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Nonlocal theory of massive gravity

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We construct a fully covariant theory of massive gravity which does not require the introduction of an external reference metric, and overcomes the usual problems of massive gravity theories (fatal ghosts instabilities, acausality and/or van Dam-Veltman-Zakharov discontinuity). The equations of motion of the theory are nonlocal but respect causality. The starting point is the quadratic action proposed in the context of the degravitation idea. We show that it is possible to extend it to a fully nonlinear covariant theory. This theory describes the 5 degrees of freedom of a massive graviton plus a scalar ghost. However, contrary to generic nonlinear extensions of Fierz-Pauli massive gravity, the ghost has the same mass m as the massive graviton, independently of the background, and smoothly goes into a nonradiative degree of freedom for $m \to 0$. As a consequence, for $m \sim H_0$ the vacuum instability induced by the ghost is irrelevant even over cosmological time scales. We finally show that an extension of the model degravitates a vacuum energy density of order $M_{\rm Pl}^4$ down to a value of order $M_{\rm Pl}^2 m^2$, which for $m = \mathcal{O}(H_0)$ is of order of the observed value of the vacuum energy density.

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I. INTRODUCTION AND SUMMARY

The search for a viable theory of massive gravity provides a longstanding challenge to theoretical physics and has a long history [1,2]. Recent years have witnessed an explosion of activity on the subject, and more generally on infrared modifications of general relativity (GR), motivated both by the intrinsic field-theoretical interest of the problem and by its potential relevance for understanding the origin of dark energy. This has led to beautiful theoretical ideas such as the Dvali-Gabadadze-Porrati (DGP) model [3], degravitation of vacuum energy [4–8], effective field theories for massive gravity based on the Stückelberg mechanism [9], Galileon theories [10], and to the construction of a ghost-free theory of massive gravity, the de Rham-Gabadadze-Tolley (dRGT) theory [11,12] (see also [13–22] and Ref. [23] for a review). Cosmological consequences of these ideas have been extensively explored. A self-accelerated solution was first found in the context of DGP [24,25] but is unfortunately plagued by a ghost instability [26–30]. Self-accelerated solutions, with the Hubble parameter determined by the graviton mass, have also been found in dRGT theory [31–38].

Despite these remarkable advances, some crucial problems remain open. In dRGT, even if the sixth ghostlike degree of freedom is absent in any background, the fluctuations of the remaining 5 degrees of freedom can become ghostlike over nontrivial backgrounds. In particular, the self-accelerating solutions of dRGT theory generically have scalar or vector instabilities [39–41]. Another important open problem is posed by the existence of superluminal modes over some backgrounds [42–49], which also appears in Galileon theories [50]. As discussed in [51] in the case of Galileon theories, this however does not necessarily imply the loss of causality, since in the attempt of constructing closed timelike curves one is forced to leave the domain of validity of the effective field theory.

At a different (and possibly more subjective) level, the need for an external reference metric in dRGT theory is disturbing. This can be seen particularly clearly in the unitary gauge, where the Stückelberg fields ϕ^a are set to zero. In this gauge the theory is constructed in terms of a field $h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is a fiducial reference metric given by a classical solution of Einstein gravity. In practice, this means that we have a different Lagrangian for the massive theory for any classical solution of the massless theory, a situation that can hardly be accepted as fundamental. Bimetric versions of ghost-free massive gravity [52–56] address this concern by assigning a dynamics to the reference metric, but this seems to spoil, or at least significantly complicate, the simple and beautiful geometric interpretation of GR.

In this paper we approach the problem of constructing a consistent theory of massive gravity from a different point of view, developing a nonlocal formulation of the theory. One might fear that nonlocality brings in new conceptual complications. However, while it is true that nonlocality can bring in technical complications (e.g., integro-differential equations of motion), conceptual issues that are sometimes raised in this context are actually rather due to some common misconceptions about nonlocal theories. For instance, one should not mix up nonlocality with lack of causality. If in an equation of motion there is a term proportional to the inverse of the d'Alembertian operator $\Box^{-1}$, this does not mean that the theory is acausal, as long as $\Box^{-1}$ is defined in terms of the retarded Green's function. Another common misconception is that nonlocal theories necessarily hide extra ghostlike degrees of freedom, much as higher-derivative theories. If the equations of motion

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involve a function $f(\Box)$ and we expand this function and truncate the expansion to a finite order N, we indeed have a higher-derivative theory with time derivatives up to order $2N$. This requires $2N$ data as initial conditions and therefore describes N degrees of freedom. The Ostrogradski theorem ensures that at least one of these extra degrees of freedom is a ghost. However, as discussed in [57] (see also [58–60]), in general the solutions of the truncated theory are spurious and do not converge to solutions of the full nonlocal theory as the order of the expansion $N \to \infty$. In particular, when $f(\Box)$ is nonanalytic, e.g., $f(\Box) = 1/(\Box - m^2)$, most of the solutions of the truncated theory have large frequencies, which lie outside the convergence radius of the derivative expansion, and for $N \to \infty$ they do not converge to solutions of the full theory. Nonlocal theories with nonanalytic functions $f(z)$ emerge for instance from the integration of degrees of freedom in a perfectly healthy theory, so in this case it is clear that they have no pathology. As a trivial example, one can consider the nonlocal Lagrangian [57]

$$L[q] = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 + \frac{1}{2}g^2\omega^2 q\left(\frac{\omega^2}{d^2/dt^2 + \omega^2}\right)q. \qquad (1.1)$$

An expansion of the nonlocal factor in powers of d^2/dt^2, followed by truncation to a finite order N, leads to a theory that requires $2N$ initial data and so describes N degrees of freedom, out of which at least one is an Ostrogradski ghost. However, the Lagrangian $L[q]$ can be obtained by integrating out the x variable from the Lagrangian

$$L'[q,x] = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 + \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 - g\omega^2 xq, \qquad (1.2)$$

which describes two coupled harmonic oscillators, and obviously has no pathologies. Similar examples also commonly occur in quantum field theory. For instance, even if the running of coupling constants is more frequently formulated in momentum space, there is also an alternative formulation, developed in gauge theories and in quantum gravity in the pioneering works [61,62] (see also [63–68]), which uses nonlocal effective actions in coordinate space. In this formalism the one-loop effective action obtained in QED by integrating out the fermionic fields can be written in the form

$$S_{\rm eff} = -\frac{1}{4}\int d^4x F_{\mu\nu}\left[\frac{1}{g_0^2} + \beta_0 \ln\left(\frac{-\Box}{\mu^2}\right)\right]F^{\mu\nu}, \qquad (1.3)$$

where β_0 is the one-loop β-function and μ an appropriately chosen mass scale.[1] Nonlocal expressions can also emerge from the reduction to four dimensions of higher-dimensional theories, as in DGP, where the reduction to four dimensions gives an action involving $\sqrt{-\Box}$ [3,23]. Thus, nonlocal theories do not necessarily have pathologies, and nonlocal modifications of gravity have been discussed in a number of different contexts, in the attempt to construct both IR [6–8] and UV modifications [69–72]. Nonlocal operators also naturally enter in the description of fields with spin $s > 2$ [73].

We will see in this paper that the use of a nonlocal formulation can be very useful in massive gauge theories and in massive gravity. Indeed, at the level of linearized theories there is a sort of duality between gauge invariance and locality, which can both be made manifest in the formalism, but in a mutually exclusive manner. The simplest example is given by massive electrodynamics, governed by the Proca Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m_\gamma^2 A_\mu A^\mu. \qquad (1.4)$$

As we will recall in Sec. II (following Refs. [7,8]), this theory is actually equivalent to a theory with Lagrangian

$$\mathcal{L}' = -\frac{1}{4}F_{\mu\nu}\left(1 - \frac{m_\gamma^2}{\Box}\right)F^{\mu\nu}. \qquad (1.5)$$

The formulation (1.4) is explicitly local but not gauge invariant. In contrast, the Lagrangian (1.5) is explicitly gauge invariant, even in the massive case, at the price of manifest locality. Observe that the equation of motion derived from Eq. (1.5) can be written as

$$(\Box - m_\gamma^2)A^\nu = \left(1 - \frac{m_\gamma^2}{\Box}\right)\partial^\nu\partial_\mu A^\mu. \qquad (1.6)$$

The nonlocal term in this equation can be eliminated by fixing the gauge $\partial_\mu A^\mu = 0$, thereby recovering the equations of motion derived directly from (1.4). Thus the theory described by (1.4), which is not gauge invariant, can be understood as the gauge fixing of a gauge-invariant but nonmanifestly local theory. Fixing the gauge we lose manifest gauge invariance, but at the same time we eliminate the nonlocal terms.

The same strategy can be used for linearized massive gravity. As we will see in Sec. III (see also [23]) the Fierz-Pauli (FP) Lagrangian

$$\mathcal{L}_{\rm FP} = \frac{1}{2}h_{\mu\nu}\mathcal{E}^{\mu\nu,\rho\sigma}h_{\rho\sigma} - \frac{m^2}{2}(h_{\mu\nu}h^{\mu\nu} - h^2), \qquad (1.7)$$

(where $\mathcal{E}^{\mu\nu,\rho\sigma}$ is the Lichnerowicz operator) is equivalent to a theory with

$$\mathcal{L}' = \frac{1}{2}h_{\mu\nu}\left(1 - \frac{m^2}{\Box}\right)\mathcal{E}^{\mu\nu,\rho\sigma}h_{\rho\sigma} - 2m^2 N\frac{1}{\Box}\partial_\mu\partial_\nu(h^{\mu\nu} - \eta^{\mu\nu}h), \qquad (1.8)$$

[1] The operator $\ln(-\Box/\mu^2)$ can be defined via a momentum space convolution or, equivalently, from

$$\ln\left(\frac{-\Box}{\mu^2}\right) = \int_0^\infty dm^2\left[\frac{1}{m^2 + \mu^2} - \frac{1}{m^2 - \Box}\right].$$

where N is an extra field that enters as a Lagrange multiplier. Since $\partial_\mu\partial_\nu(h^{\mu\nu}-\eta^{\mu\nu}h)$ is the linearization of the Ricci scalar, and $(1/2)h_{\mu\nu}\mathcal{E}^{\mu\nu,\rho\sigma}h_{\rho\sigma}$ is the quadratic Einstein-Hilbert action, the formulation (1.8) is manifestly invariant under linearized diffeomorphisms even in the massive case, but is nonlocal, and again the local formulation (1.7) can be obtained by imposing a gauge fixing at the level of the equations of motion. The theory governed by the first term in Eq. (1.8),

$$\mathcal{L}_{\rm nonloc}=\frac{1}{2}h_{\mu\nu}\left(1-\frac{m^2}{\Box}\right)\mathcal{E}^{\mu\nu,\rho\sigma}h_{\rho\sigma}, \tag{1.9}$$

has been first proposed in the context of the degravitation idea in [6,8]. In Ref. [8] it is argued that this theory only describes 2 degrees of freedom, corresponding to the states with helicities ± 2 of a massive graviton. A detailed analysis, performed in Sec. IV, will show however that it actually describes 6 radiative degrees of freedom, that make up the five components of a massive spin-2 particle plus a scalar, and the scalar is a ghost. Indeed, the constraint imposed by N in Eq. (1.8) eliminates the ghost, leaving us with the 5 degrees of freedom of the massive graviton of the linearized FP theory. Furthermore we will show that the scalar, the helicity zero and the helicity ± 1 modes decouple smoothly in the $m\to 0$ limit, so this theory has no van Dam-Veltman-Zakharov (vDVZ) discontinuity.

The advantage of this nonlocal reformulation of the linearized theory will become apparent in Sec. V, where we will look for a generally covariant generalization of the equations of motion derived either from Eq. (1.8) or from Eq. (1.9). As already observed in [74], the covariantization of Eq. (1.8) cannot produce a viable theory, since the constraint imposed by N is promoted to the fully covariant constraint $R=0$, which is not present in Einstein theory. Therefore such a theory has a vDVZ discontinuity that persists at the fully nonlinear level, and is ruled out. We will then turn our attention to the covariantization of Eq. (1.9), which in its simplest form is

$$G_{\mu\nu}-m^2(\Box_g^{-1}G_{\mu\nu})^{\rm T}=8\pi G T_{\mu\nu}, \tag{1.10}$$

where we used the fact that any symmetric tensor $S_{\mu\nu}$ (here $S_{\mu\nu}=\Box_g^{-1}G_{\mu\nu}$, where $\Box_g$ is the d'Alembertian in curved space) can be decomposed as $S_{\mu\nu}=S^{\rm T}_{\mu\nu}+(1/2)(\nabla_\mu S_\nu+\nabla_\nu S_\mu)$, where $\nabla^\mu S^{\rm T}_{\mu\nu}=0$ (see also [74] for a similar approach, applied however to FP theory). The extraction of the transverse part can in principle be performed with nonlocal operators, which fits well with our general approach. We will then turn to a discussion of the virtues, as well as of the potential problem, of the classical theory defined by Eq. (1.10).

The first virtue is that it provides a fully covariant theory of massive gravity, without the need of introducing an external reference metric. Once again, the advantage of having full general covariance even in a massive theory comes at the price of nonlocality. A second important point is that this theory has no vDVZ discontinuity, since the four extra states (the two modes in the scalar sector and the states with helicity ± 1) smoothly decouple in the $m\to 0$ limit. Thus, the Vainshtein mechanism [75] is not needed here. Another bonus is that this theory does not have the acausality problem identified in [45], since the latter comes from the same constraint that removes the Boulware-Deser ghost. At the linearized level this is simply the constraint imposed by N in Eq. (1.8), which is absent in Eq. (1.9). However, these encouraging results seem to come at a disastrous price, namely the existence of a sixth ghostlike mode, which is already present in the linearized theory (1.9). We will tackle the ghost issue in Sec. VI, where we will see that this ghost is quite different from the Boulware-Deser ghost that appears in generic nonlinear extensions of FP theory. In our case the ghost has the same mass m as the spin-2 graviton, so for $m\sim H_0$ it is very light. At the same time, in the limit $m\to 0$ it decouples from the theory and reduces to a nonradiative degree of freedom of GR. In contrast, the Boulware-Deser ghost is not smoothly connected, in the $m\to 0$ limit, to a harmless nonradiative field, and is not light in a generic background. Rather on the contrary, one usually tries to get rid of it by tuning the parameters of the theory so that its mass goes to infinity. We will see that, as a consequence of the fact that the ghost present in Eq. (1.9) decouples in the $m\to 0$ limit, the decay rate of the vacuum due to associated production of ghosts plus positive-energy states is negligible, even over a cosmological time scale. Thus, despite the ghost, the classical theory described by Eq. (1.10) can be perfectly acceptable. In Sec. VII we will examine a variant of Eq. (1.10) of the form

$$G_{\mu\nu}-m^2\left(\frac{1}{\Box_g-\mu^2}G_{\mu\nu}\right)^{\rm T}=8\pi G T_{\mu\nu}, \tag{1.11}$$

with $\mu=\mathcal{O}(m^2/M_{\rm Pl})\ll m$. This is basically the same as Eq. (1.10) on field configurations for which $\Box_g\gg\mu^2$, i.e., for modes that change on a length scale (or on a time scale) L such that $L\ll\mu^{-1}$, but strongly deviates from it in the far IR, when $L\gtrsim\mu^{-1}$. The introduction of μ is particularly interesting since, if we put on the right-hand side a vacuum energy-momentum tensor $T_{\mu\nu}=-\rho_{\rm vac}g_{\mu\nu}$, Eq. (1.11) admits a de Sitter solution $G_{\mu\nu}=-\Lambda g_{\mu\nu}$ with

$$\Lambda=8\pi G\frac{\mu^2}{m^2+\mu^2}\rho_{\rm vac}. \tag{1.12}$$

Taking now $\mu\to 0$ we see that $\Lambda\to 0$. This can be seen as an extreme form of degravitation, in which even in the presence of an arbitrarily large vacuum energy, the effective cosmological constant $\Lambda=\mathcal{O}(\mu^2)\to 0$. More generally, for finite μ the vacuum energy $\rho_{\rm vac}$ is degravitated so that the quantity that actually contributes to the observed acceleration of the Universe is

$$\rho_\Lambda = \frac{\mu^2}{m^2+\mu^2}\rho_{\rm vac}. \tag{1.13}$$

In order to reproduce the observed value $\rho_\Lambda = \mathcal{O}(M_{\rm Pl}^2 H_0^2)$ from a vacuum energy $\rho_{\rm vac} = \mathcal{O}(M_{\rm Pl}^4)$ we need $\mu = \mathcal{O}(H_0 m/M_{\rm Pl})$ In particular, for $m = \mathcal{O}(H_0)$, the vacuum energy that drives the observed acceleration of the Universe is reproduced by a value

$$\mu = \mathcal{O}\left(\frac{m^2}{M_{\rm Pl}}\right), \tag{1.14}$$

which could be naturally generated by gravitational loop corrections. We conclude in Sec. VIII, where we show that this nonlocal theory of massive gravity, specialized to a Friedmann-Robertson-Walker (FRW) background, provides a specific model of nonlocal cosmology. Nonlocal cosmological models have been much studied recently. In particular, in Ref. [76] a nonlocal Friedmann equation has been proposed, obtained by adding to the Einstein-Hilbert action an extra term of the form $Rf(\Box^{-1}R)$. Its theoretical structure and cosmological consequences for different choices of the function $f(\Box^{-1}R)$ have been discussed in a number of papers; see e.g., [77–87]. In this approach there is no basic principle that fixes the function $f(\Box^{-1}R)$, which is therefore chosen on purely phenomenological grounds, and can be reconstructed to fit any given expansion history [88,89]. In our case, in contrast, the nonlocal Friedmann equation follows from Eq. (1.11), and there is no arbitrary function corresponding to f; the only free parameter is the graviton mass (and μ, which however could in principle be determined in terms of m from the loop corrections).

