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Spin interactions in mesons in strong magnetic field

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Spin interactions in a relativistic quark-antiquark system in a magnetic field is considered in the framework of the relativistic Hamiltonian, derived from the QCD path integral. The formalism allows us to separate spin-dependent terms from the basic spin-independent interaction contained in the Wilson loop, and produce confining and gluon exchange interaction. As a result, one obtains a relativistic spin-spin interaction V_{ss} generalizing its nonrelativistic analog. It is shown that in a large magnetic field eB, V_{ss} modifies and produces hyperfine shifts which grow linearly with eB and preclude the use of perturbation theory. We also show that tensor forces for $eB \neq 0$ are active in all meson states but do not grow with eB.

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I. INTRODUCTION

The spin-dependent (SD) terms in the interaction of two fermions in QED have a long history [1–4], where spin terms were deduced based on Pauli or Dirac form of Hamiltonians. In particular the hyperfine (hf) interaction term was introduced by Fermi in [3] in the nonrelativistic form. Later on, the QED perturbation theory was used to consider hf interaction in higher orders [5] (see [6] for a review), and also in the presence of a magnetic field (m.f.) [7].

An important tool for the higher order relativistic treatment of bound states in QED was the Bethe-Salpeter equation (BSE) [8], or its instantaneous Salpeter form [9].

It is the purpose of the present paper to consider spin-dependent effects in the dynamics of the quark-antiquark ($q\bar{q}$) mesons in strong m.f. The physical interest of such systems is due to high magnetic fields, which are expected in neutron stars, early universe, and heavy-ion collisions; see [10] for discussion and references.

In such QCD systems, as mesons or baryons, the main part of dynamics is of nonperturbative (np) origin, e.g., the confinement interaction $V_{\rm conf}$, which cannot be described by perturbative propagator particle exchanges. In addition, the relative time problem existing in BSE is more fundamental in QCD, where the vacuum field correlations with period λ, yielding confinement (and all strong interactions, including spin-dependent ones), are shorter in time than physical particle exchanges, so that each quark in a meson propagates in the already time-averaged vacuum [11]. Therefore, to establish the fine and hf structure of strong interactions, one must use, instead of BSE, another relativistic formalism, namely, the relativistic path-integral method, based on the so-called Fock-Feynman-Schwinger representation (FFSR) [12,13]. Recently, a new form of this method was derived in [14], where a new integral representation of the $q\bar{q}$ Green's function was given.

Basically, the main part of the strong dynamics (non-spin-dependent) is given by the Wilson loop average and the methods, based on FFSR, give all meson, baryon, glueball, and hybrid properties in good agreement with experimental data; see [15,16] for reviews.

Of special importance for us here are spin-dependent forces in QCD (which we shall test applying high m.f.). The latter have been derived for large quark masses in [17], and in the framework of our method for any masses in [11,18–22].

The most important development in the study of the SD interactions consists of three advancements:

(1) Definition of perturbative and np, SD contributions and expressions for them in terms of the standard field correlators. This was done for large masses in [11,18].
(2) Derivation of SD terms in the relativistic systems for low and zero mass quarks, done in [15,19,20].
(3) Definition of SD terms for nonzero temperature and in the deconfined QCD phase, which was done in [21].
(4) Reexamining of the SD forces and suppression of spin-orbit forces in hadrons by the string motion, [22].

In the present paper we perform a calculation of SD forces in the relativistic $q\bar{q}$ system in the presence of m.f. eB, which strongly modifies these forces, and in particular yields a linear growth in eB for the hf term. We also show that m.f. creates a deformation of the meson shape and thus stipulates tensor forces in the originally spherical meson.

A possible contradiction of these effects with the positivity property of the Green's function is formulated and possible outcomes are discussed.

In the same formalism we calculate the self-energy correction to the meson mass in strong m.f.

The paper is organized as follows: in Sec. II we define spectral properties of the Green's function in m.f. In Sec. III we derive the $q\bar{q}$ Green's function and relativistic Hamiltonians in m.f. from FFSR. In Sec. IV we concentrate on SD forces in m.f. and write down explicit expressions for them. Section V is devoted to tensor forces in m.f. The last section contains a discussion of the results and an outlook.

II. ABSENCE OF PAIR CREATION IN COLOR EUCLIDEAN AND MAGNETIC FIELDS

Consider a quark-antiquark system in the external magnetic field interacting with color vacuum Euclidean and perturbative fields, so that the covariant derivative is

$$D_\mu \equiv D_\mu(A_\mu^{(e)}, A_\mu^a t^a) = \partial_\mu - ieA_\mu^{(e)}(x) - igA_\mu^a t^a, \qquad (1)$$

where $A_\mu^{(e)}$ can be decomposed into the magnetic ("Euclidean") part $A_\mu^{(B)}$ and the electric part, e.g.,

$$A_\mu^{(e)} = A_\mu^{(B)} + A_\mu^{(e)}(\text{pert}), \quad \mathbf{A}^{(B)} = \frac{1}{2}(\mathbf{x}\times\mathbf{B}), \qquad A_4^{(B)} = 0, \qquad (2)$$

for the constant magnetic field $\mathbf{B}$ along the z axis. Then the partition function averaged over the nonperturbative vacuum can be written as

$$\langle Z\rangle_A = \left\langle \int D\bar\psi D\psi \exp(-i\bar\psi(m+\hat D)\psi)\right\rangle_A, \qquad (3)$$

where the averaging over gluonic fields is implied,

$$\langle K\rangle_A = \int K \exp\left(-\frac{1}{4}\int (F^a_{\mu\nu}F^a_{\mu\nu})d^4x\right)DA, \qquad (4)$$

and we disregard for simplicity gauge fixing and ghost terms.

Integration over $D\bar\psi D\psi$ in (3) yields the standard answer (where the proper renormalization is implied):

$$\langle Z\rangle_A = \left\langle \exp\frac{1}{2}\operatorname{tr}\ln(m^2-\hat D^2)\right\rangle_A = \left\langle \exp\left(\frac{1}{2}\operatorname{tr}\int_0^\infty \frac{ds}{s}e^{-s(m^2-\hat D^2)}\right)\right\rangle_A. \qquad (5)$$

Now the question of the stability of the vacuum in the given external fields can be associated with the non-negativity of the operator $(m^2-\hat D^2)$, since otherwise negative eigenvalues of this operator would provide imaginary part in the exponent of (5), which implies finite probability of pair creation, as it is clearly seen in the Schwinger expression for the pair creation in the constant electric (non-Euclidean) field [23].

In what follows, we show that in purely Euclidean (color-electric or color-magnetic) fields and in the magnetic field (external and perturbative) the operator $(m^2-\hat D^2)$ is non-negative and hence pair creation is absent and vacuum stability is ensured.

To this end, consider the Euclidean operator

$$iD_\mu\gamma_\mu = \gamma_\mu(i\partial_\mu + eA_\mu^{(e)}(x) + gA_\mu^a t^a), \qquad (6)$$

with Euclidean and Hermitian γ matrices, $\gamma_4\equiv\beta$; $\gamma_i = -i\beta\alpha_i$; $\gamma_i^+ = \gamma_i$, and with Hermitian $A_i^{(e)}$ and $A_4 \equiv A_4^a t^a$, $A_4^+ = A_4$, while $A_4^{(e)}\equiv 0$, so that

$$(iD_\mu\gamma_\mu)^+ = iD_\mu\gamma_\mu, \qquad (7)$$

and for the eigenfunctions u_n and eigenvalues λ_n we have

$$iD_\mu\gamma_\mu u_n = \lambda_n u_n, \qquad \lambda_n \text{ real}. \qquad (8)$$

Hence,

$$-\hat D^2 u_n = (iD_\mu\gamma_\mu)^2 u_n = \lambda_n^2 u_n, \qquad \lambda_n^2 \ge 0, \qquad (9)$$

and $\|m^2-\hat D^2\| \ge m^2$, which implies vacuum stability and no pair creation for any $m\ge 0$.

In terms of the quark propagator $S=\frac{i}{\hat D+m}$, one has representations

$$iS = \sum_n \frac{u_n(x)u_n^+(y)}{\lambda_n - im}, \quad \left(\frac{1}{m^2-\hat D^2}\right)_{xy} = \sum \frac{u_n(x)u_n^+(y)}{m^2+\lambda_n^2}. \qquad (10)$$

Writing

$$u_n = \begin{pmatrix}\varphi_n\\ \chi_n\end{pmatrix},$$

one obtains the following equations:

$$-\tilde D^2 + \boldsymbol{\sigma}(g\mathbf{E} - g\mathbf{H} - e\mathbf{B})(\varphi_n - \chi_n) = \lambda_n^2(\varphi_n-\chi_n),$$
$$-\tilde D^2 - \boldsymbol{\sigma}(g\mathbf{E} + g\mathbf{H} + e\mathbf{B})(\varphi_n + \chi_n) = \lambda_n^2(\varphi_n+\chi_n), \qquad (11)$$

where $\tilde D^2 = (\partial_\mu - ieA_\mu^{(e)} - igA_\mu)^2$, and A_μ, $\mathbf{E}$, $\mathbf{H}$ correspond to the color fields, e.g., $A_\mu \equiv A_\mu^a t^a_{\alpha\beta}$, while $A_\mu^{(e)}$ is the electromagnetic field, and $\mathbf{B}$ is the external magnetic field. From the system (11) one can see that for Hermitian Euclidean fields $A_\mu^{(e)}$, A_μ, $\mathbf{E}$, $\mathbf{H}$, $\mathbf{B}$, the eigenvalues λ_n^2 are real and the Green's function $\frac{1}{m^2-\hat D^2}$ has only a positive set of eigenvalues.

