Investigations of the torque anomaly in an annular sector. II. Global calculations, electromagnetic case

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Recently, it was suggested that there was some sort of breakdown of quantum field theory in the presence of boundaries, manifesting itself as a torque anomaly. In particular, Fulling *et al.* used the finite energy-momentum-stress tensor in the presence of a perfectly conducting wedge, calculated many years ago by Deutsch and Candelas, to compute the torque on one of the wedge boundaries, where the latter was cut off by integrating the torque density down to minimum lower radius greater than zero. They observed that this torque is not equal to the negative derivative of the energy obtained by integrating the energy density down to the same minimum radius. This motivated a calculation of the torque and energy in an annular sector obtained by the intersection of the wedge with two coaxial cylinders. In a previous paper we showed that for the analogous scalar case, which also exhibited a torque anomaly in the absence of the cylindrical boundaries, the point-split regulated torque and energy indeed exhibit an anomaly, unless the point splitting is along the axis direction. In any case, because of curvature divergences, no unambiguous finite part can be extracted. However, that ambiguity is linear in the wedge angle; if the condition is imposed that the linear term be removed, so that the energy goes to zero for large angles, the resulting torque and energy are finite, and exhibit no anomaly. In this paper, we demonstrate that the same phenomenon takes place for the electromagnetic field, so there is no torque anomaly present here either. This is a nontrivial generalization, since the anomaly found by Fulling *et al.* is linear for the Dirichlet scalar case, but nonlinear for the conducting electromagnetic case.

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I. INTRODUCTION

Recently, Fulling *et al.* [1] suggested that a quantum torque anomaly exists in field theories in the presence of boundaries. This is related, but somewhat distinct from that group's earlier discussion of a pressure anomaly [2], since the latter explicitly depended on taking seriously the distance dependence of stress tensor components below the cutoff scale. In the new torque anomaly, the stress tensor employed is the completely finite one (cutoff independent) for an ideal wedge calculated first by Dowker and Kennedy [3] for the Dirichlet scalar case, and then given for electromagnetic fields subject to perfectly conducting boundaries by Deutsch and Candelas [4]. These computations were later revisited by Brevik and Lygren [5] and by Saharian and Tarloyan [6]. It should, however, be borne in mind that in computing those completely finite vacuum expectation values of the stress tensor, regularization, such as that afforded by point splitting in the angular or the radial direction, is necessary, before the subtraction of the free-space vacuum stress tensor is effected. So the distinction between the two types of anomalies is not so sharp.

Naturally, the stress tensor computed for the wedge is singular at the apex of the wedge. Therefore, it is not possible to compute the total energy of the wedge, or the torque exerted by quantum fluctuations of the interior fields on one of the sides of the wedge. So what is proposed in Ref. [1] is to integrate only from some nonzero inner radius a from the apex, for both the torque and the energy. That is, let the torque per unit length be

$$\tau(a, \alpha) = \int_a^\infty d\rho \rho \langle T^\theta{}_\theta \rangle, \tag{1.1}$$

where the integral is over one of the wedge sides, θ is the axial angle, and α is the angle of the wedge. The corresponding energy per unit length is

$$\mathcal{E}(a, \alpha) = \int_a^\infty d\rho \rho \int_0^\alpha d\theta \langle T^{00} \rangle. \tag{1.2}$$

It is immediately seen from the Deutsch-Candelas stress tensor that

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$$\tau(a, \alpha) \neq -\frac{\partial}{\partial \alpha}\mathcal{E}(a, \alpha). \tag{1.3}$$

This is Fulling's torque anomaly.

A possible resolution of this anomaly has been suggested by Dowker [7]. It would appear that what is necessary is more than simply putting in spatial cutoffs on the integrals. This, in effect, equates the force on a semi-infinite plate, not touching a second semi-infinite plate, with the negative derivative of the quantum vacuum energy contained in only the open region between those plates, rather than the energy in all of space. Therefore, we here are considering a region completely bounded by conducting surfaces: the two radial wedge boundaries and two circular cylindrical boundaries sharing a common axis, as shown in Fig. 1. In Ref. [8] we considered such a geometry for a massless scalar field, with Dirichlet boundaries. We regulate the integral by point splitting in the time or the axial direction. For the former, the divergent expressions indeed exhibit an anomaly, in that the torque is not equal to the negative derivative of the energy contained within the sector. This anomaly disappears for point splitting in the axial direction, consistent with the findings of Ref. [2], since that is a neutral direction, not referring to the stress tensor components involved in either the energy density or the torque density. Introducing the cylindrical boundaries, however, causes another problem by generating divergences associated with curvature. These curvature divergences generate logarithmic terms in the cutoff parameter, which means that it is impossible to extract a finite energy. However, all the divergences encountered are linear functions of the wedge angle, so if we demand that the "renormalized" observable energy approach zero as the wedge angle gets large, we can remove such terms, yielding a finite energy which indeed has the correct balance with the torque. These results are consistent with the annular piston results calculated a few years ago [9], using the multiple-scattering technique.

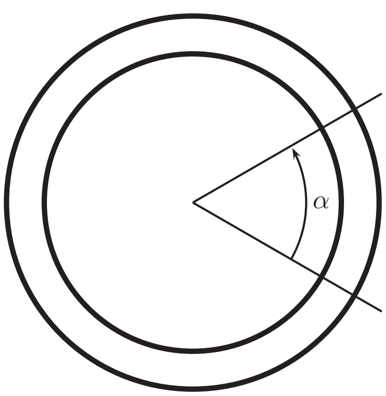

FIG. 1. The Casimir energy and torque are calculated for the region bounded between two perfectly conducting cylinders, of radius a and b, bounded in the angular direction by two perfectly conducting radial planes, making an angle α between them.