Some extra material is discussed in the appendices. We use the signature $\eta_{\mu\nu} = (-,+,+,+)$ and units $\hbar = c = 1$, and we define $\kappa = (32\pi G)^{1/2}$. We use $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ to denote the flat-space d'Alembertian, and $\Box_g$ for the d'Alembertian with respect to a metric $g_{\mu\nu}$.

II. NONLOCAL FORMULATION OF MASSIVE ELECTRODYNAMICS

Before moving to massive gravity, let us first discuss how massive electrodynamics can be written in a gauge-invariant but nonlocal form. This will be useful to pave the way for the gravitational case, and also has an intrinsic interest. Part of these results have already been presented in [7,8] (see also [23] for review). We will however discuss in more detail some technically subtle points involved in the derivation. We start from the Proca action with an external conserved current j^μ

$$S = \int d^4x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m_\gamma^2 A_\mu A^\mu - j_\mu A^\mu\right]. \tag{2.1}$$

The equations of motion obtained from (2.1) are

$$\partial_\mu F^{\mu\nu} - m_\gamma^2 A^\nu = j^\nu. \tag{2.2}$$

Acting with ∂_ν on both sides and using $\partial_\nu j^\nu = 0$, Eq. (2.2) gives

$$m_\gamma^2 \partial_\nu A^\nu = 0. \tag{2.3}$$

Thus, if $m_\gamma \neq 0$, we get the condition $\partial_\nu A^\nu = 0$ dynamically, as a consequence of the equation of motion, and we have eliminated one degree of freedom. Making use of Eq. (2.3), in the vacuum Eq. (2.2) becomes

$$(\Box - m_\gamma^2)A^\mu = 0. \tag{2.4}$$

Equations (2.3) and (2.4) together describe the 3 degrees of freedom of a massive photon. In this formulation Lorentz invariance and locality are manifest, while the $U(1)$ gauge invariance of the massless theory is lost, because of the non-gauge-invariant term $m_\gamma^2 A_\mu A^\mu$ in the Lagrangian.

A. Nonlocal equations of motion

An equivalent formulation of massive electrodynamics that preserves both Lorentz and gauge invariance by giving up manifest locality can be obtained as follows [7,8]. One begins by performing the "Stückelberg trick", i.e., one introduces a scalar field φ and replaces

$$A_\mu \to A_\mu + \frac{1}{m_\gamma}\partial_\mu\varphi, \tag{2.5}$$

in the Lagrangian. Under this replacement $F_{\mu\nu}$ is unchanged, while the term $A_\mu j^\mu$ only produces a boundary term, since we are assuming that j^μ is conserved. Therefore only the mass term changes, and the new action is

$$S = \int d^4x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m_\gamma^2 A_\mu A^\mu - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - m_\gamma A^\mu\partial_\mu\varphi - j_\mu A^\mu\right]. \tag{2.6}$$

By construction A_μ and φ only appear in this Lagrangian in the combination $(A_\mu + m_g^{-1}\partial_\mu\varphi)$ (apart from boundary terms, that we will always assume to vanish, setting appropriate boundary conditions at infinity). Thus the theory is trivially invariant under the local transformation

$$A_\mu \to A_\mu - \partial_\mu\theta, \qquad \varphi \to \varphi + m_\gamma\theta. \tag{2.7}$$

The equations of motion obtained by taking the variation of the action (2.6) with respect to A_ν and φ are, respectively,

$$\partial_\mu F^{\mu\nu} = m_\gamma^2 A^\nu + m_\gamma\partial^\nu\varphi + j^\nu, \tag{2.8}$$

$$\Box\varphi + m_\gamma\partial_\mu A^\mu = 0. \tag{2.9}$$

Of course these equations of motion are invariant under the gauge symmetry (2.7) of the action. The Stückelberg field φ can then be eliminated from the action by making use of its own equation of motion, that can be written formally as

$$\varphi(x) = -m_\gamma \Box^{-1}(\partial_\mu A^\mu), \tag{2.10}$$

where, for any integrable function $f(x)$,

$$(\Box^{-1} f)(x) \equiv \int d^4x' G(x;x') f(x'), \tag{2.11}$$

and $G(x;x')$ is a Green's function of the $\Box$ operator, which for the moment we keep generic. Some basic facts about the inversion of the d'Alembertian operator in flat and in curved space are recalled in Appendix A. Substituting Eq. (2.10) into Eq. (2.8) we get

$$(\Box - m_\gamma^2)A^\nu = \left(1 - \frac{m_\gamma^2}{\Box}\right)\partial^\nu \partial_\mu A^\mu + j^\nu. \tag{2.12}$$

Since Eqs. (2.8) and (2.9) are invariant under the transformation (2.7), the equation of motion (2.12), which involves only A_μ, must be invariant under the gauge transformation $A_\mu \to A_\mu - \partial_\mu \theta$. We can check this immediately observing that, under $A_\mu \to A_\mu - \partial_\mu \theta$, the right-hand side of Eq. (2.12) changes by a factor

$$-\left(1 - \frac{m_\gamma^2}{\Box}\right)\partial^\nu \Box\theta = -(\Box - m_\gamma^2)\partial^\nu \theta, \tag{2.13}$$

which is local, and cancels the change of the left-hand side. Alternatively, we can display the gauge invariance explicitly observing that Eq. (2.12) can be rewritten as [8,23]

$$\left(1 - \frac{m_\gamma^2}{\Box}\right)\partial_\nu F^{\mu\nu} = j^\nu. \tag{2.14}$$

B. Nonlocal action principle

We now wish to find an action whose variation gives Eq. (2.14). It is natural to expect that this is obtained performing the substitution (2.10) directly into action (2.6), which gives

$$S = -\frac{1}{4}\int d^4x \left[F_{\mu\nu}\left(1 - \frac{m_\gamma^2}{\Box}\right)F^{\mu\nu} - j_\mu A^\mu\right]. \tag{2.15}$$

In fact, the issue is more subtle. Writing Eq. (2.15) explicitly, we have

$$S = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu\right] + \frac{m_\gamma^2}{4}\int d^4x d^4x' F_{\mu\nu}(x) G(x;x') F^{\mu\nu}(x'). \tag{2.16}$$

Taking the functional derivatives with respect to A_ν and to $\partial_\mu A_\nu$ we see that the corresponding equation of motion is

$$\partial_\mu\left[F^{\mu\nu}(x) - \frac{m_\gamma^2}{2}\int d^4x' [G(x;x') + G(x';x)] F^{\mu\nu}(x')\right] = j^\nu. \tag{2.17}$$

Therefore, as observed also in [90], the variational principle automatically symmetrizes the Green's function, so it gives back Eq. (2.14) only if $G(x;x') = G(x';x)$, i.e., if $\Box^{-1}$ is defined either using the symmetric combination $G_+ = (1/2)(G_{\rm ret} + G_{\rm adv})$, or the Feynman Green's function G_F; see Appendix A. There are two possible solutions to this problem:

(1) We indeed use $G_+(x;x')$ [or $G_F(x;x')$] in Eq. (2.10) and therefore in Eq. (2.14). In this case the nonlocal action that provides the equations of motion is indeed given by Eq. (2.15). At first sight, the fact that one uses $G_+(x;x')$ or $G_F(x;x')$, which are combinations of $G_{\rm ret}(x;x')$ and $G_{\rm adv}(x;x')$, might seem to pose problems of causality. However, we see from Eq. (2.12) that the acausal behavior can be eliminated choosing the gauge $\partial_\mu A^\mu = 0$, and is therefore a gauge artifact that does not affect gauge-invariant observables. This point of view is indeed tenable in a nonlocal formulation of massive electrodynamics, but is potentially dangerous in a nonlocal formulation of non-Abelian theories or in the nonlocal formulation of fully nonlinear massive gravity that we will study in Sec. V, since nonlinear interactions could communicate the acausal behavior to the physical sector.

(2) Alternatively, we can take the point of view that the classical theory is defined by its equations of motion, while the action is simply a convenient "device" that, through a set of well-defined rules, allows us to compactly summarize the equations of motion. We can then take the point of view that the action is given by Eq. (2.15) where $\Box^{-1}$ is defined using the symmetric Green's function G_+. Then the $\Box^{-1}$ operator is self-adjoint and this allows us to perform standard manipulations such as the integration by parts; see Appendix A. The Euler-Lagrange equations obtained from this action are then given by Eq. (2.14), where again $\Box^{-1} = \Box_+^{-1}$. We then add the rule that the physical equations of motion are obtained replacing now $\Box_+^{-1}$ with the inverse d'Alembertian computed with the Green's function of our choice, in particular with $\Box_{\rm ret}^{-1}$, which ensures causality. This is indeed the procedure used in [76,91], in the context of nonlocal gravity theories with a Lagrangian of the form $Rf(\Box^{-1}R)$.

A similar procedure can be used at the quantum level, in the computation of in-in matrix elements $\langle{\rm in, vac}|\hat{\varphi}|{\rm in, vac}\rangle$ of a quantum field $\hat{\varphi}$, for a Poincaré-invariant in-vacuum state in the asymptotic past. In this case one can first work in Euclidean space, computing the Euclidean effective action in an asymptotically flat space-time. In Euclidean space the Green's function that enters in the $\Box^{-1}$ operator is defined imposing vanishing boundary conditions at infinity, and the $\Box^{-1}$ operator is unambiguously defined. One can then prove that the nonlocal effective equations for $\langle{\rm in, vac}|\hat{\varphi}|{\rm in, vac}\rangle$ can be obtained from the Euclidean equations of motion by an analytic continuation, with the prescription that the Euclidean $\Box^{-1}$ operator becomes the retarded inverse d'Alembertian $\Box_{\rm ret}^{-1}$ in Lorentzian signature [62] (see also the discussion in [66]).

An equivalent formulation of the latter procedure is obtained as follows, adapting a construction developed in [92] in the case of $Rf(\Box^{-1}R)$ theories. We take the action (2.15), with a generic $\Box^{-1}$, and rewrite it introducing a Lagrange multiplier field $\xi_{\mu\nu}$ as well as an auxiliary field $\psi_{\mu\nu}$, as

$$S = \int d^4x\Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m_\gamma^2}{4}F_{\mu\nu}\psi^{\mu\nu} + \xi_{\mu\nu}(\Box\psi^{\mu\nu} - F^{\mu\nu}) - j_\mu A^\mu\Big]. \tag{2.18}$$

The variation with respect to $\xi_{\mu\nu}$ enforces the constraint $\psi^{\mu\nu} = \Box^{-1}F^{\mu\nu}$, and therefore Eq. (2.18) is formally equivalent to Eq. (2.15), independently of the Green's function used in the definition of $\Box^{-1}$. The variations with respect to A_μ and $\psi_{\mu\nu}$ give, respectively,

$$\partial_\mu\Big(F^{\mu\nu} - \frac{m_\gamma^2}{2}\psi^{\mu\nu} + 2\xi^{\mu\nu}\Big) = j^\nu, \tag{2.19}$$

$$\Box\xi^{\mu\nu} + \frac{m_\gamma^2}{4}F^{\mu\nu} = 0. \tag{2.20}$$

Substituting $\xi^{\mu\nu} = -(m_\gamma^2/4)\Box^{-1}F^{\mu\nu}$ and $\psi^{\mu\nu} = \Box^{-1}F^{\mu\nu}$ into Eq. (2.19) we get Eq. (2.14), independently of the definition of $\Box^{-1}$. These manipulations are somewhat formal, since we saw that a proper treatment of the variation of the action (2.15) should rather give Eq. (2.17). However, in the spirit of point (2) above, they can be used as a well-defined set of rules that allows us to obtain the equation of motion (2.14) from an action.[2]

In conclusion, the equation of motion (2.14) or (with the above qualifications) the action (2.15) provides a formulation of massive electrodynamics in which only the field A_μ appears (i.e., Stückelberg fields are no longer present), and which is both manifestly Lorentz invariant and gauge invariant. The price that we pay is the lack of manifest locality, since the equation of motion (2.12) involves the nonlocal operator $\Box^{-1}$. It should be stressed, however, that the theory *is* local, even if not manifestly so, since we have seen that the Lagrangian (2.15) is equivalent to the original Proca Lagrangian, which is local. Observe that we could now use gauge invariance to fix the Lorentz gauge $\partial_\mu A^\mu = 0$. In this way the equation of motion (2.12) simply becomes $(\Box - m_\gamma^2)A^\nu = j^\nu$ and the nonlocal term disappears. We therefore get back Eqs. (2.3) and (2.4) that define Proca theory, except that now the equation $\partial_\mu A^\mu = 0$ emerges as the gauge-fixing condition of an underlying gauge theory. In other words, the nonlocality only affects pure gauge modes and can be removed by a suitable gauge fixing.

[2]At the quantum level, an approach similar in spirit consists in stating that the nonlocal action is not the fundamental quantity for determining whether the theory is causal. Rather, one must consider the quantum effective action, which is a functional of the expectation value of the quantum fields. The boundary conditions for the nonlocal effective action are now fixed by the choice of initial and final quantum states, and can be dealt with using the Schwinger-Keldysh technique. A breakdown of causality in the variational equation for the classical fields does not necessarily imply an inconsistency in the computation of $\langle \text{in}|\text{out}\rangle$ matrix elements; see the discussion in [90].

III. NONLOCAL FORMULATION OF FIERZ-PAULI MASSIVE GRAVITY

We now consider FP massive gravity linearized over Minkowski space. The action is $S_{\rm FP} + S_{\rm int}$, where

$$S_{\rm FP} = \frac{1}{2}\int d^4x[h_{\mu\nu}\mathcal{E}^{\mu\nu,\rho\sigma}h_{\rho\sigma} - m^2(h_{\mu\nu}h^{\mu\nu} - h^2)], \tag{3.1}$$

is the Fierz-Pauli action, $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$, and indices are raised and lowered with the flat metric. The Lichnerowicz operator $\mathcal{E}^{\mu\nu,\rho\sigma}$ is defined as

$$\mathcal{E}^{\mu\nu,\rho\sigma} \equiv \frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - 2\eta^{\mu\nu}\eta^{\rho\sigma})\Box + (\eta^{\rho\sigma}\partial^\mu\partial^\nu + \eta^{\mu\nu}\partial^\rho\partial^\sigma) - \frac{1}{2}(\eta^{\mu\rho}\partial^\sigma\partial^\nu + \eta^{\nu\rho}\partial^\sigma\partial^\mu + \eta^{\mu\sigma}\partial^\rho\partial^\nu + \eta^{\nu\sigma}\partial^\rho\partial^\mu), \tag{3.2}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the flat-space d'Alembertian. Therefore

$$\mathcal{E}^{\mu\nu,}{}_{\rho\sigma}h^{\rho\sigma} = \Box h^{\mu\nu} - \eta^{\mu\nu}\Box h + \eta^{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \partial^\mu\partial^\nu h - \partial_\rho\partial^\nu h^{\mu\rho} - \partial_\rho\partial^\mu h^{\nu\rho}. \tag{3.3}$$