Note that in the real electric field $A_0^{(e)}$, $A_4^{(e)} = iA_0^{(e)}$ and the property of non-negativity of $(-\hat D^2)$ is violated, implying possible quark pair creation, while in the absence of $A_0^{(e)}$, but with a real vacuum color field $A_4^+ = A_4$ and hence, a real color-electric vacuum field $E_i^{\rm vac}\sim\partial_i A_4$, the vacuum is stable with a known vacuum condensate $((E_i^{\rm vac})^2 + (H_i^{\rm vac})^2)$. However, the quasizero modes with small λ_n can accumulate, implying chiral symmetry breaking, signalled by the Banks-Casher formula [24]; see [25] for a review and discussion of chiral symmetry breaking from this point of view. As we shall see, perturbative color-magnetic interactions in the lowest order, in the external magnetic field can violate the positivity condition (9), signalling the divergence of the perturbative series.

III. RELATIVISTIC $q\bar{q}$ GREEN'S FUNCTION IN A MAGNETIC FIELD IN THE PATH-INTEGRAL FORM

The quark Green's function in a magnetic field can be written as

$$S_q(x,y) = (m+\hat{D})^{-1}_{xy} = (m-\hat{D})_x(m^2-\hat{D}^2)^{-1}_{xy}, \quad (12)$$

where

$$m^2-\hat{D}^2 = m^2 - D^2_\mu - g\sigma_{\mu\nu}F_{\mu\nu} - e\sigma_{\mu\nu}F^{(e)}_{\mu\nu}, \quad (13)$$

and D_μ is given in (6), while

$$\sigma_{\mu\nu}F_{\mu\nu} = \begin{pmatrix} \boldsymbol{\sigma}\mathbf{H} & \boldsymbol{\sigma}\mathbf{E} \\ \boldsymbol{\sigma}\mathbf{E} & \boldsymbol{\sigma}\mathbf{H} \end{pmatrix}, \qquad \sigma_{\mu\nu}F^{(e)}_{\mu\nu} = \begin{pmatrix} \boldsymbol{\sigma}\mathbf{B} & 0 \\ 0 & \boldsymbol{\sigma}\mathbf{B} \end{pmatrix}. \quad (14)$$

One can use for S_q the path-integral form [12,14]

$$S_q(x,y) = (m-\hat{D})_x \int_0^\infty ds D^4 z \Phi^{(F)}_z(x,y)e^{-K}, \quad (15)$$

where

$$\Phi^{(F)}_z(x,y) = P_A \exp\Big(ie\int_y^x A^{(e)}_i dz_i + ig\int_y^x A_\mu dz_\mu + e\sigma_{\mu\nu}\int_0^s F^{(e)}_{\mu\nu}d\tau + g\sigma_{\mu\nu}\int_0^s F_{\mu\nu}d\tau\Big), \quad (16)$$

$$K = \int_0^s \Big[m^2 + \frac{1}{4}\Big(\frac{dz_\mu}{d\tau}\Big)^2\Big]d\tau. \quad (17)$$

Then the $q\bar{q}$ Green's function can be written as

$$G_{q_1\bar{q}_2}(x,y) = \int_0^\infty ds_1 \int_0^\infty ds_2 (D^4z^{(1)})_{xy}(D^4z^{(2)})_{xy} \times \langle \hat{T} W_\sigma(A)\rangle_A W_\sigma(A^{(e)})\rangle, \quad (18)$$

where

$$W_\sigma(A) = P\exp\Big(ig\int_C A_\mu(z)dz_\mu + g\int_0^{s_1}\sigma_{\mu\nu}F_{\mu\nu}(\tau_1)d\tau_1 - g\int_0^{s_2}\sigma_{\mu\nu}F_{\mu\nu}(\tau_2)d\tau_2\Big), \quad (19)$$

$$W_\sigma(A^{(e)}) = \exp\Big(ie_1\int_y^x A^{(e)}_\mu dz^{(1)}_\mu - ie_2\int_y^x A^{(e)}_\mu dz^{(2)}_\mu + e_1\int_0^{s_1}d\tau_1(\sigma_{\mu\nu}F^{(e)}_{\mu\nu}) - e_2\int_0^{s_2}d\tau_2(\sigma_{\mu\nu}F^{(e)}_{\mu\nu})\Big), \quad (20)$$

$$\hat{T} = \frac{1}{4}\,\mathrm{tr}(\Gamma_1(m_1-\hat{D}_1)\Gamma_2(m_2-\hat{D}_2))\exp(-K_1-K_2), \quad (21)$$

and $\Gamma_1=\gamma_\mu$, $\Gamma_2=\gamma_\nu$ for vector currents, while $\Gamma_1=\Gamma_2=\gamma_5$ for pseudoscalars, and the symbol tr implies summation over color and Dirac indices and refers to all terms.

At this point we introduce new variables ω_i in the path integral (15), defined via the connection between the proper time τ_i and the real Euclidean time $t^E_i = z_4(\tau_i)$ (see details in Appendix 1 of [14]), and integrating over time fluctuations using

$$s_i = \frac{T}{2\omega_i}, \qquad ds_i = -\frac{Td\omega_i}{2\omega_i^2}; \qquad d\tau_i = \frac{dt^E_i}{2\omega_i},$$

$$\int_0^\infty ds_i(D^4z^{(i)})\Phi_z(x,y)e^{-K} = T\int_0^\infty \frac{d\omega_i}{2\omega_i^2}(D^3z^{(i)})_{\mathbf{xy}}e^{-K(\omega_i)}\langle\Phi_z(x,y)\rangle_{\Delta z_4}, \quad (22)$$

where $\langle\rangle_{\Delta z_4}$ means the averaging over time fluctuations, which can be written in terms of the averaged Wilson line [14]

$$\langle\Phi_z(x,y)\rangle_{\Delta z_4} = \sqrt{\frac{\omega_i}{2\pi T}}\Phi_z(x,y), \quad (23)$$

and $K(\omega)$ is

$$K(\omega) = \int_0^T dt_E\Big(\frac{\omega}{2} + \frac{m^2}{2\omega} + \frac{\omega}{2}\Big(\frac{d\mathbf{z}}{dt_E}\Big)^2\Big), \qquad T = |x_4 - y_4|. \quad (24)$$

In this way the path integral in $ds_iD^4z^{(i)}$ is replaced by $d\omega_i(D^3z^{(i)})$, and the $q\bar{q}$ Green's function (18) can be written as

$$G_{q_1\bar{q}_2}(x,y) = \frac{T}{2\pi}\int_0^\infty \frac{d\omega_1}{\omega_1^{3/2}}\int_0^\infty\frac{d\omega_2}{\omega_2^{3/2}}(D^3z^{(1)}D^3z^{(2)})_{\mathbf{xy}} \times \overline{\langle \hat{T}W_\sigma(A)W_\sigma(A^{(e)})\rangle}, \quad (25)$$

where

$$\overline{\langle\hat{T}W\rangle} = 4\,\mathrm{tr}Y\overline{\langle WW\rangle}\exp(-K(\omega_1)-K(\omega_2)), \quad (26)$$

$\bar{W}$ is the averaged over time fluctuations Wilson loop (see [14]) and $Y = \frac{1}{4}\Gamma_1(m_1 - i\hat{P}_1)\Gamma_2(m_2 - i\hat{p}_2)$. Note that the operator ordering is not taken into account in (26) and we shall take care of it below computing SD terms.

Using cluster expansion and the non-Abelian Stokes theorem [26] one can rewrite $\overline{W_\sigma}$ as

$$\langle \mathrm{Tr}W(C)\rangle = \Big\langle \mathrm{Tr}\exp ig\int d\pi_{\mu\nu}(z)F_{\mu\nu}(z)\Big\rangle = \exp\sum_{n=1}^\infty \frac{(ig)^n}{n!}\int d\pi(1)\ldots \times\int d\pi(n)\langle\langle F(1)\ldots F(n)\rangle\rangle, \quad (27)$$

where $d\pi_{\mu\nu} \equiv ds_{\mu\nu} + \sigma^{(1)}_{\mu\nu} d\tau_1 - \sigma^{(2)}_{\mu\nu} d\tau_2$, and $ds_{\mu\nu}$ is an area element of the minimal surface, which can be constructed using straight lines, connecting the points $z^{(1)}_\mu(t)$ and $z^{(2)}_\nu(t)$ on the paths of q_1 and $\bar{q}_2$ at the same time t [11,27]. Note that $z^{(1)}_4(t) = z^{(2)}_4(t) = t$. Then the spin-independent part of the exponent reduces to the confinement term $V_{\rm conf}(r)$ plus the color Coulomb potential $V_{\rm Coul}$, while the spin-dependent part $V_{\rm SD}$ depends also on proper time variables τ_1 τ_2, (see [11,18] for derivation and discussion). For the case of zero quark orbital momenta with the minimal surface, discussed above, one obtains a simple answer for $\langle W_\sigma(A)\rangle_A$, which we shall derive below.