In the present paper, we generalize the result of Ref. [8], hereafter referred to as paper I, to the electromagnetic situation, with perfectly conducting boundary conditions. In the next section we set up the general Green's dyadic formulation, for the situation of cylindrical symmetry, where, with perfectly conducting boundaries, we have the complete decomposition between transverse electric (TE) and transverse magnetic (TM) modes. This means that the TM modes are the Dirichlet modes calculated in paper I, while the TE modes are scalar Neumann modes. In Sec. III we derive formulas for the energy in the sector, as well as the torque on one of the radial planes. These quantities are regulated by point splitting either in the temporal or the axial direction. All the divergent terms are extracted for the energy in Sec. IV, corresponding to the volume, the surface area, corners, and curvature corrections. These correspond to known terms in the heat kernel expansion for this problem [10–12]. The finite part is extracted in Sec. V, which arises from the uniform asymptotic expansion of the Bessel functions appearing in the Green's functions, and the remainder, which is computed numerically in Sec. VI. Just as in the scalar case, the numerical results exhibit a linear dependence on the wedge angle for sufficiently (not very) large angles. So it is proposed to remove this linear dependence completely, by a renormalization process that eliminates all the divergent terms, leaving finite results which satisfy the expected balance between energy and torque. Concluding remarks are offered in Sec. VII.

II. GREEN'S DYADIC

The electromagnetic Feynman Green's dyadic, which corresponds to the vacuum expectation value of the time-ordered product of electric fields, satisfies the differential equation

$$\left(\frac{1}{\omega^2}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times -1\right)\boldsymbol{\Gamma}(\mathbf{r}, \mathbf{r}'; \omega) = \mathbf{1}\delta(\mathbf{r}-\mathbf{r}'), \tag{2.1}$$

or, for the divergenceless dyadic $\boldsymbol{\Gamma}' = \boldsymbol{\Gamma} + 1$,

$$\left(\frac{1}{\omega^2}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times -1\right)\boldsymbol{\Gamma}'(\mathbf{r}, \mathbf{r}'; \omega) = \frac{1}{\omega^2}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times 1)\delta(\mathbf{r}-\mathbf{r}'). \tag{2.2}$$

Here, and in the following, we have taken a Fourier transform in time. Henceforth, we will suppress the explicit reference to the frequency dependence. For a situation with cylindrical symmetry, and perfectly conducting boundary conditions, the modes decouple into TE and TM modes, and we can write

$$\boldsymbol{\Gamma}' = \boldsymbol{E}G^E + \boldsymbol{H}G^H, \tag{2.3}$$

in terms of TE and TM Green's functions, where the polarization tensor operators have the structure (for example, see Ref. [5])

$$\boldsymbol{E} = -\nabla^2(\boldsymbol{\nabla} \times \hat{z})(\boldsymbol{\nabla}' \times \hat{z}), \tag{2.4a}$$

$$\boldsymbol{H} = (\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \hat{z}))(\boldsymbol{\nabla}' \times (\boldsymbol{\nabla}' \times \hat{z})), \tag{2.4b}$$

where z is the translationally invariant direction. Acting on a completely translationally invariant function,

$$\boldsymbol{E} + \boldsymbol{H} = -\nabla_\perp^2(\boldsymbol{\nabla}\boldsymbol{\nabla} - 1\nabla^2), \tag{2.5}$$

where

$$\nabla^2 = \nabla_\perp^2 + \frac{\partial^2}{\partial z^2}. \tag{2.6}$$

Further useful properties of $\boldsymbol{E}$ and $\boldsymbol{H}$ are

$$\boldsymbol{\nabla} \times \boldsymbol{E} \times \overleftarrow{\boldsymbol{\nabla}}' = \boldsymbol{H}\nabla^2, \qquad \boldsymbol{\nabla} \times \boldsymbol{H} \times \overleftarrow{\boldsymbol{\nabla}}' = \boldsymbol{E}\nabla'^2, \tag{2.7a}$$

where it is understood that both gradients still act on the $\mathbf{r}$ and $\mathbf{r}'$ dependent functions to the right, and

$$\boldsymbol{E}(\mathbf{r}, \mathbf{r}') \cdot \boldsymbol{H}(\mathbf{r}'', \mathbf{r}''') = \boldsymbol{H}(\mathbf{r}, \mathbf{r}') \cdot \boldsymbol{E}(\mathbf{r}'', \mathbf{r}''') = 0, \tag{2.7b}$$

$$\boldsymbol{E}(\mathbf{r}, \mathbf{r}') \cdot \boldsymbol{E}(\mathbf{r}'', \mathbf{r}''') = \boldsymbol{E}(\mathbf{r}, \mathbf{r}''')\nabla_\perp'^2\nabla''^2, \tag{2.7c}$$

$$\boldsymbol{H}(\mathbf{r}, \mathbf{r}') \cdot \boldsymbol{H}(\mathbf{r}'', \mathbf{r}''') = \boldsymbol{H}(\mathbf{r}, \mathbf{r}''')\nabla_\perp'^2\nabla''^2, \tag{2.7d}$$

where we will understand that after differentiation, the intermediate coordinates $\mathbf{r}'$ and $\mathbf{r}''$ become identified.