The interaction with the matter energy-momentum tensor is given by

$$S_{\rm int} = \frac{\kappa}{2}\int d^4x h_{\mu\nu}T^{\mu\nu}. \tag{3.4}$$

We take $T^{\mu\nu}$ conserved, so at the linearized level $\partial_\nu T^{\mu\nu} = 0$. In order to obtain a gauge-invariant but nonlocal formulation of the theory one can introduce a Stückelberg vector field A^μ through

$$h_{\mu\nu} \to h_{\mu\nu} + \frac{1}{m}(\partial_\mu A_\nu + \partial_\nu A_\mu), \tag{3.5}$$

and then integrate it out using its own equations of motion [7,8,23,74]. By construction, the theory is trivially invariant under the gauge transformation

$$h_{\mu\nu} \to h_{\mu\nu} - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu), \qquad A_\mu \to A_\mu + m\xi_\mu, \tag{3.6}$$

that corresponds to a linearized diffeomorphism. It is often useful to perform a further Stückelberg transformation $A_\mu \to A_\mu + (1/m)\partial_\mu \varphi$ that introduces a $U(1)$ symmetry and explicitly extracts the helicity-0 mode. For the purpose of obtaining the nonlocal form of massive gravity this step is not really necessary, so we will only make the replacement (3.5). Then the action becomes

$$S_{\rm FP} + S_{\rm int} = \int d^4x \Big[\frac{1}{2} h_{\mu\nu}\mathcal{E}^{\mu\nu,\rho\sigma}h_{\rho\sigma} - \frac{m^2}{2}(h_{\mu\nu}h^{\mu\nu} - h^2) - \frac{1}{2}F_{\mu\nu}F^{\mu\nu}\Big] + \int d^4x \Big[\frac{\kappa}{2} h_{\mu\nu}T^{\mu\nu} + 2mA_\nu j^\nu\Big], \quad (3.7)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and

$$j^\nu \equiv \partial_\mu (h^{\mu\nu} - \eta^{\mu\nu} h). \quad (3.8)$$

Observe that we could obtain the standard normalization $(-1/4)F_{\mu\nu}F^{\mu\nu}$ for the kinetic term of A^μ by rescaling $A^\mu \to A^\mu/\sqrt{2}$. This would however produce a number of $\sqrt{2}$ factors that would clutter many subsequent formulas, so we prefer to keep a nonstandard normalization for the $F_{\mu\nu}F^{\mu\nu}$ term. The variation with respect to A^μ gives

$$\partial_\mu F^{\mu\nu} = -mj^\nu. \quad (3.9)$$

Applying ∂_ν to Eq. (3.9) we also get the condition

$$\partial_\nu j^\nu = 0. \quad (3.10)$$

One can now eliminate the Stückelberg field A_μ through its equations of motion. To solve Eq. (3.9) we separate A^ν into its transverse and longitudinal parts,

$$A^\nu = A^\nu_T - \partial^\nu \alpha, \quad (3.11)$$

where $\partial_\nu A^\nu_T = 0$, and we get $\Box A^\nu_T = -mj^\nu$. Thus, the equation of motion of the Stückelberg field allows us to fix the transverse part to the value $A^\nu_T = -m\Box^{-1}j^\nu$, while the longitudinal part remains arbitrary. This is a peculiarity of the FP mass term, which is such that after the Stückelberg replacement the kinetic term for the Stückelberg field A^μ happens to depend only on the $U(1)$-invariant combination $F_{\mu\nu}F^{\mu\nu}$. Therefore the longitudinal part, which has the form of a $U(1)$ gauge transformation, remains arbitrary. Thus, the most general solution of Eq. (3.9) is [7,8]

$$A^\nu = -m\Box^{-1}j^\nu - \partial^\nu \alpha, \quad (3.12)$$

where α is an arbitrary scalar field. Note that, because of Eq. (3.10), $\partial_\nu(\Box^{-1}j^\nu) = 0$ so the term $\Box^{-1}j^\nu$ is indeed transverse. The transformation properties of the field α under linearized diffeomorphisms can be obtained observing that $\Box\alpha = -\partial_\mu A^\mu$. Since under linearized diffeomorphisms $A^\mu \to A^\mu + m\xi^\mu$, we get

$$\Box\alpha \to \Box\alpha - m\partial_\mu \xi^\mu. \quad (3.13)$$

Observe that the transformation property of $(\Box\alpha)/m$ is the same as that of $h/2$. We then find convenient to trade α for a new field N,

$$N \equiv \frac{h}{2} - \frac{\Box\alpha}{m}, \quad (3.14)$$

which is invariant under linearized diffeomorphisms.[3] Performing the replacement (3.12) in the action (3.7) and trading α for N we find

$$S_{\rm FP} + S_{\rm int} = \int d^4x \Big[\frac{1}{2} h_{\mu\nu}\Big(1 - \frac{m^2}{\Box}\Big)\mathcal{E}^{\mu\nu,\rho\sigma}h_{\rho\sigma} - 2m^2 N \frac{1}{\Box}\partial_\mu\partial_\nu(h^{\mu\nu} - \eta^{\mu\nu}h) + \frac{\kappa}{2} h_{\mu\nu}T^{\mu\nu}\Big]. \quad (3.15)$$

Observe that N enters the action as a Lagrange multiplier.[4] Taking the variation with respect to N we get

$$\partial_\mu\partial_\nu(h^{\mu\nu} - \eta^{\mu\nu}h) = 0, \quad (3.16)$$

which, in terms of j^ν, can be rewritten as $\partial_\nu j^\nu = 0$. The variation with respect to $h_{\mu\nu}$ gives

$$\Big(1 - \frac{m^2}{\Box}\Big)\mathcal{E}^{\mu\nu,}{}_{\rho\sigma}h^{\rho\sigma} = -\frac{\kappa}{2}T^{\mu\nu} - 2m^2\Big(\eta_{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\Box}\Big)N. \quad (3.17)$$

The field N can be determined algebraically by taking the trace of Eq. (3.17) and using $\eta_{\mu\nu}\mathcal{E}^{\mu\nu,\rho\sigma}h_{\rho\sigma} = (d-1)\partial_\nu j^\nu$. Thus, upon use of the equation of motion (3.10), the trace of the left-hand side of Eq. (3.17) vanishes, and

$$N = -\frac{\kappa}{4dm^2}T. \quad (3.18)$$

Plugging Eq. (3.18) into Eq. (3.17) we finally obtain

[3]Here our treatment departs from that in [7,8,74], where α is fixed to some given value, e.g., $\alpha = 0$. Actually, α (or, equivalently N), is an independent field that will enter the action, and we will see that it plays the role of a Lagrange multiplier. This gives a more transparent derivation of the constraint associated to FP massive gravity, which otherwise emerges as a consistency condition on the equations of motion.

[4]Equation (3.15) agrees with the result found with a somewhat different route in [23]; see Eq. (4.48) in that work. Our κ corresponds to 2κ and our m^2N to N in the notation of [23].

$$\left(1-\frac{m^2}{\Box}\right)\mathcal{E}^{\mu\nu,}{}_{\rho\sigma}h^{\rho\sigma} = -\frac{\kappa}{2}T^{\mu\nu} + \frac{\kappa}{2d}\left(\eta^{\mu\nu}-\frac{\partial^\mu\partial^\nu}{\Box}\right)T. \tag{3.19}$$

Observe that the right-hand side is divergenceless (consistently with the linearized Bianchi identity $\partial_\mu[\mathcal{E}^{\mu\nu,}{}_{\rho\sigma}h^{\rho\sigma}]=0$) and traceless. Therefore Eq. (3.19) fully summarizes the two equations (3.16) and (3.17) and provides a nonlocal formulation of FP massive gravity.

Observe also that one could diagonalize the action (3.15) with a nonlocal field redefinition [23],

$$h'_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\nu}\frac{m^2}{\Box - m^2}N, \tag{3.20}$$

$$N' = \sqrt{6}\frac{m^2}{\Box - m^2}N. \tag{3.21}$$

The action (3.15) then becomes

$$S_{\rm FP} + S_{\rm int} = \int d^4x\left[\frac{1}{2}h'_{\mu\nu}\left(1-\frac{m^2}{\Box}\right)\mathcal{E}^{\mu\nu,\rho\sigma}h'_{\rho\sigma} + \frac{1}{2}N'(\Box - m^2)N'\right] + \frac{\kappa}{2}\int d^4x\left(h'_{\mu\nu}T^{\mu\nu} + \frac{1}{\sqrt{6}}N'T\right). \tag{3.22}$$

However one should be aware that, in general, it is not legitimate to perform nonlocal field redefinitions, such as that given in Eqs. (3.20) and (3.21), and use the action written in terms of these nonlocal fields. The basic point is that operators such as $1/\Box$ or $1/(\Box - m^2)$ are nonlocal not only in space but even in time, and therefore there is no one-to-one correspondence between the initial conditions on the original fields and on the redefined fields. This is important in particular when one wants to clearly identify the true dynamical degrees of freedom of the theory. In Appendix B we discuss some examples of the apparent paradoxes in which one can run (even in massless GR) when performing nonlocal field redefinitions.

IV. DEGREES OF FREEDOM OF THE NONLOCAL ACTION

We now consider the action

$$S_{\rm nonloc} \equiv \int d^4x\frac{1}{2}h_{\mu\nu}\left(1-\frac{m^2}{\Box}\right)\mathcal{E}^{\mu\nu,\rho\sigma}h_{\rho\sigma}. \tag{4.1}$$

This action was first introduced in the context of the degravitation idea [6,8], and we have seen that it also enters in FP massive gravity. However, to obtain FP massive gravity, it must be supplemented by the constraint imposed by the field N, as shown in Eq. (3.15). The action $S_{\rm nonloc}$ will be our starting point for the construction of a fully nonlinear theory of massive gravity in Sec. V, so we will discuss it now in more detail. In particular, we want to understand what degrees of freedom it describes. This is an issue which hides some subtleties, and on which there seems to be some confusion in the literature.

The propagating degrees of freedom of a theory can be read from the propagator. In GR one starts from the quadratic Einstein-Hilbert action

$$S^{(2)}_{\rm EH} = \frac{1}{2}\int d^4x h_{\mu\nu}\mathcal{E}^{\mu\nu,\rho\sigma}h_{\rho\sigma}. \tag{4.2}$$

To obtain the propagator one must add a gauge-fixing term. A convenient choice is

$$S_{\rm gf} = -\int d^4x(\partial^\nu\bar{h}_{\mu\nu})(\partial_\rho\bar{h}^{\rho\mu}), \tag{4.3}$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - (1/2)h\eta_{\mu\nu}$. Then

$$S^{(2)}_{\rm EH} + S_{\rm gf} = \int d^4x\left[-\frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + \frac{1}{4}\partial^\mu h\partial_\mu h\right]. \tag{4.4}$$

Inverting this quadratic form one finds the propagator of massless gravitons,

$$\tilde{D}^{\mu\nu\rho\sigma}(k) = \frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma})\left(\frac{-i}{k^2 - i\epsilon}\right), \tag{4.5}$$

where the $i\epsilon$ prescription selects the Feynman propagator. Consider now the nonlocal action (4.1). This action is gauge invariant, so we need again a gauge fixing. We find convenient to use as a gauge-fixing term

$$S_{\rm gf} = -\frac{1}{\xi}\int d^4x(\partial^\nu\bar{h}_{\mu\nu})\left(1-\frac{m^2}{\Box}\right)(\partial_\rho\bar{h}^{\rho\mu}), \tag{4.6}$$

and use the gauge $\xi = 1$. After some integration by parts we get

$$S_{\rm nonloc} + S_{\rm gf} = \int d^4x\left[-\frac{1}{2}\partial_\rho h_{\mu\nu}\left(1-\frac{m^2}{\Box}\right)\partial^\rho h^{\mu\nu} + \frac{1}{4}\partial^\mu h\left(1-\frac{m^2}{\Box}\right)\partial_\mu h\right] = \frac{1}{2}\int d^4x h^{\mu\nu}A_{\mu\nu\rho\sigma}(\Box - m^2)h^{\rho\sigma}, \tag{4.7}$$

where

$$A_{\mu\nu\rho\sigma} = \frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}). \tag{4.8}$$

Observe that in gauge $\xi = 1$ the nonlocal terms in $S_{\rm nonloc} + S_{\rm gf}$ cancel. This gives again an example of the interplay between gauge invariance and nonlocality. We can write the action in a gauge-invariant form at the price of nonlocality, or in a local form at the price of fixing a suitable gauge. The propagator in this gauge is obtained by inverting this quadratic form, which gives

$$\tilde{D}^{\mu\nu\rho\sigma}(k) = \frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}) \times \left(\frac{-i}{k^2 + m^2 - i\epsilon}\right). \tag{4.9}$$

Thus, the tensor structure is the same as in massless GR, and the only change is in the overall factor $-i/(k^2 - i\epsilon)$, which becomes $-i/(k^2 + m^2 - i\epsilon)$. Observe that in the theory defined by S_{nonloc} there is no vDVZ discontinuity, and in the $m \to 0$ limit the propagator (4.9) smoothly reduces to the massless propagator (4.5).

Naively one might think that, since the tensor structure of the propagator in massless GR and in S_{nonloc} are the same, the radiative degrees of freedom are the same, too. If this were the case, S_{nonloc} would only contain two massive states with helicities ± 2. However, this reasoning is incorrect. This can be first illustrated comparing massless and massive electrodynamics. The propagator of the massless photon is

$$\tilde{D}^{\mu\nu}(k) = \frac{-i}{k^2 - i\epsilon}\left[\eta^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right]. \tag{4.10}$$

For conserved currents, in momentum space we have $k^\mu \tilde{j}_\mu(k) = 0$, so the term proportional to $k^\mu k^\nu$ in the propagator does not contribute, and the saturated propagator is

$$\begin{aligned}\tilde{j}_\mu(-k)\tilde{D}^{\mu\nu}(k)\tilde{j}_\nu(k) &= \frac{-i}{k^2 - i\epsilon}\eta_{\mu\nu}\tilde{j}^\mu(-k)\tilde{j}^\nu(k) \\ &= \frac{-i}{k^2 - i\epsilon}\Big[-\tilde{j}^0(-k)\tilde{j}^0(k) \\ &\quad + \tilde{j}^3(-k)\tilde{j}^3(k) + \sum_{i=1,2}\tilde{j}^i(-k)\tilde{j}^i(k)\Big].\end{aligned} \tag{4.11}$$

In the massless case this tensor structure, proportional to $\eta^{\mu\nu}$, describes the exchange of only the states with helicities ± 1. In fact, for an on-shell photon we can write $k_\mu = \omega(-1, 0, 0, 1)$, and then current conservation implies $\tilde{j}^0(k) = \tilde{j}^3(k)$. Thus the first two terms in Eq. (4.11) cancel, and the interaction mediated by an on-shell massless photon is proportional to

$$\sum_{i=1,2}\tilde{j}^i(-k)\tilde{j}^i(k) = \tilde{j}^+(-k)\tilde{j}^-(k) + \tilde{j}^-(-k)\tilde{j}^+(k), \tag{4.12}$$

where $\tilde{j}^\pm = (\tilde{j}^1 \pm i\tilde{j}^2)/\sqrt{2}$. This shows that the interaction of on-shell massless photons only involves the operators $\tilde{j}^\pm$, which have helicities ± 1, and therefore is of the form $\tilde{A}_+(-k)\tilde{j}_-(k) + \tilde{A}_-(-k)\tilde{j}_+(k)$, where $\tilde{A}_\pm$ are fields with helicities ± 1.

The propagator of the massive photon is instead

$$\tilde{D}^{\mu\nu}(k) = \frac{-i}{k^2 + m_\gamma^2 - i\epsilon}\left(\eta^{\mu\nu} + \frac{k^\mu k^\nu}{m_\gamma^2}\right). \tag{4.13}$$

Again, for a conserved current the term $k^\mu k^\nu/m_\gamma^2$ does not contribute so the massive photon propagator can be taken to be

$$\tilde{D}^{\mu\nu}(k) = \frac{-i}{k^2 + m_\gamma^2 - i\epsilon}\eta^{\mu\nu}, \tag{4.14}$$

so its tensor structure is effectively given simply by $\eta^{\mu\nu}$, just as for the massless propagator. However, now $k_\mu = (-\omega, 0, 0, k)$ with $\omega = (k^2 + m_\gamma^2)^{1/2}$, and the current conservation equation $k_\mu \tilde{j}^\mu = 0$ gives $\tilde{j}^0(k) = (k/\omega)\tilde{j}^3(k)$. Then the terms $-\tilde{j}^0(-k)\tilde{j}^0(k) + \tilde{j}^3(-k)\tilde{j}^3(k)$ no longer cancel. Rather, now

$$\begin{aligned}&\tilde{j}_\mu(-k)\tilde{D}^{\mu\nu}(k)\tilde{j}_\nu(k) \\ &= \frac{-i}{k^2 - i\epsilon}\Big[\tilde{j}^+(-k)\tilde{j}^-(k) + \tilde{j}^-(-k)\tilde{j}^+(k) \\ &\quad + \frac{m_\gamma^2}{\omega^2}\tilde{j}^3(-k)\tilde{j}^3(k)\Big],\end{aligned} \tag{4.15}$$

showing that there is an extra term that describes the coupling of the longitudinal polarization. Thus, even if the tensor structure of the propagator (4.14) is the same as that of the massless propagator, still it describes two transverse and one longitudinal degrees of freedom, as of course should be for a massive photon. Observe also that, for $m_\gamma \to 0$, the longitudinal mode smoothly decouples.

