PHYSICAL REVIEW D **88,** 045026 (2013)

Revisiting the Faddeev-Skyrme model and Hopf solitons

A. Gorsky,[1,2,4] M. Shifman,[2] and A. Yung[2,3]

[1]*Theory Department, ITEP, Moscow 117218, Russia*
[2]*William I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, Minnesota 55455, USA*
[3]*Petersburg Nuclear Physics Institute, Gatchina, St. Petersburg 188300, Russia*
[4]*Moscow Institute for Physics and Technology, Dolgoprudny 141700, Russia*
(Received 27 June 2013; published 26 August 2013)

We observe that the Faddeev-Skyrme model emerges as a low-energy limit of scalar QED with two charged scalar fields and a self-interaction potential of a special form (inspired by supersymmetric QED). Then we discuss possible Hopf solitons of the "twisted-toroidal" type.

DOI: 10.1103/PhysRevD.88.045026 PACS numbers: 11.15.Kc, 11.27.+d

I. INTRODUCTION

Many field theories, in particular, supersymmetric Yang–Mills theories, support topologically stable solitons. Their stability is due to the existence of certain topological charges (in the case of supersymmetry, they are usually related to central charges of the relevant superalgebra [1]). In such cases one can perform the Bogomol'nyi completion [2] for the energy functional (in the instanton case, for the action), which selects the filed configuration corresponding to the minimal energy in the sector with the given topological charge. Well-known examples are the Abrikosov–Nielsen–Olesen (ANO) strings [3], for which the topological stability is due to $\pi_1(\mathrm{U}(1)) = Z$; instantons in the two-dimensional $CP(1)$ model [4], for which the topology is determined by $\pi_2(\mathrm{SU}(2)/\mathrm{U}(1)) = Z$; and the Belavin–Polyakov–Schwartz–Tyupkin instantons [5] in four-dimensional Yang–Mills theory, for which the topological classification is based on $\pi_3(\mathrm{SU}(2)) = Z$. Faddeev and Niemi discovered [6] a novel class of solitons, of the knot type, for which the stability is due to the existence of the Hopf topological invariant.

The model with the solitonic knots considered by Faddeev and Niemi is a deformed O(3) nonlinear sigma model in four dimensions,

$$\mathcal{L} = \frac{F^2}{2}\partial_\mu \vec{S}\partial^\mu \vec{S} - \frac{\lambda}{4}(\partial_\mu \vec{S} \times \partial_\nu \vec{S}) \cdot (\partial^\mu \vec{S} \times \partial^\nu \vec{S}), \tag{1.1}$$

where the three-component field $\vec{S}$ is an "isotopic" vector subject to the constraint

$$\vec{S}^{\,2} = 1. \tag{1.2}$$

The second term in Eq. (1.1) presents a deformation of the O(3) model. Sometimes it is referred to as the Skyrme–Faddeev, or Faddeev–Hopf, model; for a review, see Ref. [7]. The constant F has dimension $[m^2]$ while λ is dimensionless.[1]

The vacuum corresponds to a constant value of $\vec{S}$, which we can choose as $(\vec{S})_{\mathrm{vac}} = (0, 0, 1)$. Because of Eq. (1.2), the target space of the sigma model at hand is S_2. Finiteness of the soliton energy implies that the vector $\vec{S}$ must tend to its vacuum value at the spatial infinity,

$$\vec{S} \to \{0, 0, 1\} \quad \text{at } |\vec{x}| \to \infty. \tag{1.3}$$

The boundary condition (1.3) compactifies the space to S_3. Since

$$\pi_3(S_2) = Z,$$

the knot solitons present topologically nontrivial maps of $S_3 \to S_2$. As was noted in Ref. [6], there is an associated integer topological charge, the Hopf invariant, which presents the soliton number. This charge cannot be the degree of mapping $S_3 \to S_2$ because dimensions of S_3 and S_2 are different.

Faddeev and Niemi conjectured [6] that the Hopf solitons can carry a twisted-toroidal structure and can form knotted configurations. Later this was confirmed in numerical studies [8–10]. The physical meaning of the Hopf invariant can be visualized in a rather transparent way if we have a large size stringlike toroidal structure with windings. Then the Hopf number should reduce to the topological number in the perpendicular slice times the number of windings, as shown in Fig. 1 illustrating that the topological stability is enforced due to a twist of a U(1) phase associated with the Belavin–Polyakov instanton (for a review, see, e.g., Ref. [11]). In other words, the Hopfion picture presumably becomes rather simple in the limit when the ratio of the periods is a large number.

The Hopf solitons were identified[2] in solid state physics [12,13] and in QCD with quarks in the adjoint representation [14]. It was argued [15] that an interpolation between a baryon number-2 Skyrmion and a Hopf soliton can be found. For a review on this subject, see Ref. [16]. Hopf-type solitons were discussed in Ref. [17], where the

[1]Below we will introduce a different parametrization in which $F^2 = \xi/2$ and $\lambda = (\beta - 1)/g^2$. Moreover, $g^2\xi \equiv m_\gamma^2$. The origin of this parametrization will become clear shortly.

[2]We will comment more on Hopfions in solid state physics at the end of Sec. V.

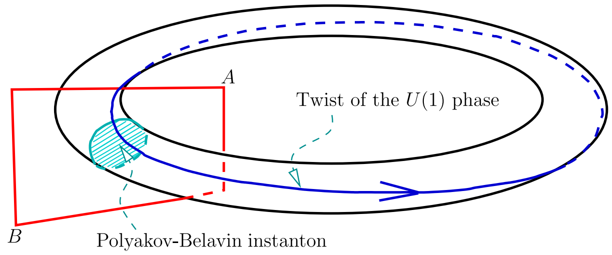

FIG. 1 (color online). The simplest Hopf soliton, in the adiabatic limit, corresponds to a Belavin–Polyakov "instanton" extended in one extra dimension and bent into a torus, with a 2π twist of the instanton phase modulus.

$\mathcal{N} = 2\,\mathrm{U}(1)$ gauge theory was considered. A theory with several discrete distinct vacua was engineered [17] supporting domain walls. Then one folds such a wall into a cylindrical structure and bends the cylinder to form a torus. It was argued that the structure thus obtained is a Hopf-type soliton on the perturbed Higgs branch of the moduli space. The soliton was stabilized by a twist inducing an Abelian charge. A resurgence of interest to Hopfions is reflected in the recent publications [9,10].