For electromagnetism, the energy density is

$$u = T^{00} = \frac{E^2 + B^2}{2}, \tag{2.8}$$

so by use of the Maxwell equations the energy contained in a volume V with perfectly conducting boundaries ∂V becomes, in terms of the imaginary frequency $\zeta = -i\omega$,

$$\int_V (d\mathbf{r}) u(\mathbf{r}) = \frac{1}{2}\int_V (d\mathbf{r})\,\mathrm{Tr}\left[\mathbf{1} + \frac{1}{\zeta^2}(\nabla^2 1 - \boldsymbol{\nabla}\boldsymbol{\nabla})\right] \cdot \mathbf{E}(\mathbf{r})\mathbf{E}(\mathbf{r}')^*|_{\mathbf{r}'=\mathbf{r}}, \tag{2.9}$$

because

$$\int_V (d\mathbf{r})\,\mathrm{Tr}\boldsymbol{\nabla} \times [(\boldsymbol{\nabla} \times \mathbf{E}(\mathbf{r}))\mathbf{E}(\mathbf{r})^*] = i\omega \oint_{\partial V} \sigma\hat{\mathbf{n}} \times \mathbf{B}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})^* = 0, \tag{2.10}$$

for perfectly conducting boundaries. Quantum mechanically, we replace the expectation value of the product of electric fields $\mathbf{E}(\mathbf{r})$ by the Green's dyadic:

$$\langle \mathbf{E}(\mathbf{r})\mathbf{E}(\mathbf{r}')^*\rangle = \frac{1}{i}\boldsymbol{\Gamma}(\mathbf{r}, \mathbf{r}'). \tag{2.11}$$

Because we will be regulating all integrals by point splitting, we can ignore delta functions (contact terms) in evaluations, so in terms of $\boldsymbol{\Gamma}'$, the quantum vacuum energy is

$$\begin{aligned} E &= \int_V (d\mathbf{r}) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t}\langle u(\mathbf{r})\rangle \\ &= \frac{1}{2i}\int_V (d\mathbf{r})\,\mathrm{Tr}\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t}\frac{1}{\zeta^2}(\nabla^2 + \zeta^2)\boldsymbol{\Gamma}'(\mathbf{r}, \mathbf{r}')\Big|_{\mathbf{r}'\to\mathbf{r}} \\ &= \int_V (d\mathbf{r}) \int \frac{d\zeta}{2\pi} e^{i\zeta t_E}\,\mathrm{Tr}\boldsymbol{\Gamma}'(\mathbf{r}, \mathbf{r}), \end{aligned} \tag{2.12}$$

where in the last equation we have performed the rotation to Euclidean space, so $-it \to t_E$ is a Euclidean time-splitting parameter, going to zero through positive values. This is a well-known formula; for example, see Ref. [13]. The energy may be written in terms of the scalar Green's functions in Eq. (2.3),

$$E = \int_V (d\mathbf{r}) \int \frac{d\zeta}{2\pi} e^{i\zeta t_E}\zeta^2\nabla_\perp^2(G^E + G^H)(\mathbf{r}, \mathbf{r}')|_{\mathbf{r}'\to\mathbf{r}}, \tag{2.13}$$

which again involves an integration by parts, and use of the perfect conducting boundary conditions on both arguments of the Green's functions (see below)

$$\oint_{\partial V} d\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}_\perp G_\perp^{E,H}(\mathbf{r}, \mathbf{r}')|_{\mathbf{r}'\to\mathbf{r}} = 0. \tag{2.14}$$

The decomposition theorems contained in this section are familiar from waveguide theory; for example, see Ref. [14].

III. ANNULAR SECTOR

We now specialize to the situation at hand, an annular sector bounded by two concentric cylinders, intercut by a coaxial wedge, as illustrated in Fig. 1. The inner cylinder has radius a, the outer b, and the wedge angle is α. The axial direction is chosen to coincide with the z axis. The explicit form for the Green's dyadic is

$$\begin{aligned} \boldsymbol{\Gamma}'(\mathbf{r}, \mathbf{r}') = -\frac{2}{\alpha}\sum_m \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(z-z')}\frac{1}{\kappa^2} &\times [\boldsymbol{E}(\mathbf{r}, \mathbf{r}')\cos\nu\theta\cos\nu\theta' g_\nu^E(\rho, \rho') \\ &+ \boldsymbol{H}(\mathbf{r}, \mathbf{r}')\sin\nu\theta\sin\nu\theta' g_\nu^H(\rho, \rho')]. \end{aligned} \tag{3.1}$$

Here $\nu = mp$ where $p = \pi/\alpha$, and $\kappa^2 = \zeta^2 + k^2$. The m summation runs from 0 to ∞ for the TE modes, but only from 1 to ∞ for the TM modes. We will see the crucial role of the TE "zero mode" in the following. The H mode vanishes on the radial planes, and on the circular arcs,

$$g_\nu^H(a, \rho') = g_\nu^H(b, \rho') = 0. \tag{3.2}$$

The normal derivative of the E mode vanishes on the radial planes, as it does on the circular arcs:

$$\frac{\partial}{\partial\rho} g_\nu^E(\rho, \rho')|_{\rho=a,b} = 0. \tag{3.3}$$

Thus, the TE mode corresponds to a scalar mode satisfying Neumann boundary conditions, while the TM modes correspond to scalar Dirichlet modes. Therefore, the latter are exactly those found in the corresponding scalar calculation in paper I. Both scalar Green's functions satisfy the same equation:

$$\left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\kappa^2+\frac{\nu^2}{\rho^2}\right)g_\nu^{E,H}=\frac{1}{\rho}\delta(\rho-\rho'). \tag{3.4}$$

Therefore, imposing the boundary conditions (3.2) and (3.3) we find

$$\begin{aligned}g_\nu^H(\rho,\rho')&=I_\nu(\kappa\rho_<)K_\nu(\kappa\rho_>)\\&\quad-\frac{K_\nu(\kappa a)K_\nu(\kappa b)}{\Delta}I_\nu(\kappa\rho)I_\nu(\kappa\rho')\\&\quad-\frac{I_\nu(\kappa a)I_\nu(\kappa b)}{\Delta}K_\nu(\kappa\rho)K_\nu(\kappa\rho')\\&\quad+\frac{I_\nu(\kappa a)K_\nu(\kappa b)}{\Delta}[I_\nu(\kappa\rho)K_\nu(\kappa\rho')\\&\quad+K_\nu(\kappa\rho)I_\nu(\kappa\rho')],\end{aligned} \tag{3.5a}$$