The situation is completely analogous when comparing the propagator of massless GR with that of S_{nonloc}. In the massless case the tensor structure in Eq. (4.5) reflects the fact that a massless graviton only has the helicities ± 2. In fact, in momentum space energy-momentum conservation reads $k_\mu \tilde{T}^{\mu\nu}(k) = 0$. For on-shell massless gravitons we can write again $k_\mu = \omega(-1, 0, 0, 1)$, and energy-momentum conservation becomes

$$\tilde{T}^{0\nu}(k) = \tilde{T}^{3\nu}(k). \tag{4.16}$$

We can now compute explicitly the saturated propagator $\tilde{T}_{\mu\nu}(-k)\tilde{D}^{\mu\nu\rho\sigma}(k)\tilde{T}_{\rho\sigma}(k)$, and eliminate all occurrences of $\tilde{T}^{0\nu}(k)$ using Eq. (4.16). Then one finds that the terms involving a spatial index $i = 3$ cancel, and

$$\begin{aligned}&\tilde{T}_{\mu\nu}(-k)\tilde{D}^{\mu\nu\rho\sigma}(k)\tilde{T}_{\rho\sigma}(k) \\ &= \tilde{T}_{-2}(-k)\frac{-i}{k^2 - i\epsilon}\tilde{T}_{+2}(k) + \tilde{T}_{+2}(-k)\frac{-i}{k^2 - i\epsilon}\tilde{T}_{-2}(k),\end{aligned} \tag{4.17}$$

where

$$\tilde{T}_{\pm 2} = \frac{1}{2}(\tilde{T}_{11} - \tilde{T}_{22} \mp 2i\tilde{T}_{12}). \tag{4.18}$$

Under rotations by an angle θ around the z axis the combinations $\tilde{T}_{\pm 2}$ transform as $\tilde{T}_{\pm 2} \to \exp\{\pm 2i\theta\}\tilde{T}_{\pm 2}$, and are therefore eigenstates of the helicity with eigenvalue ± 2. This shows that this propagator describes a massless particle with helicities ± 2.[5]

In contrast, for massive gravitons one must again write $k_\mu = (-\omega, 0, 0, k)$ with $\omega = (k^2 + m^2)^{1/2}$, and the conservation equation $k_\mu \tilde{T}^{\mu\nu}(k) = 0$ no longer reduces the saturated propagator to a form that only involves the helicity-2 operators. Rather, we now have $0 = k_\mu \tilde{T}^{\mu\nu}(k) = -\omega \tilde{T}^{0\nu}(k) + k\tilde{T}^{3\nu}(k)$, so

$$\tilde{T}^{0\nu}(k) = (k/\omega)\tilde{T}^{3\nu}(k). \tag{4.19}$$

We use this relation to eliminate all occurrences of $\tilde{T}^{0\nu}$, and we introduce

$$\tilde{T}_{\pm 1} = \tilde{T}_{13} \mp i\tilde{T}_{23}, \tag{4.20}$$

$$\tilde{T}_0 = 3\tilde{T}_{33}. \tag{4.21}$$

The five quantities $\tilde{T}_q(k)$, with $q = -2, \ldots, 2$, are helicity eigenstates with eigenvalue q, and their normalizations have been chosen for later convenience. The four-dimensional trace $\tilde{T}(k) = \eta^{\mu\nu}\tilde{T}_{\mu\nu}(k)$ is instead a Lorentz scalar. From Eq. (4.19) we have $\tilde{T}^{00}(k) = (k/\omega)\tilde{T}^{30}(k)$ and $\tilde{T}^{30}(k) = \tilde{T}^{03}(k) = (k/\omega)\tilde{T}^{33}(k)$, so $\tilde{T}^{00}(k) = (k/\omega)^2\tilde{T}^{33}(k)$. This gives

$$\tilde{T}(k) = \eta^{\mu\nu}\tilde{T}_{\mu\nu}(k) = \tilde{T}_{11}(k) + \tilde{T}_{22}(k) + \frac{m^2}{\omega^2}\tilde{T}_{33}(k). \tag{4.22}$$

Eliminating $\tilde{T}^{0\nu}(k)$ through Eq. (4.19) and trading the six quantities $\tilde{T}_{ij}(k)$ for the five components of a spin-2 operator $\tilde{T}_q(k)$, $q = -2, \ldots, 2$ plus the scalar T, we get

$$\begin{aligned} &\tilde{T}_{\mu\nu}(-k)\tilde{D}^{\mu\nu\rho\sigma}(k)\tilde{T}_{\rho\sigma}(k) \\ &= \sum_{q=-2,2} \tilde{T}_{-q}(-k)\frac{-i}{k^2 + m^2 - i\epsilon}\tilde{T}_q(k) \\ &\quad + \frac{m^2}{\omega^2}\sum_{q=-1,1} \tilde{T}_{-q}(-k)\frac{-i}{k^2 + m^2 - i\epsilon}\tilde{T}_q(k) \\ &\quad + \frac{m^2}{6\omega^2}(T_0(-k), T(-k))\frac{-iD}{k^2 + m^2 - i\epsilon}\begin{pmatrix} T_0(k) \\ T(k) \end{pmatrix}, \end{aligned} \tag{4.23}$$

where the matrix D is given by

$$D = \begin{pmatrix} m^2/\omega^2 & -1 \\ -1 & 0 \end{pmatrix}. \tag{4.24}$$

The eigenvalues of D are $\lambda_\pm = \epsilon \pm \sqrt{1 + \epsilon^2}$ where $\epsilon = m^2/(2\omega^2)$, so $\lambda_+ > 0$ and $\lambda_- < 0$, corresponding to a particle with the good sign in the propagator and a ghost, respectively. The eigenvectors are the combinations $t_\pm = T_0 + \lambda_\mp T$, which to lowest order in ϵ reduce to $T_0 \mp T$. The fields that diagonalize the propagator in the scalar sector are therefore the corresponding combinations of the helicity-0 mode and of the scalar field. We see that the propagator of $S_{\rm nonloc}$ describes six dynamical fields: besides the expected massive states with helicities $q = \pm 2$, there are two states with helicities $q = \pm 1$, a state with helicity $q = 0$ (which, together, form the states of a spin-2 massive particle), and a scalar field. In the limit $m \to 0$ the contribution of the helicities ± 1 goes smoothly to zero, because it is multiplied by an overall factor m^2/ω^2. The same happens in the scalar sector.[6]

This counting of degrees of freedom is confirmed observing that we have been able to rewrite FP massive gravity in the form (3.15). Here the field N enters as a Lagrange multiplier, so it is not dynamical and it enforces the single constraint (3.16). This constraint therefore removes 1 scalar degree of freedom from the 6 described by $S_{\rm nonloc}$, and we remain with 5 degrees of freedom, in

[5]Indeed, the graviton propagator (4.5) can be found without performing explicitly the inversion of the quadratic form in the action, simply observing that it must be symmetric in (μ, ν) and in (ρ, σ). Thus, it can only depend on the combinations $(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho})$ and $\eta_{\mu\nu}\eta_{\rho\sigma}$ (apart from the term involving $k_\mu k_\nu$, $k_\rho k_\sigma$, etc. that gives zero when contracted with the energy-momentum tensor; the particular choice of gauge-fixing used in Eq. (4.3) actually sets these terms to zero). Requiring that the combination $(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} + a\eta_{\mu\nu}\eta_{\rho\sigma})$ selects the helicity-2 part of the energy-momentum tensor and using $k_\mu \tilde{T}^{\mu\nu}(k) = 0$, with $k_\mu = \omega(-1, 0, 0, 1)$, fixes $a = -1$ [2].

[6]This shows that the statement in [8] that the action $S_{\rm nonloc}$ describes only the states with helicity ± 2, once the helicities 0 and ± 1 have been integrated out, is incorrect. The integration over the Stückelberg field A_μ should not be confused with the integration over the helicity-0 and helicity-1 modes. When we perform the Stückelberg replacement (3.5) we are formally increasing the number of fields in the theory, and this increase is compensated by the appearance of a gauge symmetry. Thus, after the replacement $h_{\mu\nu} \to h_{\mu\nu} + (1/m)(\partial_\mu A_\nu + \partial_\nu A_\mu)$, the field $h_{\mu\nu}$ still contains its helicity-0, helicity-1, and helicity-2 states and, furthermore, we have introduced extra helicity-0 and helicity-1 states associated to A_μ. When we eliminate the latter (either integrating out A_μ, or for instance just choosing the gauge $A_\mu = 0$) we still remain with the helicity-0, helicity-1, and helicity-2 states associated with $h_{\mu\nu}$, and the action $S_{\rm nonloc}$ still contains two scalars, two states with helicities ± 1, and two states with helicities ± 2.

agreement with the fact that Eq. (3.15) is just a rewriting of linearized FP massive gravity.

Of course, the above counting of degrees of freedom can also be derived from the invariance of the action (4.1) under linearized diffeomorphisms $h_{\mu\nu} \to h_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)$. Using this invariance, out of the 10 components of $h_{\mu\nu}$ we can eliminate 4 (and only 4) degrees of freedom from the action (4.1). Following the steps that in the massless case lead to the transverse-traceless (TT) gauge we can in fact use the four functions ξ^μ to fix the Lorentz gauge

$$\partial_\nu \bar{h}^{\mu\nu} = 0, \tag{4.25}$$

where $\bar{h}^{\mu\nu} = (h^{\mu\nu} - (1/2)\eta^{\mu\nu} h)$. Under gauge transformations

$$\partial_\nu \bar{h}^{\mu\nu} \to \partial_\nu \bar{h}^{\mu\nu} - \Box \xi^\mu, \tag{4.26}$$

so fixing the Lorentz gauge leaves a residual gauge invariance parametrized by four functions ξ^μ that satisfy $\Box \xi^\mu = 0$. In linearized *massless* gravity, after fixing the Lorentz gauge, the metric satisfies $\Box h^{\mu\nu} = 0$, so the residual gauge invariance can be used to set to zero four more components of $h_{\mu\nu}$, namely one transverse vector and two scalars. Thus, out of the original ten components of $h_{\mu\nu}$, four are eliminated by Eq. (4.25) and four more by the residual gauge invariance, and we remain with the 2 degrees of freedom of a massless graviton, corresponding to the helicities ± 2. In the massive case the residual gauge symmetry cannot be used to eliminate further degrees of freedom. Indeed, the equation of motion derived from Eq. (4.1) is

$$\left(1 - \frac{m^2}{\Box}\right) \mathcal{E}^{\mu\nu,}{}_{\rho\sigma} h^{\rho\sigma} = 0. \tag{4.27}$$

Observe that

$$\mathcal{E}^{\mu\nu,}{}_{\rho\sigma} h^{\rho\sigma} = \Box \bar{h}^{\mu\nu} - \partial^\mu \partial_\rho \bar{h}^{\rho\nu} - \partial^\nu \partial_\rho \bar{h}^{\rho\mu} + \eta^{\mu\nu} \partial_\rho \partial_\sigma \bar{h}^{\rho\sigma}. \tag{4.28}$$

Thus, in the Lorentz gauge we have $\mathcal{E}^{\mu\nu,}{}_{\rho\sigma} h^{\rho\sigma} = \Box \bar{h}^{\mu\nu}$ and Eq. (4.27) becomes local, and is just a massive Klein-Gordon (KG), $(\Box - m^2)\bar{h}_{\mu\nu} = 0$. Contracting with $\eta_{\mu\nu}$ we also have $(\Box - m^2) h = 0$, and therefore in the end, after fixing the gauge (4.25), the equation of motion for $h_{\mu\nu}$ becomes

$$(\Box - m^2) h^{\mu\nu} = 0. \tag{4.29}$$

Using functions ξ^μ which are constrained to obey $\Box \xi^\mu = 0$ we cannot eliminate components of $h^{\mu\nu}$ that satisfy $\Box h^{\mu\nu} \neq 0$. Thus, we find again the action (4.1) describes 6 degrees of freedom, which corresponds to the five components of a massive spin-2 particle, plus a Lorentz scalar.[7] Further insight into the structure of the nonlocal action $S_{\rm nonloc}$ can be gained by introducing nonlocal variables in terms of which the action takes a local form. This provides a rather elegant formulation, which is discussed in Appendix C.

V. A COVARIANT FULLY NONLINEAR THEORY OF MASSIVE GRAVITY

We now show how to construct a viable covariant, fully nonlinear theory of massive gravity using this nonlocal formulation.

A. Covariantization of FP theory

Consider first the FP action, in the form (3.15). To perform the covariantization we begin by observing that, linearizing around flat space, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, we have $R = R^{(1)} + \mathcal{O}(h^2)$, where

$$R^{(1)} = \partial_\mu \partial_\nu (h^{\mu\nu} - \eta^{\mu\nu} h). \tag{5.1}$$

Thus, the simplest covariant generalization of the term $N \Box^{-1} \partial_\mu \partial_\nu (h^{\mu\nu} - \eta^{\mu\nu} h)$ in the action (3.15) is just $N \Box_g^{-1} R$, where the field N is promoted to be a scalar under full diffeomorphisms, and the covariantization of Eq. (3.16) is simply $R = 0$. This condition was already found, with a different route, in [74], where it was also

[7]This situation is completely analogous to what happens in the nonlocal formulation of massive electrodynamics discussed in Sec. II. The action (2.15) describes three (rather than two) radiative degrees of freedom. This is evident from the fact that it is just a rewriting of the original Proca theory, which describes a massive photon and therefore 3 radiative degrees of freedom. One might be puzzled by the fact that Eq. (2.15) describes 3 propagating degrees of freedom because this action is gauge invariant, and we are used to the fact that a $U(1)$ gauge invariance removes 2 degrees of freedom. However, again the point is that the single function θ that parametrizes the gauge transformation $A_\mu \to A_\mu - \partial_\mu \theta$ allows us to eliminate 2 degrees of freedom only in the massless case. This can be seen for instance using the $U(1)$ symmetry to fix the Lorentz gauge $\partial_\mu A^\mu = 0$. This leaves us with the freedom of performing a residual gauge transformation $A_\mu \to A_\mu - \partial_\mu \theta$ with $\Box \theta = 0$. In massless electrodynamics, in the Lorentz gauge the equation of motion in vacuum $\partial_\mu F^{\mu\nu} = 0$ becomes $\Box A^\nu = 0$ and we can use the residual gauge freedom to set $A^0 = 0$, and then $\partial_\mu A^\mu = 0$ becomes $\nabla \cdot \mathbf{A} = 0$. We have therefore reached the radiation gauge $A^0 = 0$, $\nabla \cdot \mathbf{A} = 0$. Thus, when $m_\gamma = 0$, the single function θ can be used to eliminate both A^0 and the longitudinal component of the photon. In contrast, when $m_\gamma \neq 0$, after fixing the Lorentz $\partial_\mu A^\mu = 0$, we remain with the equation $(\Box - m_\gamma^2) A^\nu = 0$, and the residual gauge invariance parametrized by a function θ with $\Box \theta = 0$ cannot be used to eliminate a further degree of freedom.

correctly observed that it provides a discontinuity with the massless theory, since in GR we rather have $R = -8\pi GT$. At the linearized level this is just the vDVZ discontinuity and we see that in this covariantization it persists at the level of the fully nonlinear theory. Such a covariantization therefore necessarily leads to a theory in conflict with the experiment, even assuming that it gives a consistent theory. Of course the covariantization procedure is not unique, and one could rather replace $N\partial_\mu\partial_\nu(h^{\mu\nu} - \eta^{\mu\nu}h)$ with $N[R + \mathcal{O}(R^2_{\mu\nu\rho\sigma})]$, which still has the correct linearized limit. However, this would still give rise to a constraint that is not present in GR, and that reduces to $R = 0$ at low curvatures. We will therefore turn our attention to $S_{\rm nonloc}$, and construct a covariant generalization of this theory, rather than of FP theory.

B. Covariantization of $S_{\rm nonloc}$

We find it convenient to work at the level of the equations of motion, so we look for a covariantization of Eq. (4.27) including also the source term,

$$\left(1 - \frac{m^2}{\Box}\right)\mathcal{E}^{\mu\nu,}{}_{\rho\sigma}h^{\rho\sigma} = -16\pi G T^{\mu\nu}, \qquad (5.2)$$

where we have rescaled $h_{\mu\nu} \to \kappa h_{\mu\nu}$, so that now in Eq. (5.2) $h_{\mu\nu}$ is dimensionless, and we used $\kappa^2 = 32\pi G$. In this section we continue to use the notation $\Box$ for the flat-space d'Alembertian, while we denote by $\Box_g$ the d'Alembertian computed with respect to a generic metric $g_{\mu\nu}$. The covariant generalization of the left-hand side can be found observing that the linearization of the Einstein tensor over Minkowski is given by $G_{\mu\nu} = G^{(1)}_{\mu\nu} + \mathcal{O}(h^2)$, with

$$G^{(1)}_{\mu\nu} = -\frac{1}{2}\mathcal{E}_{\mu\nu,\rho\sigma}h^{\rho\sigma}. \qquad (5.3)$$

Thus, a generally covariant expression that reduces to the right-hand side of Eq. (5.2) is $-2(1 - m^2/\Box_g)G^{\mu\nu}$. Of course, this is not the only possible expression that has the correct linearized limit. However, we must further require that the correct generalization of the left-hand side is a covariantly conserved tensor, in order to be consistent with $\nabla_\mu T^{\mu\nu} = 0$. We then proceed as in [74], and observe that any symmetric tensor $S_{\mu\nu}$ can be decomposed as

$$S_{\mu\nu} = S^{\rm T}_{\mu\nu} + \frac{1}{2}(\nabla_\mu S_\nu + \nabla_\nu S_\mu), \qquad (5.4)$$

where $\nabla^\mu S^{\rm T}_{\mu\nu} = 0$. One could further decompose $S^{\rm T}_{\mu\nu}$ into a transverse-traceless part $S^{\rm TT}_{\mu\nu}$ and the trace part, and similarly $S_\mu = S^{\rm T}_\mu + \nabla_\mu\Sigma$, where $\nabla^\mu S^{\rm T}_{\mu\nu} = 0$ and $\nabla^\mu S^{\rm T}_\mu = 0$. The various components can be extracted explicitly with the use of nonlocal operators, and in flat space the explicit expressions are given in Eqs. (B1)–(B6). For a general metric the explicit expressions are more complicated, basically because the covariant derivative does not commute with $\Box_g$.

In terms of this decomposition, a natural covariantization of Eq. (5.2) is

$$\left[\left(1 - \frac{m^2}{\Box_g}\right)G_{\mu\nu}\right]^{\rm T} = 8\pi G T_{\mu\nu}, \qquad (5.5)$$

i.e.,

$$G_{\mu\nu} - m^2(\Box_g^{-1}G_{\mu\nu})^{\rm T} = 8\pi G T_{\mu\nu}, \qquad (5.6)$$

where the superscript T denotes the operation of taking the transverse part. By construction the divergence of the left-hand side vanishes, so we still have $\nabla^\mu T_{\mu\nu} = 0$.[8] The classical theory defined by Eq. (5.6) is a covariant, fully nonlinear theory of massive gravity defined without introducing a reference metric. As discussed in the Introduction, this is conceptually quite satisfying, since the introduction of a reference metric basically means that we have a different definition of the massive theory for every background of the massless theory. Furthermore this theory has no vDVZ discontinuity since its propagator, given by Eq. (4.9), reduces smoothly to the GR propagator as $m \to 0$.

Note also that, if we use the retarded Green's function in the $\Box_g^{-1}$ operators that appear in Eq. (5.6), the theory is nonlocal but only involves an integration over the past light cone, and therefore preserves causality. Observe also that the problems of superluminal propagation discussed in [45] are generated by the same constraint that eliminates the ghost in FP theory.[9] Since this constraint is absent in the theory defined by $S_{\rm nonloc}$, this particular example of superluminality is also absent. Thus causality problems, at least in the form identified to date in nonlinear extensions of FP massive gravity, are not present (although a detailed analysis is needed to study whether other forms of superluminality might emerge in some specific background).

