PHYSICAL REVIEW D **88,** 044046 (2013)

Entropy-product rules for charged rotating black holes

M. Cvetič,[1,2] H. Lü,[3] and C. N. Pope[4,5]

[1]*Department of Physics and Astronomy, University of Pennsylvania, Philadelphia, Pennsylvania 19104, USA*
[2]*Center for Applied Mathematics and Theoretical Physics, University of Maribor, SI 2000 Maribor, Slovenia*
[3]*Department of Physics, Beijing Normal University, Beijing 100875, China*
[4]*George and Cynthia Woods Mitchell Institute for Fundamental Physics and Astronomy, Texas A&M University, College Station, Texas 77843-4242, USA*
[5]*DAMTP, Centre for Mathematical Sciences, Cambridge University, Wilberforce Road, Cambridge CB3 OWA, United Kingdom*

(Received 31 July 2013; published 26 August 2013)

We study the universal nature of the product of the entropies of all horizons of charged rotating black holes. We argue, by examining further explicit examples, that when the maximum number of rotations and/or charges are turned on, the entropy product is expressed in terms of angular momentum and/or charges only, which are quantized. (In the case of gauged supergravities, the entropy product depends on the gauge-coupling constant also.) In two-derivative gravities, the notion of the "maximum number" of charges can be defined as being sufficiently many nonzero charges that the Reissner-Nordström black hole arises under an appropriate specialization of the charges. (The definition can be relaxed somewhat in charged anti-de Sitter black holes in $D \geq 6$.) In higher-derivative gravity, we use the charged rotating black hole in Weyl-Maxwell gravity as an example for which the entropy product is still quantized, but it is expressed in terms of the angular momentum only, with no dependence on the charge. This suggests that the notion of maximum charges in higher-derivative gravities requires further understanding.

DOI: 10.1103/PhysRevD.88.044046 PACS numbers: 04.50.−h

I. INTRODUCTION

Understanding black-hole entropy at the microscopic level has been a major focus of research in string theory and M theory in the past years. While the microscopics of asymptotically flat Bogomol'ni-Prasad-Sommerfield (BPS) black holes in four and five dimensions is by now well understood [1] (for a review, see, for example, [2] and references therein), the internal properties of general nonextremal black holes are less clear. However, it has been known for a long time that general asymptotically flat multicharged rotating black holes of supergravities in four [3] and five [4] dimensions[1] have a tantalizing entropy formula [3], and a first law of thermodynamics [7–9] associated with both the inner and the outer black-hole horizons, which are highly suggestive of a possible microscopic interpretation in terms of a two-dimensional conformal field theory (CFT). Specifically, the entropies $S_\pm$ of the outer and inner horizons are of the form [3,8,9]: $S_\pm = 2\pi(\sqrt{N_L} \pm \sqrt{N_R})$, where the quantities N_L and N_R may be viewed as the excitation numbers of the left and right moving modes of a weakly coupled two-dimensional conformal field theory. The product $S_+S_- = 4\pi^2(N_L - N_R)$ should therefore by quantized in integer multiples of $4\pi^2$ [7–9] (and reemphasized in [10]). Indeed, one finds

$$S_+S_- = 4\pi^2\Big(J^2 + \prod_{i=1}^{4} Q_i\Big), \tag{1.1}$$

$$S_+S_- = 4\pi^2\Big(J_1J_2 + \prod_{i=1}^{3} Q_i\Big), \tag{1.2}$$

for four- and five-dimensional black holes, respectively. (These results were implicit in [8,9], although not explicitly evaluated.) These expressions are modulus independent, and are expressed solely in terms of the quantized duality-invariant quartic (cubic) charge form and the quantized angular momenta.

In parallel developments Ansorg and collaborators (see, for example, [11,12] and references therein) studied axisymmetric solutions of Einstein-Maxwell gravity, with sources external to the outer horizon. They obtained striking universal formulas expressing the entropy products of the outer and inner Killing horizons in terms of the total angular momentum J and total charge Q. For the Reissner-Nordström black hole [11], these products reduce to (1.1) with all $Q_i = Q$. The quantized nature of the entropy-product formulas for other asymptotically flat solutions, such as general ring and string solutions, was recently verified in [13], and for static black holes and rings of $N = 2$ supergravity in four and five dimensions in [14] and [15,16], respectively.

[1]These black holes can be used as generating solutions for the maximally supersymmetric $\mathcal{N} = 4$ ($\mathcal{N} = 8$) supergravities obtained by toroidally compactifying the heterotic string (or type IIA string or M theory). In addition to the mass M, these solutions are specified in four dimensions by four charges Q_i ($i = 1, 2, 3, 4$) and one angular momentum J. In five dimensions they are specified by the mass and three charges, Q_i ($i = 1, 2, 3$), and two angular momenta, J_1 and J_2. It turns out that in four dimensions the complete generating solution is specified by an additional fifth charge, which has been obtained only in the BPS [5] and static [6] cases.

Another approach that has brought considerable insights into the internal structure of general black holes is the study of absorption coefficients or greybody factors for fields in the black-hole background. This involves solving the wave equations for external fields in the black-hole geometry. A remarkable feature of many black-hole metrics is that the wave equations, such as the Klein-Gordon equation for a minimally coupled massless scalar field, are typically separable, and this greatly simplifies the study of the scattering problem. The core of the calculation is reduced to the investigation of the solutions of the radial equation, whose complexity is governed by the nature of its singular points. As well as having singular points at the origin and at infinity, additional singularities occur at all of the zeros of the metric functions that determine the number, and the locations, of the horizons. Thus it can be that important features of the scattering process are governed not only by the properties of the metric outside and on the outer horizon, but also by its properties at interior horizons and at other singular points of the metric radial functions.

Specifically, the radial part of the Klein-Gordon equation for massless probe scalars in the background of general asymptotically flat black holes in four and five dimensions exhibits an approximate $SL(2,\mathbf{R})\times SL(2,\mathbf{R})$ conformal symmetry, associated with the poles at the inner and outer black-hole horizons [8,9,17]. The terms that break this symmetry are associated with features of the asymptotic geometry and can be neglected in an appropriate low-energy regime for the probe scalars. This raises the expectation [17] that at least the low-energy dynamics of general black holes could be described by a two-dimensional CFT.[2]

Recently, this proposal was developed further in [20,21], by identifying an explicit part of the general multicharged rotating black-hole geometry that exhibits a manifest $SL(2,\mathbf{R})\times SL(2,\mathbf{R})$ conformal symmetry of the wave equation. The metrics of these conformal backgrounds differ from the original black-hole metrics by the removal of certain terms in the warp factor only, and they were accordingly dubbed the "subtracted geometries."[3] The key global structure and the thermodynamic properties of these subtracted geometries, such as the areas of the two horizons and the angular periodicities, remain the same, and so the subtracted geometry is expected to preserve the information about the internal structure of the black hole. The subtracted geometry is, however, asymptotically conical [21,22], rather than asymptotically flat. A physical interpretation of the subtraction is the removal of the ambient asymptotically Minkowski spacetime in a way that extracts the "intrinsic" $SL(2,\mathbf{R})\times SL(2,\mathbf{R})$ symmetry of the black hole. A lift of the subtracted metric on a circle gives rise to $AdS_3\times$ Sphere geometries, and thus the microscopic interpretation of the general black-hole entropy can be deduced via an AdS_3/CFT_2 correspondence [20,21]. Further studies of the properties of the dual CFT operators that parametrize deformations from the subtracted geometry were carried out in [24,29].