$$\begin{aligned}g_\nu^E(\rho,\rho')&=I_\nu(\kappa\rho_<)K_\nu(\kappa\rho_>)\\&\quad-\frac{K'_\nu(\kappa a)K'_\nu(\kappa b)}{\hat\Delta}I_\nu(\kappa\rho)I_\nu(\kappa\rho')\\&\quad-\frac{I'_\nu(\kappa a)I'_\nu(\kappa b)}{\hat\Delta}K_\nu(\kappa\rho)K_\nu(\kappa\rho')\\&\quad+\frac{I'_\nu(\kappa a)K'_\nu(\kappa b)}{\hat\Delta}[I_\nu(\kappa\rho)K_\nu(\kappa\rho')\\&\quad+K_\nu(\kappa\rho)I_\nu(\kappa\rho')],\end{aligned} \tag{3.5b}$$

where

$$\Delta_\nu(\kappa a,\kappa b)=I_\nu(\kappa b)K_\nu(\kappa a)-I_\nu(\kappa a)K_\nu(\kappa b), \tag{3.6a}$$

$$\hat\Delta_\nu(\kappa a,\kappa b)=I'_\nu(\kappa b)K'_\nu(\kappa a)-I'_\nu(\kappa a)K'_\nu(\kappa b). \tag{3.6b}$$

A. Energy

Now using Eq. (2.13) we have for the energy per length in the z direction

$$\mathcal{E}=-\int\frac{d\zeta}{2\pi}\frac{dk}{2\pi}\zeta^2e^{i\zeta t_E}e^{ikZ}\times\sum_m\int_a^b d\rho\rho[g_\nu^E(\rho,\rho)+g_\nu^H(\rho,\rho)]. \tag{3.7}$$

In paper I we showed that

$$\int_a^b d\rho\rho g_\nu^H(\rho,\rho)=\frac{1}{2\kappa}\frac{\partial}{\partial\kappa}\ln\Delta, \tag{3.8}$$

and in just the same way we can show [15]

$$\int_a^b d\rho\rho g_\nu^E(\rho,\rho)=\frac{1}{2\kappa}\frac{\partial}{\partial\kappa}\ln\kappa^2\hat\Delta, \tag{3.9}$$

in terms of the quantities defined in Eq. (3.6). Therefore, the energy per unit length is given by

$$\mathcal{E}=-\frac{1}{4\pi}\int_0^\infty d\kappa\kappa^2f(\kappa\delta,\phi)\sum_m\frac{\partial}{\partial\kappa}\ln\kappa^2\Delta\hat\Delta. \tag{3.10}$$

Here, to explore the effects of different point-splitting schemes, we write

$$\zeta=\kappa\cos\gamma,\qquad k=\kappa\sin\gamma,\qquad t_E=\delta\cos\phi,\qquad Z=\delta\sin\phi, \tag{3.11}$$

where $Z=z-z'$ is an infinitesimal point splitting in the z direction, and then we define the regulator function

$$f(\kappa\delta,\phi)=\int_0^{2\pi}\frac{d\gamma}{2\pi}\cos^2\gamma e^{i\kappa\delta\cos(\gamma-\phi)}, \tag{3.12}$$

which equals $1/2$ for $\delta=0$. For finite δ, temporal splitting corresponds to

$$f(\kappa\delta,0)=J_0(\kappa\delta)-\frac{1}{\kappa\delta}J_1(\kappa\delta), \tag{3.13a}$$

while z splitting corresponds to

$$f(\kappa\delta,\pi/2)=\frac{1}{\kappa\delta}J_1(\kappa\delta). \tag{3.13b}$$

B. Torque

To compute the torque on one of the radial planes, we need to compute the angular component of the stress tensor,

$$\begin{aligned}\langle T^\theta{}_\theta\rangle&=-\frac12\langle E_\theta^2-B_\rho^2-B_z^2\rangle\\&=-\frac{1}{2i}\Big[\hat{\boldsymbol\theta}\cdot\boldsymbol\Gamma'\cdot\hat{\boldsymbol\theta}+\frac{1}{\omega^2}\hat{\boldsymbol\rho}\cdot\boldsymbol\nabla\times\boldsymbol\Gamma'\times\overleftarrow{\boldsymbol\nabla}'\cdot\hat{\boldsymbol\rho}\\&\quad+\frac{1}{\omega^2}\hat{\boldsymbol z}\cdot\boldsymbol\nabla\times\boldsymbol\Gamma'\times\overleftarrow{\boldsymbol\nabla}'\cdot\hat{\boldsymbol z}\Big]\Big|_{\mathbf r'\to\mathbf r}.\end{aligned} \tag{3.14}$$

The torque then is immediately obtained by integrating the first moment of this over one radial side of the annular region, that is, for $\theta=0$ or α:

$$\begin{aligned}\tau&=\int_a^b d\rho\rho\int_{-\infty}^\infty\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\langle T^\theta{}_\theta\rangle\\&=\frac{1}{\alpha}\sum_m\nu^2\int_0^\infty\frac{d\kappa\kappa}{2\pi}J_0(\kappa\delta)\int_a^b\frac{d\rho}{\rho}[g_\nu^E(\rho,\rho)+g_\nu^H(\rho,\rho)].\end{aligned} \tag{3.15}$$

In paper I we gave the radial integral for the TM part:

$$\int_a^b\frac{d\rho}{\rho}g_\nu^H(\rho,\rho)=-\frac{\alpha}{2\nu^2}\frac{\partial}{\partial\alpha}\ln\Delta; \tag{3.16}$$

and we can show the same form holds for the TE part [15]:

$$\int_a^b\frac{d\rho}{\rho}g_\nu^E(\rho,\rho)=-\frac{\alpha}{2\nu^2}\frac{\partial}{\partial\alpha}\ln\hat\Delta. \tag{3.17}$$

Thus the electromagnetic torque on one of the planes is

$$\tau=-\frac{\partial}{\partial\alpha}\frac{1}{4\pi}\sum_m\int_0^\infty d\kappa\kappa J_0(\kappa\delta)\ln\kappa^2\Delta\hat\Delta. \tag{3.18}$$

Using integration by parts in Eq. (3.10), and Bessel's equation, we see this is indeed the negative derivative with respect to the wedge angle of the interior energy provided $\phi = \pi/2$; that is, for point splitting in the z direction. We will now proceed to evaluate the energy, by explicitly isolating the divergent contributions as $\delta \to 0$, and extract the finite parts. Will it be true, as in the scalar case, that after renormalization the finite torque is equal to the negative derivative of the finite energy with respect to the wedge angle?