[8] Of course, one can always add (the transverse part of) quantities quadratic in the Riemann tensor to the left-hand side, since these do not affect the linearized limit. By dimensional reasons, these terms must be suppressed by the inverse of a mass squared. If they are suppressed by $1/M^2_{\rm Pl}$, these terms are irrelevant much below the Planck scale. However, having at our disposal both m and $M_{\rm Pl}$, one can also in principle write a theory where such terms are suppressed by $1/\Lambda^2$ with $\Lambda^n = m^{n-1}M_{\rm Pl}$ for some n (i.e., one of the scales that appear in the Stückelberg description of the local formulation of massive gravity). In this case, Eq. (5.6) should be regarded as the IR limit of this more general class of theories, and these extra terms could be important for the UV completion of the theory.

[9] However, as mentioned in the Introduction, the observation of Ref. [45] does not yet imply the loss of causality in dRGT, since in the attempt of constructing closed timelike curves one might be forced to leave the domain of validity of the effective field theory [51], so in dRGT the causality problem is rather postponed to the UV completion. See [93] for a review of the issue.

VI. THE GHOST PROBLEM

As we found in Sec. IV, 1 degree of freedom in $S_{\rm nonloc}$ is a ghost. At first this seems to doom the theory (5.6) to failure. In particular, one might fear that the vacuum decays quickly through associated production of positive-energy massive gravitons and negative-energy ghosts; see e.g., the discussion in [94–97]. In our case, as discussed in Sec. IV, in the scalar sector we have a healthy state ψ and a ghost state ϕ, which are linear combinations of the helicity-0 component of the massive spin-2 graviton and of the scalar degree of freedom. In the covariantization of $S_{\rm nonloc}$ we have for instance a trilinear gravitational vertex proportional to $h\partial h\partial h$ (where h denotes symbolically the five components of the massive graviton plus the scalar), which induces processes such as (vacuum) $\to \psi\psi\phi\phi$ through diagrams such as that on the left of Fig. 1. The four-point interaction $hh\partial h\partial h$ is instead responsible for the diagram on the right of Fig. 1.

Observe however that in our case the mass of the ghost has the same value m as the mass of the spin-2 graviton. This result is a consequence of the structure of the propagator in $S_{\rm nonloc}$ and is protected by the diffeomorphism invariance of the covariantization of $S_{\rm nonloc}$. For cosmological applications the graviton mass m must be very small, of the order of the present value of the Hubble parameter H_0. As a consequence, our ghost is extremely light, too. Furthermore, and quite crucially, we have seen that in this theory there is no vDVZ discontinuity, and the extra scalar and vector polarizations decouple smoothly in the limit $m \to 0$; see Eq. (4.23). Indeed, we prove in Appendix C 2 that in the $m \to 0$ limit the ghost smoothly goes into a nonradiative degree of freedom of GR. We therefore expect that, for $m = \mathcal{O}(H_0)$, instabilities associated to the ghost only develop at most on cosmological time scales. In this sense, the existence of the ghost could even be welcome, since a phase of accelerated expansion of the Universe can be seen as an instability. We will now put this physical intuition on a more formal basis by estimating the probability of ghost-induced vacuum decay.[10]

[10]It should be stressed that the ghost that appears in $S_{\rm nonloc}$ and in its covariantization is quite different from the Boulware-Deser ghost that appears in nontuned nonlinear extensions of FP theory. The Boulware-Deser ghost on a generic background gets a mass fixed by the scales of the background [98], and is not smoothly connected, in the $m \to 0$ limit, to a harmless nonradiative field. Rather on the contrary, one tries to get rid of it by making it very heavy. Indeed, at the linearized level the FP tuning sends the ghost mass to infinity. The problem is that this procedure is in general unstable against the introduction of nonlinearities, and when expanding the theory over a generic background a finite ghost mass reappears. The tuning of the potential in the dRGT theory [11,12] is indeed performed so to send again the mass of the ghost above the cutoff scale of the effective theory.

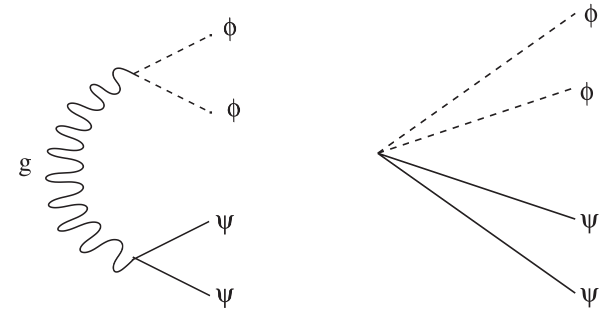

FIG. 1. Left: the Feynman graph describing (vacuum) $\to \psi\psi\phi\phi$. The wavy line denotes any of the six states described by $h_{\mu\nu}$. Right: the same process, mediated directly by the four-point vertex.

A. Ghost-induced vacuum decay rate

Before examining the computation in the nonlocal formulation of massive gravity, let us consider a simpler theory with action

$$S = \int d^4x \Big[\frac{1}{2}(-\partial_\mu\psi\partial^\mu\psi - m_\psi^2\psi^2) + \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m_\phi^2\phi^2) + \frac{\lambda}{4}\phi^2\psi^2 \Big]. \qquad (6.1)$$

With our "mostly plus" signature ψ is a healthy scalar while ϕ is a ghost. We first recall how the computation is performed in this simpler case [95–97]. We consider for definiteness the graph on the right of Fig. 1 and we denote by k_1, k_2 the momenta of the normal particles, and by p_1, p_2 the momenta of the ghosts. For the normal particles the energies are positive, $k_1^0 > 0$, $k_2^0 > 0$, while for the ghosts $p_1^0 < 0$, $p_2^0 < 0$. We also introduce the notation $\omega_i = k_i^0$, and $E_i = -p_i^0$ $(i = 1, 2)$. The amplitude associated to the graph on the right-hand side of Fig. 1 is given by

$$i\mathcal{M}_{fi} = i\lambda(2\pi)^4\delta^{(4)}(p_1 + p_2 + k_1 + k_2). \qquad (6.2)$$

Regularizing the theory in a spatial volume V and a time interval T, and using

$$|(2\pi)^4\delta^{(4)}(p)|^2 = VT(2\pi)^4\delta^{(4)}(p), \qquad (6.3)$$

the differential probability of vacuum decay, dw, is given by

$$dw = \lambda^2(2\pi)^4\delta^{(4)}(p_1 + p_2 + k_1 + k_2)VT\frac{1}{2!}\frac{d^3p_1}{(2\pi)^3 2E_1} \times \frac{d^3p_2}{(2\pi)^3 2E_2}\frac{1}{2!}\frac{d^3k_1}{(2\pi)^3 2\omega_1}\frac{d^3k_2}{(2\pi)^3 2\omega_2}, \qquad (6.4)$$

(where the $1/2!$ are the factors for identical particles). This is the probability that the decay happens anywhere in space and at any time, so the decay probability per unit volume and unit time is

$$\Gamma = \frac{w}{VT} = \frac{\lambda^2}{2!2!}\int \frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{d^3k_1}{(2\pi)^3 2\omega_1} \times \frac{d^3k_2}{(2\pi)^3 2\omega_2}(2\pi)^4\delta^{(4)}(p_1+p_2+k_1+k_2). \quad (6.5)$$

This can be conveniently manipulated introducing the identities

$$1 = \int d^4P\,\delta^{(4)}(P-p_1-p_2), \qquad 1 = \int d^4K\,\delta^{(4)}(K-k_1-k_2). \quad (6.6)$$

Then

$$\begin{aligned}\Gamma &= \frac{\lambda^2}{(2\pi)^4}\int d^4P d^4K \delta^{(4)}(P+K) \\ &\quad\times\left[\frac{1}{2!}\int\frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2}(2\pi)^4\delta^{(4)}(P-p_1-p_2)\right] \\ &\quad\times\left[\frac{1}{2!}\int\frac{d^3k_1}{(2\pi)^3 2\omega_1}\frac{d^3k_2}{(2\pi)^3 2\omega_2}(2\pi)^4\delta^{(4)}(K-k_1-k_2)\right] \\ &= \frac{\lambda^2}{(2\pi)^4}\int d^4Pd^4K\delta^{(4)}(P+K)\Phi^{(2)}_\phi(-P^2)\Phi^{(2)}_\psi(-K^2) \\ &= \frac{\lambda^2}{(2\pi)^4}\int d^4P\Phi^{(2)}_\phi(-P^2)\Phi^{(2)}_\psi(-P^2), \end{aligned} \quad (6.7)$$

where $\Phi^{(2)}_\phi$ is the two-body phase space for two identical particles of mass m_ϕ, and depends on P only through the Lorentz scalars $-P^2$, and similarly for $\Phi^{(2)}_\psi(-P^2)$ (observe that, with our signature, $-P^2$ and $-K^2$ are positive). The two-body phase space for two identical particles of mass m is

$$\Phi^{(2)}(s) = \theta(s-4m^2)\frac{1}{16\pi}\sqrt{1-\frac{4m^2}{s}}, \quad (6.8)$$

and goes to a constant in the large s limit. Thus, the integral over d^4P diverges. To better understand this divergence we can further manipulate Eq. (6.7) inserting the identity in the form

$$\int_0^\infty ds\,\delta(s+P^2) = 1, \quad (6.9)$$

where $P^2 = -(P^0)^2 + \mathbf{P}^2$. Then

$$\begin{aligned}\Gamma &= \frac{\lambda^2}{(2\pi)^4}\int d^4P\Phi^{(2)}_\phi(-P^2)\Phi^{(2)}_\psi(-P^2)\int_0^\infty ds\,\delta(s+P^2) \\ &= \frac{\lambda^2}{(2\pi)^4}\int_0^\infty ds\Phi^{(2)}_\phi(s)\Phi^{(2)}_\psi(s)\int dP^0d^3\mathbf{P}\,\delta(s-(P^0)^2+\mathbf{P}^2) \\ &= \frac{\lambda^2}{(2\pi)^4}\int_0^\infty ds\Phi^{(2)}_\phi(s)\Phi^{(2)}_\psi(s) \\ &\quad\times\int dP^0d^3\mathbf{P}\frac{1}{2P^0}\delta(P^0-\sqrt{\mathbf{P}^2+s}) \\ &= \frac{\lambda^2}{(2\pi)^4}\int_0^\infty ds\Phi^{(2)}_\phi(s)\Phi^{(2)}_\psi(s)\int\frac{d^3\mathbf{P}}{2\sqrt{\mathbf{P}^2+s}}. \end{aligned} \quad (6.10)$$

Both the integral over s and that over the three-momentum $\mathbf{P}$ diverge. Putting a cutoff $|\mathbf{P}| < \Lambda$ as well as $s < \Lambda^2$ and using the asymptotic form $\Phi^{(2)}(s) \simeq 1/(16\pi)$ one obtains

$$\Gamma \sim \lambda^2\left(\frac{\Lambda}{8\pi}\right)^4. \quad (6.11)$$

Observe that, despite the fact that $d^3\mathbf{P}/(2\sqrt{\mathbf{P}^2+s})$ is a Lorentz-invariant measure, it cannot be regularized preserving Lorentz invariance. Putting a cutoff $|\mathbf{P}| < \Lambda$ breaks Lorentz invariance, and therefore such a cutoff should come from new Lorentz-violating physics [95,96].[11] It has been shown in [97] that, in the presence of nonlocal interactions that become soft in the UV (which is indeed our case), the momentum integral can become convergent, and only the integral over s needs a regularization. A cutoff $s < \Lambda^2$ does not spoil Lorentz invariance, so in this case Lorentz-violating physics is no longer required. The whole issue of Lorentz violations becomes irrelevant if we put the cutoff Λ at the Planck scale, since in any case we expect that beyond $M_{\rm Pl}$ a Lorentz-invariant and local quantum field theory description will no longer be appropriate. However, for $\Lambda \sim M_{\rm Pl}$, Eq. (6.11) provides a decay rate of about one event per Planck volume per Planck time so (unless one chooses a ridiculously small value of λ) the vacuum decay is basically instantaneous. This means that, in the theory (6.1), with no new physics until $\Lambda \sim M_{\rm Pl}$, the ghost instability is a fatal one.

Consider next what happens if a ghost interacts only gravitationally, but still in a local theory. The situation is then somewhat different because $G = 1/M^2_{\rm Pl}$ enters in the coupling and there is a natural UV cutoff $\Lambda \sim M_{\rm Pl}$, beyond which one might reasonably assume that string theory or

[11]One might consider the possibility of regularizing the original integral over d^4P by rotating into Euclidean space and putting a Lorentz-invariant cutoff Λ over the modulus of the Euclidean momentum, $(P^0_E)^2 + \mathbf{P}^2 < \Lambda^2$. However, the usual Wick rotation is an operation that is performed on the integrals that enter in the loop corrections to the amplitudes, and is justified by the analyticity properties of the amplitudes. Here the Euclidean rotation would rather be performed on the phase space integrals, and it is not obvious that it makes any sense.

any other UV completion takes over and softens the gravitational interaction. In normal GR the structure of the Lagrangian is schematically of the form

$$\mathcal{L} \sim \frac{1}{G}[\partial h \partial h + (h \partial h \partial h) + (hh \partial h \partial h)]. \quad (6.12)$$

After rescaling $h \to G^{1/2} h$ to get a canonically normalized kinetic term, we get a Lagrangian of the form

$$\mathcal{L} \sim \partial h \partial h + G^{1/2}(h \partial h \partial h) + G(hh \partial h \partial h), \quad (6.13)$$

and therefore the four-graviton vertex gives a contribution proportional to GE^2, where E is an energy scale of the process. In a local theory of massive gravity in which $h_{\mu\nu}$ contains the 5 degrees of freedom of the massive graviton plus a scalar ghost, in order of magnitude the computation will therefore be the same as above, with the replacement

$$\lambda \to \lambda_{\rm eff}(s) \sim Gs. \quad (6.14)$$

Thus

$$\Gamma \sim \frac{1}{(2\pi)^4} \frac{1}{(16\pi)^2} \int^{\Lambda^2} ds (Gs)^2 \int^{\Lambda} 2\pi |\mathbf{P}| d|\mathbf{P}| \sim \left(\frac{\Lambda}{8\pi}\right)^4 \frac{\Lambda^4}{M_{\rm Pl}^4}. \quad (6.15)$$

A contribution of the same order comes from the graphs on the left-hand side of Fig. 1, due to a factor $(G^{1/2}s)$ from each vertex and a factor $\sim 1/s$ from the graviton propagator. Again, for $\Lambda \sim M_{\rm Pl}$, Eq. (6.15) gives a production rate of order one event per Planck time, per Planck volume, and we therefore get a catastrophic decay that immediately destabilizes the vacuum. This is indeed what happens if we consider the vacuum decay induced by the ghost that is present in the local formulation of nonlinear generalizations of FP massive gravity, in agreement with the discussion in [99].

Consider now the vacuum decay rate in the nonlocal theory of massive gravity that we are proposing. To obtain an order-of-magnitude estimate of the vacuum decay rate we can argue as follows. The interaction involving the ghost can only come from the nonlocal sector (given that for $m = 0$ the ghost decouples and reduces to a nonradiative degree of freedom). Thus, compared to the standard gravitational case, the interaction is softened by a factor $m^2/\Box$. This factor reflects the fact that the ghost matches smoothly a nonradiative degree of freedom in the massless limit, and there is no vDVZ discontinuity in our theory. Independently of the details of the action, this should contribute an extra factor $\mathcal{O}(m^2/s)$ to the amplitude, and hence a factor m^4/s^2 to the probability. Thus, in order of magnitude, the vacuum decay rate in the nonlocal theory can be estimated as[12]

$$\Gamma \sim \frac{1}{(2\pi)^4} \frac{1}{(16\pi)^2} \int^{\Lambda^2} ds (Gs)^2 \frac{m^4}{s^2} \int^{\Lambda} 2\pi |\mathbf{P}| d|\mathbf{P}| \sim \left(\frac{m}{8\pi}\right)^4 \frac{\Lambda^4}{M_{\rm Pl}^4}. \quad (6.16)$$

We see that, even for $\Lambda \simeq M_{\rm Pl}$, the rate does not exceed a value of order $[m/(8\pi)]^4$. Taking $m \sim 8\pi H_0$, for the production of a $\psi\psi\phi\phi$ final state out of the vacuum, this gives a rate of one event in a volume equal to the present Hubble volume H_0^{-3}, over the whole age of the Universe $t \sim H_0^{-1}$. Such a rate is totally irrelevant. To get a sense for it, consider the energy density ρ_ψ in ψ particles produced per unit time by this process (of course, an equal and opposite energy density is produced in ghosts, and the total energy density is conserved). This is obtained multiplying the rate Γ (number of events per unit time per unit volume) by the energy carried by each event. Since the integral in Eq. (6.16) is dominated by the UV cutoff region, setting $\Lambda \sim M_{\rm Pl}$ this is simply $\dot{\rho}_\psi \sim (m/8\pi)^4 M_{\rm Pl}$. We can compare it with the evolution of the energy density of the Universe due to the standard cosmological expansion; writing $\rho_{\rm tot} \sim H^2 M_{\rm Pl}^2$, we have $\dot{\rho}_{\rm tot} \sim H\dot{H} M_{\rm Pl}^2 \sim H^3 M_{\rm Pl}^2$. Thus,

$$\frac{\dot{\rho}_\psi}{\dot{\rho}_{\rm tot}} \sim \left(\frac{m}{8\pi H}\right)^4 \frac{H}{M_{\rm Pl}}. \quad (6.17)$$

This quantity is an increasing function of time, and for $m = \mathcal{O}(8\pi H_0)$ even at the present epoch it is minuscule, of order $H_0/M_{\rm Pl} \sim 10^{-60}$. Indeed, to make it of order one at the present epoch, we would need a mass m parametrically larger than H_0, $m \sim 8\pi H_0 (M_{\rm Pl}/H_0)^{1/4} \sim 10^{16} H_0$ (i.e., $m^{-1} \sim 0.1$ au). For smaller values of m, and in particular for the values $m \sim H_0$ of cosmological interest, the process of ghost-induced vacuum decay is irrelevant.