The intriguing internal properties of general asymptotically flat black holes in four and five dimensions, and their potential dual two-dimensional CFT descriptions, are intimately related to geometrical properties of the two horizons. On the other hand, black holes in asymptotically anti-de Sitter (AdS) spacetime, and rotating black holes in dimensions larger than five, have the property that the radial metric function has more than two zeros.[4] The wave equations in these backgrounds will have dominant contributions associated with poles at each of these zeros. One can therefore again expect that the thermodynamics associated with *each* pole will play a role in governing the properties of the black hole at the microscopic level. This could potentially be suggestive of a microscopic behavior of such black holes in terms of a dual field theory in more than two dimensions. One specific (mesoscopic) test of these ideas is the calculation of the product of *all* the horizon entropies [30]. It turns out that these entropy products are also universal; they depend only on quantized charges, quantized angular momenta and the cosmological (or gauge-coupling) constant, which is also quantized in the context of compactifications of string theory.

Most of the black-hole examples that have been investigated arise as solutions of conventional gravity or supergravities with second-order equations of motion, for which the horizon area and entropy are related by the Bekenstein-Hawking formula $S=\frac{1}{4}A$. In these cases the quantization of the product of entropies is therefore synonymous with the quantization of the product of horizon areas. In higher-derivative gravities, by contrast, the entropy is no longer, in general, proportional to the area of the horizon but is instead given by the Wald formula which involves the variation of the action with respect to the Riemann tensor.

[2]These terms can also be neglected for special black-hole backgrounds, including the near-supersymmetric limit (the AdS/CFT correspondence) [8,9,18] and the near-extreme rotating limit (the Kerr/CFT correspondence) [10,19].

[3]The sources for the subtracted geometry were obtained in [22] as a certain scaling limit of another black hole. The full solution for the subtracted geometry can also be obtained by acting on the original black-hole solution with specific Harrison transformations [22–25] within the STU model, which is a consistent truncation of maximally supersymmetric supergravity to $\mathcal{N}=2$ supergravity coupled to three vector multiplets. For related works, see [26–28].

[4]To be more precise, the singular points we are referring to correspond to all the (real or complex) values r_i of the radial coordinate r at which the norm of some Killing vector vanishes. The metric on the surface at the fixed radius r_i may have signature $(0,+,+,+,\cdots,+)$, in which case it is an ordinary horizon; or signature $(0,-,+,+,\cdots,+)$, in which case the surface is a timelike "pseudo-horizon" with imaginary area; or r_i may be complex (such roots arise in conjugate pairs). For the sake of brevity, in this paper we shall refer to all of the surfaces defined by the roots of the relevant metric radial function as "horizons."

Entropy-product formulas in higher-derivative gravities were studied recently in [31]. A question arises as to whether in higher-derivative theories it is the product of the entropies or the product of the areas (or neither) that is quantized. Subtleties can arise when trying to answer this question. In particular, the definition of entropy can be ambiguous in any even spacetime dimension, since one can always add a purely topological Euler integrand to the action which, while not affecting the equations of motion, does change the Wald entropy by a purely numerical additive constant. For example, in four dimensions one can add a Gauss-Bonnet term to the standard Einstein-Hilbert plus matter Lagrangian, such that the original entropy S_0^i at the ith horizon is modified to

$$S^i = S_0^i + \alpha. \quad (1.3)$$

It is clear that if the original entropy product $\prod_i S_0^i$ were quantized and expressible purely in terms of the charges and angular momenta, then the modified entropy product $\prod_i S^i$ would not be. As we shall discuss in detail later, there is in fact a natural way to remove the ambiguity in the definition of the entropy, by requiring that the black hole should have zero entropy in the case where its mass is sent to zero.

In Sec. II we study the entropy-product formulas for some further examples of charged rotating black holes in four and five dimensions that had not previously been examined. These include the four-dimensional dyonic rotating black-hole solutions of the Einstein-Maxwell-dilaton theory obtained by Kaluza-Klein (KK) reduction of pure five-dimensional gravity [32,33], and also the recently constructed general three-charged rotating black holes of five-dimensional gauged supergravity [34]. We also consider the charged rotating black holes of four-dimensional $f(R)$-Maxwell theory [35]. Although $f(R)$ gravity is ostensibly a higher-derivative theory, the known black-hole solutions have constant Ricci scalar R, and hence the equations of motion are effectively reduced to second-order ones.

Turning now to black holes in theories involving higher-derivative gravity in a more nontrivial way, exact solutions are rather hard to come by. For higher-derivative theories whose Lagrangians are built from polynomial curvature invariants, if each Riemann tensor is contracted with at least two Ricci tensors, then Einstein metrics continue to be solutions. The Wald formula then implies that the entropy is proportional to the area of the horizon, and the previous entropy-product formulas still hold for any such black holes that are Einstein metrics. Exact solutions for static charged black holes have also been found in Lovelock-Maxwell theory, and for these it was argued that the entropy-product rule seemingly breaks down [31]. However, we argue that this may just be an artefact of considering the rather degenerate special case of nonrotating black holes. The relevant point here, as we shall discuss in detail later, is that the total number of horizons for a black hole with generic nonvanishing charges and angular momenta can be greater than the number of horizons in special cases where charges and/or angular momenta vanish. It is in the generic case with the maximal number of horizons that one can expect the product of horizon areas to be quantized.

A simple illustrative example is provided by the Reissner-Nordström solution, which has two horizons located at the roots $r_\pm$ of $r^2 - 2Mr + Q^2 = 0$. The product of the horizon areas is $A_+A_- = (4\pi r_-^2)(4\pi r_+^2) = 16\pi^2 Q^4$, which is indeed quantized and independent of the mass M. Taking the limit when Q goes to zero gives the area product $A_+A_- = 0$, which, although trivial because of the factor $A_- = 0$, is still quantized. If, however, we were to consider the Schwarzschild solution in isolation, we would say it has just one horizon, at $r_+ = 2M$, and the "area product" would simply be $A_+ = 4\pi r_+^2 = 16\pi M^2$, which is not quantized and does depend on M. Thus one sees that the Schwarzschild black hole itself, having only one rather than two horizons, is not a sufficiently generic solution to reveal the underlying nature of the quantized area-product formula for the Reissner-Nordström family.

Returning to the black-hole solutions of Lovelock-Maxwell theory examined in [31], it is quite plausible that the failure of the area-product rule for the static black holes is again a consequence of not considering the most generic situation, in this case with rotation included. Unfortunately the more general rotating solutions in the Lovelock-Maxwell theory are not presently known, and so it is not possible at this time to settle the question definitively.

In Sec. III we consider an example that is rather analogous, and where we are able to explicitly illustrate a similar phenomenon, namely for charged rotating black holes in the conformally invariant Weyl-Maxwell theory in four dimensions. We demonstrate that in this example the entropy-product rule holds for rotating black-hole solutions but that it would fail if one considered just the static nonrotating solutions in isolation.