The average $\langle\langle\ldots\rangle\rangle$ stands for connected correlators, for example, for the bilocal correlator, $\langle\langle F(1)F(2)\rangle\rangle = \langle F(1)F(2)\rangle - \langle F(1)\rangle\langle F(2)\rangle$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ is the vacuum field strength. Obviously, due to the $O(4)$ rotational invariance and color neutrality of the vacuum, $\langle\langle F\rangle\rangle = \langle F\rangle = 0$.

In the Gaussian approximation for the vacuum, when only the lowest, bilocal correlator is retained, one has, with the accuracy of a few percent (see Refs. [20,22] for the discussion),

$$\langle {\rm Tr} W(C)\rangle \propto \exp\left[-\frac{1}{2}\int_S d\pi_{\mu\nu}(x) d\pi_{\lambda\rho}(x') D_{\mu\nu\lambda\rho}(x-x')\right], \quad (28)$$

where

$$D_{\mu\nu\lambda\rho}(x-x') \equiv \frac{g^2}{N_c}\langle\langle {\rm Tr} F_{\mu\nu}(x)\Phi(x,x')F_{\lambda\rho}(x')\Phi(x',x)\rangle\rangle. \quad (29)$$

This bilocal correlator of gluonic fields can be expressed through only two gauge-invariant scalar functions $D(u)$ and $D_1(u)$ as [28]

$$D_{\mu\nu\lambda\rho}(u) = (\delta_{\mu\lambda}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\lambda})D(u) + \frac{1}{2}\left[\frac{\partial}{\partial u_\mu}(u_\lambda\delta_{\nu\rho} - u_\rho\delta_{\lambda\nu}) + \begin{pmatrix}\mu\leftrightarrow\nu\\ \lambda\leftrightarrow\rho\end{pmatrix}\right]D_1(u). \quad (30)$$

The correlator $D(u) = D(u_0, |\mathbf{u}|)$ contains a nonperturbative part and it is responsible for confinement: the QCD string formation at large interquark separations. Indeed, $\int ds_{i4}(u)\int ds_{i4}(v) D(u-v) = S\int d^2(u-v)D(u-v)$, where S is the total area between the averaged quark trajectories. The fundamental string tension can be calculated from the area law asymptotics of (28) for a large area loop and is expressed as a double integral:

$$\sigma = \iint d^2(u-v)D(u-v) = 2\int_0^\infty d\nu\int_0^\infty d\lambda D(\nu,\lambda). \quad (31)$$

Using (19) with $\sigma_{\mu\nu}F_{\mu\nu} \equiv 0$, one obtains spin-independent terms in the $q\bar{q}$ interaction [28].

$$V_0(r) = V_{\rm conf}(r) + V_{\rm OGE}(r), \quad (32)$$

$$V_{\rm conf}(r) = 2r\int_0^r d\lambda\int_0^\infty d\nu D(\lambda,\nu) \to \sigma r, \qquad (r\to\infty), \quad (33)$$

$$V_{\rm OGE} = \int_0^r \lambda d\lambda \int_0^\infty d\nu D_1^{\rm pert}(\lambda,\nu) = -\frac{4}{3}\frac{\alpha_s}{r}, \quad (34)$$

where we keep only the perturbative part of D_1 and to the lowest order $D_1^{\rm pert}(\lambda,\nu) = \frac{16\alpha_s}{3\pi(\lambda^2+\nu^2)^2}$. We now turn to the spin-dependent terms of the $q\bar{q}$ interaction, $V_{\rm SD}$, and we shall be interested only in the zero orbital moment states for simplicity; hence, only the spin-spin interaction term V_{ss} will be treated below.

The spin-dependent terms in the interquark interaction are generated by the combination $\sigma_{\mu\nu}F_{\mu\nu}$ present in Eq. (14) and therefore one needs correlators of the color-electric and color-magnetic fields, as well as mixed terms, separately. They immediately follow from the general expression (30) and read [29]

$$\frac{g^2}{N_c}\langle\langle {\rm Tr} E_i(x)\Phi E_j(y)\Phi^\dagger\rangle\rangle = \delta_{ij}\left(D^E(u) + D_1^E(u) + u_4^2\frac{\partial D_1^E}{\partial u^2}\right) + u_i u_j\frac{\partial D_1^E}{\partial u^2},$$

$$\frac{g^2}{N_c}\langle\langle {\rm Tr} H_i(x)\Phi H_j(y)\Phi^\dagger\rangle\rangle = \delta_{ij}\left(D^H(u) + D_1^H(u) + \mathbf{u}^2\frac{\partial D_1^H}{\partial \mathbf{u}^2}\right) - u_i u_j\frac{\partial D_1^H}{\partial u^2},$$

$$\frac{g^2}{N_c}\langle\langle {\rm Tr} H_i(x)\Phi E_j(y)\Phi^\dagger\rangle\rangle = \varepsilon_{ijk}u_4 u_k\frac{\partial D_1^{EH}}{\partial u^2}, \quad (35)$$

where $u_\mu = x_\mu - y_\mu$, $u^2 = u_\mu u_\mu$. We keep here the superscripts E and H in the correlators D and D_1 in order to distinguish, in principle, the electric and magnetic parts of the correlators and thus to be able to consider a nonzero temperature T and to distinguish Euclidean and Minkowskian contributions. Indeed, while $D^E = D^H$ and $D_1^E = D_1^H$ at $T = 0$, at higher temperatures they behave differently. In particular, above the deconfinement temperature $T > T_c$, the electric correlator D^E disappears, whereas D_1^E and the magnetic correlators survive.

It is clear from (14), that in the nonrelativistic limit, when in $\int \sigma_{\mu\nu}F_{\mu\nu}d\tau$, $d\tau_i = \frac{dt_i}{2m_i}$, only the upper left corner of (14), i.e., $(\boldsymbol{\sigma}\mathbf{H})$, will contribute to V_{ss}. The corresponding derivation was done in [11,18] and gives

$$V_{ss}(r) = \frac{\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2}{12m_1m_2}V_4(r) + \frac{3(\boldsymbol{\sigma}_1\mathbf{r})(\boldsymbol{\sigma}_2\mathbf{r}) - \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 r^2}{12m_1m_2r^2}V_3(r), \quad (36)$$

where

$$V_3(r) = -\int_{-\infty}^{\infty} d\nu r^2 \frac{\partial D_1^{\rm pert}(r,\nu)}{\partial r^2} = \frac{4\alpha_s}{r^3}, \qquad (37)$$

$$V_4(r) = \int_{-\infty}^{\infty} d\nu\left(3D_1^{\rm pert}(r,\nu) + 2r^2\frac{\partial D_1^{\rm pert}(r,\nu)}{\partial r^2}\right) = \frac{32\pi\alpha_s}{3}\delta^{(3)}(\mathbf{r}), \qquad (38)$$

It is our purpose below to calculate V_{ss} in the relativistic $q\bar{q}$ system and in an arbitrarily large magnetic field B, and to this end we shall use below, first, the path-integral formalism [13,14], deriving the general structure of the $q\bar{q}$ Green's function, and in the next section we shall determine how relativistic V_{ss} expressions depend on B.

First, we need to find the Euclidean action $S^E_{q_1\bar{q}_2}$ in terms of ω_1, ω_2 and common time t^E of the $q\bar{q}$ system at $t_1^E = t_2^E = t^E$. To this end, we define the Euclidean Lagrangian $L^E_{q_1\bar{q}_2}$. We write $\frac{dz^{(i)}}{d\tau_i} = 2\omega_i\frac{dz_k^{(i)}}{dt^E} = 2\omega_i\dot{z}_k$, $k = 1, 2, 3$. Then all terms in the exponents in (19)–(21) can be represented as $\exp(-\int dt^E L^E_{q_1\bar{q}_2})$ and thus we arrive at the following action:

$$S^E_{q_1\bar{q}_2} = \int_0^{T_E} dt^E\Bigg[\frac{\omega_1+\omega_2}{2} + \sum_i\left(\frac{\omega_i}{2}(\dot{z}_k^{(i)})^2\right) - ie_iA_k^{(e)}\dot{z}_k^{(i)} + \frac{m_1^2}{2\omega_1} + \frac{m_2^2}{2\omega_2} + e_1\frac{\boldsymbol{\sigma}_1\mathbf{B}}{2\omega_1} + e_2\frac{\boldsymbol{\sigma}_2\mathbf{B}}{2\omega_2} + \sigma|\mathbf{z}^{(1)}-\mathbf{z}^{(2)}| - \frac{4}{3}\frac{\alpha_s}{|\mathbf{z}^{(1)}-\mathbf{z}^{(2)}|}\Bigg] + S_F^E, \qquad (39)$$

where S_F^E contains (σF) terms. Here $A_k^{(e)}$ is the kth component of the QED vector potential, σ is the QCD string tension, and the contribution of terms $(\sigma_1 F)$, $(\sigma_2 F)$ is separated in S_F^E. The next step is the transition to the Minkowski metric and the construction of the Hamiltonian. This is easy, since confinement is already expressed in terms of the string tension. We have $\exp(-\int L^E dt_E) \to \exp(i\int L^M dt_M)$, $t_E \to it_M$, and

$$H_{q_1\bar{q}_2} = \sum_i \dot{z}_k^{(i)}p_k^{(i)} - L_M, \qquad p_k^{(i)} = \frac{\partial L^M}{\partial\dot{z}_k^{(i)}} = \omega_i\dot{z}_k^{(i)} + e_iA_k^{(e)}. \qquad (40)$$