Observe that, to get this result, it was crucial that the ghost interaction is softened by the term $m^2/\Box$, giving an extra factor m^2/E^2 in the amplitude and finally an extra factor m^4/Λ^4 in the rate. Without this factor, the decay rate would have rather been given by Eq. (6.15). The crucial difference with the Boulware-Deser ghost that appears in

[12]Observe that, for the purpose of this order-of-magnitude estimate, it is irrelevant whether the term $m^2/\Box$ contributes to the integral over s with a factor m^4/s^2, or to the integral over $|\mathbf{P}|$ with a factor $m^4/|\mathbf{P}|^4$. In the latter case it would render UV finite the integral over d^3P, as in [97]. If the boost integral is not regularized by the $m^2/\Box$ term, then we need to rely on physics beyond the Planck scale for its regularization. However, this does not necessarily mean that physics beyond the Planck scale must violate Lorentz invariance in order to regularize the vacuum decay rate (indeed, Lorentz violations even at the Planck scale are severely constrained [100]). Rather, a UV completion such as string theory could provide an effective nonlocality that regularizes the boost integral similarly to what happens in [97].

generic nonlinear extensions of FP gravity can therefore be traced back to the fact that the nonlocal theory (5.6) has no vDVZ discontinuity, and all extra degrees of freedom, including the ghost, decouple in the $m \to 0$ limit, as shown by Eq. (4.23).

VII. DE SITTER SOLUTIONS AND DEGRAVITATION

It is interesting to observe that Eq. (5.6) does not admit exact de Sitter solutions $G^{\rm dS}_{\mu\nu} = -\Lambda g_{\mu\nu}$ (observe that, with our signature, de Sitter corresponds to $\Lambda > 0$). In fact, in this case $(\Box_g^{-1} G_{\mu\nu})^T = -\Lambda(\Box_g^{-1} g_{\mu\nu})^T = -\Lambda \Box_g^{-1} g_{\mu\nu}$ (because $g_{\mu\nu}$ is already transverse). Since $\Box_g g_{\mu\nu} = 0$, we have $\Box_g^{-1} g_{\mu\nu} = \infty$. Thus, in de Sitter, the term $(\Box_g^{-1} G_{\mu\nu})^T$ diverges. To understand this problem, let us first introduce a new parameter μ, and consider the equation

$$G_{\mu\nu} - m^2\left(\frac{1}{\Box_g - \mu^2} G_{\mu\nu}\right)^{\rm T} = 8\pi G T_{\mu\nu}. \quad (7.1)$$

We have chosen the sign of μ^2 in Eq. (7.1) so that $\mu^2 > 0$ corresponds to a nontachyonic mass term. We can think of μ as a regulator that will eventually be set to zero, but it is in fact quite interesting to consider the theory with finite μ. In particular, will see that it is especially interesting to take a value $\mu = \mathcal{O}(m^2/M_{\rm Pl})$. In this case $\mu \ll m$, and Eq. (5.6) is now seen as an approximation which is only valid for modes with typical spatial or temporal variations much smaller than $1/\mu$, so that on these modes $\Box_g \gg \mu^2$.

Equation (7.1) admits de Sitter solutions, which show interesting degravitation properties. Consider an energy-momentum tensor of the form $T_{\mu\nu} = -\rho_{\rm vac} g_{\mu\nu} = (\rho_{\rm vac}, -a^2\rho_{\rm vac}\delta_{ij})$, so $p_{\rm vac} = -\rho_{\rm vac}$, and look for a solution $G_{\mu\nu} = G^{\rm dS}_{\mu\nu} = -\Lambda g_{\mu\nu}$. Now this solution exists, and Λ is fixed by

$$\Lambda = 8\pi G \frac{\mu^2}{m^2 + \mu^2} \rho_{\rm vac}. \quad (7.2)$$

Taking now $\mu \to 0$ at fixed m we see that $\Lambda \to 0$. This can be seen as an extreme form of degravitation in which, even in the presence of an arbitrarily large vacuum energy, the effective cosmological constant $\Lambda = \mathcal{O}(\mu^2) \to 0$. More generally, for finite μ the vacuum energy $\rho_{\rm vac}$ is degravitated so that the quantity that actually contributes to the observed acceleration of the Universe is

$$\rho_\Lambda = \frac{\mu^2}{m^2 + \mu^2} \rho_{\rm vac}. \quad (7.3)$$

In order to reproduce the observed value $\rho_\Lambda = \mathcal{O}(M_{\rm Pl}^2 H_0^2)$ from a vacuum energy $\rho_{\rm vac} = \mathcal{O}(M_{\rm Pl}^4)$ we need

$$\mu = \mathcal{O}\left(\frac{H_0 m}{M_{\rm Pl}}\right). \quad (7.4)$$

In particular, if $m = \mathcal{O}(H_0)$, we need

$$\mu = \mathcal{O}\left(\frac{m^2}{M_{\rm Pl}}\right). \quad (7.5)$$

Such a value, which would provide a natural solution for the cosmological constant problem, is just of the size that can be expected from gravitational loop corrections: in fact, in the bubble graph giving the one-loop graviton self-energy diagram, which provides the correction δm^2 to the graviton mass, each of the two trilinear vertices gives a factor $\sqrt{G} \sim 1/M_{\rm Pl}$, so $\delta m^2 \sim 1/M_{\rm Pl}^2$ (and the same for the one-loop correction to the propagator involving a single four-graviton vertex). In the limit $m \to 0$ we must have $\delta m^2 \to 0$ since in this case mass renormalization is protected by general covariance, so it is natural to expect $\mu^2 \sim \delta m^2 \sim m^4/M_{\rm Pl}^2$.

The fact that the only exact de Sitter solution has the above value of Λ is not a problem for inflation because, if we take m of order of the present Hubble rate H_0, in the early Universe Eq. (7.1) admits quasi-de Sitter solutions which are practically indistinguishable from the usual slow-roll inflationary solutions. Indeed, in a spatially uniform time-dependent background $\Box \sim d^2/dt^2 = \mathcal{O}(\omega^2)$, where ω is the characteristic frequency of variation of the background, and the nonlocal term in Eq. (7.1) is negligible if $\omega^2 \gg m^2$ (which also implies $\omega^2 \gg \mu^2$). For a FRW metric with Hubble parameter $H(t)$, in particular, the characteristic frequency is $\omega = |\dot{H}|/H$. In terms of the slow-roll parameter $\epsilon = -\dot{H}/H^2$ the condition $\omega \gg m$ reads $\epsilon \gg m/H(t)$. If we take $m = \mathcal{O}(H_0)$, in the early Universe when inflation takes place, $m/H(t)$ is a ridiculously small number (e.g., of order 10^{-57} for grand unified theory scale inflation), much smaller than the typical values, say $\epsilon = \mathcal{O}(10^{-2})$, of the slow-roll parameter. More generally, for $m = \mathcal{O}(H_0)$, the nonlocal term is irrelevant in the early Universe and only becomes important in the recent cosmological epoch. In other words, Eq. (7.1) works as a high-pass filter that degravitates all sources with a typical frequency ω smaller than μ, or typical length scales larger than μ^{-1} [which, for $m = \mathcal{O}(H_0)$ and $\mu = \mathcal{O}(m^2/M_{\rm Pl})$, is parametrically larger than the horizon size]. In particular, an exactly constant vacuum energy is totally degravitated. However, the source term due to an inflaton field slowly rolling into a potential, even in the "slow"-roll regime, still evolves with a characteristic frequency which is huge compared to μ, and is not affected at all by the nonlocal terms.[13]

[13]Observe also that, linearizing Eq. (7.1), we get a propagator $\tilde{D}(p) = -i(p^2 + \mu^2)/[p^2(p^2 + m^2 + \mu^2)]$. Beside the pole at $p^2 = -(m^2 + \mu^2) \simeq -m^2$ we therefore now have an extra pole at $p^2 = 0$. Its residue is however proportional to $\mu^2/(m^2 + \mu^2)$ which, for $\mu \sim m^2/M_{\rm Pl}$ and $m \sim H_0$, is order $H_0^2/M_{\rm Pl}^2$. These extra states are therefore totally decoupled on the subhorizon and even on horizon scales, and their only role is to degravitate the vacuum energy to the value (7.3) rather than down to zero.

Finally it is interesting to observe that, if we rather take a tachyonic value $\mu^2 = -m^2$, Eq. (7.1) becomes

$$G_{\mu\nu} - m^2\left(\frac{1}{\Box_g + m^2} G_{\mu\nu}\right)^{\mathrm{T}} = 8\pi G T_{\mu\nu}, \qquad (7.6)$$

which is equivalent to

$$\left(\frac{\Box_g}{\Box_g + m^2} G_{\mu\nu}\right)^{\mathrm{T}} = 8\pi G T_{\mu\nu}. \qquad (7.7)$$

In this case, degravitation is lost. However, this equation is interesting because it admits a family of self-inflationary solutions, i.e., de Sitter solutions $G_{\mu\nu} = -\Lambda g_{\mu\nu}$ with arbitrary Λ, in the absence of any external source, $T_{\mu\nu} = 0$. Again, as long as $|\dot{H}/H| \gg m$, the nonlocal term in Eq. (7.6) is negligible. Thus, taking m of order H_0, the early Universe cosmology is unaffected, and in particular we have the standard radiation dominated and matter dominated (MD) era. However, when $H(t)$ drops to values comparable to H_0, even in a theory without a explicit cosmological constant term in the action, we expect that the matter dominated solution will be attracted by one of these self-inflationary de Sitter solutions.

VIII. NONLOCAL COSMOLOGY FROM MASSIVE GRAVITY

The nonlocal modification of GR that we are proposing induces a nonlocal modification of the Friedmann equation. To derive the equations of nonlocal cosmology we specialize Eq. (7.1) to a flat FRW metric. We use coordinates $(t, \mathbf{x})$ where t is cosmic time, so $g_{\mu\nu} = (-1, a^2(t)\delta_{ij})$ and $T_{\mu\nu} = (\rho, a^2 p \delta_{ij})$, and we work for generality in d spatial dimensions. The evolution of the scale factor is determined by the nonlocal generalization of the Friedmann equation, i.e., by the (00) component of Eq. (7.1), together with energy-momentum conservation, which is ensured by the fact that the left-hand side of Eq. (7.1) is transverse. Introducing

$$S_{\mu\nu} \equiv \frac{1}{\Box_g - \mu^2} G_{\mu\nu}, \qquad (8.1)$$

and splitting $S_{\mu\nu}$ as in Eq. (5.4), we can rewrite Eq. (7.1) as the coupled system of equations

$$G_{\mu\nu} - m^2 S^{\mathrm{T}}_{\mu\nu} = 8\pi G T_{\mu\nu}, \qquad (8.2)$$

$$(\Box_g - \mu^2) S_{\mu\nu} = G_{\mu\nu}. \qquad (8.3)$$

To extract the transverse part from $S_{\mu\nu}$ we take the divergence of Eq. (5.4). Then $S^{\mathrm{T}}_{\mu\nu}$ drops and we get

$$\nabla^\mu(\nabla_\mu S_\nu + \nabla_\nu S_\mu) = 2\nabla^\mu S_{\mu\nu}. \qquad (8.4)$$

These four equations determine the four components of S_μ in terms of $S_{\mu\nu}$. Then $S^{\mathrm{T}}_{\mu\nu}$ is obtained in terms of $S_{\mu\nu}$ by

$$S^{\mathrm{T}}_{\mu\nu} = S_{\mu\nu} - \frac{1}{2}(\nabla_\mu S_\nu + \nabla_\nu S_\mu). \qquad (8.5)$$

On a generic background it can be nontrivial to find the solution of Eq. (8.4). However, in FRW the solution can be obtained very simply observing that in this case there is no preferred spatial direction, so the only possible solution of Eq. (8.4) for the spatial vector S^i is $S^i = 0$. Equation (8.4) with $\nu = 0$ then suffices to determine S_0,

$$\begin{aligned} S_0 &= -\frac{1}{\partial_0^2 + dH\partial_0 - dH^2} \nabla_\mu S^\mu_0 \\ &= -\frac{1}{\partial_0^2 + dH\partial_0 - dH^2}(\dot{u} + dHu - Hv), \end{aligned} \qquad (8.6)$$

where $H(t) = \dot{a}/a$, the dot is the derivative with respect to cosmic time t, and we have introduced the variables

$$u(t) = S^0_0(t), \qquad v(t) = S^i_i(t), \qquad (8.7)$$

(where sum over $i = \{1, \dots d\}$ is understood). From Eq. (8.5) we have $S^{\mathrm{T}}_{00} = S_{00} - \nabla_0 S_0 = S_{00} - \dot{S}_0$ since, in the coordinates $(t, \mathbf{x})$ we have $\Gamma^\mu_{00} = 0$. Therefore

$$\begin{aligned} (S^0_0)^{\mathrm{T}} &= S^0_0 + \partial_0 S_0 \\ &= u - \partial_0 \frac{1}{\partial_0^2 + dH\partial_0 - dH^2}(\dot{u} + dHu - Hv), \end{aligned} \qquad (8.8)$$

and of course $H = H(t)$ so we must be careful with the ordering of ∂_0 and $[\partial_0^2 + dH\partial_0 - dH^2]^{-1}$. This allows us to write Eq. (8.2) in terms of $H(t)$, $u(t)$, and $v(t)$. We now turn to Eq. (8.3). Evaluating the $\Box$ operator on a rank-(1, 1) tensor we get

$$\Box S^0_0 = -\ddot{S}^0_0 - dH\dot{S}^0_0 + 2dH^2 S^0_0 - 2H^2 S^i_i, \qquad (8.9)$$

$$\Box S^i_i = -\ddot{S}^i_i - dH\dot{S}^i_i - 2dH^2 S^0_0 + 2H^2 S^i_i. \qquad (8.10)$$

Finally, in d spatial dimensions for the Einstein tensor in a FRW background we have

$$\begin{aligned} G^0_0 &= -\frac{d(d-1)}{2} H^2, \\ G^i_i &= -d(d-1)\dot{H} - \frac{d^2(d-1)}{2} H^2. \end{aligned} \qquad (8.11)$$

Putting these results together we get a system of three coupled equations for the three variables $\{H(t), u(t), v(t)\}$: the (00) component of Eq. (8.2) gives the modified Friedmann equation

$$\begin{aligned} &\frac{d(d-1)}{2} H^2 + m^2\Big[u - \partial_0 \frac{1}{\partial_0^2 + dH\partial_0 - dH^2} \\ &\quad \times (\dot{u} + dHu - Hv)\Big] = 8\pi G\rho, \end{aligned} \qquad (8.12)$$

while the (00) and (ii) components of Eq. (8.3) give, respectively,

$$\ddot{u} + \mu^2 u + dH\dot{u} - 2dH^2 u + 2H^2 v = \frac{d(d-1)}{2} H^2, \tag{8.13}$$

$$\ddot{v} + \mu^2 v + dH\dot{v} + 2dH^2 u - 2H^2 v = d(d-1)\dot{H} + \frac{d^2(d-1)}{2} H^2. \tag{8.14}$$

Equations (8.13) and (8.14) can be decoupled introducing

$$U = u + v, \qquad V = u - \frac{1}{d} v. \tag{8.15}$$

Observe that $U = S^0_0 + S^i_i = g^{\mu\nu} S_{\mu\nu}$. Then we get the system of equations

$$\frac{d(d-1)}{2} H^2 + \frac{m^2}{d+1}\Big[U + dV - \partial_0 \frac{1}{\partial_0^2 + dH\partial_0 - dH^2} (\dot{U} + d\dot{V} + d(d+1)HV)\Big] = 8\pi G\rho, \tag{8.16}$$

$$\ddot{U} + \mu^2 U + dH\dot{U} = d(d-1)\Big[\dot{H} + \frac{1}{2}(d+1)H^2\Big], \tag{8.17}$$

$$\ddot{V} + \mu^2 V + dH\dot{V} - 2(d+1)H^2 V = -(d-1)\dot{H}, \tag{8.18}$$

which provides the generalization of the Friedmann equation to nonlocal massive gravity.[14] Observe that U and V enter as new propagating degrees of freedom, corresponding to the two dynamical degrees of freedom in the scalar sector, and therefore one must also impose appropriate initial conditions on them. Further initial data are required for the inversion of the operator $(\partial_0^2 + dH\partial_0 - dH^2)$. A detailed study of the solutions of these equations will be presented in subsequent work.