IV. DIVERGENT TERMS FOR THE TE ENERGY

We now turn to the examination of the Neumann or TE contribution to the Casimir energy of the annular region, which is

$$\hat{\mathcal{E}} = -\frac{1}{4\pi}\int_0^\infty d\kappa\kappa^2 f(\kappa\delta,\phi)\sum_{m=0}^\infty \frac{\partial}{\partial\kappa}\ln\kappa^2\hat{\Delta}, \tag{4.1}$$

where $\hat{\Delta}$ is given by Eq. (3.6b). As in the Dirichlet case, we expand the Bessel functions according to the uniform asymptotic expansion, which here reads [16]

$$I'_\nu(\nu\xi) \sim \frac{1}{\sqrt{2\pi\nu t}}\frac{1}{\xi}e^{\eta\nu}\left(1+\sum_{k=1}^\infty \frac{v_k(t)}{\nu^k}\right), \tag{4.2a}$$

$$K'_\nu(\nu\xi) \sim -\sqrt{\frac{\pi}{2\nu t}}\frac{1}{\xi}e^{-\eta\nu}\left(1+\sum_{k=1}^\infty (-1)^k\frac{v_k(t)}{\nu^k}\right), \tag{4.2b}$$

where[1] $t=(1+\xi^2)^{-1/2}$, $d\eta/d\xi = 1/(\xi t)$, and the polynomials $v_k(t)$ are generated from those for the functions I_ν and K_ν by

$$v_0(t) = 1,$$
$$v_k(t) = u_k(t) + t(t^2-1)\left[\frac{1}{2}u_{k-1}(t) + tu'_{k-1}(t)\right]. \tag{4.3}$$

Because of this behavior, the second product of Bessel functions in Eq. (3.6b) is exponentially subdominant. Thus the logarithm in Eq. (4.1) is asymptotically

$$\ln\kappa^2\hat{\Delta} \sim \text{constant} + \nu[\eta(\xi)-\eta(\tilde{\xi})] + (t^{-1/2}+\tilde{t}^{-1/2}) + \ln\left(1+\sum_{k=1}^\infty\frac{v_k(t)}{\nu^k}\right) + \ln\left(1+\sum_{k=1}^\infty(-1)^k\frac{v_k(\tilde{t})}{\nu^k}\right), \tag{4.4}$$

where $\xi = \kappa b/\nu$, $\tilde{\xi} = \xi a/b$, $\tilde{t} = (1+\tilde{\xi}^2)^{-1/2}$. Here the constant means a term independent of κ, which will not survive differentiation. Note that the $1/\xi$ behavior seen in the prefactors in Eq. (4.2) is canceled by the multiplication of $\hat{\Delta}$ by κ^2. In the following, we will consider the z-splitting regulator, $\phi = \pi/2$, since the result for time splitting may be obtained by differentiation:

$$\hat{\mathcal{E}}(0) = \frac{\partial}{\partial\delta}[\delta\hat{\mathcal{E}}(\pi/2)]. \tag{4.5}$$

We now extract the divergences, that is the terms proportional to nonpositive powers of δ, just as in paper I. We label those terms by the corresponding power of $1/\delta$. The calculation closely parallels that in paper I, except for the additional zero mode, $m=0$. Except for that term, the leading divergence is exactly that found in paper I,

$$\hat{\mathcal{E}}_4^{m>0} = -\frac{\alpha(b^2-a^2)}{4\pi^2\delta^4} + \frac{b-a}{8\pi\delta^3}. \tag{4.6}$$

However, the $m=0$ term yields

$$\hat{\mathcal{E}}_4^{m=0} = -\frac{b-a}{4\pi\delta^3}, \tag{4.7}$$

thereby (correctly) reversing the sign of the second term in Eq. (4.6). Thus the leading divergence is again the expected Weyl volume divergence:

$$\hat{\mathcal{E}}^{(4)} = -\frac{A}{2\pi^2\delta^4}, \qquad A = \frac{1}{2}\alpha(b^2-a^2). \tag{4.8}$$

Evidently, the $O(\nu^{-3})$ term, for $m>0$, is exactly reversed in sign from that for the Dirichlet term,

$$\hat{\mathcal{E}}_3^{m>0} = -\frac{\alpha(a+b)}{16\pi\delta^3} + \frac{1}{8\pi\delta^2}, \tag{4.9}$$

but again the sign of the subleading term is reversed by including $m=0$:

$$\hat{\mathcal{E}}_3^{m=0} = -\frac{1}{4\pi\delta^2}. \tag{4.10}$$

Thus, we get the correct surface area and corner terms:

$$\hat{\mathcal{E}}^{(3)} = -\frac{P}{16\pi\delta^3}, \qquad P = \alpha(a+b) + 2(b-a), \tag{4.11a}$$

$$\hat{\mathcal{E}}^{(2)} = -\frac{C}{48\pi\delta^2}, \qquad C = 4\left(\frac{\pi}{\pi/2} - \frac{\pi/2}{\pi}\right) = 6. \tag{4.11b}$$