In Sec. IV we comment on a general phenomenon for rotating black holes, namely that if the metric is written as a timelike bundle over a Euclidean-signature base space, with warp factors multiplying the base and the fiber metrics, then the expression for the area of any horizon is independent of the warp factor. However, if one takes the static limit, the area becomes dependent on the warp factor. This observation could have further implications for the study of the microscopic properties of general rotating black holes in gravity theories in diverse dimensions, along the lines of the "subtracted geometry."

We conclude our paper in Sec. V.

II. FURTHER AREA-PRODUCT EXAMPLES IN $D = 4$ AND $D = 5$

In this section we consider three further examples of black-hole solutions in four and five dimensions for which area-product relations had not previously been studied. The first is the four-dimensional rotating dyonic black-hole solution of the Einstein-Maxwell-dilaton theory that can be obtained as the dimensional reduction of five-dimensional pure gravity [32,33]. Next, we look at the general solution for a three-charge rotating black hole in five-dimensional gauged supergravity [34]. The third example is the charged rotating black-hole solution of Maxwell theory coupled to $f(R)$ gravity in four dimensions [35].

A. Entropy-product formula for the dyonic KK black hole

The solution for the dyonic rotating Kaluza-Klein black hole carrying electric and magnetic charges can be embedded in the four-dimensional $\mathcal{N} = 2$ supergravity STU theory. The electric and magnetic charges are both carried by just one of the four gauge fields in the theory. It should be noted that although there exists a discrete duality symmetry in the KK reduction of five-dimensional gravity, under which the electric and magnetic charges are exchanged, there is no continuous duality symmetry, and so this dyonic black hole cannot be rotated into a purely electric or purely magnetic one. Using the notation and conventions of [36], the solution can be written as

$$d\hat{s}_4^2 = -e^{\varphi_4}(dt - \omega d\phi)^2 + e^{-\varphi_4} ds_3^2, \tag{2.1}$$

where

$$\begin{aligned} ds_3^2 &= (\rho^2 - 2mr)\left(\frac{dr^2}{\Delta} + d\theta^2\right) + \Delta \sin^2\theta d\phi^2, \\ \Delta &= r^2 + a^2 - 2mr, \qquad \rho^2 = r^2 + a^2\cos^2\theta. \end{aligned} \tag{2.2}$$

The functions φ_4 and ω are given by

$$\begin{aligned} \omega &= \frac{2amc_4[(r-m)\Xi + mc_5]\sin^2\theta}{\rho^2 - 2mr}, \\ e^{2\varphi_4} &= \frac{(\rho^2 - 2mr)^2 \Xi}{(f_1\Xi + 2mc_4^2 U_-)(f_2 + 2m\Xi U_+)}, \end{aligned} \tag{2.3}$$

where we have defined

$$\begin{aligned} U_\pm &= (r-m)c_5 \pm as_4 s_5 \cos\theta, \\ f_1 &= \rho^2 - 2mr + 2m^2 c_4^2, \\ f_2 &= \rho^2 - 2mr + 2m^2 \Xi^2, \\ c_i &= \cosh\delta_i, \\ s_i &= \sinh\delta_i, \\ \Xi &= \sqrt{1 + c_4^2 s_5^2}. \end{aligned} \tag{2.4}$$

The other nonvanishing fields in the four-dimensional supergravity theory are

$$\begin{aligned} e^{2\varphi_1} &= e^{2\varphi_2} = e^{2\varphi_3} = \frac{f_1\Xi + 2mc_4^2 U_-}{\Xi(f_2 + 2m\Xi U_+)}, \\ \hat{A} &= \nu d\phi + \sigma_4(dt - \omega d\phi), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \nu &= \frac{2ms_4c_4\Delta\cos\theta + 2amc_4s_5[c_4^2c_5(r-m) + m\Xi]\sin^2\theta}{\Xi(\rho^2 - 2mr)}, \\ \sigma_4 &= \frac{2m^2c_4^2s_5c_5 + 2m\Xi[(r-m)s_5 + as_4c_5\cos\theta]}{f_2 + 2m\Xi U_+}. \end{aligned} \tag{2.6}$$

There are two horizons, which are located at the two roots $r_\pm$ of $\Delta = 0$. The entropies are given by

$$S_\pm = 2\pi mc_4 |r_\pm \Xi - m(\Xi - c_5)|. \tag{2.7}$$

[The absolute value must be used here because $r_-\Xi - m(\Xi - c_5)$ can be negative under appropriate conditions; see Sec. 4.] The angular momentum J and the electric and magnetic charges Q and P are given by

$$J = amc_4\Xi, \qquad Q = 2ms_5\Xi, \qquad P = \frac{2ms_4c_4}{\Xi}, \tag{2.8}$$

from which it follows that

$$S_+S_- = 4\pi^2 m^2 c_4^2 |a^2\Xi^2 - m^2 s_4^2 s_5^2|, \tag{2.9}$$

and hence, as can be seen from (2.8),

$$S_+S_- = 4\pi^2 \left| J^2 - \frac{1}{16}P^2Q^2 \right|. \tag{2.10}$$

It might, at first sight, seem surprising that with only one field strength active, the charges contribute to the entropy-product formula. However, bearing in mind that the charge contribution in a general black hole must be invariant under $SL(2,R)^3$, and that for the usual "four-charge" black hole the charge contribution is of the form $P_1Q_2P_3Q_4$, this is in fact correct. See, for example, Eq. (6.22) in [37][5]:

[5] In [37] the notation where a "standard" four-charge black hole has one electric charge q_0 and three magnetic charges (p^1, p^2, p^3) is used. By contrast, in the notation of [36] the standard four-charge black hole has magnetic charges p^1 and p^3, and electric charges q_2 and q_4. Our presentation of the Rasheed black hole has electric and magnetic charges q_4 and p^4. In the notation of [37] a simple choice for the Rasheed black hole would be to take q_0 and p^0 to be nonzero, in which case the last term in (2.11) gives $-(q_0)^2(p^0)^2$, in contrast to $+4q_0p^1p^2p^3$ for the standard four-charge black hole [3]. See also [5], Eq. (69), where the manifestly (S- and T-) duality invariant quartic charge form was first derived, as it appeared in the entropy formula of the most general BPS black hole of four-dimensional $\mathcal{N} = 4$ ungauged supergravity. For related U-duality invariant charge forms in four and five dimensions, see, e.g., [38].

$$D(p,q) = 4[(p^1q_1)(p^2q_2) + (p^1q_1)(p^3q_3) + (p^2q_2)(p^3q_3) - p^0q_1q_2q_3 + q_0p^1p^2p^3] - (p^\mu q_\mu)^2. \quad (2.11)$$

For comparison, we may consider the standard four-charge black hole. In the notation of [36] this has magnetic charges P_1 and P_3, and electric charges Q_2 and Q_4. Evaluating the entropy-product formula, we find

$$S_+S_- = 4\pi^2\left(J^2 + \frac{1}{4}P_1Q_2P_3Q_4\right). \quad (2.12)$$

This is indeed consistent with (2.10) and, after making the exchange $p^2 \leftrightarrow q_2$, (2.11).

