK, 1996).
[9] Y. Meurice, R. Perry, and S.-W. Tsai, Phil. Trans. R. Soc. A **369**, 2602 (2011).
[10] A. A. Migdal, Sov. Phys. JETP **42**, 743 (1975).
[11] L. Kadanoff, Ann. Phys. (N.Y.) **100**, 359 (1976).
[12] K. Wilson, Phys. Rev. B **4**, 3184 (1971).
[13] G. Baker, Phys. Rev. B **5**, 2622 (1972).
[14] Y. Meurice, J. Phys. A **40**, R39 (2007).
[15] T. Nishino, J. Phys. Soc. Jpn. **64**, 3598 (1995).
[16] M. Levin and C. P. Nave, Phys. Rev. Lett. **99**, 120601 (2007).

[17] Z.-C. Gu, M. Levin, B. Swingle, and X.-G. Wen, Phys. Rev. B **79**, 085118 (2009).

[18] U. Schollwoeck, Phil. Trans. R. Soc. A **369**, 2643 (2011).

[19] Z. Y. Xie, H. C. Jiang, Q. N. Chen, Z. Y. Weng, and T. Xiang, Phys. Rev. Lett. **103**, 160601 (2009).

[20] H. H. Zhao, Z. Y. Xie, Q. N. Chen, Z. C. Wei, J. W. Cai, and T. Xiang, Phys. Rev. B **81**, 174411 (2010).

[21] Z. Y. Xie, J. Chen, M. P. Qin, J. W. Zhu, L. P. Yang, and T. Xiang, Phys. Rev. B **86**, 045139 (2012).

[22] R. Savit, Rev. Mod. Phys. **52**, 453 (1980).

[23] Y. Meurice, Phys. Rev. B **87**, 064422 (2013).

[24] E. Efrati, Z. Wang, A. Kolan, and L. Kadanoff, arXiv:1301.6323.

[25] E. Fradkin and L. Susskind, Phys. Rev. D **17**, 2637 (1978).

[26] P. Orland, Nucl. Phys. **B163**, 275 (1980).

[27] G. G. Batrouni and M. B. Halpern, Phys. Rev. D **30**, 1775 (1984).

[28] R. Anishetty, S. Cheluvaraja, H. Sharatchandra, and M. Mathur, Phys. Lett. B **314**, 387 (1993).

[29] D. Varshalovich, A. Moskalev, and V. Khersonsky, *Quantum Theory of Angular Momentum* (World Scientific, Singapore, 1988).

[30] J.-M. Drouffe and J.-B. Zuber, Phys. Rep. **102**, 1 (1983).

[31] M. Bañuls, K. Cichy, K. Jansen, and J. Cirac, arXiv:1305.3765.

[32] E. T. Tomboulis, Mod. Phys. Lett. A **24**, 2717 (2009).

[33] A. Denbleyker, Y. Liu, Y. Meurice, M. Qin, J. Unmuth-Yockey, T. Xiang, Z. Xie, J. Yu, and H. Zou (to be published).

[34] P. N. Meisinger and M. C. Ogilvie, arXiv:1306.1495.

[35] D. Banerjee and S. Chandrasekharan, Phys. Rev. D **81**, 125007 (2010).

[36] N. Prokof'ev and B. Svistunov, Phys. Rev. Lett. **87**, 160601 (2001).