As a result one obtains (back in the Euclidean time $T = |x_4 - y_4|$)

$$G(x,y) = \frac{T}{2\pi}\int_0^\infty\frac{d\omega_1}{\omega_1^{3/2}}\int_0^\infty\frac{d\omega_2}{\omega_2^{3/2}}4\,{\rm tr}Y\langle\mathbf{x}|e^{-H_{q_1\bar{q}_2}T}|\mathbf{y}\rangle, \qquad (41)$$

$$H_{q,\bar{q}_2} = H_0 + H_\sigma + W, \qquad (42)$$

$$W = V_{\rm conf} + V_{\rm OGE} + \Delta M_{\rm SE} + \Delta M_{ss}, \qquad (43)$$

where

$$H_0 = \sum_{i=1}^2\frac{(\mathbf{p}^{(i)} - \frac{e_i}{2}(\mathbf{B}\times\mathbf{z}^{(i)}))^2 + m_i^2 + \omega_i^2}{2\omega_i}, \qquad (44)$$

$$H_\sigma = -\frac{e_1\boldsymbol{\sigma}_1\mathbf{B}}{2\omega_1} - \frac{e_2\boldsymbol{\sigma}_2\mathbf{B}}{2\omega_2}. \qquad (45)$$

Here the terms $\Delta M_{\rm SE}$ and ΔM_{ss} are produced by S_F^E, and we shall find them as a first order correction. But before that we must treat the ω_i dependence either in the path integral (25) or the Hamiltonian (42). In the path integral ω_i play the role of quark energy parameters, and one can use the spectral decomposition in (41) to rewrite it as

$$G(x,y) = \frac{T}{2\pi}\int_0^\infty\frac{d\omega_1}{\omega_1^{3/2}}\int_0^\infty\frac{d\omega_2}{\omega_2^{3/2}}\sum_{n=0}^\infty 4\,{\rm tr}Y\langle\mathbf{x}|n\rangle e^{-M_nT}\langle n|\mathbf{y}\rangle. \qquad (46)$$

At large T one can use the stationary point method, and one defines $\omega_i^{(0)}$ from the extremum values of $M_n(\omega_1,\omega_2)$, namely, for the Hamiltonian $\bar{H}$,

$$H_0 + H_\sigma + V_{\rm conf} + V_{\rm OGE} = \bar{H}; \quad \bar{H}\Psi = M_n^{(0)}\Psi, \qquad (47)$$

and $\omega_i^{(0)}$ is defined from the condition

$$\frac{\partial M_n^{(0)}(\omega_1,\omega_2)}{\partial\omega_i}\Bigg|_{\omega_i=\omega_i^{(0)}} = 0, \qquad i = 1, 2. \qquad (48)$$

To have an idea of the possible meson masses and the values of $\omega_i^{(0)}$, which we shall use below, it is instructive to consider, as in [11], the main part of the Hamiltonian, i.e.,

$$\tilde{H} = H_0 + H_\sigma + V_{\rm conf}, \qquad \tilde{H}\tilde{\psi} = \tilde{M}\tilde{\psi}, \qquad (49)$$

and replace $V_{\rm conf}$ by the quadratic term,

$$V_{\rm conf}(r) = \sigma r \to \tilde{V}_{\rm conf}(r) = \frac{\sigma}{2}\left(\frac{r^2}{\gamma} + \gamma\right), \qquad (50)$$

where γ is the variational parameter, yielding some 5% accuracy in the replacement (50). Then the resulting mass $\tilde{M}$ can be found explicitly as

$$\tilde{M}(\omega_1,\omega_2,\gamma) = \varepsilon_{n_\perp,n_z} + \frac{m_1^2 + \omega_1^2 - e_1\mathbf{B}\boldsymbol{\sigma}_z}{2\omega_1} + \frac{m_2^2 + \omega_2^2 - e_2\mathbf{B}\boldsymbol{\sigma}_z}{2\omega_2}, \qquad (51)$$

where

$$\varepsilon_{n_\perp,n_z} = \frac{1}{2\tilde{\omega}}\left[\sqrt{e^2B^2 + \frac{4\sigma\tilde{\omega}}{\gamma}}(2n_\perp + 1) + \sqrt{\frac{4\sigma\tilde{\omega}}{\gamma}}\left(n_z + \frac{1}{2}\right)\right] + \frac{\gamma\sigma}{2}. \qquad (52)$$

As a result, one can estimate the masses $\tilde{M}$ and $\omega_i^{(0)}$ at large m.f., since the basic pattern is defined by relative signs of eB terms in $\varepsilon_{n_\perp,n_z}$ and H_σ.

Indeed, for $eB \gg \sigma$ one can write $\tilde{M} \cong \sum_{i=1,2} \frac{m_i^2+\omega_i^2+|e_iB|-e_i\boldsymbol{\sigma}_i\mathbf{B}}{2\omega_i}$ and

$$\omega_i^{(0)} \approx \Omega_i \equiv \sqrt{m_i^2 + |e_iB| - e_i\boldsymbol{\sigma}_i\mathbf{B}} + O(\sqrt{\sigma}),\quad \tilde{M} \cong \Omega_1 + \Omega_2. \tag{53}$$

Thus, for the neutral meson with $e_2 = -e_1$, and σ_{1z}, $\sigma_{2z} = (++)$, ω_{++} is growing as $\sqrt{e_1 B}$, while for the $(+-)$ state ω_{+-} is tending to a constant. One can also find the wave function

$$\tilde{\psi}(\boldsymbol{\eta}) = \frac{1}{\sqrt{\pi^{3/2} r_\perp^2 r_0}} \exp\left(-\frac{\eta_\perp^2}{2r_\perp^2} - \frac{\eta_z^2}{2r_0^2}\right),\quad \boldsymbol{\eta} = \mathbf{z}^{(1)} - \mathbf{z}^{(2)}, \tag{54}$$

and

$$r_\perp^2 = \frac{2}{eB}\left(1 + \frac{4\sigma\tilde{\omega}}{\gamma_0 e^2 B^2}\right)^{-1/2}, \qquad r_0 = \left(\frac{\gamma}{\sigma\tilde{\omega}}\right)^{1/4}. \tag{55}$$

At large $eB \gg \sigma$, one has $r_\perp^2 \approx \frac{2}{eB}$, $r_0 \approx \text{const} \approx \frac{1}{\sqrt{\sigma}}$, and hence

$$|\tilde{\psi}(0)|^2 \approx \frac{eB \cdot \sqrt{\sigma}}{2\pi^{3/2}}, \qquad (eB \gg \sigma). \tag{56}$$

This is the focusing effect of m.f., which is most important in SD forces as well as in other processes [30].

It is important to stress that we have kept in $\bar{H}$ only those terms which are the main part of interaction, and therefore in $M_n^{(0)}$ and $\omega^{(0)}$ they are treated to all orders, i.e., exactly. However, the terms V_{ss} and $\Delta M_{\rm SE}$ are considered only as a perturbation, and therefore one should substitute there the values $\omega_i^{(0)}$ obtained from (47) and (48), where V_{ss}, $\Delta M_{\rm SE}$ do not enter. The Hamiltonians $\bar{H}$ are considered in [14], and below we shall derive both V_{ss} and $\Delta M_{\rm SE}$ in the relativistic $q\bar{q}$ system.

IV. THE QUARK-ANTIQUARK SPIN-DEPENDENT INTERACTION IN A STRONG MAGNETIC FIELD

The advantage of representation (18) and (25) lies in the fact that the only place where the Dirac γ matrices enter is the local term $(m - \hat{D})$, and it can be assembled in the factor Y with due care, while all the rest nontrivial spin dependence is contained in the (σF) and (σB) factors (14). We shall demonstrate below in this section that the correlators $(\sigma^{(i)}F)(\sigma^{(k)}F)$ with $i = k$ define $\Delta M_{\rm SE}$, while those with $i \neq k$, define V_{ss}.