B. The three-charge solution in $D = 5$ gauged supergravity

Area-product formulas for a variety of rotating black holes in gauged supergravities were studied in [30]. However, only more recently has the general three-charge rotating black hole in five-dimensional gauged supergravity been constructed [34], and so we are now in a position to check the area-product relation for this example. It has horizons determined by the roots of a sixth-order polynomial in the radial coordinate r. Since only even powers of r occur, we can define $x = r^2$ and reduce the problem to one with three horizons, at the roots $x = x_1$, $x = x_2$ and $x = x_3$. The radial function is then given by [34]

$$\begin{aligned}\Delta = {}& (x+a^2)(x+b^2)(1+g^2x) - 2mx + 2mg^2\{(s_1^2+s_2^2+s_3^2)x^2 - (s_1^2s_2^2+s_1^2s_3^2+s_2^2s_3^2)[(a^2+b^2-2m)x \\ & + a^2b^2(2+g^2x)] + s_1^2s_2^2s_3^2([(a+b)^2-2m)][(a-b)^2-2m] - 2g^2a^2b^2(2x+2m+a^2+b^2) + g^4a^4b^4) \\ & + 2mg^2a^2b^2[s_1^4s_2^4+s_1^4s_3^4+s_2^4s_3^4 - 2s_1^2s_2^2s_3^2(s_1^2+s_2^2+s_3^2)]\}. \end{aligned} \quad (2.13)$$

The entropy, angular momenta and charges are also given in [34]. After transforming to the variable $x = r^2$, the entropy at the ith horizon is given by

$$S_i = \frac{\pi^2}{2\chi_a\chi_b}\sqrt{\frac{W_i}{x_i}}, \quad (2.14)$$

where

$$\begin{aligned}W_i = {}& [(x_i+a^2)(x_i+b^2) + 2mx_i(s_1^2+s_2^2+s_3^2)]\{(x_i+a^2)(x_i+b^2) + 2mg^2[(a+b)^2 - g^2a^2b^2][(a-b)^2 \\ & - g^2a^2b^2]s_1^2s_2^2s_3^2 - 4mg^2a^2b^2(s_1^2s_2^2+s_1^2s_3^2+s_2^2s_3^2)\} + 8m^2x_ic_1c_2c_3s_1s_2s_3ab\sqrt{\Xi_{1a}\Xi_{2a}\Xi_{3a}\Xi_{1b}\Xi_{2b}\Xi_{3b}} \\ & + 4m^2x_i[x_i + g^2a^2b^2(s_1^2+s_2^2+s_3^2)](s_1^2s_2^2+s_1^2s_3^2+s_2^2s_3^2) - 4m^2\{(a^2+b^2)(1+g^2a^2)(1+g^2b^2)x_i \\ & + g^2[(a^4+b^4)x_i + g^2a^4b^4(2+g^2x_i)](s_1^2+s_2^2+s_3^2) + g^2a^2b^2(a^2+b^2+g^2a^2b^2) \\ & \times [2 + g^2x_i(s_1^2s_2^2+s_1^2s_3^2+s_2^2s_3^2)]\}s_1^2s_2^2s_3^2 + 4m^2g^4a^4b^4(s_1^4s_2^4+s_1^4s_3^4+s_2^4s_3^4) - 8m^2x_ig^6a^4b^4s_1^4s_2^4s_3^4 \\ & + 8m^3(x_i + g^2a^2b^2)s_1^2s_2^2s_3^2, \end{aligned} \quad (2.15)$$

and

$$\begin{aligned} \chi_a &= 1 - g^2a^2, \qquad \chi_b = 1 - g^2b^2, \\ \Xi_{ia} &= 1 + g^2a^2s_i^2, \qquad \Xi_{ib} = 1 + g^2b^2s_i^2. \end{aligned} \quad (2.16)$$

The angular momenta and electric charges are given by [34]

$$J_a = \frac{\pi m}{2\chi_a^2\chi_b}\left(ac_1c_2c_3\sqrt{\Xi_{1a}\Xi_{2a}\Xi_{3a}} - b\chi_a^2s_1s_2s_3\sqrt{\Xi_{1b}\Xi_{2b}\Xi_{3b}}\right), \quad (2.17)$$

$$\begin{aligned} J_b &= \frac{\pi m}{2\chi_a\chi_b^2}\left(bc_1c_2c_3\sqrt{\Xi_{1b}\Xi_{2b}\Xi_{3b}} - a\chi_b^2s_1s_2s_3\sqrt{\Xi_{1a}\Xi_{2a}\Xi_{3a}}\right), \\ Q_i &= \frac{\pi m}{2\chi_a\chi_b}\left(\frac{c_is_i\sqrt{\Xi_{1a}\Xi_{2a}\Xi_{3a}\Xi_{1b}\Xi_{2b}\Xi_{3b}}}{sqrt\Xi_{ia}\Xi_{ib}} - g^2ab\frac{c_1c_2c_3s_1s_2s_3}{c_is_i}\sqrt{\Xi_{ia}\Xi_{ib}}\right). \end{aligned} \quad (2.18)$$

From these it is a straightforward matter to compute the product of the entropies and express it in terms of the conserved charges. We find that it can be written as

$$\prod_{i=1}^{3} S_i = \pm \frac{2i\pi^3}{g^3}(\pi J_a J_b + 4Q_1Q_2Q_3). \qquad (2.19)$$

This result reduces to the special cases presented previously in [30] if two or more charges are set equal. [The $\pm$ sign on the right-hand side reflects the fact that there is a sign ambiguity in taking the square roots in (2.14). This would not be seen if we took the product over all six horizons in the language of the original radial variable r, in which case the right-hand side would be squared.]

C. Charged rotating black hole in $f(R)$ theory

Although $f(R)$ gravities involve, in general, higher derivatives in their equations of motion, their solutions include those for which the Ricci scalar is constant. In this case, the effective equations of motion are then reduced to second order. In four dimensions, it happens that the trace of the Maxwell energy-momentum tensor vanishes, and so one can still construct analytic charged solutions in this case for which R is constant.

The four-dimensional Lagrangian for $f(R)$ gravity coupled to a Maxwell field is given by

$$\mathcal{L}_4 = \sqrt{-g}\Big(f(R) - \frac{1}{4}F^2\Big). \qquad (2.20)$$

The Einstein equations of motion are then

$$f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)f'(R) = \frac{1}{2}\Big(F^2_{\mu\nu} - \frac{1}{4}g_{\mu\nu}F^2\Big), \qquad (2.21)$$

where $f'(R)$ means $\partial f(R)/\partial R$. The trace of the Einstein equation is independent of the Maxwell field,

$$f'(R)R - 2f(R) - 3\Box f'(R) = 0, \qquad (2.22)$$

and so it admits solutions where the Ricci scalar is constant, $R = R_0$, where

$$f'(R_0)R_0 = 2f(R_0). \qquad (2.23)$$

For solutions with $R = R_0$, the Einstein equations (2.21) then reduce to

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R - 2\Lambda) = \frac{1}{2}\Big(\tilde{F}^2_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\tilde{F}^2\Big), \qquad (2.24)$$

where $\Lambda = \frac{1}{4}R_0$ is the effective cosmological constant and $\tilde{F}_{\mu\nu} = F_{\mu\nu}/\sqrt{f'(R_0)}$. This is precisely the same equation as in Einstein-Maxwell theory, which admits the well-known Reissner-Nordström and Kerr-Newman solutions, except that now the charge is scaled by the $1/\sqrt{f'(R_0)}$ factor. This gives rise to the static [39] and rotating [35] charged black holes in four-dimensional $f(R)$ gravity coupled to the Maxwell field. Note that the entropy is also given by one-quarter the area of the horizon, scaled by the $f'(R_0)$ factor. Thus the entropy-product formula for the charged rotating black hole in $f(R)$ gravity is similar to that for the Kerr-Newman solution in Einstein-Maxwell theory, but for an overall scaling by an $f'(R_0)$-dependent factor. The entropy-product formula for the static case was discussed in [31].