In this paper we address some aspects of the Hopf solitons and the Skyrme–Faddeev model. We start with the proof of the following statement: The Skyrme–Faddeev model is the low-energy limit of scalar QED with a potential of a certain type (inspired by supersymmetric QED). Then we present arguments (valid provided two key parameters are large and based on the Skyrme–Faddeev model *per se* and the underlying parent theory, scalar QED) that the Hopfions of the twisted-toroidal type do exist. We also briefly discuss similar constructions in four-dimensional cylinder geometry.

The paper is organized as follows. In Sec. II we outline a general picture behind the emergence of the twisted-toroidal solitons and their relation with the Hopf invariant. In Sec. III we present our basic model: scalar QED with two charged flavors. Although this model is not supersymmetric, the form of the scalar field interacting potential is prompted by supersymmetry. The U(1) gauge group is Higgsed, and in the low-energy limit ($m_\gamma \to \infty$), we demonstrate the emergence of the Skyrme–Faddeev model for the massless fields. Thus, the suggested renormalizable model can be viewed as an ultraviolet completion of the Skyrme–Faddeev model. Section IV is devoted to peculiarities of the "straight" strings in this model. In Sec. V we explain how to construct a Hopfion of the twisted-toroidal type by introducing windings and why our analytical consideration is applicable. The applicability requires choosing two free parameters to be large. In Sec. VI we fractionalize the Belavin–Polyakov instanton by compactifying one spatial dimension onto S_1. This automatically fixes the size modulus in terms of the size of S_1.

II. PRELIMINARIES

First, let us note that if a Hopf soliton is found in the model with the energy functional

$$\mathcal{E} = \int d^3x \left[\frac{F^2}{2} \partial_i \vec{S} \partial_i \vec{S} + \frac{1}{4} (\partial_i \vec{S} \times \partial_j \vec{S})(\partial_i \vec{S} \times \partial_j \vec{S}) \right], \tag{2.1}$$

one can homogeneously inflate all spatial dimensions by passing to

$$\mathcal{E} = \int d^3x \sqrt{\lambda} \left[\frac{F^2}{2} \partial_i \vec{S} \partial_i \vec{S} + \frac{\lambda}{4} (\partial_i \vec{S} \times \partial_j \vec{S})(\partial_i \vec{S} \times \partial_j \vec{S}) \right], \tag{2.2}$$

and then performing the transformations

$$x \to \sqrt{\lambda} x. \tag{2.3}$$

The Hopf invariant cannot be written as an integral of a local density—local in the field $\vec{S}$. However, if one uses a U(1) gauged formulation of the $CP(1)$ sigma model in terms of the doublet fields n^i, $\bar{n}_i$ ($i = 1, 2$) and

$$\bar{n}_i n^i = 1, \qquad \vec{S} = \bar{n}\,\vec{\tau}\, n, \tag{2.4}$$

(for a review, see Refs. [11,18]), then the Hopf invariant reduces to the Chern–Simons term for the above gauge field [6],

$$\mathcal{H} = \frac{1}{4\pi^2} \int d^3x \epsilon^{\mu\nu\rho} (A_\mu \partial_\nu A_\rho), \quad A_\mu = -\frac{i}{2} \bar{n} \overleftrightarrow{\partial}_\mu n. \tag{2.5}$$

The Hopf solitons that were found numerically [8] have an intricate knotlike shape. At the same time, the Hopf invariant seemingly has a transparent meaning in the Skyrme–Faddeev model. Namely, for the toruslike configurations (Fig. 1), it reduces to the instanton number (or, alternatively, magnetic flux) in the perpendicular slice times a winding along the torus large cycle.

To illustrate this interpretation, consider a four-dimensional gauge theory which can be obtained from Witten's superconducting string model [19] by its reduction. Namely, let us downgrade one of two U(1)'s of the Witten model to a global symmetry, rather than local,

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + |\mathcal{D}_\mu \phi|^2 - \frac{\lambda_\phi}{4} (\phi^2 - v_\phi^2)^2 + |\partial_\mu \chi|^2 - \frac{\lambda_\chi}{4} (\chi^2 - v_\chi^2)^2 - \beta \phi^2 \chi^2. \tag{2.6}$$

This model is a crossbreed between those used in Refs. [20,21]. If the constants $\lambda_{\phi,\chi}$ and β are appropriately chosen, the field ϕ condenses in the vacuum, Higgsing the gauge U(1) symmetry and, simultaneously, stabilizing the field χ. Then in the vacuum, $\chi_{\mathrm{vac}} = 0$, which implies that the global U(1) associated with the χ phase rotations remains unbroken. The theory (2.6) obviously supports a string which is almost the ANO string. There is an important distinction, however. In the string core, $\phi = 0$, and the $\beta \phi^2 \chi^2$ term stabilizing χ is switched off. Having $\chi = 0$

inside the string is energetically inexpedient. Thus, the string solution has $\chi \neq 0$ in the core [19]. This spontaneously breaks the global U(1) on any given string solution. As a result, a massless phase field $\in$ U(1) is localized on the string. The world sheet theory becomes

$$S = \int dtdz\left\{\frac{T}{2}[(\partial_\mu x_0)^2 + (\partial_\mu y_0)^2] + f^2(\partial_\mu \alpha)^2\right\}, \quad (2.7)$$

where T is the string tension, f is a (dimensionless) constant which can be expressed in terms of the bulk parameters, t is time, z is the coordinate along the string, while x_0 and y_0 are perpendicular coordinates. They can be combined as $x_\perp = \{x_1, x_2\}$, where $x_\perp$ depends on t and z,

$$x_\perp = x_\perp(t, z).$$

Moreover, $\alpha(t, z)$ is the phase field on the world sheet, $\alpha \leftrightarrow \alpha \pm 2\pi \leftrightarrow \alpha \pm 4\pi \ldots$. In other words, the target space of α is the unit circle.