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APPENDIX A: PROPERTIES OF $\Box^{-1}$ AND $\Box_g^{-1}$

In this appendix we recall some elementary facts on the inverse d'Alembertian in flat and in curved space. We begin with the flat-space d'Alembertian, that we denote simply as $\Box$, while we reserve the notation $\Box_g$ for the d'Alembertian in the metric $g_{\mu\nu}$. The general solution of an equation of the form $\Box\varphi = j$ is

$$\varphi(x) \equiv (\Box^{-1} j)(x) = \varphi_{\rm hom}(x) + \int d^4x' G(x-x') j(x'), \tag{A1}$$

where $\varphi_{\rm hom}(x)$ is a solution of the homogeneous equation and $G(x-x')$ is a Green's function of the d'Alembertian operator,

$$\Box_x G(x-x') = \delta^{(4)}(x-x'). \tag{A2}$$

Observe that, with the signature $(-,+,+,+)$, the propagator $D(x)$ of the quantum theory is defined by $\Box_x D(x-x') = i\delta^{(4)}(x-x')$, so $D(x) = iG(x)$. Solutions corresponding to different Green's functions differ by a solution of the homogeneous equation. In all the formal manipulations involving $\Box^{-1}$ we will set to zero the homogeneous solution. In this way the kernel of the $\Box$ operator becomes trivial, its inversion is well defined, and we can perform a number of formal operations, such as integrating $\Box^{-1}$ by parts; see below. The choice of the Green's function is determined by the physics of the problem. At the classical level, causality requires the use of the retarded Green's function

$$G_{\rm ret}(x;x') = -\frac{1}{4\pi} \frac{1}{|\mathbf{x}-\mathbf{x}'|} \delta(t - t' - |\mathbf{x}-\mathbf{x}'|). \tag{A3}$$

The advanced Green's function is instead

$$G_{\rm adv}(x;x') = -\frac{1}{4\pi} \frac{1}{|\mathbf{x}-\mathbf{x}'|} \delta(t - t' + |\mathbf{x}-\mathbf{x}'|). \tag{A4}$$

In flat space, all Green's functions are actually a function of $x - x'$ only. However, of course, $G_{\rm ret}(x;x')$ and $G_{\rm adv}(x;x')$ are not symmetric in x, x'. Rather exchanging x with x', $G_{\rm ret}(x;x')$ becomes $G_{\rm adv}(x;x')$, and vice versa. A Green's function invariant under $x \leftrightarrow x'$ is obtained taking the symmetric combinations

$$G_+(x;x') = \frac{1}{2}[G_{\rm ret}(x;x') + G_{\rm adv}(x;x')] = -P \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2} e^{-ip(x-x')}, \tag{A5}$$

where P denotes the principal part. Another Green's function invariant under $x \leftrightarrow x'$ is the Feynman Green's function

$$G_F(x;x') = -\int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - i\epsilon} e^{-ip(x-x')}. \tag{A6}$$

The two differ by an imaginary term, according to the relation $1/(y \pm i\epsilon) = P(1/y) \mp i\pi\delta(y)$. Observe that the operator $\Box^{-1}$ is self-adjoint only if it is defined using a symmetric Green's function, i.e., G_+ or G_F, since in this

[14] Note that, for $\mu^2 = 0$, Eq. (8.17) can be rewritten as $a^{-d}\partial_0(a^d\dot{U}) = d(d-1)a^{-(d+1)/2}\partial_0(a^{(d+1)/2}H)$, which integrates to $\dot{U}(t) = d(d-1)a^{2d}(t)H(t) - \frac{d(d-1)^2}{2} a^d(t)\int_0^t dt' a^d(t')H^2(t')$.

case for any two differentiable and square integrable functions $A(x)$ and $B(x)$ we have (using for definiteness G_+)

$$\int d^4xA(x)(\Box_+^{-1}B)(x) = \int d^4xd^4x'A(x)G_+(x;x')B(x') = \int d^4x(\Box_+^{-1}A)(x)B(x), \tag{A7}$$

or, in other words, $\Box_+^{-1}$ can be integrated by parts. Observe also that in flat space, for a generic Green's function, ∂_μ commutes with $\Box^{-1}$. This is a consequence of the fact that in flat space $G(x;x') = G(x-x')$. Thus

$$\begin{aligned}\partial_\mu(\Box^{-1}f)(x) &= \int d^4x'\left[\frac{\partial}{\partial x^\mu}G(x-x')\right]f(x') \\ &= -\int d^4x'\left[\frac{\partial}{\partial x'^\mu}G(x-x')\right]f(x') \\ &= +\int d^4x'G(x-x')\frac{\partial}{\partial x'^\mu}f(x') \\ &= \Box^{-1}(\partial_\mu f)(x),\end{aligned} \tag{A8}$$

where in the second line we integrated $\partial/\partial x'^\mu$ by parts.

We next consider the inverse d'Alembertian in curved space. On a scalar function f the inverse of $\Box_g$ is defined by

$$(\Box_g^{-1}f)(x) = \int dx'\sqrt{-g(x')}G_g(x;x')f(x'), \tag{A9}$$

where

$$(\Box_g)_xG_g(x;x') = \frac{1}{\sqrt{-g(x)}}\delta(x-x'), \tag{A10}$$

and

$$\Box_g = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu), \tag{A11}$$

is the d'Alembertian on a scalar function. The notation $(\Box_g)_x$ indicates that the derivatives are with respect to x^μ. The factor $1/\sqrt{-g(x)}$ on the right-hand side of Eq. (A10) is chosen because under coordinate transformation $\delta(x-x')/\sqrt{-g(x)}$ is a scalar (rather than a scalar density), as it is clear from the fact that its integral over $dx\sqrt{-g}$ is equal to one. Thus, with this definition even $G_g(x;x')$ and hence $\Box_g^{-1}f$ are scalar under coordinate transformations.

Observe that in a generic space-time the Green's function is no longer a function of the difference $x-x'$. However, $G_F(x;x')$ and $G_+(x;x')$ are still symmetric under the exchange of x and x'. The operator $\Box_g^{-1}$ defined using $G_{g,+}(x;x')$ or $G_{g,\mathrm{F}}(x;x')$ is therefore self-adjoint and we can integrate it by parts, since in this case

$$\begin{aligned}&\int d^4x\sqrt{-g(x)}A(x)(\Box_g^{-1}B)(x) \\ &= \int d^4x\sqrt{-g(x)}d^4x'\sqrt{-g(x')}A(x)G_g(x;x')B(x') \\ &= \int d^4x\sqrt{-g(x)}d^4x'\sqrt{-g(x')}A(x)G_g(x';x)B(x') \\ &= \int d^4x\sqrt{-g(x)}B(x)\int d^4x'\sqrt{-g(x')}G_g(x;x')A(x') \\ &= \int d^4x\sqrt{-g(x)}(\Box_g^{-1}A)(x)B(x).\end{aligned} \tag{A12}$$

Of course the $\Box_g$ operator depends on whether it acts on a scalar, a scalar density, a four-vector, a tensor, etc., and the same is true for $\Box_g^{-1}$. Observe that, since $g_{\mu\nu}$ commutes with ∇_μ, it commutes also with $\Box$. Therefore, for any tensor $T_{\mu\nu}$,

$$g^{\mu\nu}T_{\mu\nu} = \Box(g^{\mu\nu}\Box^{-1}T_{\mu\nu}). \tag{A13}$$

Applying $\Box^{-1}$ to both sides,

$$\Box^{-1}(g^{\mu\nu}T_{\mu\nu}) = g^{\mu\nu}\Box^{-1}T_{\mu\nu}. \tag{A14}$$

Now $g^{\mu\nu}T_{\mu\nu}$ is a scalar, so

$$\Box^{-1}(g^{\mu\nu}T_{\mu\nu}) = \int dx'\sqrt{-g(x')}G(x;x')g^{\mu\nu}(x')T_{\mu\nu}(x'), \tag{A15}$$

where $G(x;x')$ is a Green's function of the $\Box_g$ operator acting on scalars. Thus, the definition of $\Box^{-1}$ on a tensor $T_{\mu\nu}$ is such that

$$g^{\mu\nu}(x)(\Box^{-1}T_{\mu\nu})(x) = \int dx'\sqrt{-g(x')}G(x;x')g^{\mu\nu}(x')T_{\mu\nu}(x'). \tag{A16}$$

The explicit form of $\Box_g^{-1}$ on a scalar density (such as $\sqrt{-g}R$) can be obtained similarly, observing that $\nabla_\mu(\sqrt{-g}) = 0$. Thus, $\sqrt{-g}$ commutes with $\Box_g = \nabla_\mu\nabla^\mu$, and this implies that it also commutes with $\Box_g^{-1}$.[15] This means that the definition of $\Box_g^{-1}$ on a scalar density, such as $\sqrt{-g}R$, is

$$[\Box_g^{-1}(\sqrt{-g}R)](x) \equiv \sqrt{-g(x)}(\Box_g^{-1}R)(x) = \sqrt{-g(x)}\int dx'\sqrt{-g(x')}G_g(x;x')R(x'). \tag{A17}$$

[15]The proof is obtained writing $\nabla_\mu(\sqrt{-g}\Box_g^{-1}R) = \sqrt{-g}\nabla_\mu(\Box_g^{-1}R)$. Applying again ∇^μ, $\Box_g(\sqrt{-g}\Box_g^{-1}R) = \sqrt{-g}\Box_g\Box_g^{-1}R = \sqrt{-g}R$. With the definition of $\Box_g$ that sets to zero the solution of the homegeneous equation we also have $\Box_g^{-1}\Box_g = 1$, so we get $\sqrt{-g}\Box_g^{-1}R = \Box_g^{-1}(\sqrt{-g}R)$.

Thus, $\Box_g^{-1}$ applied to a scalar density gives back a scalar density. We can therefore write equivalently $\sqrt{-g}\Box_g^{-1}R$ or $\Box_g^{-1}(\sqrt{-g}R)$, taking however into account the different definitions of $\Box_g^{-1}$ on scalars and on scalar densities.

APPENDIX B: NONLOCAL FIELD REDEFINITIONS AND PROPAGATING DEGREES OF FREEDOM

Nonlocal field theories are certainly less familiar than the usual local field theories, and when manipulating them one must be aware of some subtleties. In particular, blind manipulations might lead one to believe that the theory has more propagating degrees of freedom than it actually has, and might even lead one to believe that the theory has ghosts when in fact it is perfectly healthy. The issue is interesting in itself, and is important for understanding the degrees of freedom in the nonlocal formulations of massive gravity, so we discuss it here in some detail.

As a first trivial example consider a scalar field ϕ which satisfies a nondynamical equation of motion such as a Poisson equation $\boldsymbol{\nabla}^2\phi = \rho$. If we define a new field $\tilde{\phi}$ from $\tilde{\phi} = \Box^{-1}\phi$, the equation of motion can be written as $\Box\tilde{\phi} = \boldsymbol{\nabla}^{-2}\rho \equiv \tilde{\rho}$, and now $\tilde{\phi}$ looks like a dynamical field. However, we certainly cannot transform a nondynamical degree of freedom into a dynamical one in this manner. A way to see where the procedure goes wrong is to realize that assigning initial conditions on a given time slice to ϕ does not provide initial conditions on $\tilde{\phi}$. To get $\tilde{\phi}$ on any single time slice, we need to know ϕ everywhere not only in space but even in time. Alternatively, we can observe that for $\rho = 0$ we have $\phi = 0$, which means that the we must set also $\tilde{\phi} = 0$ in order not to introduce spurious degrees of freedom. In other words, among the solutions of the equation $\Box\tilde{\phi} = 0$, we must discard all the plane-wave solutions, and only retain $\tilde{\phi} = 0$.

An example of this sort appears even in linearized massless GR, when we decompose the metric perturbation in terms of quantities which are transverse or longitudinal with respect to the Lorentz group. The decomposition reads

$$h_{\mu\nu} = h^{\rm TT}_{\mu\nu} + \frac{1}{2}(\partial_\mu\epsilon^{\rm T}_\nu + \partial_\nu\epsilon^{\rm T}_\mu) + \partial_\mu\partial_\nu\alpha + \frac{1}{d}\eta_{\mu\nu}s, \qquad \text{(B1)}$$

where $h^{\rm TT}_{\mu\nu}$ is transverse and traceless with respect to the Lorentz indices,

$$\partial^\mu h^{\rm TT}_{\mu\nu} = 0, \qquad \eta^{\mu\nu}h^{\rm TT}_{\mu\nu} = 0, \qquad \text{(B2)}$$

and $\partial^\mu\epsilon^{\rm T}_\mu = 0$. The factor $1/d$ in front of s is an unconventional normalization that will be useful later. Thus, in $d = 3$, $h^{\rm TT}_{\mu\nu}$ carries 5 degrees of freedom, $\epsilon^{\rm T}_\mu$ 3, and 2 scalar degrees of freedom are carried by α and s. Observe that $\epsilon^{\rm T}_\mu$ and α come from the decomposition of a generic four-vector $\epsilon_\mu = \epsilon^{\rm T}_\mu + \partial_\mu\alpha$. It is straightforward to invert Eq. (B1) and express α, s, $\epsilon^{\rm T}_\mu$, and $h^{\rm TT}_{\mu\nu}$ in terms of $h_{\mu\nu}$, but the inversion involves the nonlocal operator $\Box^{-1}$. The explicit expression of s and α in terms of $h_{\mu\nu}$ can be found taking the trace of Eq. (B1), which gives $h = [(d+1)/d]s + \Box\alpha$, and contracting Eq. (B1) with $\partial^\mu\partial^\nu$, which gives $\partial^\mu\partial^\nu h_{\mu\nu} = \Box[(s/d) + \Box\alpha]$. Combining these equations we get

$$s = \left(\eta^{\mu\nu} - \frac{1}{\Box}\partial^\mu\partial^\nu\right)h_{\mu\nu}, \qquad \text{(B3)}$$

$$\alpha = -\frac{1}{d}\frac{1}{\Box}\left(\eta^{\mu\nu} - \frac{d+1}{\Box}\partial^\mu\partial^\nu\right)h_{\mu\nu}. \qquad \text{(B4)}$$

We can now extract $\epsilon^{\rm T}_\mu$ by applying ∂^μ to Eq. (B1) and using the above expressions for α and s. This gives

$$\epsilon^T_\mu = \frac{2}{\Box}\left(\delta^\rho_\mu - \frac{\partial_\mu\partial^\rho}{\Box}\right)\partial^\sigma h_{\rho\sigma}. \qquad \text{(B5)}$$

Finally, substituting these expressions into Eq. (B1) we get

$$\begin{aligned} h^{\rm TT}_{\mu\nu} = {}& h_{\mu\nu} - \frac{1}{d}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right)h \\ & - \frac{1}{\Box}(\partial_\mu\partial^\rho h_{\nu\rho} + \partial_\nu\partial^\rho h_{\mu\rho}) + \frac{1}{d}\eta_{\mu\nu}\frac{1}{\Box}\partial^\rho\partial^\sigma h_{\rho\sigma} \\ & + \frac{d-1}{d}\frac{1}{\Box^2}\partial_\mu\partial_\nu\partial^\rho\partial^\sigma h_{\rho\sigma}. \end{aligned} \qquad \text{(B6)}$$

Observe also that $h^{\rm TT}_{\mu\nu}$ and s are invariant under linearized diffeomorphisms, while the four-vector $\epsilon_\mu = \epsilon^{\rm T}_\mu + \partial_\mu\alpha$ transforms as $\epsilon_\mu \to \epsilon_\mu - \xi_\mu$. Thus we can choose the gauge so that $\epsilon_\mu = 0$, and this leaves no residual gauge symmetry.

The crucial point is that this inversion involves $\Box^{-1}$ and is therefore nonlocal both in space and time,[16] and a blind use of these variables can lead to some apparent paradox. Indeed, substituting Eq. (B1) in the quadratic Einstein-Hilbert action, ϵ_μ drops because of the invariance under linearized diffeomorphisms, and one finds

$$\begin{aligned} S^{(2)}_{\rm EH} &= \frac{1}{2}\int d^{d+1}x h_{\mu\nu}\mathcal{E}^{\mu\nu,\rho\sigma}h_{\rho\sigma} \\ &= \frac{1}{2}\int d^{d+1}x\left[h^{\rm TT}_{\mu\nu}\Box(h^{\mu\nu})^{\rm TT} - \frac{d-1}{d}s\Box s\right]. \end{aligned} \qquad \text{(B7)}$$

[16] This should be contrasted with the usual (3 + 1) decomposition of the metric, which involves only the inversion of the Laplacian $\boldsymbol{\nabla}^2$ (see e.g., [101,102]), and is therefore nonlocal in space but local in time.

Performing the same decomposition in the energy-momentum tensor, the interaction term can be written as[17]

$$S_{\rm int} = \frac{\kappa}{2}\int d^{d+1}x h_{\mu\nu}T^{\mu\nu} = \frac{\kappa}{2}\int d^{d+1}x\Big[h^{\rm TT}_{\mu\nu}(T^{\mu\nu})^{\rm TT} + \frac{1}{d}sT\Big], \quad \text{(B8)}$$

so the equations of motion derived from $S^{(2)}_{\rm EH} + S_{\rm int}$ are

$$\Box h^{\rm TT}_{\mu\nu} = -\frac{\kappa}{2}T^{\rm TT}_{\mu\nu}, \qquad \Box s = +\frac{\kappa}{2(d-1)}T. \quad \text{(B9)}$$

Thus, using these variables one might be induced to conclude that linearized GR has 6 radiative degrees of freedom, because both $h^{\rm TT}_{\mu\nu}$ and s are governed by a KG equation. Observe furthermore that these degrees of freedom are gauge invariant, so they cannot be gauged away. Furthermore, for all $d > 1$ the scalar s has the "wrong" sign of the kinetic term in the action, so one might be induced to conclude that it is a ghost.

Of course these conclusions are wrong, and linearized GR is a ghost-free theory with only 2 radiative degrees of freedom, corresponding to the ± 2 helicities of the graviton. The loophole in the above argument is exactly the same as in the trivial example presented at the beginning of this section, where a nondynamical field ϕ was transformed into an apparently dynamical field $\tilde\phi$ through the redefinition $\tilde\phi = \Box^{-1}\phi$. Indeed, expressing s in terms of the variables entering the (3 + 1) decomposition, we find (specializing the above results to $d = 3$) $s = 6\Phi - 2\Box^{-1}\nabla^2(\Phi + \Psi)$, where Φ and Ψ are the scalar Bardeen's variable defined in flat space (see [102]). Since Φ and Ψ are nonradiative, s is nonradiative too. The fact that it is obtained applying the $\Box^{-1}$ operator to the nonradiative field $\nabla^2(\Phi + \Psi)$ gives to its equation of motion the appearance of a dynamical equation, but nevertheless s does not represent a dynamical degree of freedom of the theory. Again, this is reflected in the fact that giving initial conditions on a given time slice for the metric does not provide the initial conditions on s.