III. ENTROPY-PRODUCT FORMULA FOR MAXWELL-WEYL THEORY

In four dimensions, owing to the fact that the Gauss-Bonnet term is a total derivative, the most general quadratic-curvature Lagrangian can be parametrized as

$$\mathcal{L}_2 = \alpha R^{\mu}{}_{\nu} R_{\mu\nu} + \beta R^2. \qquad (3.1)$$

Any Einstein metric with cosmological constant Λ, which is a solution for the theory described by the Lagrangian $\mathcal{L}_0 = R - 2\Lambda$, will continue to be a solution of the theory described by $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_2$. For any such black-hole solution, the Wald formula implies that the entropy will just be a constant multiple of the area of the horizon, and hence entropy-product results for the Kerr-AdS metric in Einstein gravity will continue to hold in the extended theory. If black-hole solutions over and above those that are Einstein metrics existed, then their entropy products would need to be investigated in their own right.

In fact, here no explicit black-hole solutions, beyond Kerr-AdS, are known for the general case of cosmological Einstein gravity augmented by the quadratic-curvature Lagrangian (3.1). The situation becomes simpler, however, if we consider pure conformal gravity, where the Lagrangian is simply a multiple of the square of the Weyl tensor, and additional non-Einstein black-hole solutions can be found. In fact, nontrivial solutions can also be found in the conformally invariant Weyl-Maxwell theory, described by the Lagrangian

$$\mathcal{L} = \sqrt{-g}\Big(\frac{1}{2}\alpha C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} + \frac{1}{3}\alpha F^2\Big) = \sqrt{-g}\Big(\alpha R^{\mu\nu}R_{\mu\nu} - \frac{1}{3}\alpha R^2 + \frac{1}{3}\alpha F^2\Big) + \alpha\mathcal{L}_{\rm GB}. \qquad (3.2)$$

(Here $\mathcal{L}_{\rm GB}$ denotes the Gauss-Bonnet Lagrangian.)

A. Charged rotating black holes

Charged rotating black holes in the four-dimensional conformally invariant Einstein-Weyl theory,

$$\mathcal{L} = \sqrt{-g}\Big(\frac{1}{2}\alpha C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} + \frac{1}{3}\alpha F^{\mu\nu}F_{\mu\nu}\Big), \qquad (3.3)$$

were studied in [40]. The solution for a dyonic black hole can be written as [40]

$$ds_4^2 = \rho^2\Big(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\Big) + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\Big(a dt - (r^2+a^2)\frac{d\phi}{\Xi}\Big)^2 - \frac{\Delta_r}{\rho^2}\Big(dt - a\sin^2\theta\frac{d\phi}{\Xi}\Big)^2,$$
$$A = \frac{qr}{\rho^2}\Big(dt - a\sin^2\theta\frac{d\phi}{\Xi}\Big) + \frac{p\cos\theta}{\rho^2}\Big(a dt - (r^2+a^2)\frac{d\phi}{\Xi}\Big), \tag{3.4}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \Delta_\theta = 1 - g^2a^2\cos^2\theta,$$
$$\Xi = 1 - g^2a^2, \tag{3.5}$$
$$\Delta_r = (r^2+a^2)(1+g^2r^2) - 2mr + \frac{(p^2+q^2)r^3}{6m}.$$

The horizons occur at the roots of $\Delta_r = 0$. In what follows we shall set $p = 0$ so that there is only an electric charge, since the inclusion of a magnetic charge adds no further features of relevance to the discussion.

The conserved energy, charge and angular momentum are given by [40]

$$E = \frac{2\alpha g^2}{\Xi^2}\Big(m + \frac{a^2q^2}{12m}\Big), \qquad Q = \frac{\alpha q}{3\Xi},$$
$$J = \frac{2a\alpha g^2}{\Xi^2}\Big(m + \frac{q^2}{12mg^2}\Big), \tag{3.6}$$

and the Wald entropy at a root r_i is given by

$$S_i = \frac{2\pi\alpha}{\Xi}\Big(1 + g^2r_i^2 + \frac{q^2r_i}{6m} - c\Xi\Big). \tag{3.7}$$

The constant c is purely numerical (i.e. parameter independent) and corresponds to adding a constant multiple of the Gauss-Bonnet invariant to the action. If we choose $c = 1$, the resulting Lagrangian involves only the Ricci tensor and Ricci scalar, and we have

$$S_i = \frac{2\pi\alpha}{\Xi}\Big(\frac{g^2}{4\pi}A_i + \frac{q^2r_i}{6m}\Big), \tag{3.8}$$

where $A_i = 4\pi(r_i^2 + a^2)$ is the area of the ith "horizon." This is a more natural definition for the entropy of the system since the entropy vanishes when the solution becomes the vacuum. Calculating the product of the entropies at the four horizons, keeping the constant c arbitrary for now, and then expressing the result in terms of the conserved charges, we find

$$\prod_{i=1}^4 S_i = (2\pi)^4\alpha^2[cJ^2 + (1-c)E^2g^{-2} + 3c(1-c)Q^2 + c^2(1-c)^2\alpha^2]. \tag{3.9}$$

Making the natural choice $c = 1$ discussed above yields the result

$$\prod_{i=1}^4 S_i = (2\pi)^4\alpha^2J^2, \tag{3.10}$$

which is indeed quantized. This is a highly nontrivial result since now the entropy is no longer simply one-quarter of the area of the horizon, in which case the quantization of the entropy product implies the geometric quantization of the product of all horizon areas.

B. Charged static black holes

In the previous subsection, we saw that with the natural choice $c = 1$ the product of entropies depended only on the angular momentum, but was independent of the charges. If we send the angular momentum to zero, the entropy of one of the horizons also goes to zero. More precisely, in the static case, the four null surfaces are reduced to only three. There are in fact more general static solutions than the obvious one resulting from setting $a = 0$. The most general static solution is given by [41]

$$ds^2 = -fdt^2 + \frac{dr^2}{f} + r^2d\Omega_2^2, \qquad A = -\frac{q}{r}dt,$$
$$f = -\frac{1}{3}\Lambda r^2 + c_1r + c_0 + \frac{d}{r}, \qquad 3c_1d + 1 + q^2 = c_0^2, \tag{3.11}$$

and the entropies are given by

$$S_i = -\frac{2\pi\alpha(3d + (c_0+2)r_i)}{3r_i}. \tag{3.12}$$

Thus we have

$$\prod_{i=1}^3 S_i = -\frac{8}{27}\pi^3\alpha^3(2 - 3c_0 + c_0^3 - 26d^2g^2) + \frac{128}{9}\alpha(c_0+2)\pi^3Q_e^2, \tag{3.13}$$

where $Q_e = \frac{1}{4}\alpha q$ is the electric charge. It is thus clear that the product of entropies is no longer expressed purely in terms of quantized charges in this static case.