Now, let us take a long Abrikosov string and bend it into a circle of circumference L. If α is constant along z (say, $\alpha = 0$), this configuration is obviously unstable. Minimizing its energy, the torus will shrink until L becomes of the order of the string thickness ℓ, and then the string will annihilate. However, one can stabilize it by forcing α to wind along z in such a way as to make the full 2π winding when z changes from 0 to L,

$$\alpha(t, z) = 2\pi z/L. \quad (2.8)$$

Note that α linearly depending on z goes through the equation of motion on the world sheet, $\partial^2\alpha = 0$. For k windings

$$\alpha_k(t, z) = 2\pi zk/L, \qquad k = 2, 3, \ldots \quad (2.9)$$

It is not difficult to estimate the value of L. Indeed, the string energy is[3]

$$E = TL + \frac{(2\pi f)^2}{L}. \quad (2.10)$$

Minimizing (2.10) with respect to L, we get

$$L = 2\pi f/\sqrt{T}. \quad (2.11)$$

Making f large enough, we can always force L to be much larger than the flux tube thickness ℓ, which is, roughly speaking, of the order of $1/\sqrt{T}$. Alternatively, we can make k large enough. For k windings $L = 2\pi fk/\sqrt{T}$.

The soliton of the type discussed above was first constructed in Ref. [20], where it goes under a special name "vorton" (in the context of cosmic strings; for a recent review and a rather extended list of references, see Ref. [23]). Needless to say, in the problem of vortons, we do not have a topological Hopf invariant in the strict mathematical sense of this word. However, the very existence of the Hopf-like solitons demonstrated above shows that, perhaps, something like a "quasi-invariant" does exist. Indeed, consider a class of field configurations in which the field χ vanishes nowhere except, perhaps, spatial infinity.[4] For such field configurations, one can define $\alpha = \mathrm{Arg}(\chi)$ at all spatial points except, perhaps, infinity. Then consider the following integral[5]:

$$h = \int d^3xc\epsilon^{ijk}F_{ij}\partial_k\alpha, \quad (2.12)$$

where c is a normalizing constant. It is obvious that h is an integral over a full derivative. For field configurations with no zeros of χ, it is well defined. Moreover, if for a given field configuration $h \neq 0$, then this field configuration describes, say, a "twisted torus" similar to the Hopf solitons. The above twisted torus is classically stable. Instability occurs through tunneling.

As was mentioned, the model considered above does *not* possess an honest-to-god Hopf invariant. Below we will demonstrate that a slightly more complicated renormalizable model—four-dimensional QED, with two scalar flavors and a potential of a special form in the Higgs regime—reduces exactly to the Skyrme–Faddeev model (1.1) in the low-energy limit $m_\gamma^2 \to \infty$ and thus supports the Hopf invariant. We then formulate a condition under which a stable vorton exists in this model. Although the condition of existence is likely to be met, the corresponding arguments are heuristic rather than rigorous. The relation between the vorton in two-flavor scalar QED and Hopfions obtained numerically, e.g., in Refs. [8–10] (see also Ref. [7]) remains unclear.

III. BASIC MODEL AND ITS LOW-ENERGY LIMIT

A. Basic model

To begin with we will analyze four-dimensional scalar QED with two flavors and the self-interaction potential of a special form. The model is nonsupersymmetric, but the form of the potential is supersymmetry-inspired.

The action can be written as follows:

$$S_0 = \int d^4x\left\{-\frac{1}{4g^2}F_{\mu\nu}^2 + |\mathcal{D}_\mu\varphi^A|^2 - \lambda(|\varphi^A|^2 - \xi)^2\right\}, \quad (3.1)$$

where $\mathcal{D}_\mu$ is the covariant derivative

$$\mathcal{D}_\mu = \partial_\mu - iA_\mu; \quad (3.2)$$

A is the flavor index, $A = 1, 2$; and ξ is a real positive parameter (which can be identified as the Fayet–Iliopoulos

[3]For an alternative idea on obtaining a similar $L + \frac{1}{L}$ formula, see Ref. [22].

[4]This is a dynamical requirement, of course, demanding the toric flux tube's length to be much larger than its thickness.

[5]Note that in this simple example, the Hopf-like charge (2.12) is nothing but the integral of the charge density component of the conventional Goldstone–Wilczek anomalous current. It can be identified as the anomalous contribution to the electric or axial charges depending on the parity of the scalar field.

term [24] in supersymmetric QED). Moreover, for what follows we will introduce a parameter β for the ratio of the coupling constants,

$$\beta = \frac{2\lambda}{g^2}. \tag{3.3}$$

In supersymmetric QED we would have $\beta = 1$. However, as we will see below, to stabilize the Hopfion in the semiclassical regime, one must require $\beta \gg 1$. The vacuum energy in Eq. (3.1) vanishes. The vacuum manifold is determined by

$$|\varphi^1|^2 + |\varphi^2|^2 = \xi. \tag{3.4}$$

The above constraint leaves us with three real parameters out of four residing in $\varphi^{1,2}$. One extra (phase) parameter can be eliminated by imposing an appropriate gauge condition. It is easy to see that the vacuum manifold is nothing other than S_2, presenting the target space of the $CP(1)$ model. The U(1) gauge boson is Higgsed. The spectrum of the model consists of a massive photon; a massive Higgs meson,

$$m_\gamma = \sqrt{2} g \sqrt{\xi}, \qquad m_H = m_\gamma \sqrt{\beta}; \tag{3.5}$$

and two massless Goldstone particle corresponding to oscillations on the vacuum manifold. Below we will be interested in the limit $\beta \gg 1$ or, alternatively, $m_H \gg m_\gamma$, known as the London (or Abrikosov) limit in the theory of the ANO strings.

The model (3.4) supports semilocal strings (see, e.g., Refs. [25,26]). Their core is provided by the Abrikosov–Nielsen–Olesen string [3], while the tail is due to the Belavin–Polyakov two-dimensional instanton [4] of the $CP(1)$ model lifted in four dimensions.

B. Low-energy limit

If excitation energies are lower than m_γ, the photon and the Higgs boson can be integrated out. Then we obtain the low-energy theory for the moduli fields, which, as was mentioned, reduces to the $CP(1)$ sigma model.

In fact, at $\beta = 1$ (i.e., in the supersymmetric limit) the answer is known. If we introduce normalized n fields,

$$n^A = \frac{\varphi^a}{\sqrt{\xi}}, \qquad \bar{n}_A n^A = 1, \tag{3.6}$$

in the gauged formulation, the low-energy model for the moduli fields is the standard $CP(1)$ model,

$$\mathcal{L} = \xi (\mathcal{D}_\mu \bar{n}_A)^\dagger (\mathcal{D}^\mu n^A). \tag{3.7}$$

The field A_μ in the definition of the covariant derivative is auxiliary; it has no kinetic term and is expressible in terms of n^A (see, e.g., Sec. 27.2 in Ref. [18]),

$$A_\mu = -\frac{i}{2} (\bar{n}_A \overleftrightarrow{\partial}_\mu n^A). \tag{3.8}$$

The relation between S^a in Eq. (1.1) and n is

$$S^a = \bar{n} \tau^a n, \qquad a = 1, 2, 3, \tag{3.9}$$

where τ^a are the Pauli matrices.