Closely following the path blazed in computing the divergent terms coming from the polynomial asymptotic corrections in the Dirichlet case in paper I, but including the $m=0$ terms, we find the first three curvature corrections

$$\hat{\mathcal{E}}_2 = \frac{3}{64\pi}\frac{1}{\delta}\left(\frac{1}{a}-\frac{1}{b}\right), \tag{4.12a}$$

$$\hat{\mathcal{E}}_1 = -\frac{5}{1024}\frac{\alpha}{\pi}\frac{1}{\delta}\left(\frac{1}{a}+\frac{1}{b}\right) + \frac{3\ln\delta}{128\pi}\left(\frac{1}{a^2}+\frac{1}{b^2}\right). \tag{4.12b}$$

A. $m = 0$ case

Before proceeding, it is time to recognize that use of the uniform asymptotic expansion is apparently inconsistent for $m=0$, because $\nu=0$ then. So let us calculate the $m=0$ contribution directly from

[1]The variable ξ is the same as that called z in paper I; we have changed the notation here to avoid confusion with the axial coordinate.

$$\hat{\mathcal{E}}_{m=0} = -\frac{1}{4\pi}\int_0^\infty d\kappa\kappa^2 \frac{J_1(\kappa\delta)}{\kappa\delta}\frac{\partial}{\partial\kappa}\ln\kappa^2[I_0'(\kappa b)K_0'(\kappa a) - I_0'(\kappa a)K_0'(\kappa b)], \tag{4.13}$$

where the divergent terms arise from the large argument expansions

$$I_0'(x) \sim \frac{e^x}{\sqrt{2\pi x}}\left(1 - \frac{3}{8x} - \frac{15}{128x^2} + \cdots\right), \tag{4.14a}$$

$$K_0'(x) \sim e^{-x}\sqrt{\frac{\pi}{2x}}\left(1 + \frac{3}{8x} - \frac{15}{128x^2} + \cdots\right). \tag{4.14b}$$

Inserting this into Eq. (4.13) we obtain

$$\begin{aligned}\hat{\mathcal{E}}_{m=0} &\sim -\frac{1}{4\pi\delta}\int_0^\infty d\kappa J_1(\kappa\delta)\bigg[\kappa(b-a) + 1 \\ &\quad + \frac{3}{8}\frac{1}{\kappa}\left(\frac{1}{b} - \frac{1}{a}\right) + \frac{3}{8}\frac{1}{\kappa^2+\lambda^2}\left(\frac{1}{b^2} + \frac{1}{a^2}\right) + \cdots\bigg] \\ &\sim -\frac{b-a}{4\pi\delta^3} - \frac{1}{4\pi\delta^2} + \frac{3}{32\pi\delta}\left(\frac{1}{a} - \frac{1}{b}\right) \\ &\quad + \frac{3\ln\lambda\delta}{64\pi}\left(\frac{1}{a^2} + \frac{1}{b^2}\right).\end{aligned} \tag{4.15}$$

Here, in the last term we introduced a mass, $\kappa^2 \to \kappa^2 + \lambda^2$, in order to eliminate the infrared divergence. These terms all agree with the corresponding terms found from the uniform asymptotic expansion by taking $m = 0$. We might note that these terms are all independent of α, so cannot contribute to the torque, but for completeness we will retain them.

There is one remaining divergent term, arising from the $1/\nu^3$ term, but here we exclude $m = 0$, because that subtraction is not necessary since the corresponding $m = 0$ contribution to the energy is already finite at $\delta = 0$. That curvature term is

$$\hat{\mathcal{E}}_0 \sim \frac{\alpha\ln\delta}{180\pi^2}\left(\frac{1}{b^2} - \frac{1}{a^2}\right). \tag{4.16}$$

Let us summarize the divergent terms for the Neumann or TE modes:

$$\begin{aligned}\hat{\mathcal{E}}_{\text{div}} &= -\frac{A}{2\pi^2\delta^4} - \frac{P}{16\pi\delta^3} - \frac{C}{48\pi\delta^2} + \frac{3}{64\pi\delta}\left(\frac{1}{a} - \frac{1}{b}\right) \\ &\quad - \frac{5\alpha}{1024\pi\delta}\left(\frac{1}{a} + \frac{1}{b}\right) + \frac{3\ln\delta/\mu}{128\pi}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) \\ &\quad - \frac{\alpha\ln\delta/\mu}{180\pi^2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right).\end{aligned} \tag{4.17}$$

Here, we have introduced an arbitrary scale μ, which will appear in the finite part given in the next section.

B. Heat-kernel expansion

This small-δ Laurent expansion (4.17) exactly agrees with that found by the heat-kernel calculation of Dowker and Apps and of Nesterenko *et al.* [10–12], who consider a wedge intercut with a single coaxial circular cylinder with radius R. From the latter heat-kernel coefficients the cylinder-kernel coefficients can be readily extracted [17]. The trace of the cylinder kernel $T(t)$ is defined in terms of the eigenvalues of the Laplacian in d dimensions,

$$T(t) = \sum_j e^{-\lambda_j t} \sim \sum_{s=0}^\infty e_s t^{s-d} + \sum_{\substack{s=d+1 \\ s-d \text{ odd}}} f_s t^{s-d}\ln t, \tag{4.18}$$

where the expansion holds as $t \to 0$ through positive values. (This t is not to be confused with the quantity appearing in the uniform asymptotic expansion.) The energy is given by

$$E(t) = -\frac{1}{2}\frac{\partial}{\partial t}T(t), \tag{4.19}$$

which corresponds to the energy computed here with $\phi = 0$, that is, time splitting. In view of Eq. (4.5) we see that the z-splitting result should be identical to that of $-\frac{1}{2t}T(t)$ with $t \to \delta$. In this way we transcribe the results of Ref. [12] for the outside cylinder kernel per unit length:

$$\begin{aligned}-\frac{1}{2t}T(t) &\sim -\frac{A}{2\pi^2 t^4} - \frac{P}{16\pi t^3} - \frac{1}{16\pi^2 t^2} \\ &\quad + \frac{3 - 5\alpha/16}{64\pi R t} + \frac{\ln t}{16\pi^2 R^2}\left(\frac{3\pi}{8} - \frac{4\alpha}{45}\right).\end{aligned} \tag{4.20}$$

This exactly agrees with Eq. (4.17) when $a \to R$ and $b \to \infty$ (except that the the latter limit is not taken in the first two terms). The reason for the factor of 2 discrepancy in the third (corner) term is that Nesterenko *et al.* have only two corners, not four.