An important lesson one learns from the Weyl-Maxwell theory is that in higher-derivative gravities, the failure of the entropy-product formula in the static case does not necessarily imply the failure of the formula in the more general rotating solutions with angular momenta as well.

After all, for Schwarzschild black holes with or without a cosmological constant, the entropy or the product of entropies depends on the mass, rather than on any quantized quantities. However, as we discussed for the Schwarzschild black hole in the Introduction, the static solution is a special case of the more general solutions with angular momenta or charges, which have a larger number of null surfaces. The static solution corresponds to the degenerate limit where the area of one or more of the null surfaces of the more general class of solutions goes to zero.

In all the examples examined so far, as long as maximally rotating solutions exist in two-derivative or higher-derivative gravities, the entropy-product formulas do work, in the sense of depending only on the products of angular momenta with (or without) charges. However, in many cases, such as in Lovelock gravities, the exact solutions for rotating black holes are unknown. We expect that the entropy-product formulas will work in these cases, even if they fail for the known, but rather degenerate, charged static solutions.

IV. WARP-FACTOR INDEPENDENCE AND THE STATIC LIMIT

Most of the charged black-hole metrics in ungauged supergravities were constructed by using solution-generating techniques, starting from an uncharged black hole as a "seed" solution. One of the most universal solution-generating techniques involves performing a dimensional reduction of the seed metric to three dimensions, and then acting with global symmetries of the associated nonlinear sigma model coupled to gravity in three dimensions. For example, in the case of constructing charged rotating solutions in four dimensions, one performs a timelike reduction using the Kaluza-Klein ansatz given in (2.1). The reduced three-dimensional metric ds_3^2 is invariant under the global symmetry transformations and thus remains the same as in the original reduction of the seed Kerr solution, as in (2.2). The specific forms of the warp factor e^{φ_4} and the function ω in (2.1) will depend on the details of the theory under consideration, and the nature of the charges that are turned on, but the general structure, and the universality of the 3-metric ds_3^2, will be common to all examples.

The horizons of the charged metric will be located at the same radii r_i as those of the original seed metric, namely at the zeros of the function Δ. It is then evident from (2.1) and (2.2) that the area of the horizon at $r = r_i$ will be given by

$$\mathcal{A}_i = 4\pi\sqrt{(2mr - \rho^2)\omega^2}|_{r=r_i}, \tag{4.1}$$

and so, in particular, it is independent of the warp factor e^{φ_4}.

The above discussion assumes that the metric is stationary, but not static. In the static case, $a = 0$ and the function ω vanishes, and so evidently for the metric to be nondegenerate at the horizons it must be that the warp factor acquires a factor of Δ that can cancel the overall factor of Δ in the 3-metric ds_3^2. The area of the horizon at $r = r_i$ is now given by

$$\mathcal{A}_i = 4\pi[e^{-\varphi_4}\Delta]_{r=r_i}, \tag{4.2}$$

which, unlike in the rotating case, *does* depend upon the warp factor e^{φ_4}.

The independence of the horizon area on the warp factor for stationary black-hole metrics has been noted in earlier works, and indeed it has formed the basis for the notion of subtracted geometries that was considered in [20,21]. In those papers, the proposal was to subtract certain terms in the warp factors of the black-hole metrics, in such a way that the massless scalar wave equation for the subtracted geometry attains a manifest $SL(2,\mathbf{R}) \times SL(2,\mathbf{R})$ conformal symmetry. It was noted in [20,21] that the areas of the two horizons and the periodicity of the azimuthal angles for the rotating solutions are unchanged in the subtracted geometries. It could, however, be troubling if the phenomenon we have noted above, in which the warp factor does enter in the area formula in the static case, were to signal a discontinuity if one approached the static situation as an $a \to 0$ limit of the rotating solution. It is of interest, therefore, to investigate this limit in detail in explicit examples.

We saw in Sec. II A that the entropy-product formula (2.10) for the rotating dyonic black hole required an absolute value of the $J^2 - P^2Q^2/16$ factor on the right-hand side, to allow for the case where $J^2 < P^2Q^2/16$. Specifically, the origin of this nonanalytic dependence on the conserved charges is that the metric function ω in (4.1) changes sign and becomes negative at the inner horizon if J^2 becomes less than $P^2Q^2/16$. A direct calculation of the entropy product for the static metric with $a = 0$ confirms that indeed

$$S_+S_- = \frac{1}{4}\pi^2P^2Q^2. \tag{4.3}$$

Thus we see in this example that the direct evaluation in the static metric is in agreement with the result obtained by taking a $J \to 0$ limit.

It is noteworthy that it is possible to cast the metric in the warped-product form (2.1) for any axisymmetric solution in four dimensions. In five dimensions, all known rotating black-hole solutions in gauged and ungauged supergravity can be cast in the analogous form:

$$ds_5^2 = -\Delta^{-1/3}(dt + \omega_\phi d\phi + \omega_\psi d\psi)^2 + \Delta^{2/3}ds_4^2, \tag{4.4}$$

where the four-dimensional base metric ds_4^2 is Kähler, and the warp factor Δ depends only on the radial coordinate r and the polar angle θ. This is the case for all known rotating black holes, including [42] the general rotating three-charge AdS black hole that was obtained in [34]. We have checked, for many of these examples, that their entropy is independent of the warp factor.

V. CONCLUSIONS

In this paper, we discussed the universal nature of the product of the entropies associated with all the horizons of a black hole and, in particular, that the product depends only on the charges and angular momenta, which are subject to quantization conditions. The entropy-product formulas are valid as long as the maximum number of rotation parameters and/or the maximum number of charges are turned on. The meaning of the former is clear, namely that there is a nonvanishing rotation in each of the $[(D-1)/2]$ orthogonal spatial 2-planes. The notion of the "maximum number of charges" requires further explanation. It is well known that Einstein-Maxwell gravity in four or five dimensions, with or without a cosmological constant, can be embedded in four- or five-dimensional supergravity. The Reissner-Nordström black-hole solution in the associated supergravity theory can be viewed as a superposition of some more basic $U(1)$-charged building blocks. For example, four-dimensional Einstein-Maxwell gravity can be embedded in the STU model, which is a consistent truncation of maximally supersymmetric ungauged supergravity that has $\mathcal{N}=2$ supersymmetry and four $U(1)$ gauge fields. The quantized entropy-product formula is valid provided that all four charges are turned on, with the Reissner-Nordström black hole then corresponding to the special case where the four charges are set equal.

This picture can be extended to higher-dimensional nonsupersymmetric theories, where the Reissner-Nordström black hole again emerges as a superposition of more fundamental ingredients [43]. Thus for a theory that supports multiply-charged black holes, if the solution specializes to the Reissner-Nordström black hole when the charges are equated, then the number of charges can be viewed as "maximal," and the entropy-product formulas will hold. In the dyonic black hole we discussed in Sec. II, although it involves only two charges, namely the electric and magnetic charges carried by a single $U(1)$ gauge field in the STU model, rather than the usual four charges carried by all four gauge fields, it still can be viewed as having the maximum number of charges since the solution can be reduced to the Reissner-Nordström black hole if the charges are equated.[6] (As discussed in the Appendix, for dilatonic AdS black holes in dimensions $D \geq 6$, the entropy-product rule can work in cases where there are fewer than the maximal number of charges.)