Now we will show that at $\beta \neq 1$ an additional term is generated in the low-energy action—the Skyrme–Faddeev term in Eq. (1.1).

The simplest way to establish the existence of the Skyrme–Faddeev term is to evaluate the two actions (microscopic and its low-energy limit) in a string background field satisfying the Bogomol'nyi equations.

In the microscopic theory (3.1) these equations are

$$F_{12} + g^2(|\varphi^A|^2 - \xi) = 0, \quad (\mathcal{D}_1 + i\mathcal{D}_2)\varphi^{1,2} = 0. \tag{3.10}$$

The string is assumed to be aligned along the z axis. If these equations are satisfied, the microscopic action takes the form

$$\Delta S = \int d^4x \frac{g^2}{2} (\beta - 1)(|\varphi^A|^2 - \xi)^2 = \int d^4x \frac{\beta - 1}{2g^2} F_{12}^2. \tag{3.11}$$

This expression can be obviously uplifted to four dimensions,

$$\Delta S = \int d^4x \frac{\beta - 1}{4g^2} F_{\mu\nu}^2. \tag{3.12}$$

Generally speaking, Eq. (3.11) could miss possible four-derivative terms that vanish on the Bogomol'nyi (anti)self-dual fields. One can see, however, that no such terms can be uplifted to four dimensions in the Lorentz-invariant way. Equation (3.12) is the only exception. Therefore, the result (3.12) is complete.

Thus, we conclude that the low-energy theory for the scalar QED (3.1) is

$$S = \int d^4x \left\{ \frac{\xi}{4} \partial_\mu S^a \partial^\mu S^a - \frac{\beta - 1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \cdots \right\}, \tag{3.13}$$

where $F_{\mu\nu}$ should be expressed in terms of S^a using (3.8) and (3.9), namely,

$$F_{\mu\nu} = \frac{1}{2} \varepsilon_{abc} S^a \partial_\mu S^b \partial_\nu S^c. \tag{3.14}$$

Then

$$F_{\mu\nu}^2 = \frac{1}{4} (\varepsilon_{abc} \partial_\mu S^b \partial_\nu S^c)^2, \tag{3.15}$$

cf. Eq. (1.1). The ellipses in Eq. (3.13) stand for higher derivative corrections (with six derivatives and higher). The theory in Eq. (3.13) is the Skyrme–Faddeev model (1.1), with a particular choice of parameters.

IV. SEMILOCAL ABELIAN STRINGS

In scalar QED with two flavors [see Eq. (3.1)], strings are no longer the conventional ANO strings with exponentially small tails of the profile functions. The presence of massless fields in the bulk makes them semilocal. The semilocal strings have a power falloff at large distances from the string axis (see Ref. [27] for a review). The semilocal Bogomol'nyi-Prasad-Sommerfield (BPS) string interpolates between the ANO string and two-dimensional O(3) sigma-model instanton uplifted to four dimensions (also known as the lump). The semilocal string possesses an additional zero mode associated with string's transverse size ρ. In the limit $\rho \to 0$, we recover the ANO string while at $\rho \gg 1/m_\gamma$ it becomes a lump.

A. Semilocal string solution

Consider first the BPS-saturated semilocal string in the theory with $\beta = 1$. The ansatz for the string solution has the following structure:

$$\varphi^1(x) = \phi_1(r)e^{i\theta}, \qquad \varphi^2(x) = \phi_2(r),$$
$$A_i(x) = -\varepsilon_{ij}\frac{x_j}{r^2}[1 - f(r)], \qquad r \equiv \sqrt{\vec{x}_\perp^2}, \tag{4.1}$$

with the boundary conditions implying that only one scalar field has a nonvanishing condensate inside the core,

$$\phi_1(0) = 0, \qquad f(0) = 1, \qquad \phi_2(\infty) = 0,$$
$$\phi_1(\infty) = \sqrt{\xi}, \qquad f(\infty) = 0. \tag{4.2}$$

Here r and θ are polar coordinates in the plane orthogonal to the string axis.

It is not difficult to find an approximate solution valid at large values of the scale modulus,

$$\phi_1(r) = \sqrt{\xi}\frac{r}{\sqrt{r^2 + |\rho|^2}}, \qquad \phi_2(r) = \sqrt{\xi}\frac{\rho}{\sqrt{r^2 + |\rho|^2}},$$
$$f = \frac{|\rho|^2}{r^2 + |\rho|^2}, \tag{4.3}$$

with the complex modulus ρ. The absolute value $|\rho|$ is the scale modulus, while $\arg\rho$ is a phase modulus inherent to the Belavin–Polyakov lump. This solution is related to the Belavin–Polyakov instanton solution in the two-dimensional O(3) sigma model uplifted to four dimensions.

B. Effective world sheet theory

Substituting the static solution in the original four-dimensional action (3.1) and assuming that the modulus ρ has a slow adiabatic dependence on the world sheet coordinates t and z, we get the following answer:

$$\mathcal{E}_{N_f=2} = 2\pi\xi \int dtdz|\partial_k\rho|^2 \ln\frac{L}{|\rho|}, \tag{4.4}$$

where $k = 0, 3$ labels the world sheet coordinates. This action has only the kinetic term and no potential term. This is because ρ is associated with the exact zero mode of the string solution for the BPS string. Note that the integral over r is logarithmically divergent in the infrared; a cutoff is provided by the string by the length L (which is supposed to be very large but finite). We work in the logarithmic approximation and assume that $L \gg |\rho|$. A similar effective low-energy theory was obtained in Ref. [25] for non-Abelian semilocal strings.

Recently logarithmic divergence of the norm of the size zero modes for a semilocal vortex in (2 + 1) dimensions was addressed in Ref. [28]. It was shown that in (2 + 1) dimensions, there is a superselection rule for vortices with different ρ. In our (3 + 1) case, the vortex become a string, and its finite length L provides a physical infrared cutoff for the logarithmic divergence. The logarithmic factor in Eq. (4.4) is large but still finite.

In what follows we will consider a more generic case with $\beta > 1$ and, in particular, the large-β limit. If $\beta \neq 1$ the string becomes non-BPS: it tends to shrink in type-I superconductors ($\beta < 1$) and expand at $\beta > 1$, which corresponds to type-II superconductivity.