Consider the Taylor expansion in powers of the color spin interaction $g\sigma_{\mu\nu}F_{\mu\nu} \equiv g(\sigma F)$, $m^2 - \hat{D}^2 = m^2 - \tilde{D}^2 - g(\sigma F)$, $\tilde{D}_\mu^2 = D_\mu^2 - e(\sigma F^{(e)})$,

$$\begin{aligned}\frac{1}{m^2 - \hat{D}^2} &= \frac{1}{m^2 - \tilde{D}^2 - g\sigma F}\\ &= \frac{1}{m^2 - \tilde{D}^2} + \frac{1}{m^2 - \tilde{D}^2} g\sigma F \frac{1}{m^2 - \tilde{D}^2}\\ &\quad + \frac{1}{m^2 - \tilde{D}^2} g(\sigma F) \frac{1}{m^2 - \tilde{D}^2} g(\sigma F) \frac{1}{m^2 - \tilde{D}^2}\\ &\quad + \cdots.\end{aligned} \tag{57}$$

One can define in (57) the self-energy correction to the mass,

$$\Delta m^2(x, y) = -g(\sigma F)_x \left(\frac{1}{m^2 - \tilde{D}^2}\right)_{xy} g(\sigma F)_y. \tag{58}$$

$$\bar{\Delta} m^2 = \int d^4(x - y) \Delta m^2(x, y);$$
$$\Delta m^2(x, y) = -g^2 \sigma_i \sigma_k (\langle H_i H_k\rangle + \langle E_i E_k\rangle)_{xy} G_0(x, y), \tag{59}$$

where

$$G_0(x, y) = \left(\frac{1}{m^2 - \tilde{D}^2}\right)_{xy}, \tag{60}$$

while

$$\frac{g^2}{N_c}\langle H_i(x)H_k(y) + E_i(x)E_k(y)\rangle = 2D(x - y)\delta_{ik}. \tag{61}$$

It was shown in [31] that in the absence of a magnetic field and in the limit of small m and small vacuum correlation length λ, one can replace $(\frac{1}{m^2 - \tilde{D}^2})_{xy}$ by the free propagator $\frac{1}{(4\pi)^2(x-y)^2}$, yielding

$$\bar{\Delta} m^2 = -\frac{3\sigma}{\pi}; \tag{62}$$

then the correction (58) yields for the total mass $M_n^{(0)}$, $\Delta M_n = \sum_i \frac{\bar{\Delta} m_i^2}{2\omega_i^{(0)}}$, for zero mass q and $\bar{q}$,

$$M_n^{(0)}(\omega_0) \rightarrow M_n^{(0)}(\omega_0) - \frac{3\sigma}{\pi\omega_0}. \tag{63}$$

Note that ω plays the role of the integration variables in (25) and (41) and is defined from the stationary point condition (48), where $M_n^{(0)}$ does not include Δm^2. However, for large m and small $|x - y| \lesssim \lambda$ one should multiply (62) with the coefficient $\eta(m\lambda) < 1$ calculated in [31].

Consider now the case of constant magnetic field $\mathbf{B}$ along the z axis. One can calculate the effect of magnetic field on the self-energy correction, to this end expand

$$(\sigma F)\frac{1}{m^2-D_\mu^2-e\sigma_3 B}(\sigma F) = (\sigma F)\frac{m^2-D_\mu^2+e\sigma_3 B}{(m^2-D_\mu^2)^2-(eB)^2}(\sigma F) \to \frac{\sigma F(G_+ + G_-)\sigma F}{2}, \quad (64)$$

where $G_{+/-}=(m^2-D_\mu^2\pm eB)^{-1}_{xy}$, and $m_i^2-D_\mu^2\approx m_i^2+|e_iB|$. As was shown in Eq. (53), for large $eB\gg\sigma$ one has a large effective mass ω_*, $\omega_*^2\approx m_i^2+2|e_iB|$ in one of the Green's functions G_+, G_- and the corresponding Green's function will contribute $\frac{3\sigma}{2\pi\omega_0}\eta(eB)$, where $\eta(eB)\equiv\eta(\sqrt{2|e_iB|+m_i^2}\lambda)$ is the coefficient, introduced in [31], e.g., $\eta(0)=1$, $\eta(5\ \mathrm{GeV})^2)=0.03$; see the Appendix of [31] for the explicit expression. The final expression for ΔM_n can therefore be written as

$$\Delta M_n=-\frac{3\sigma}{2\pi\omega_0}(1+\eta(eB)), \quad (65)$$

where ω_0 corresponds to the Green's function G of a given quark when spin terms (σF) are absent. One can see that at large eB, the self-energy correction numerator decreases approximately twice, as compared to $eB=0$, $m=0$, since $\eta(eB\gg\sigma)\to 0$.

We now turn to the $(q\bar q)$ Green's function and write it in the form

$$\int G_{q_1\bar q_2}(x,y)d^3(x-y) = \mathrm{tr}\langle[\Gamma_1(m_1-i\hat p_1)e^{-HT}\Gamma_2(m_2-i\hat p_2)]\rangle, \quad (66)$$

where we have taken into account that $(\hat\partial-ig\hat A)$, acting in (66) on the Wilson loop, can be replaced by the momentum operator [32].

For the case when H does not contain γ_μ matrices, noncommuting with $(m_i-i\hat p_i)$, one can rewrite (66) as in [32], but now taking into account spin and isospin nonconservation in m.f., one must keep the possible eigenvalue dependence on the spin projections,

$$\int G_{q_1\bar q_2}(x,y)d^3(x-y)=\sum_{n,\nu}(\varepsilon_r\otimes\varepsilon_r)_\nu\frac{(M_n^{(\nu)}f_\Gamma^n)^2e^{-M_n^{(\nu)}T}}{2M_n}, \quad (67)$$

with $\varepsilon_{\gamma_5}=\varepsilon_1=1$, $\varepsilon_V\equiv\varepsilon_\mu^{(k)}$, and the index ν denotes a specific polarization and charge component of quark and antiquark, e.g., $\nu=1$ for the $(\bar u u)$, $\langle -+|$ component. As a result the quark decay constant of the γ_ν state is

$$(f^n_{\Gamma\nu})^2=\frac{N_c\langle Y_{\Gamma\nu}\rangle|\psi_n^{(\nu)}(0)|^2}{\bar\omega_1\bar\omega_2M_n^{(\nu)}\xi_\nu}, \quad (68)$$

where ξ_ν occurs due to ω_i integrations in (44) (see the Appendix of [14]) and

$$\langle Y_{\Gamma\nu}\rangle=\frac14\mathrm{tr}(\Gamma^\nu(m_1-i\hat p_1)\Gamma^\nu(m_2-i\hat p_2)). \quad (69)$$

In the case when H contain spin-dependent terms and, in addition, depends on magnetic field B, one should be more careful with the ordering of operators (σF), $\boldsymbol{\sigma}\mathbf{B}$ in H and projectors $(m_1-i\hat p_1)$, $(m_2-i\hat p_2)$.

Correspondingly, $(m-\hat D)$ can be rewritten as

$$(m_q-\hat D)_x\to(m_1-i\hat p_1), \qquad (m_{\bar q}-\hat{\bar D})_x=m_2-i\hat p_2, \quad (70)$$

where $p_1=(i\omega_1,\mathbf{p})$, $p_2=-(i\omega_2,-\mathbf{p})$ and we take into account that $D_\mu(x)$ acting on $\Phi_z(x,y)$ yields $\partial_\mu\to ip_\mu$.

$$m_1-i\hat p_1=m_1+\omega_1\gamma_4-i\mathbf{p}\boldsymbol{\gamma} = \begin{pmatrix} m_1+\omega_1 & -i\boldsymbol{\sigma}\mathbf{p}\\ i\boldsymbol{\sigma}\mathbf{p} & m_1-\omega_1\end{pmatrix}. \quad (71)$$

At this point we have at least two possibilities for the relative ordering of factors $(m_{\bar q}^2-\hat{\bar D}^2)$ and $(m_{\bar q}-\hat{\bar D})$ in $G_{q\bar q}$. We shall define this ordering as

$$G_{q\bar q}(x,y)=\langle\mathrm{tr}[\Gamma_1S_q(x,y)\Gamma_2S_{\bar q}(y,x)]\rangle_A = \langle\mathrm{tr}[\Gamma_1(m_q-\hat D)_x(m_q^2-\hat D^2)^{-1}_{xy} \times\Gamma_2(m_{\bar q}-\hat{\bar D})_y(m_{\bar q}^2-\hat{\bar D}^2)^{-1}_{yx}]\rangle_A. \quad (72)$$

We shall show below that this ordering yields correct results for spinless and spin-dependent parts, which can be checked in the nonrelativistic limit, whereas other orderings lead to wrong answers. Now

$$m_2-i\hat p_2=m_2-\omega_2\gamma_4-i\mathbf{p}\boldsymbol{\gamma} = \begin{pmatrix} m_2-\omega_2 & -i\boldsymbol{\sigma}\mathbf{p}\\ i\boldsymbol{\sigma}\mathbf{p} & m_2+\omega_2\end{pmatrix}. \quad (73)$$