The situation changes in higher-derivative gravity. The changes are twofold: First, in general higher-derivative gravity, the entropy may no longer be proportional to the area of the horizon, and thus the quantization of the product of entropies may no longer be a purely geometric property. Second, the concept of "maximal charges" has to be refined. In the explicit example of the conformally invariant Maxwell-Weyl gravity, we demonstrated that the entropy-product formula still works, but the result is expressed in terms of the angular momentum only, with no dependence on the charge. This implies that the Maxwell charge is no longer "maximal," in the sense discussed above. In order to obtain an entropy-product formula that is quantized, one must therefore necessarily turn on the angular momentum.

The example of charged static black holes in another higher-derivative theory, Maxwell-Lovelock gravity, was studied in [31], and it was found there that the product of the entropies did not satisfy a quantization rule. This suggests that the phenomenon we found in Maxwell-Weyl gravity may be more widespread in higher-derivative theories; one must have nonzero angular momentum in order to obtain a quantized entropy-product formula.

Of course, the higher-derivative examples that we have been discussing all have in common that the gravitational action is a higher-derivative action, while the Maxwell gauge field action is left unaltered. A natural conjecture, therefore, is that in higher-derivative theories the Maxwell field should have appropriate higher-derivative terms also, in order to acquire the "maximal" status. In fact, such terms are natural in higher-derivative extensions of supergravities. The success of the entropy product formulas for the rotating black hole in Maxwell-Weyl gravity suggests this possibility. In a spherical dimensional reduction, the angular momentum can be viewed as the electric charge of some Kaluza-Klein vector associated with an isometry of the sphere. The reduction of such a higher-derivative theory will clearly give rise to a Kaluza-Klein vector with higher-derivative terms also.

To conclude, there appears to be a robust rule that the product of entropies at all horizons of a black hole can be expressed purely in terms of quantized quantities provided that the maximum numbers of angular momenta and/or charges are turned on. The key point in all cases is that the maximal number of distinct "horizons" should be attained. In two-derivative theories, the notion of maximum number of charges can be restated as the condition that, by equating them appropriately, the solutions can be reduced to the Reissner-Nordström black hole. In higher-derivative theories, the notion is not yet entirely clear and is worthy of further investigation.

ACKNOWLEDGMENTS

M. C. and H. L. are grateful to the George and Cynthia Woods Mitchell Institute for Fundamental Physics and Astronomy at Texas A&M University for hospitality during the early stages of this work. M. C. thanks the Department of Physics at Beijing Normal University, where initial ideas were discussed, for hospitality, and the Theory Division of CERN for hospitality during completion of the project. H. L. is grateful to ICTS-USTC for hospitality during the later stages of this work. We are grateful to Gary Gibbons for useful discussions. M. C.

[6]Note that in ungauged supergravity theories in four and five dimensions, the product of horizon areas is governed, respectively, by quartic or cubic charge forms, which are modulus independent and invariant under duality transformations. (See, for example, [5] for the quartic charge form in $\mathcal{N}=4$ ungauged supergravity.) Note that in four dimensions both the four-charge black holes [3] and the Kaluza-Klein black hole [32,33] are special examples where the duality-invariant quartic charge form is nonzero.

would like to thank Tolga Birkandan, Monica Guica, Finn Larsen and Zain Saleem for extensive discussions and collaborations on related topics. M. C. is supported in part by DOE Grant No. DE-SC0007901, the Fay R. and Eugene L. Langberg Endowed Chair and the Slovenian Research Agency (ARRS). The research of H. L. is supported in part by the NSFC Grants No. 11175269 and No. 11235003. C. N. P. is supported in part by DOE Grant No. DE-FG02-95ER40917.

APPENDIX: ON MAXIMAL CHARGES

Earlier in the paper we introduced the concept of a "maximal" number of charges in a charged black-hole solution. The concept is, for the most part, simply discussed in the context of static solutions.

In general, charged black holes are constructed as superpositions of multiple charges associated with different vector gauge fields in the theory. If the solution reduces to the Reissner-Nordström black hole under an appropriate specialization of the charges, then we refer to such multiply-charged black holes as having the maximum number of charges. For these solutions, the entropy product at all horizons is expressed in terms of charges only. However, the converse is not necessarily true. For example, gauged supergravities in seven or six dimensions cannot be truncated to Einstein-Maxwell theory, and hence the charged AdS black holes cannot be reduced to Reissner-Nordström-AdS black holes by specializing the charges. Nevertheless, for charged black holes in these theories, built from two basic ingredients where one or the other of two independent gauge fields is excited, the entropy-product rule still holds. In this Appendix, we shall comment further on these observations.

Let us consider the Lagrangian proposed in [43] as the focus for our discussion, since the theory contains all the essential properties of supergravities in terms of the structure of its relevant solutions. The Lagrangian is given by

$$e^{-1}\mathcal{L}_D = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{a_1\phi}F_1^2 - \frac{1}{4}e^{a_2\phi}F_2^2 - V(\phi), \tag{A1}$$

$$V(\phi) = -\frac{g^2 N_1}{4}[2(D-3)^2(N_1-1)e^{-a_1\phi} + 2a_1^2(D-3) \times (D-2)N_1 e^{-\frac{1}{2}(a_1+a_2)\phi} - a_1^2(D-2)((D-3)N_1 - (D-1))e^{-a_2\phi}], \tag{A2}$$

where the constants (a_1, a_2) satisfy the constraint

$$a_1 a_2 = -\frac{2(D-3)}{D-2}. \tag{A3}$$

The theory admits charged AdS black holes [43], given by

$$\begin{aligned} ds^2 &= -(H_1^{N_1}H_2^{N_2})^{-\frac{(D-3)}{D-2}} f dt^2 + (H_1^{N_1}H_2^{N_2})^{\frac{1}{D-2}}(f^{-1}dr^2 + r^2 d\Omega_{D-2}^2), \\ A_1 &= \frac{\sqrt{N_1}c_1}{s_1} H_1^{-1}dt, \\ A_2 &= \frac{\sqrt{N_2}c_2}{s_2} H_2^{-1}dt, \\ \phi &= \frac{1}{2}N_1 a_1 \log H_1 + \frac{1}{2}N_2 a_2 \log H_2, \\ f &= 1 - \frac{\mu}{r^{D-3}} + g^2 r^2 H_1^{N_1} H_2^{N_2}, \\ H_1 &= 1 + \frac{\mu s_1^2}{r^{D-3}}, \\ H_2 &= 1 + \frac{\mu s_2^2}{r^{D-3}}, \end{aligned} \tag{A4}$$

where $s_i = \sinh\delta_i$, $c_i = \cosh\delta_i$, and (N_1, N_2) are given by

$$N_1 + N_2 = \frac{2(D-2)}{D-3}, \qquad a_1^2 = \frac{4}{N_1} - \frac{2(D-3)}{D-2}. \tag{A5}$$

The solution becomes the Reissner-Nordström-AdS black hole if we set $\delta_1 = \delta_2$.