To calculate the part of the world sheet energy functional due to violation of "BPS-ness" we use the low-energy action (3.13), taking into account the four-derivative correction. Substituting the gauge potential (4.1) and (4.3) into the second term in Eq. (3.13), we arrive at the following effective world sheet theory on the semilocal string:

$$\mathcal{E}_{\rm eff} = 2\pi\int dtdz\left\{\xi|\partial_k\rho|^2\ln\frac{L}{|\rho|} + \frac{1}{3}\frac{(\beta-1)}{g^2}\frac{1}{|\rho|^2}\right\}. \tag{4.5}$$

Note that this ρ dependence emerges from the lower limit of integration of the profile function over the radial coordinate transverse to the string. Since the instanton profile function presented above is invalid at small ρ, Eq. (4.5) cannot be trusted at $\rho \to 0$. In fact we need

$$|\rho| \gg \frac{1}{m_\gamma} \tag{4.6}$$

to justify the last term in Eq. (4.5).

The second term in the action above shows that the string is unstable. The thickness ρ of the semilocal string in the type-II superconductor ($\beta > 1$) expands. Note that this term was obtained in Ref. [29] in the limit $(\beta - 1) \ll 1$ where the stability issue was first discussed. The formula (4.5) is valid for all β as long as $|\rho|^2 \gg (\beta - 1)m_\gamma^{-2}$. In Sec. V we will show that, in fact, the $|\rho|^{-2}$ term in Eq. (4.5) is the first term of expansion of a function of which some general features can be established.

V. MAKING A HOPFION

Previously, the topologically stable solutions in the model (1.1) saturating the Hopf invariant were found by numerical calculations. Here we would like to outline some analytic argument regarding the stability of a "large" Hopfion of the twisted-toroidal type.

As was explained in Sec. IV, a semilocal string solution of scalar QED [see Eq. (3.1)] has a complex modulus $\rho \equiv |\rho| e^{i\alpha}$. The size $|\rho|$ is arbitrary. Now, we bend this string in the form of a torus and let the phase α wind along the torus as shown in Fig. 1. More exactly, this figure shows one winding while in fact it can be any integer number, to be denoted by k. To justify the analytic consideration below, we need to stabilize $|\rho|$ at a large value such that $|\rho| \gg m_\gamma^{-1}$ and, in addition, to stabilize the circumference L at $L \gg |\rho|$. There is an interplay of various factors to be analyzed.

First, the string tension $T = 2\pi\xi$ produces a term linear in L in the twisted-toroidal soliton mass, $\delta M_1 = 2\pi\xi L$. Second, the winding

$$\alpha(z) = \frac{2\pi k z}{L} \tag{5.1}$$

generates the term

$$\delta M_2 = 2\pi\xi \int_0^L dz |\rho|^2 \left(\frac{\partial \alpha}{\partial z}\right)^2 \ln\frac{L}{|\rho|} = \frac{\xi(2\pi|\rho|k)^2}{L} \ln\frac{L}{|\rho|}. \tag{5.2}$$

If δM_1 tends to shrink L, the second term δM_2 tends to make L larger, provided that the value of $|\rho|$ is fixed. The problem is that it is not fixed for the time being.

The question of the $|\rho|$ stabilization depends on the $|\rho|^{-2}$ term in Eq. (4.5). Now we will assume $\beta \neq 1$ and analyze its impact. It turns out that at $\beta \sim 1$ the size $|\rho|$ is stabilized at $|\rho| \sim 1/m_\gamma$ where our approximation is invalid. In order to achieve stabilization at larger scales we must assume that $\beta \gg 1$.

However, at large β the second term in Eq. (4.5) is no longer a small correction to the first one. At large β we need to sum up all higher derivative corrections enhanced by powers of β. It is not difficult to see that calculating higher order terms, were such a calculation possible, would produce a series of the type

$$\delta M_3 = \frac{\beta\xi L}{3m_\gamma^2|\rho|^2} \sum_{\ell=0}^{\infty} c_\ell \left(\frac{\beta}{|\rho|^2 m_\gamma^2}\right)^\ell \equiv \frac{\beta\xi L}{3m_\gamma^2|\rho|^2} F\left(\frac{\beta}{|\rho|^2 m_\gamma^2}\right), \tag{5.3}$$

where F is some function of its argument

$$\kappa = \frac{\beta}{|\rho|^2 m_\gamma^2}, \tag{5.4}$$

and we dropped unity compared to β in $\beta - 1$. What is known about the function F?

At $\kappa \to 0$ the function $F \to 1$. One can argue that F stays finite for all values of κ. Moreover, using the scaling analysis, it is easy to see that if stabilization is possible at all, it should occur at $\kappa \sim 1$. Finally, we expect that at large κ

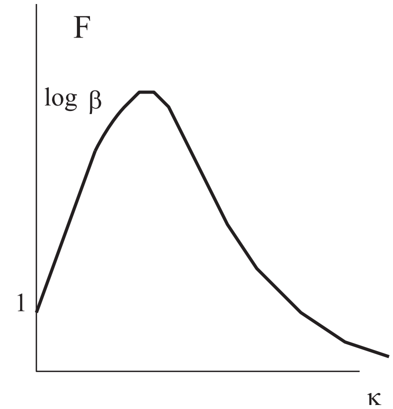

FIG. 2. Qualitative shape of the function $F(\kappa)$.

$$F(\kappa) \sim \frac{3}{2}\frac{\ln\beta}{\kappa}, \qquad \kappa \to \infty \tag{5.5}$$

to match the Abrikosov formula for the string tension in the London limit. Indeed, then at large κ (small ρ), the tension of the untwisted string will approach the Abrikosov formula $T = 2\pi\xi \log\beta$ for the tension of the ANO string in the limit $\beta \gg 1$ [3]. This tension is much larger than the first term $2\pi\xi$ (tension of the ANO BPS string) because of the $\log\beta$ enhancement.

Strictly speaking, we cannot descend down to $\kappa \lesssim 1$ with Eq. (5.5). However, qualitatively, it seems safe to say that $F(\kappa \sim 1) \sim \log\beta \gg 1$ being enhanced by $\log\beta$. Since $F(0) = 1$ one can conclude then that F increases as κ varies from zero to unity, so that F' is parametrically larger (on average)[6] than unity at $\kappa \lesssim 1$. Once κ becomes $\gtrsim 1$, the function F starts decreasing, as dictated by Eq. (5.5). The expected qualitative behavior of the function $F(\kappa)$ is shown in Fig. 2.