V. EXTRACTION OF FINITE PART

Just as in the Dirichlet case considered in paper I, the divergent terms have finite remainders, which we state here:

$$\begin{aligned}\hat{\mathcal{E}}_f &= -\frac{\pi^2}{2880\alpha^3}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) - \frac{\zeta(3)}{64\pi\alpha^2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) + \frac{1}{576\alpha}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + \left\{\frac{3}{128\pi b^2}\left[-\frac{11}{12} + \gamma + \ln\frac{b\alpha}{\mu} + 2\ln\mu\lambda\right] + (b \to a)\right\} \\ &\quad + \left\{\frac{\alpha}{\pi b^2}\left(-\frac{1}{180\pi}\ln\frac{b\alpha}{\pi\mu} + \frac{1079}{69120}\right) - (b \to a)\right\} + \frac{29}{46080}\frac{\alpha^2}{\pi}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) - \frac{5}{12012}\frac{\alpha^3}{\pi^4}\zeta(3)\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + \hat{\mathcal{E}}_R.\end{aligned} \tag{5.1}$$

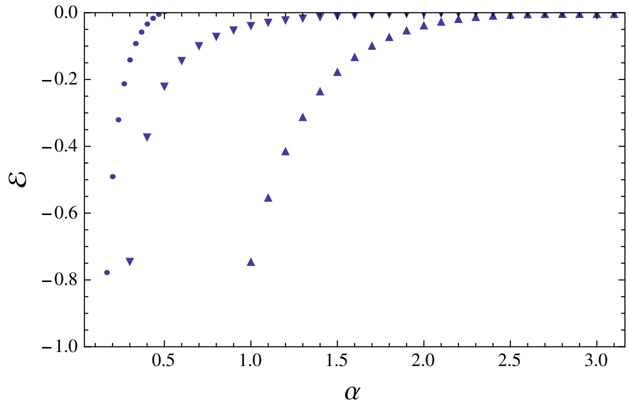

FIG. 2 (color online). Renormalized energy for TE modes for $a/b = 0.1$ (triangles), $a/b = 0.5$ (inverted triangles), and $a/b = 0.9$ (dots). Here the energies per unit length, in units of $1/b^2$, are plotted as a function of the wedge angle α.

The last two explicitly given terms are what come from the next two terms in the uniform expansion for $m > 0$. Note that we have made no approximation here; we have merely added and subtracted the leading terms in the uniform asymptotic expansion of the integrand for the energy. The remainder, therefore, consists of two parts: that arising from $m = 0$,

$$\hat{\mathcal{E}}_{R0} = -\frac{1}{8\pi}\int_0^\infty d\kappa\kappa\left[\kappa\frac{\partial}{\partial\kappa}\ln\kappa^2\tilde{\Delta}_{m=0} - \kappa(b-a) - 1 - \frac{3}{8\kappa}\left(\frac{1}{b}-\frac{1}{a}\right) - \frac{3}{8(\kappa^2+\lambda^2)}\left(\frac{1}{b^2}+\frac{1}{a^2}\right)\right], \quad (5.2)$$

and the rest coming from the terms with $m > 0$,

$$\hat{\mathcal{E}}'_R = -\frac{1}{8\pi b^2}\sum_{m=1}^\infty \nu^3\int_0^\infty d\xi\xi^2\left[\hat{f}(\nu,\xi,a/b) + \sum_{n=4}^{-2}\hat{f}_n(\nu,\xi,a/b)\right]. \quad (5.3)$$

Here, with the abbreviations $I = I_\nu(\nu\xi)$, $\tilde{I} = I_\nu(\nu\xi a/b)$, etc., the original integrand in Eq. (4.1) is

$$\hat{f} = \frac{(1+\frac{1}{\xi^2})(I\tilde{K}' - K\tilde{I}') + \frac{a}{b}(1+\frac{b^2}{a^2\xi^2})(I'\tilde{K} - K'\tilde{I})}{I'\tilde{K}' - K'\tilde{I}'}. \quad (5.4)$$

The subtractions are easily read off:

$$\hat{f}_4 = -\frac{1}{\xi t} + \frac{a}{b}\frac{1}{\tilde{\xi}\tilde{t}}, \quad (5.5a)$$

$$\hat{f}_3 = -\frac{1}{2\nu}\left(\xi t^2 + \frac{a}{b}\tilde{\xi}\tilde{t}^2\right), \quad (5.5b)$$

$$\hat{f}_2 = \frac{1}{8\nu^2}\xi t^3(-3+7t^2) - \frac{a}{b}(\xi\to\tilde{\xi}), \quad (5.5c)$$

$$\hat{f}_1 = \frac{1}{8\nu^3}\xi t^4(-3+20t^2-21t^4) + \frac{a}{b}(\xi\to\tilde{\xi}), \quad (5.5d)$$

$$\hat{f}_0 = \frac{1}{5760\nu^4}\xi t^5(-2835+39105t^2-99225t^4+65835t^6) - \frac{a}{b}(\xi\to\tilde{\xi}), \quad (5.5e)$$

$$\hat{f}_{-1} = \frac{1}{128\nu^5}\xi t^6(-108+2616t^2-11728t^4+17640t^6-8484t^8) + \frac{a}{b}(\xi\to\tilde{\xi}), \quad (5.5f)$$

$$\hat{f}_{-2} = \frac{1}{32560\nu^6}\xi t^7(-598185+22680945t^2-156073050t^4+393353730t^6-415212525t^8+156010365t^{10}) - \frac{a}{b}(\xi\to\tilde{\xi}). \quad (5.5g)$$

The last two subtractions, and the associated terms in Eq. (5.1), are not necessary, but they improve convergence.