The thermodynamic quantities are all calculated in [43]. For our purposes, we shall give only the charges and entropy:

$$\begin{aligned} Q_i &= \frac{(D-3)\omega_{D-2}}{16\pi}\mu N_i c_i s_i, \\ S &= \frac{1}{4}r_0^{D-2}H_1(r_0)^{N_1/2}H_2(r_0)^{N_2/2}\Omega_{D-2}. \end{aligned} \tag{A6}$$

For the case when $g = 0$, the general solution has two real horizons; the outer horizon is located at $r_0 = \mu^{1/(D-3)}$, and the inner horizon is at $r = 0$. The entropy-product formula, ignoring inessential numerical constants, is given by [43]

$$S_+S_- \sim Q_1^{N_1}Q_2^{N_2}. \tag{A7}$$

If we turn off one of the charges, then $r = 0$ is no longer a horizon, but a singularity with zero area. Thus this example demonstrates the validity of our definition of "maximal" charges.

For nonvanishing g, the situation can be more subtle. The horizons are located at all the roots of the metric function f. For rational values of N_i, the number of roots is finite. The general formula determining these roots is rather involved. For the Reissner-Nordström-AdS black holes, it can be shown that

$$\prod_i S_{i=1}^{2(D-2)} \sim (g^{-1}Q)^{2(D-2)}. \tag{A8}$$

It was conjectured in [43] that the general product formula, for unequal charges, is

$$\prod_{i=1}^{2(D-2)} S_i \sim [(g^{-1}Q_1)^{N_1}(g^{-1}Q_2)^{N_2}]^{D-3}. \quad \text{(A9)}$$

The entropy-product formula appears to be suggesting that it fails to work if we set one charge to zero, say $Q_2 = 0$. This is clearly the case for $g = 0$, as we have discussed. However, if we let $N_1 = 2$ and $Q_2 = 0$, we find that for $D \geq 6$, the entropy formula continues to work:

$$\prod_{i=1}^{2(D-3)} S_i \sim (g^{-1}Q_1)^{2(D-3)}, \quad \text{for } D \geq 6. \quad \text{(A10)}$$

That is to say, the product of entropies depends only on the charges and the gauge coupling constant g. Note that the total number of horizons is reduced from $2(D-2)$ to $2(D-3)$. On the other hand, the entropy-product formula indeed fails to work when $D = 4$ or 5 with this charge specification. In the cases of $D = 4$, 5, 6 or 7, the theory can be embedded in gauged supergravity, and the results were found already in [30].

It is somewhat surprising that when we have less than the maximal number of charges, the entropy-product formula still works. The most likely explanation can be found in the superposition rule

$$N_1 + N_2 = \frac{2(D-2)}{D-3}. \quad \text{(A11)}$$

This shows that N_i can only take integer values in $D = 4$ and 5. In dimensions higher than five, the largest integer value that N_1 can take is $N_1 = 2$, in which case N_2 is fractional and smaller than 1. In other words, $N_1 = 2$ is the maximum integer ingredient, and that seems to be sufficient for the entropy-product rule to work, even if Q_2 is turned off. Of course, it is worth emphasizing again that for the "ungauged" supergravity theories with $g = 0$, the entropy-product formula requires that both charges Q_i are turned on.

[1] A. Strominger and C. Vafa, Phys. Lett. B **379**, 99 (1996).
[2] A. Sen, Gen. Relativ. Gravit. **40**, 2249 (2008).
[3] M. Cvetič and D. Youm, Phys. Rev. D **54**, 2612 (1996).
[4] M. Cvetič and D. Youm, Nucl. Phys. **B476**, 118 (1996).
[5] M. Cvetič and A. A. Tseytlin, Phys. Rev. D **53**, 5619 (1996); **55**, 3907(E) (1997).
[6] M. Cvetič and D. Youm, Nucl. Phys. **B472**, 249 (1996).
[7] F. Larsen, Phys. Rev. D **56**, 1005 (1997).
[8] M. Cvetič and F. Larsen, Phys. Rev. D **56**, 4994 (1997).
[9] M. Cvetič and F. Larsen, Nucl. Phys. **B506**, 107 (1997).
[10] M. Cvetič and F. Larsen, J. High Energy Phys. 09 (2009) 088.
[11] M. Ansorg, J. Hennig, and C. Cederbaum, Gen. Relativ. Gravit. **43**, 1205 (2010).
[12] J.L. Jaramillo, N. Vasset, and M. Ansorg, arXiv:0712.1741.
[13] A. Castro and M.J. Rodriguez, Phys. Rev. D **86**, 024008 (2012).
[14] P. Galli, T. Ortin, J. Perz, and C.S. Shahbazi, J. High Energy Phys. 07 (2011) 041.
[15] P. Meessen and T. Ortin, Phys. Lett. B **707**, 178 (2012).
[16] P. Meessen, T. Ortin, J. Perz, and C. S. Shahbazi, J. High Energy Phys. 09 (2012) 001.
[17] A. Castro, A. Maloney, and A. Strominger, Phys. Rev. D **82**, 024008 (2010).
[18] J. M. Maldacena and A. Strominger, Phys. Rev. D **55**, 861 (1997).
[19] M. Guica, T. Hartman, W. Song, and A. Strominger, Phys. Rev. D **80**, 124008 (2009).
[20] M. Cvetič and F. Larsen, J. High Energy Phys. 02 (2012) 122.
[21] M. Cvetič and F. Larsen, J. High Energy Phys. 09 (2012) 076.
[22] M. Cvetič and G. W. Gibbons, J. High Energy Phys. 07 (2012) 014.
[23] A. Virmani, J. High Energy Phys. 07 (2012) 086.
[24] M. Cvetič, M. Guica, and Z. H. Saleem, arXiv:1302.7032.
[25] A. Sahay and A. Virmani, J. High Energy Phys. 07 (2013) 089.
[26] A. Chakraborty and C. Krishnan, arXiv:1212.1875.
[27] A. Chakraborty and C. Krishnan, arXiv:1212.6919.
[28] S. Jana and C. Krishnan, arXiv:1303.3097.
[29] M. Baggio, J. de Boer, J.I. Jottar, and D.R. Mayerson, J. High Energy Phys. 04 (2013) 084.
[30] M. Cvetič, G.W. Gibbons, and C.N. Pope, Phys. Rev. Lett. **106**, 121301 (2011).
[31] A. Castro, N. Dehmami, G. Giribet, and D. Kastor, arXiv:1304.1696.
[32] D. Rasheed, Nucl. Phys. **B454**, 379 (1995).
[33] F. Larsen, Nucl. Phys. **B575**, 211 (2000).
[34] S.-Q. Wu, Phys. Lett. B **707**, 286 (2012).
[35] A. Larranaga, Pramana **78**, 697 (2012).
[36] Z.-W. Chong, M. Cvetič, H. Lü, and C.N. Pope, Nucl. Phys. **B717**, 246 (2005).
[37] L. Borsten, D. Dahanayake, M.J. Duff, H. Ebrahim, and W. Rubens, Phys. Rep. **471**, 113 (2009).
[38] M. Cvetič and C.M. Hull, Nucl. Phys. **B480**, 296 (1996).
[39] T. Moon, Y.S. Myung, and E.J. Son, Gen. Relativ. Gravit. **43**, 3079 (2011).
[40] H.-S. Liu and H. Lü, J. High Energy Phys. 02 (2013) 139.
[41] R.J. Riegert, Phys. Rev. Lett. **53**, 315 (1984).
[42] T. Birkandan and M. Cvetič (unpublished).
[43] H. Lü, arXiv:1306.2386.