Assembling all three terms together, we get for the energy of the twisted semilocal string

$$\mathcal{E} = 2\pi\left\{\xi L + \frac{\xi(2\pi|\rho|k)^2}{L}\ln\frac{L}{|\rho|} + \frac{\beta}{3g^2}\frac{L}{|\rho|^2} F\left(\frac{\beta}{|\rho|^2 m_\gamma^2}\right)\right\}. \tag{5.6}$$

The system of the extremization equations with respect to L and ρ is

$$\begin{aligned} &1 - 4\pi^2 k^2 \frac{|\rho|^2}{L^2}\ln\frac{L}{|\rho|} + \frac{2\beta}{3m_\gamma^2|\rho|^2}F(\kappa) = 0, \\ &4\pi^2 k^2 \frac{|\rho|^2}{L^2}\ln\frac{L}{|\rho|} - \frac{2\beta}{3m_\gamma^2|\rho|^2}F(\kappa) - \frac{2\beta^2}{3m_\gamma^4|\rho|^4}F'(\kappa) = 0, \end{aligned} \tag{5.7}$$

where we assume that the logarithmic factor $\log L/|\rho|$ is large and need not be differentiated. Adding these two equations, we find

[6] The prime in F' above denotes differentiation with respect to κ.

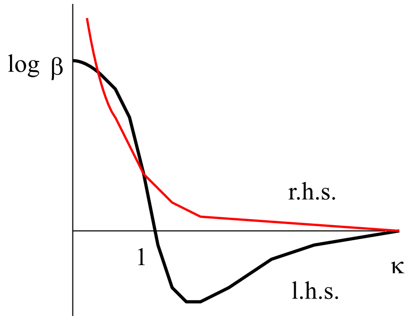

FIG. 3 (color online). Left- and right-hand sides of Eq. (5.11).

$$\kappa^2 F'(\kappa) = \frac{3}{2}. \tag{5.8}$$

Provided the solution of this equation is found, the value of L is stabilized at

$$\frac{L^2}{|\rho|^2} = \frac{4\pi^2 k^2 \ln k}{1 + \frac{2}{3}\kappa F(\kappa)}. \tag{5.9}$$

The energy of the Hopfion is

$$\mathcal{E} = \frac{4\sqrt{2}\pi^2\sqrt{\beta}}{g}\sqrt{\xi}k\sqrt{\log k}\sqrt{\frac{1 + \frac{2}{3}\kappa F(\kappa)}{\kappa}}. \tag{5.10}$$

The argument presented above tells us that Eq. (5.8) has two solutions, one at $\kappa_* \sim 1/\sqrt{\log\beta}$ and another one at $\kappa_{**} \sim 1$. The qualitative behavior of left- and right-hand sides of the equation

$$F'(\kappa) = \frac{3}{2}\frac{1}{\kappa^2} \tag{5.11}$$

are shown in Fig. 3.

It is easy to see that the second solution corresponds to the maximum of energy, while the minimum is achieved at

$$\kappa_* \sim \frac{1}{\sqrt{\log\beta}}, \quad |\rho_*| \sim \frac{\sqrt{\beta}}{m_\gamma}(\log\beta)^{1/4}, \quad \frac{L_*}{|\rho_*|} \sim k\sqrt{\log k}. \tag{5.12}$$

The energy of this solution is

$$\mathcal{E} \sim \frac{\sqrt{\beta}}{g}\sqrt{\xi}k\sqrt{\log k}(\log\beta)^{1/4}. \tag{5.13}$$

The conditions of applicability of the approximations we made are met provided $k \gg 1$ and $\beta \gg 1$.

This concludes our consideration demonstrating the existence of the twisted-toroidal Hopfions with $L \gg \rho$ in the scalar QED (3.1).

Before passing to some extensions, we would like to pause here to make a comment on related studies carried out in the context of condensed matter physics. An earlier discussion relevant to our consideration carried out in the two-flavor scalar QED [see Eq. (3.1)] can be found in an important paper [13]. This paper demonstrates, in particular, that two-flavor scalar QED is equivalent to the Faddeev–Skyrme model (1.1) coupled to a massive vector field. A perturbative procedure was suggested in Ref. [13], which allows one to integrate out the above vector field. This leads to certain higher derivative corrections to the Faddeev–Skyrme model (six derivatives or higher). The result for higher derivatives in Ref. [13] matches our Eq. (3.13) obtained in a totally different way. This shows that the Faddeev–Skyrme model is a low-energy limit of two-flavor scalar QED.

Moreover, it was also argued in Ref. [13] that a large Hopfion could exist as a local minimum in the full theory (3.1), while in the Faddeev–Skyrme model *per se*, the Hopfion soliton is a topological soliton and represents a global minimum. The latter statement also matches our analysis demonstrating the presence of the Hopfion solution at $L \gg \rho \gg 1/m_\gamma$.

Details of our analysis and that of Ref. [13] are quite different. In particular, in Ref. [13] it is the kinetic term that plays a crucial role, while our focus is on both the kinetic and potential terms. We plan to elaborate on this elsewhere. The important step in our Hopfion solution analysis is the construction of the two-dimensional effective theory (4.5) on the semilocal string.

VI. REPLACING R_4 BY $R_3 \times S_1$: "COMPOSITE" TWISTED SEMILOCAL STRINGS

As was discussed above, the semilocal string under discussion can be viewed as an interpolation between the local ANO string in the core and the two-dimensional Belavin–Polyakov instanton uplifted in four dimensions. By compactifying one of spatial dimension into S_1, one can split the Belavin–Polyakov $CP(1)$ instanton in two "constituents," which, in turn, could provide one with possibilities for additional winding numbers for such closed strings.

Let us recall a well-known fact: a four-dimensional Belavin-Polyakov-Schwarz-Tyupkin instanton melts upon compactification of one of coordinates [30] (see also Refs. [31,32]); it dissociates into constituents with fractional topological charge $1/N$ [two constituents with the topological charge $1/2$ in SU(2)]. The constituents carry the instanton number $1/N$ as well as a monopole number. The gauge field holonomy is generated at one-loop level; the eigenvalues of the holonomy fix the distance between the constituents along the compact direction. As a result, the instanton in the SU(N) gauge theory with one compactified dimension (the so-called caloron; see Ref. [33]) turns out to be a composite object built from $N-1$ "conventional" monopoles (with the $1/N$ instanton charge) and the so called Kaluza–Klein (KK) monopole [30]. The latter "wraps around" the compact dimension.