If is clear that in the nonrelativistic situation, $p\ll m$, $\omega=\sqrt{p^2+m^2}\to m$, the product $(\Gamma(m_1-i\hat p_1)\Gamma(m_2-i\hat p_2))$ tends to the nonrelativistic projector

$$\begin{pmatrix} 4m_1m_2, & 0\\ 0, & 0\end{pmatrix}. \quad (74)$$

We now turn to the spin-dependent terms and make an expansion of $m_i^2-\hat D^2=m_i^2-D_\mu^2-e\sigma B-g(\sigma F)$ in powers of $g(\sigma F)$:

$$(m_1^2-\hat D^2)^{-1}\cong\Delta_B+\Delta_Bg(\sigma F)\Delta_B, \qquad \Delta_B\equiv(m_1^2-D_\mu^2-e_1\sigma B)^{-1}, \quad (75)$$

$$(m_2^2-\hat D^2)^{-1}\cong\bar\Delta_B-\bar\Delta_Bg(\sigma F)\bar\Delta_B, \qquad \bar\Delta_B=(m_2^2-\bar D_\mu^2-e_2\sigma B)^{-1}, \quad (76)$$

and (72) can be rewritten keeping only one spin-spin term as

$$G_{q\bar{q}}(x,y) = G_{q\bar{q}}^{(0)}(x,y) - \langle \mathrm{tr}[\Gamma(m_1 - i\hat{p}_1)\Lambda_B \bar{g}(\sigma_1 F)\Delta_B \Gamma(m_2 - i\hat{p}_2)\bar{\Delta}_B g(\sigma_2 F)\bar{\Delta}_B]\rangle_A. \tag{77}$$

One readily obtains for $\Gamma = \gamma_5$,

$$\mathcal{M} \equiv \mathrm{tr}[\gamma_5(m_1 - i\hat{p}_1)(\sigma_1 F)\gamma_5(m_2 - i\hat{p}_2)(\sigma_2 F)] = \mathrm{tr}[(m_1 - i\hat{p}_1)(\sigma_1 F)((m_2 - i\hat{p}_2)(\sigma_2 F))^T], \tag{78}$$

$$(m_1 - i\hat{p}_1)(\sigma F_1) = \begin{pmatrix} (m_1+\omega_1)\sigma H_1 - i\boldsymbol{\sigma}\mathbf{p}\boldsymbol{\sigma}\mathbf{E}_1, & (m_1+\omega_1)\sigma\mathbf{E}_1 - i\boldsymbol{\sigma}\mathbf{p}\boldsymbol{\sigma}\mathbf{H}_1 \\ i\boldsymbol{\sigma}\mathbf{p}\boldsymbol{\sigma}\mathbf{H}_1 + (m_1-\omega_1)\boldsymbol{\sigma}\mathbf{E}_1, & i\boldsymbol{\sigma}\mathbf{p}\boldsymbol{\sigma}\mathbf{E}_1 + (m_1-\omega_1)\sigma\mathbf{H}_1 \end{pmatrix}, \tag{79}$$

$$((m_2 - i\hat{p}_2)(\sigma F_2))^T = \begin{pmatrix} i\boldsymbol{\sigma}\mathbf{p}\boldsymbol{\sigma}\mathbf{E}_2 + (m_2+\omega_2)\boldsymbol{\sigma}\mathbf{H}_2, & i\boldsymbol{\sigma}\mathbf{p}\boldsymbol{\sigma}\mathbf{H}_2 + (m_2+\omega_2)\boldsymbol{\sigma}\mathbf{E}_2 \\ (m_2-\omega_2)\boldsymbol{\sigma}\mathbf{E}_2 - i\boldsymbol{\sigma}\mathbf{p}\boldsymbol{\sigma}\mathbf{H}_2, & (m_2-\omega_2)\boldsymbol{\sigma}\mathbf{H}_2 - i\boldsymbol{\sigma}\mathbf{p}\boldsymbol{\sigma}\mathbf{E}_2 \end{pmatrix}. \tag{80}$$

Combining (79) and (80), one obtains ($\mathbf{H}_i \equiv \mathbf{H}(x_i)$, $i = 1, 2$).

$$\begin{aligned}\mathcal{M} = \mathrm{tr}_\sigma\{&\boldsymbol{\sigma}\mathbf{H}_1\boldsymbol{\sigma}\mathbf{H}_2(2m_1m_2 + 2\omega_1\omega_2) - 2(\boldsymbol{\sigma}\mathbf{H}_1)\boldsymbol{\sigma}\mathbf{p}(\boldsymbol{\sigma}\mathbf{H}_2)\boldsymbol{\sigma}\mathbf{p}) + \boldsymbol{\sigma}\mathbf{E}_1\boldsymbol{\sigma}\mathbf{E}_2(2m_1m_2 - 2\omega_1\omega_2)\\ &+ 2(\boldsymbol{\sigma}\mathbf{E}_1)\boldsymbol{\sigma}\mathbf{p}(\boldsymbol{\sigma}\mathbf{E}_2)\boldsymbol{\sigma}\mathbf{p}) + \boldsymbol{\sigma}\mathbf{E}_1\boldsymbol{\sigma}\mathbf{H}_2(-2i\boldsymbol{\sigma}\mathbf{p}(\omega_1+\omega_2)) + \boldsymbol{\sigma}\mathbf{H}_1\boldsymbol{\sigma}\mathbf{E}_2(2i\boldsymbol{\sigma}\mathbf{p}(\omega_1+\omega_2))\}.\end{aligned} \tag{81}$$

In what follows we disregard first the terms containing $(\boldsymbol{\sigma}\mathbf{p})$ or $(\boldsymbol{\sigma}\mathbf{p})(\boldsymbol{\sigma}\mathbf{p})$. In the nonrelativistic limit, $|\mathbf{p}| \to 0$, $\omega_i \to m_i$, one has

$$\langle\mathcal{M}\rangle_A = 8m_1m_2\langle H_i(x_1)H_i(x_2)\rangle_A. \tag{82}$$

Field correlators are expressed via two scalar correlators $D(x_1 - x_2)$ and $D_1(x_1 - x_2)$ as in (35).

Comparing with the standard definition for the nonrelativistic hf term, one has

$$\begin{aligned} V_{hf} &= \frac{\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)}}{12m_1m_2}V_4^{(H)}(r),\\ V_4^{(H)}(r) &= \int_{-\infty}^{\infty} d\nu \frac{g^2}{N_c}\langle H_i(x)H_i(y)\rangle, \qquad \nu \equiv x_4 - y_4, \end{aligned} \tag{83}$$

where

$$\begin{aligned} u_\mu &= x_\mu - y_\mu, \qquad \mu = 1,2,3,4;\\ r &= |\mathbf{x} - \mathbf{y}| = |\mathbf{u}|, \qquad \nu \equiv u_4. \end{aligned} \tag{84}$$

One can separate perturbative part in D_1,

$$\begin{aligned} D_1(x) &= D_1^{\mathrm{pert}}(x) + D_1^{\mathrm{np}}(x),\\ D_1^{\mathrm{pert}}(x) &= \frac{16\alpha_s}{3\pi x^4} + O(\alpha_s^2), \end{aligned} \tag{85}$$

and define the potentials

$$\begin{aligned} V_4^{(H)}(r) &= \int_{-\infty}^{\infty} d\nu \frac{g^2}{N_c}\langle H_i(x)H_i(y)\rangle\\ &= \int_{-\infty}^{\infty} d\nu\left(3D(r,\nu) + 3D_1(r,\nu) + 2r^2\frac{\partial D_1(r,\nu)}{\partial r^2}\right)\\ &= V_4^{(D)}(r) + V_4^{(1)}(r), \end{aligned} \tag{86}$$

$$\begin{aligned} V_4^{(E)}(r) &= \int_{-\infty}^{\infty} d\nu\bigg(3D(r,\nu) + 3D_1(r,\nu)\\ &\quad + (3\nu^2 + r^2)\frac{\partial D_1(r,\nu)}{\partial r^2}\bigg)\\ &= V_4^{(D)} - V_4^{(1)}(r). \end{aligned} \tag{87}$$

As it is known [20], the nonperturbative part of D_1 and D yield much lower input in $V_4^{(E,H)}$, the leading part is due to D_1^{pert}, i.e., $V_4^{(1)}$. Inserting this, one obtains

$$V_4^{(H)} \equiv V_4^{(H)\mathrm{pert}}(r) = \frac{32\pi\alpha_s}{3}\delta^{(3)}(\mathbf{r}), \tag{88}$$

and the result, (83) and (88), coincides with the known nonrelativistic limit.