VI. NUMERICS

The extraction of the finite part follows the same procedure described in paper I. The total finite energy given in Eq. (5.1) is the sum of the explicitly given finite terms plus the remainder:

$$\hat{\mathcal{E}}_f = \sum_{n=4}^{-2}\hat{\mathcal{E}}_n^f + \hat{\mathcal{E}}_R, \quad (6.1)$$

where $\hat{\mathcal{E}}_R$ is the sum of Eqs. (5.2) and (5.3).

The total energy becomes a linear function of α for sufficiently large wedge angles. But because of the logarithmically divergent parts in the energy, such linear terms are undetermined. That is, we can add to the energy an arbitrary counterterm of the form

$$\hat{\mathcal{E}}_{ct} = A + B\alpha. \quad (6.2)$$

We subtract off the linear behavior found numerically from Eq. (6.1), because the energy should approach zero for sufficiently (but not very) large α. In this way, we get the Neumann (TE) energies seen in Fig. 2, very similar to what we found for the Dirichlet (TM) contribution.

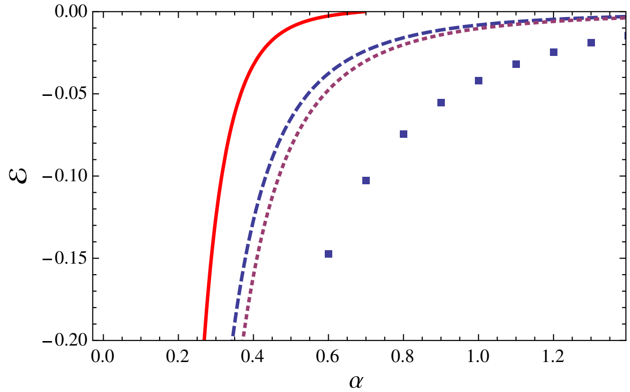

FIG. 3 (color online). Comparison of the renormalized energy per length (in units of $1/b^2$) for TE modes (squares) and TM modes (upper red curve), with the Casimir limit (6.3) (blue dashes) and the PFA approximation [first term in Eq. (5.1)] (red dots), the PFA being larger than the Casimir limit in magnitude by 27%, for $a/b = 0.5$. The TE mode is always much larger in magnitude than the TM contribution.

The TE and TM contributions are both displayed in Fig. 3, as well as the result expected for either TE or TM modes for parallel plates, which is approached as $\alpha \to 0$:

$$\mathcal{E}_C = -\frac{\pi^2}{180b^2}\frac{1-a/b}{(1+a/b)^3}\frac{1}{\alpha^3}. \tag{6.3}$$

This formula is valid in the regime $\alpha \ll 1$, $1 - a/b \ll 1$, in which case it agrees with the leading finite term in Eq. (5.1), which is the same as the α^{-3} term for the Dirichlet case. Those leading terms are, in fact, the proximity force approximation (PFA). Figure 3 shows that this limit is indeed approached for both TE and TM modes, from opposite sides, but that the TE mode is always considerably larger in magnitude than the TM mode, which is a phenomenon observed previously in a related context [18].

VII. CONCLUSIONS

Because of curvature divergences, it is impossible to extract a unique finite part of the energy. However, the divergences are all constant or linear in the wedge angle α. Therefore, we can renormalize the energy by subtracting the linear dependence for large angles, to impose a physical requirement that the energy go to zero when the separation between the wedge planes is large. The resulting energy is completely finite, independent of regularization scheme, and exhibits no torque anomaly:

$$\tau(\alpha) = -\frac{\partial}{\partial\alpha}\mathcal{E}(\alpha). \tag{7.1}$$

These results, of course, are consistent with, and generalize to electromagnetism, the annular piston work of Ref. [9]. It is remarkable how similar the electromagnetic calculation is to that for the Dirichlet scalar.

So, as with the scalar, Dirichlet, case, there is no sign of a torque anomaly. Here, this is even more surprising, because in the Dirichlet situation, the anomaly is manifested by linear terms in α in the energy, which would be canceled by the corresponding exterior ($\theta \in [\alpha, 2\pi]$) contribution for an annular piston, as well as being removed by our ‘‘renormalization’’ procedure. As emphasized in Ref. [1], for the electromagnetic wedge, there is an additional anomalous term in the energy $\sim\alpha^{-1}$ [4], which would not disappear if the exterior contribution were included, and should not be removed by renormalization. The reason we do not see this effect here will be explored further as we study the local regulated stress tensor.

To summarize, in these two papers, we have explored the torque τ (per unit length) on one side of an annular sector, formed by the intersection of two planes, and two coaxial cylinders. The question we asked was whether the torque was somehow anomalous, in that

$$\tau \neq -\frac{\partial}{\partial\alpha}\mathcal{E}. \tag{7.2}$$

Here $\mathcal{E}$ is the energy (per unit length) contained within the sector, and α is the dihedral angle between the planes. In the first paper, the quantum vacuum energy and torque were computed for a massless scalar field subject to Dirichlet boundary conditions on all the surfaces, and in the present paper, the boundaries are perfect conductors, and the fluctuating field is the electromagnetic one. In both cases we computed the divergent and finite parts of the energy, obtained by point splitting in either the (Euclidean) time or the axial direction. The physical normalization requirement that the energy of the annular sector go to zero for sufficiently large wedge angles allows us to define a finite, nonanomalous renormalized energy. The possibility of doing so, however, depends on the existence of an inner cylindrical boundary. Without that boundary it is not possible to define a torque or an energy, and ambiguities such as the torque anomaly can appear.

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