A similar type of behavior for two-dimensional instantons in $CP(1)$ was discussed, too [34,35] (see also Ref. [36]). In particular, if the spatial dimension is compactified, then the CP($N-1$) sigma-model instanton (generalizing the Belavin–Polyakov instanton) splits into N constituents. The scale instanton modulus acquires a new interpretation: it represents the distance between two constituents [for $CP(1)$], while the sizes of the constituents are fixed by the radius of the compact dimension.

Now, while uplifting the instanton from two to four dimensions, let us make an intermediate stop at three dimensions. The two-dimensional CP($N-1$) instantons are the baby Skyrmion particles in three dimensions. Recently it was recognized [37] (see also Ref. [38]) that these baby Skyrmions have a transparent composite structure: they consist of N "ultraviolet" degrees of freedom, which, naively, had been integrated out. The picture is similar to that of the baryon in the four-dimensional chiral Lagrangian (the Skyrme model) [39]. A composite structure of the Skyrmion was argued in a number of complementary ways which did not necessarily include the compact dimension.

The important question is: what are the quantum numbers of the "partons" discussed in Ref. [37]? It was argued that, just like in other similar considerations, the constituents [37] are ordered along the roots of SU(N). In addition, there is the KK constituent, which corresponds to the affine root. These constituents have the topological charges $Q = 1/N$ with respect to π_2 and nontrivial charges with respect to the magnetic Abelian group. The presence of the additional charges can be most easily seen upon dualizing the BPS equation [37]. To this end, it is convenient to dualize one scalar field through the Polyakov relation [40]

$$\partial_\nu \phi = \epsilon_{\nu\mu\rho} F_{\mu\rho}. \tag{6.1}$$

Taking into account the explicit form of the instanton solution, we obtain an analog of the Gauss law,

$$d^* F = (\delta(z - z_+) - \delta(z - z_-)), \tag{6.2}$$

where $z_\pm$ stand for the position of two constituents.

Thus, two constituents of the $CP(1)$ instanton with the opposite charges are clearly seen in the dual formulation. These constituents in the dual formulation are pointlike. The total charge of the Skyrmion with respect to the dual photon vanishes. This explains the origin of the permanent binding of two constituents inside the Skyrmion. Indeed, the "electric" energy of each parton diverges logarithmically (in $2+1$ dimensions), while the energy of the electrically neutral Skyrmion compound is finite.

With this knowledge in hand, we can make the final uplift from three to four dimensions. Since we start from the composite Skyrmion in $D = 3$, we will finish with the composite semilocal string in $D = 4$. These composite strings should have N partonic "substrings," with fractional fluxes $1/N$. This corresponds to uplifted topological charges, as well as additional "flavor" fluxes. There is also a special KK substring corresponding to the uplifting of the KK Skyrmion. The total "flavor flux" of the semilocal string vanishes while the total magnetic flux $Q = 1$.

For instance, the composite semilocal string in uplifted $CP(1)$ involves one fractional substring while the second substring is of the Kaluza–Klein type. The topological charges of each of the two constituents are $1/2$. Hence, the total tension of the composite string comes from two equal parts. It is convenient to write down the instanton solution in the following form [41]:

$$\omega = \frac{z - z_+}{z - z_-}, \tag{6.3}$$

where the moduli z_+ and z_- correspond to the positions of the partons. The center of mass and the scale moduli are defined as

$$Z = \frac{1}{2}(z_+ + z_-), \qquad \rho = |\rho| e^{i\theta} = \frac{1}{2}(z_+ - z_-). \tag{6.4}$$

It is clearly seen that the scale modulus represents the distance between the partons. To reveal the additional quantum numbers of these two constituents, it is convenient to use an analog of the trick with dualization of the BPS equation [37]. In four dimensions the scalar field is dual to a rank-two field $B_{\mu\nu}$,

$$\partial_\rho \phi = \epsilon_{\rho\alpha\mu\nu} \partial_\alpha B_{\mu\nu}. \tag{6.5}$$

As a result, from the BPS equation, we immediately get

$$d^* H = (\delta(z - z_+) - \delta(z - z_-)), \tag{6.6}$$

where $H_{\alpha\mu\nu} = \partial_\alpha B_{\mu\nu}$. Equation (6.6) clearly shows that the two constituent substrings have the opposite charges with respect to the B field.

Next, we fold the composite semilocal string into a toroidal structure with the simultaneous twist of the complex scale modulus. Since the modulus $|\rho|$ is now interpreted as the distance between the two constituents, its nontrivial twisting corresponds to a kind of linking of the constituents. This "linking number" plays the role of the topological number responsible for the classical stability of the soliton.[7]

To conclude this section, note that the instanton splitting naturally happens for the twisted-toroidal string provided we compactify one spatial dimension. The Belavin–Polyakov instanton splits into composites with fractional topological charges. The closed strings under discussion are twisted, and the twist amounts to a nontrivial holonomy along the compact dimension.

VII. CONCLUSIONS

We demonstrated that the Faddeev–Skyrme model emerges as a low-energy limit of scalar QED with two charged scalar fields and a self-interaction potential of a special form (inspired by supersymmetric QCD).

[7]Note that one could also imagine a twist with a Z_N rotation since Z_N does not act on the instanton solution in the CP($N-1$) model.

Our conclusion parallels that previously made in the condensed matter literature [13], although both, motivations and derivations, are different. Then we discuss possible Hopf solitons of the "twisted-toroidal" type. We need to stabilize both the size of the Belavin–Polyakov instanton (appearing in the perpendicular slice) and the length L. We presented analytical arguments that such stabilization is achieved provided $\beta \gg 1$ under the condition that the number of windings is large, too.

Then we briefly discussed a similar twisted-toroidal construction in four dimensions with one spatial dimension compactified.

ACKNOWLEDGMENTS

We are grateful to E. Babayev, A. Niemi, M. Nitta, and Ya. Shnir for useful remarks. The work of M. S. is supported in part by DOE Grant No. DE-FG02- 94ER-40823. The work of A. Y. is supported by FTPI, University of Minnesota, by RFBR Grant No. 13-02-00042a, and by Russian State Grant for Scientific Schools No. RSGSS-657512010.2. A. G. thanks FTPI at the University of Minnesota where a part of the work was carried out, for hospitality and support. The work of A. G. was supported in part by the Grants No. RFBR-12-02-00284 and No. PICS-12-02-91052.