Now we turn to the relativistic case, $\omega_i \gg m_i$. First of all we note that

$$V_4^{(E)} \equiv V_4^{(E)\mathrm{pert}}(r) = \frac{-32\pi\alpha_s}{3}\delta^{(3)}(\mathbf{r}). \tag{89}$$

Looking at (81), one can see that in the relativistic case when $\omega_i \gg m_i$, there is a cancellation in the spin-spin interaction, in the combination

$$\begin{aligned} &2(m_1m_2 + \omega_1\omega_2)V_4^{(H)\mathrm{pert}} + 2(m_1m_2 - \omega_1\omega_2)V_4^{(E)\mathrm{pert}}\\ &\quad = 2\omega_1\omega_2\cdot 2V_4^{(1)} + 2m_1m_2 2V_4^{(D)}, \end{aligned} \tag{90}$$

and multiplying this result with Δ_B, $\bar{\Delta}_B$ as in (77), in the $B = 0$ case we obtain for $\Delta_B \approx \frac{1}{2\Omega_1^2}$, $\bar{\Delta}_B = \frac{1}{2\Omega_2^2}$,

$$V_{hf} = \frac{\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)}}{12\bar{\omega}_1\bar{\omega}_2}\left\{V_4^{(H)}(r)\left(1 + \frac{\mathbf{p}^2}{3\omega^2}\right) + \frac{m^2}{\omega^2}V_4^D(r)\right\}. \tag{91}$$

Here $\bar{\omega}_i = \frac{\Omega_i^2}{\omega_i}$, and Ω_i is defined in (53); one can derive that $\bar{\omega}_i \geq \omega_i^{(0)}$, e.g., in the nonrelativistic limit $\omega_i^{(0)} \to m_i$, $\Omega_i \to m_i$, and also $\bar{\omega}_i \to m_i$. The same happens in relativistic case, when $eB \gg \sigma$. In what follows for $\bar\omega_1 \neq \bar\omega_2$ it is implied, e.g., that $\bar\omega^2_{+-} = \bar\omega_1(+-)\bar\omega_2(+-)$.

We now turn to the case of nonzero B and now take into account the noncommutative (2×2) terms in H_σ and in V_{hf}, which we write in the total mass as

$$M = \bar M - \mu_1\sigma_{1z} + \mu_2\sigma_{2z} + a\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2, \qquad (92)$$

where $\mu_i = \frac{eB}{2\omega}$, $a = \frac{1}{12\omega^2}\langle V_4^H\rangle$.

For π^0, $\rho^0(s_z=0)$ states, one obtains a standard mixing of $\langle +-|$ and $\langle -+|$ states of $\langle\sigma_{1z},\sigma_{2z}|$, with V_{hf}, where now for $B\neq 0$ we distinguish

$$a_{11} = \langle +-|a\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2|+-\rangle = \frac{1}{12\bar\omega^2_{+-}}\langle V_4^{(H)}\rangle, \qquad (93)$$

$$a_{22} = \langle -+|a\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2|-+\rangle = \frac{1}{12\bar\omega^2_{-+}}\langle V_4^{(H)}\rangle, \qquad (94)$$

$$2a_{12} = 2a_{21} = \langle -+|a\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2|+-\rangle = \frac{2}{12\bar\omega_{-+}\bar\omega_{+-}}\langle V_4^H\rangle. \qquad (95)$$

We also define M_{11}, M_{22} as follows:

$$M_{11} = (\bar M - (\mu_1+\mu_2) - a_{11}))_{\omega_{+-}}; \quad M_{22} = (\bar M + (\mu_1+\mu_2) - a_{22}))_{\omega_{-+}}. \qquad (96)$$

Finally, from $\det(M-E)=0$, we obtain the eigenvalues of M,

$$E_{1,2} = \frac{1}{2}(M_{11}+M_{22}) \pm \sqrt{\left(\frac{M_{22}-M_{11}}{2}\right)^2 + 4a_{12}a_{21}}. \qquad (97)$$

Here ω_{+-} and ω_{-+} correspond to the diagonal states of $\Delta_B\bar\Delta_B$, i.e., for neutral mesons to the stationary values of $\omega^{(0)}$ in the states with spin projections $\langle +-|$ and $\langle -+|$, respectively. It is clear that at large eB the values of ω behave differently, i.e., $\omega_{+-}\sim$ const, while $\omega_{-+}\sim\sqrt{eB}$ and $M_{22}\sim\sqrt{eB}$; hence, at large eB the nondiagonal part of M in (97) is decreasing, and M tends to its diagonal eigenvalues.

$$E_1(eB\to\infty)\to M_{11}, \qquad E_2(eB\to\infty)\to M_{22}. \qquad (98)$$

One can notice that for the $\langle +-|$ state M_{11} contains at large eB the fast decreasing part, $-a_{11}\sim -\psi^2(0)$, and the latter is large in the modulus, $\psi^2(0)\sim\sim eB$, hence leading to the negative mass E_1.

This happens already for only color-magnetic contribution $\langle H_iH_k\rangle$ to $V_4^{(H)}$, while our conclusion in Sec. II was that eigenvalues of $(m_i^2-\hat D_i^2)$ are positive together with the total mass eigenvalues of the operator $\langle\frac{1}{m_2^2-\hat D_1^2}\frac{1}{m_2^2-\hat D_2^2}\rangle$. Hence, we conclude that the perturbation theory in V_{hf} breaks down at large eB and one has to replace it with some modified form. However, as was understood already in [2,3], even at $B=0$ the perturbation theory with the potential $V_{ss}^{(0)}(r)\sim c\delta^{(3)}(\mathbf{r})$ is diverging since $V_{ss}^{(0)}$ for any $c<0$ ensures an infinite number of bound states, which are physically irrelevant. Therefore, one should, in any case, take into account the relativistic smearing of the hf interaction, which appears due to the time integration in (38), which is taken in (38) along the straight line, instead of the complicated relativistic trajectory of the quark with time fluctuations; see [14]. The resulting smearing length is $\lambda \gtrsim \lambda_{\rm conf}$, where $\lambda_{\rm conf}$ is the scale of $D(x)$, connected to the gluelump mass [33], $\lambda_{\rm conf}\approx 0.1$–$0.15$ fm. On the lattice $\lambda\geq a$, a is the lattice unit, $a\approx 0.1$–0.24 fm; therefore, we replace $V_{ss}^{(0)}(r)$ by a smeared out version, e.g.,

$$\tilde\delta^{(3)}(r) = \left(\frac{\lambda}{\sqrt{\pi}}\right)^3 e^{-\mu^2r^2}; \quad \tilde V_{ss}(r) = c\tilde\delta^{(3)}(\mathbf{r}), \quad \mu = \frac{1}{\lambda}\cong (1\text{–}2)\ \text{GeV}. \qquad (99)$$

Using the wave function $\tilde\psi(\eta)$ from (54) one obtains for $\langle\tilde V_{ss}\rangle$, in the pseudoscalar meson

$$\langle\tilde V_{ss}\rangle = \frac{c\mu^3}{\pi^{3/2}\sqrt{1+\mu^2r_z^2}(1+\mu^2r_\perp^2)}, \qquad c = -\frac{8\pi\alpha_s}{3\omega_1\omega_2}. \qquad (100)$$

For $\mu\to\infty$, one regains the original answer, $\langle\tilde V_{ss}\rangle\to c\psi^2(0)$.

Since $r_\perp^2 = \frac{2}{eB}(1+\frac{(\bar c\sigma)^2}{(eB)^2})^{-1/2}$, $r_\perp(eB\to\infty)\to 0$, and $\langle\tilde V_{ss}\rangle$ tends to a constant limit at large eB, preventing in this way the breakdown of the vacuum due to the vanishing of the meson mass.

V. THE TENSOR FORCES IN A MAGNETIC FIELD

As was established in the previous section, the hf interaction has the form (91) and in the m.f. the coefficients $\frac{1}{\bar\omega_1\bar\omega_2}$ transform into $\frac{1}{\bar\omega_{ik}\bar\omega_{i'k'}}$, where (ik), $(i'k')=(\sigma_{1z},\sigma_{2z})=(+-)$ or $(-+)$; for $S_z=0$ see Eqs. (93)–(95). We now turn to the tensor forces and discuss how they are transformed in the m.f. As was found in [15,18] the total spin-dependent forces without m.f. can be written as

$$V_3^{ss}+V_4^{ss} \equiv \frac{\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)}}{12\omega_1\omega_2}V_4(r) + \frac{1}{12\omega_1\omega_2}(3(\boldsymbol{\sigma}^{(1)}\mathbf{n})(\boldsymbol{\sigma}^{(2)}\mathbf{n}) - \boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)})V_3(r). \qquad (101)$$

It is clear from (101) that for the spherically symmetric $q\bar q$ states the term $V_3(r)$ is irrelevant, and therefore requires a nonzero angular momentum.

The situation is drastically changing in the m.f., since in this case the form of the wave function is distorted as in the elongated ellipsoid.

This fact implies the appearance of tensor forces in m.f. even in the ground $q\bar{q}$ state with the zero angular momentum. This situation was studied for the hydrogen atom case in m.f. in the nonrelativistic treatment of the tensor forces in the $q\bar{q}$ system.

We are using again the correlator technic and the perturbative correlator $D_1^{\rm pert}$, in terms of which the tensor interaction can be written as in (37).

The expectation value of the tensor term in the ground state with the wave function $\tilde{\psi}(\rho, z)$ (54) can be written as

$$\langle V_3^{ss}\rangle = \frac{8\alpha_s \sigma_i^{(1)}\sigma_k^{(2)}}{9\pi\omega_i\omega_2}\int_{-\infty}^{\infty} d\nu\rho d\rho dz d\varphi(3n_i n_k - \delta_{ik}) \times \psi^2(\rho, z)(-r^2)\frac{\partial}{\partial r^2}\frac{1}{(r^2+\nu^2)^2}. \quad (102)$$

Using the relations

$$\frac{1}{(r^2+\nu^2)^3} = \frac{1}{2}\frac{\partial^2}{\partial\alpha^2}\bigg|_{\alpha=0}\frac{1}{r^2+\nu^2+\alpha}; \qquad \frac{1}{r^2+\nu^2+\alpha} = \int_0^\infty d\beta e^{-\beta(r^2+\nu^2+\alpha)}, \quad (103)$$

and the $q\bar{q}$ wave function in the zeroth approximation

$$\psi^2(\rho, z) = Ne^{-\frac{\rho^2}{r_\perp^2}-\frac{z^2}{r_3^2}}; \qquad N = \frac{1}{\pi^{3/2}r_\perp^2 r_3^2}, \quad (104)$$

one can calculate the sum

$$\langle V_3^{ss}\rangle + \langle V_4^{ss}\rangle = \frac{8\pi\alpha_s}{9\omega_1\omega_2}N\bigg\{\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 + \frac{K}{4}\left(\frac{1}{r_\perp^2}-\frac{1}{r_3^2}\right) \times\left(-\sigma_x^{(1)}\sigma_x^{(2)} - \sigma_y^{(1)}\sigma_y^{(2)} + 2\sigma_z^{(1)}\sigma_z^{(2)}\right)\bigg\}. \quad (105)$$

Here K is

$$K = r_\perp^2\int_0^\infty \frac{2u^4 du}{(u^2+1)^2(u^2+\frac{r_\perp^2}{r_3^2})^{3/2}}. \quad (106)$$

Note that we have omitted here for simplicity in $\langle V_4\rangle^{ss} \sim \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$ the additional terms which appear in (91). The integral K in (106) can be done explicitly, using the relation

$$\int_0^\infty \frac{du}{(u^2+p^2)(u^2+q^2)^{1/2}} = \frac{1}{2p\sqrt{p^2-q^2}}\ln\left(\frac{p+\sqrt{p^2-q^2}}{p-\sqrt{p^2-q^2}}\right). \quad (107)$$

It is important that $K \le r_\perp^2$; therefore, since $r_\perp^2 \to 0$ for $eB \to \infty$ [see Eq. (55)] the relative role of the tensor term in (105) is diminishing with growing eB, and the absolute value of $\langle V_3^{ss}\rangle$ depends only on the values of ω_1, ω_2. To find these values, we, again as in (92), write the total mass, but now with the addition of the tensor term and for any quark charges

$$M = \bar{M} + a\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 + c^{(1)}\sigma_{1z} + c_{2z}^{(2)} + b(-\sigma_x^{(1)}\sigma_x^{(2)} - \sigma_y^{(1)}\sigma_y^{(2)} + 2\sigma_z^{(1)}\sigma_z^{(2)}) \equiv \bar{M} + h, \quad (108)$$

where

$$a = \frac{8\pi\alpha_s}{9\bar{\omega}_1\bar{\omega}_2}N, \qquad c^{(1)} = -\frac{e_1 B}{2\omega_1}, \qquad c^{(2)} = -\frac{e_2 B}{2\omega_2}, \quad (109)$$

$$b = \frac{8\pi\alpha_s}{9\bar{\omega}_1\bar{\omega}_2}N\frac{K}{4}\left(\frac{1}{r_\perp^2}-\frac{1}{r_3^2}\right). \quad (110)$$

Moreover, one must distinguish the values of coefficients a, b, c_i in different spin projection states $(\sigma_{1z}\sigma_{2z})$, namely, the values of ω_i in a, b, c_i in different $(\sigma_z^{(1)}, \sigma_z^{(2)})$ states. For example, for $S_z = +1(-1)$ one must write $\bar{\omega}_1\bar{\omega}_2 \to \bar{\omega}_{++}^2(\bar{\omega}_{--}^2)$, while for $S_z = 0$ one has the matrix elements as in (93)–(95). In a similar way for our general case with tensor forces in (105), one can write

$$\langle +-|h|+-\rangle = -a_{11} - 2b_{11} + c_{11}^{(1)} - c_{11}^{(2)},$$
$$\langle -+|h|-+\rangle = -a_{22} - 2b_{22} + c_{22}^{(1)} - c_{22}^{(2)},$$
$$\langle +-|h|-+\rangle = \langle -+|h|+-\rangle = 2(a_{12} - b_{12}) = 2(a_{21} - b_{21}), \quad (111)$$

and all diagonal terms have the corresponding $\bar{\omega}_i$, e.g., $\bar{\omega}_1 = \bar{\omega}_2 = \bar{\omega}_\pm$ for a_{11}, b_{11}, $c_{11}^{(i)}$, and in nondiagonal matrix elements $\bar{\omega}_1\bar{\omega}_2 = \bar{\omega}_{+-}\bar{\omega}_{-+}$.

From $\det(h - E) = 0$ with elements in (111) one obtains two eigenvalues of the spin-dependent part E_1, E_2.

The resulting expressions for E_1, E_2 coincide with those in (97) when one makes the following replacements:

$$a_{11} \to a_{11} + 2b_{11}, \quad a_{22} \to a_{22} + 2b_{22}, \quad a_{12} \to a_{12} - b_{12}, \quad (112)$$

and $\mu_1 \to -c^{(1)}$.

Equations (51) and (52) contain a prescription for the values of ω_{ij}, $i, j = +, -$, entering in M_{11}, M_{22}, which is valid also in the case of nonzero tensor forces.

VI. CONCLUSIONS AND PROSPECTIVES

We have obtained explicit relativistic expressions for the spin-spin interaction terms in the $q\bar{q}$ system in the arbitrarily strong m.f. As a by-product we also obtained in the same formalism expressions for the np self-energy corrections in m.f. These formulas are generalizations of the previously found expressions for the $q\bar{q}$ mesons in the absence of m.f. (see, e.g., [15]), and we also found corrections to those expressions [see Eq. (91)], where the terms $\frac{\mathbf{p}^2}{3\omega^2} V_4^{(H)}(r)$ and $\frac{m^2}{\omega^2} V_4^{(D)}(r)$ are new.

It is remarkable how m.f. changes the spin-spin forces. First of all, the matrix element of the hf term $V_4^{(H)}(r) \sim \delta^{(3)}(\mathbf{r})$ in the strong m.f. is proportional to the $\psi^2(0)$—the probability of coming together of q and $\bar{q}$, which in strong m.f. grows as eB.

This effect is known in nonrelativistic case, where the hf term is $\langle V_{hf}\rangle \sim \frac{\psi^2(0)}{M_1 M_2} \sim \frac{eB}{M_1 M_2}$ and was discussed recently in [30] for the hydrogen case. However, in this case $M_2 = M_p$ is very large and $\langle V_{hf}(hydr)\rangle$ is small and this growth of $\langle V_{hf}\rangle$ was thought to cease in relativistic limit, $eB \sim m_e^2$.

However, as we have shown here in the paper, the growth of $\langle V_{hf}\rangle$ in the relativistic case for $eB \to \infty$ is possible whenever ω^2 in the denominator of $\langle V_{hf}\rangle \sim \frac{\psi^2(0)}{\omega^2}$ does not grow with eB, which occurs for the $q\bar{q}$ states, where magnetic moment terms compensate the growth of the rest part of the mass, e.g., for the $\langle + - |$ states of the neutral mesons like π^0. Then hf terms yield a negative $(-3\langle V_{hf}\rangle)$ contribution to the mass, linearly growing with eB in the modulus, thus giving the absurd negative mass result.

To disprove this result, we have given in Sec. II the proof, that the term $V_4^{(H)}$, which causes this problem, cannot generate negative mass in any m.f., and we are coming to the conclusion that this discrepancy occurs due to the use of the perturbation theory for the hf term, which is proportional to the delta function. Therefore, we have suggested in Sec. IV the smearing procedure of the hf term, which takes into account the relativistic Zitterbewegung of quarks and should be supplemented with a rigorous procedure of the summation or estimation of the whole perturbative series. We stress that this situation occurs solely due to m.f., which cannot, as shown in Sec. II, provide the pair creation and vacuum reconstruction, which is in contrast to the real Minkowskian electric fields, which are capable of the pair creation (as in the Schwinger famous formula [23]) and of the vacuum reconstruction with the emission of positrons (as in superheavy atoms with $Z > Z_{\mathrm{crit}}$).

Another interesting new effect is the appearance of the tensor force effects at nonzero m.f., which can be tested both in atoms and mesons. However, as we have shown above, at large m.f. the tensor force contribution does not grow with eB, unlike the hf term, and is always smaller than the latter.

In this way we have completed the main part of the strong dynamics of mesons in a magnetic field for the zero angular momentum, which was started in [14].

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