[1] E. Witten and D. I. Olive, Phys. Lett. **78B**, 97 (1978).
[2] E. B. Bogomol'nyi, Sov. J. Nucl. Phys. **24**, 449 (1976).
[3] A. Abrikosov, Sov. Phys. JETP **32**, 1442 (1957); H. Nielsen and P. Olesen, Nucl. Phys. **B61**, 45 (1973).
[4] A. M. Polyakov and A. A. Belavin, JETP Lett. **22**, 245 (1975).
[5] A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu. S. Tyupkin, Phys. Lett. **59B**, 85 (1975).
[6] L. D. Faddeev and A. J. Niemi, Nature (London) **387**, 58 (1997); Phys. Rev. Lett. **82**, 1624 (1999).
[7] N. Manton and P. Sutcliffe, *Topological Solitons* (Cambridge University Press, Cambridge, England, 2004), Sect. 9.11.
[8] R. A. Battye and P. M. Sutcliffe, Phys. Rev. Lett. **81**, 4798 (1998); Proc. R. Soc. A **455**, 4305 (1999).
[9] M. Kobayashi and M. Nitta, arXiv:1304.4737.
[10] M. Kobayashi and M. Nitta, arXiv:1304.6021.
[11] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Phys. Rep. **116**, 103 (1984).
[12] E. Babaev, L. D. Faddeev, and A. J. Niemi, Phys. Rev. B **65**, 100512 (2002).
[13] E. Babaev, Phys. Rev. B **79**, 104506 (2009).
[14] S. Bolognesi and M. Shifman, Phys. Rev. D **75**, 065020 (2007); R. Auzzi, S. Bolognesi, and M. Shifman, Phys. Rev. D **77**, 125029 (2008).
[15] R. S. Ward, Phys. Rev. D **70**, 061701 (2004).
[16] E. Radu and M. S. Volkov, Phys. Rep. **468**, 101 (2008); L. D. Faddeev, in *Statistical Physics, High Energy, Condensed Matter and Mathematical Physics*, edited by M.-L. Ge and K. K. Phua (World Scientific, Singapore, 2008), p. 18.
[17] S. Bolognesi and M. Shifman, Phys. Rev. D **76**, 125024 (2007).
[18] M. Shifman, *Advanced Topics in Quantum Field Theory* (Cambridge University Press, Cambridge, England, 2012).
[19] E. Witten, Nucl. Phys. **B249**, 557 (1985).
[20] R. L. Davis and E. P. S. Shellard, Phys. Lett. B **207**, 404 (1988); **209**, 485 (1988).
[21] Y. Lemperiere and E. P. S. Shellard, Phys. Rev. Lett. **91**, 141601 (2003).
[22] M. Miettinen, A. J. Niemi, and Y. Stroganov, Phys. Lett. B **474**, 303 (2000).
[23] E. Radu and M. S. Volkov, Phys. Rep. **468**, 101 (2008).
[24] P. Fayet and J. Iliopoulos, Phys. Lett. **51B**, 461 (1974).
[25] M. Shifman and A. Yung, Phys. Rev. D **73**, 125012 (2006).
[26] M. Shifman and A. Yung, Rev. Mod. Phys. **79**, 1139 (2007); *Supersymmetric Solitons* (Cambridge University Press, Cambridge, England, 2009).
[27] A. Achucarro and T. Vachaspati, Phys. Rep. **327**, 347 (2000).
[28] K. Intriligator and N. Seiberg, arXiv:1305.1633.
[29] M. Hindmarsh, Nucl. Phys. **B392**, 461 (1993).
[30] N. M. Davies, T. J. Hollowood, V. V. Khoze, and M. P. Mattis, Nucl. Phys. **B559**, 123 (1999); T. J. Hollowood, V. V. Khoze, W. J. Lee, and M. P. Mattis, Nucl. Phys. **B570**, 241 (2000).
[31] M. Ünsal, Phys. Rev. D **80**, 065001 (2009); M. Shifman and M. Ünsal, Phys. Rev. D **78**, 065004 (2008); Phys. Lett. B **681**, 491 (2009).
[32] G. V. Dunne and M. Ünsal, J. High Energy Phys. 11 (2012) 170; Phys. Rev. D **87**, 025015 (2013).
[33] T. C. Kraan and P. van Baal, Nucl. Phys. **B533**, 627 (1998); Nucl. Phys. **A642**, c299 (1998); Phys. Lett. B **435**, 389 (1998).
[34] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, Phys. Rev. D **72**, 025011 (2005).
[35] F. Bruckmann, Phys. Rev. Lett. **100**, 051602 (2008); W. Brendel, F. Bruckmann, L. Janssen, A. Wipf, and C. Wozar, Phys. Lett. B **676**, 116 (2009); D. Harland, J. Math. Phys. (N.Y.) **50**, 122902 (2009).
[36] M. Eto, T. Fujimori, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta, and N. Sakai, Phys. Rev. D **73**, 085008 (2006).
[37] B. Collie and D. Tong, J. High Energy Phys. 08 (2009) 006.
[38] M. Eto, T. Fujimori, S. Gudnason, K. Konishi, T. Nagashima, M. Nitta, K. Ohashi, and W. Vinci, Phys. Rev. D **80**, 045018 (2009).
[39] T. H. R. Skyrme, Proc. R. Soc. A **260**, 127 (1961); Nucl. Phys. **31**, 556 (1962); *Selected Papers, with Commentary,*

of Tony Hilton Royle Skyrme, edited by G. E. Brown (World Scientific, Singapore, 1994); E. Witten, Nucl. Phys. **B223**, 433 (1983); in *Proceedings of the Lewes Workshop on Solitons in Nuclear and Elementary Particle Physics*, edited by A. Chodos, E. Hadjimichael, and C. Tze (World Scientific, Singapore, 1984); S. B. Teiman, R. Jackiw, B. Zumino, and E. Witten, *Current Algebra and Anomalies* (Princeton University, Princeton, NJ, 1985), p. 529.

[40] A. M. Polyakov, Nucl. Phys. **B120**, 429 (1977).

[41] A. A. Belavin, V. A. Fateev, A. S. Schwarz, and Yu. S. Tyupkin, Phys. Lett. **83B**, 317 (1979).