Another example of the apparent puzzles that can arise from nonlocal field redefinitions is obtained diagonalizing the action (3.15). Following [23], the quadratic term that mixes N and $h_{\mu\nu}$ can be removed defining[18]

$$h'_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\nu}\frac{m^2}{\Box - m^2}N, \quad \text{(B10)}$$

$$N' = \sqrt{6}\frac{m^2}{\Box - m^2}N. \quad \text{(B11)}$$

The action (3.15) then becomes [23]

$$S_{\rm FP} + S_{\rm int} = \int d^4x\Big[\frac{1}{2}h'_{\mu\nu}\Big(1 - \frac{m^2}{\Box}\Big)\mathcal{E}^{\mu\nu,\rho\sigma}h'_{\rho\sigma} + \frac{1}{2}N'(\Box - m^2)N'\Big] + \frac{\kappa}{2}\int d^4x\Big(h'_{\mu\nu}T^{\mu\nu} + \frac{1}{\sqrt{6}}N'T\Big), \quad \text{(B12)}$$

where $T = \eta^{\mu\nu}T_{\mu\nu}$. At first sight something very strange happened here, since the term $h'_{\mu\nu}(1 - m^2/\Box)\mathcal{E}^{\mu\nu,\rho\sigma}h'_{\rho\sigma}$ has the same functional form as the term $h_{\mu\nu}(1 - m^2/\Box)\mathcal{E}^{\mu\nu,\rho\sigma}h_{\rho\sigma}$, so one would think that the two describe the same number of dynamical degrees of freedom. However, the field N, which in Eq. (3.15) entered as a Lagrange multiplier and removed a scalar degree of freedom from $h_{\mu\nu}(1 - m^2/\Box)\mathcal{E}^{\mu\nu,\rho\sigma}h_{\rho\sigma}$, has now been traded for a field N' which looks fully dynamical. Thus, we have apparently lost a scalar constraint, and furthermore we have gained a dynamical scalar field, so the number of scalar degrees of freedom apparently increased by 2.

Again, the solution to this apparent puzzle is that the correct counting of radiative degrees of freedom can only be done using the original variables $h_{\mu\nu}$ and N.[19] For instance, the field N' is a fake dynamical field, just as the field $\tilde\phi$ discussed at the beginning of this section. Indeed, N is determined algebraically by Eq. (3.18), $N = cT$, with c a constant, so Eq. (B11) gives $(\Box - m^2)N' = c'T$. However, the initial conditions on $\{h_{\mu\nu}, N\}$ do not fix the initial conditions on $\{h'_{\mu\nu}, N'\}$. Rather, $h'_{\mu\nu}$ and N' at a given time slice can only be determined if we know $h_{\mu\nu}$ and N at all times, i.e., if we have already solved the equations of motion. Of course, nothing forbids one from considering the theory (B12) for its own sake, and solving its equations of motion by assigning initial conditions on $\{h'_{\mu\nu}, N'\}$. However, in this way we define a different theory, which has nothing to do with FP massive gravity, even as far as the number of dynamical degrees of freedom is concerned.

One should also be careful in the use of an action such as (B7) and of its nonlinear extension, when computing the S-matrix elements. Indeed, while standard theorems assure the invariance of the S-matrix under local field redefinitions, its invariance under nonlocal field redefinitions is in general not assured.

[17] Writing $T_{\mu\nu} = T^{\rm TT}_{\mu\nu} + (1/2)(\partial_\mu S^{\rm T}_\nu + \partial_\nu S^{\rm T}_\mu) + \partial_\mu\partial_\nu\Sigma + \eta_{\mu\nu}S$, energy-momentum conservation implies $(1/2)\Box S^{\rm T}_\nu + \partial_\nu(\Box\Sigma + S) = 0$. The transverse and longitudinal parts of this expression must vanish separately. For a localized source, from $\Box S^{\rm T}_\nu = 0$ it follows that $S^{\rm T}_\nu = 0$. Eliminating S from $S = -\Box\Sigma$ and expressing Σ in terms of T using $T = \Box\Sigma + (d+1)S = -d\Box\Sigma$ it follows that $T_{\mu\nu} = T^{\rm TT}_{\mu\nu} + (1/d)(\eta_{\mu\nu} - \Box^{-1}\partial_\mu\partial_\nu)T$, which gives Eq. (B8).

[18] In generic d spatial dimensions, all equations written in Sec. III simply go through with the trivial replacement $d^4x \to d^{d+1}x$. In contrast, the space-time dimension enters in the diagonalization, and in d spatial dimensions the factor $\sqrt{6}$ in the equations below must be replaced by $\sqrt{d(d-1)}$.

[19] Observe that N is a combination of h and of $\Box\alpha = -\partial_\mu A^\mu$, so it does not involve nonlocal operators.

APPENDIX C: THE ACTION OF THE MASSIVE THEORY AS A LOCAL FUNCTIONAL OF NONLOCAL FIELDS

We have seen in the previous appendix that nonlocal transformations of the field must be used with care, particularly when one wishes to study what are the dynamical and nondynamical degrees of freedom of the theory. Having understood this point, it is however still interesting to observe that there exist nonlocal transformations of the fields that bring the nonlocal actions that we have discussed into simple and elegant local forms, which can be useful for obtaining a further understanding of the structure of these theories. In particular we will see explicitly how, for $m \to 0$, the ghost of the massive theory smoothly reduces to a nonradiative degree of freedom of GR. We start again from electrodynamics, where the construction is simpler.

1. Nonlocal variables in electrodynamics

In electrodynamics the gauge field A_μ can be separated into a transverse and a longitudinal part,

$$A_\mu = A_\mu^{\mathrm{T}} + \partial_\mu \alpha, \tag{C1}$$

where

$$\partial^\mu A_\mu^{\mathrm{T}} = 0. \tag{C2}$$

To invert Eq. (C1) we take its divergence, which gives $\partial^\mu A_\mu = \Box\alpha$, so that

$$\alpha = \Box^{-1}\partial^\mu A_\mu. \tag{C3}$$

Substituting this into $A_\mu^{\mathrm{T}} = A_\mu - \partial_\mu\alpha$ we get

$$A_\mu^{\mathrm{T}} = A_\mu - \frac{1}{\Box}\partial_\mu\partial^\nu A_\nu = P_\mu^\nu A_\nu, \tag{C4}$$

where we introduced the nonlocal operator

$$P_\mu^\nu \equiv \delta_\mu^\nu - \frac{\partial_\mu\partial^\nu}{\Box}. \tag{C5}$$

Observe that $P_\mu^\rho P_\rho^\nu = P_\mu^\nu$. Furthermore, applying P_μ^ν to a pure gauge configuration we get zero,

$$P_\mu^\nu \partial_\nu \theta = 0. \tag{C6}$$

Since P_μ^ν is linear, this implies that A_μ^{T} is gauge invariant,

$$A_\mu^{\mathrm{T}} \to P_\mu^\nu(A_\nu - \partial_\nu\theta) = A_\mu^{\mathrm{T}}. \tag{C7}$$

Thus, under a gauge transformations $A_\mu \to A_\mu - \partial_\mu\theta$, we have $\alpha \to \alpha - \theta$ and $A_\mu^{\mathrm{T}} \to A_\mu^{\mathrm{T}}$. Thus P_μ^ν is a projector that associates to a gauge orbit (of which A_μ is a representative) a gauge-invariant vector field A_μ^{T} that satisfies the Lorentz condition. Observe that, because of Eq. (C2), A_μ^{T} describes 3 degrees of freedom. In the case of massive electrodynamics these are the three spin states of a massive photon. Thus, A_μ^{T} provides a *gauge-invariant* description of the 3 physical degrees of freedom of massive electrodynamics. We see here again the interplay between gauge invariance and locality in the massive gauge theory. If we insist on manifest locality we must use the gauge field A^μ, which is not gauge invariant, and we cannot construct with it a local gauge-invariant mass term. In contrast, if we give up manifest locality, we have at our disposal a field A_μ^{T} which is gauge invariant. Using this field it is straightforward to write an action with a mass term that does not spoil gauge invariance,

$$S_{\text{gauge-inv}} = \int d^4x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m_\gamma^2 A_\mu^{\mathrm{T}} A^{T\mu}\right) - j^\mu A_\mu. \tag{C8}$$

We could have also replaced $A_\mu \to A_\mu^{\mathrm{T}}$ in the kinetic term. However, we see from Eq. (C1) that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu A_\nu^T - \partial_\nu A_\mu^T. \tag{C9}$$

Similarly, upon integration by parts, for a conserved current we can write equivalently $j^\mu A_\mu$ or $j^\mu A_\mu^{\mathrm{T}}$. We now insert into (C8) the nonlocal expression of A_μ^{T} in terms of A_μ given in (C4). Performing some integration by parts and using the identity

$$\begin{aligned} F_{\mu\nu}\frac{1}{\Box}F^{\mu\nu} &= 2(\partial_\mu A_\nu)\frac{1}{\Box}(\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &= -2A_\nu A^\nu - 2(\partial_\mu A^\mu)\frac{1}{\Box}(\partial_\nu A^\nu), \end{aligned}$$

(in which again the second equality has been obtained integrating by parts) we find that

$$A_\mu^{\mathrm{T}} A^{T\mu} = -\frac{1}{2}F_{\mu\nu}\frac{1}{\Box}F^{\mu\nu}, \tag{C10}$$

and therefore

$$S_{\text{gauge-inv}} = -\frac{1}{4}\int d^4x F_{\mu\nu}\left(1 - \frac{m_\gamma^2}{\Box}\right)F^{\mu\nu}. \tag{C11}$$

Thus, the nonlocal action (2.15) is equivalent to (C8). In the former action the nonlocality is explicitly displayed. In the latter, it is hidden in the nonlocal relation between A_μ^{T} and A_μ. We stress again that this nonlocality has no physical consequences, since it only affects pure gauge modes and can be gauged away giving up explicit gauge invariance by fixing the Lorentz gauge $\partial^\mu A_\mu = 0$, since in this gauge A_μ^T reduces to the local field A_μ, as we see from Eq. (C4). This illustrates again the interplay between gauge invariance and locality in massive electrodynamics.

It is also interesting to note that the projection onto the transverse part allows one to define a scalar product in the physical configuration space, i.e., in the space of gauge orbits. Indeed, the following bilinear functional is independent of the orbit representative

$$\langle A, B\rangle \equiv \frac{1}{2}\int d^4x A_\mu^{\mathrm{T}} B^{T\mu}, \tag{C12}$$

since we have seen that $A^{\rm T}_\mu$ and $B^{T\mu}$ are gauge invariant. Using Eq. (C10) we see that the Proca action can be written as the "expectation value" of the Klein-Gordon operator with respect to that scalar product

$$S_{\text{gauge-inv}} = \langle A, (\Box - m^2)A\rangle. \quad \text{(C13)}$$

In terms of this scalar product the inclusion of a mass term is therefore trivial.

2. Nonlocal variables in linearized gravity

We now generalize to the spin-2 case the construction of nonlocal variables discussed above. We begin by introducing a projector $P_{\mu\nu}^{\rho\sigma}$ which is just the symmetrization of the square of the projector P_μ^ν defined in Eq. (C5),

$$\begin{aligned} P_{\mu\nu}^{\rho\sigma} &\equiv \frac{1}{2}(P_\mu^\rho P_\nu^\sigma + P_\nu^\rho P_\mu^\sigma) \\ &= \frac{1}{2}(\delta_\mu^\rho \delta_\nu^\sigma + \delta_\nu^\rho \delta_\mu^\sigma) \\ &\quad - \frac{1}{2\Box}(\delta_\mu^\rho \partial_\nu \partial^\sigma + \delta_\nu^\sigma \partial_\mu \partial^\rho + \delta_\nu^\rho \partial_\mu \partial^\sigma + \delta_\mu^\sigma \partial_\nu \partial^\rho) \\ &\quad + \frac{1}{\Box^2}\partial_\mu \partial_\nu \partial^\rho \partial^\sigma. \end{aligned} \quad \text{(C14)}$$

We can then define a projected field

$$\hat{h}_{\mu\nu} \equiv P_{\mu\nu}^{\rho\sigma} h_{\rho\sigma}. \quad \text{(C15)}$$

Observe that $\hat{h}_{\mu\nu}$ can be obtained starting from $h_{\mu\nu}$ and performing a gauge transformation $h_{\mu\nu} \to h_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)$ with

$$\xi_\nu = \frac{1}{\Box}\partial^\rho h_{\nu\rho} - \frac{1}{2\Box^2}\partial_\nu(\partial^\rho \partial^\sigma h_{\rho\sigma}). \quad \text{(C16)}$$

Similarly to the spin-1 case, the projector gives zero on pure-gauge configurations,

$$P_{\mu\nu}^{\rho\sigma}(\partial_\rho \xi_\sigma + \partial_\sigma \xi_\rho) = 0, \quad \text{(C17)}$$

and therefore $\hat{h}_{\mu\nu}$ is gauge invariant under linearized diffeomorphisms,

$$\hat{h}_{\mu\nu} \to P_{\mu\nu}^{\rho\sigma}[h_{\rho\sigma} - (\partial_\rho \xi_\sigma + \partial_\sigma \xi_\rho)] = \hat{h}_{\mu\nu}. \quad \text{(C18)}$$

Thus, in full analogy with the case of electrodynamics, $P_{\mu\nu}^{\rho\sigma}$ sends $h_{\mu\nu}$ into a gauge-invariant field $\hat{h}_{\mu\nu}$, that satisfies the transversality condition $\partial^\mu \hat{h}_{\mu\nu} = 0$, and which is a gauge-invariant representative of the gauge orbit to which $h_{\mu\nu}$ belongs. The field $\hat{h}_{\mu\nu}$ is transverse but not traceless. Defining $\hat{h} = \eta^{\mu\nu}\hat{h}_{\mu\nu}$, we have the identity

$$h_{\mu\nu}^{\rm TT} = \hat{h}_{\mu\nu} - \frac{1}{d}\left(\eta_{\mu\nu} - \frac{1}{\Box}\partial_\mu \partial_\nu\right)\hat{h}. \quad \text{(C19)}$$

Contracting with $\eta^{\mu\nu}$ or with ∂^μ, and using $\partial^\mu \hat{h}_{\mu\nu}$, we see that the right-hand side of Eq. (C19) is indeed transverse and traceless so it is clear that it must be equal to $h_{\mu\nu}^{\rm TT}$. This can indeed be immediately checked using the explicit expressions of $\hat{h}_{\mu\nu}$ given in Eqs. (C14) and (C15) and comparing with the explicit expression of $h_{\mu\nu}^{\rm TT}$ given in Eq. (B1). Since $\hat{h}_{\mu\nu}$ is invariant under linearized diffeomorphisms, Eq. (C19) nicely shows that $h_{\mu\nu}^{\rm TT}$ is also invariant under diffeomorphisms.

From Eq. (B3) we see that also s is invariant under linearized diffeomorphisms. In contrast, as it is clear from Eq. (B1), the vector $\epsilon_\mu = \epsilon_\mu^{\rm T} + \partial_\mu \alpha$ transforms as $\epsilon_\mu \to \epsilon_\mu - \xi_\mu$, and can be set to zero by a gauge transformation. Observe that choosing the gauge so that $\epsilon_\mu = 0$ leaves no residual gauge freedom. Thus, ϵ_μ describes the four pure-gauge modes, while the 5 degrees of freedom of $h_{\mu\nu}^{\rm TT}$, together with the scalar s, describe 6 physical degrees of freedom of the gravitational field. As we learned in Appendix B, these variables are not appropriate for identifying which degrees of freedom are radiative and which are not. However, we already saw in Sec. IV that, in the theory defined by the action $S_{\rm nonloc}$, all 6 gauge-invariant degrees of freedom are radiative.

Just as in the case of massive electrodynamics, it is straightforward to write a gauge-invariant action for linearized massive gravity using the gauge-invariant variable $\hat{h}_{\mu\nu}$. We can in fact construct the gauge-invariant action

$$S_{\text{gauge-inv}} = \frac{1}{2}\int d^{d+1}x[h_{\mu\nu}\mathcal{E}^{\mu\nu,\rho\sigma}h_{\rho\sigma} - m^2(\hat{h}_{\mu\nu}\hat{h}^{\mu\nu} - \hat{h}^2)], \quad \text{(C20)}$$

with a FP mass term constructed using $\hat{h}_{\mu\nu}$ and $\hat{h} \equiv \eta^{\mu\nu}\hat{h}_{\mu\nu}$. Observe that

$$h_{\mu\nu}\mathcal{E}^{\mu\nu,\rho\sigma}h_{\rho\sigma} = \hat{h}_{\mu\nu}\mathcal{E}^{\mu\nu,\rho\sigma}\hat{h}_{\rho\sigma}, \quad \text{(C21)}$$

since this term is gauge invariant and $h_{\mu\nu}$ and $\hat{h}_{\mu\nu}$ are related by a gauge transformation. From the explicit expression of $\hat{h}_{\mu\nu}$ in terms of $h_{\mu\nu}$ given by Eq. (B6) we find that

$$\int d^{d+1}x(\hat{h}_{\mu\nu}\hat{h}^{\mu\nu} - \hat{h}^2) = \int d^{d+1}x\left(h_{\mu\nu}\frac{1}{\Box}\mathcal{E}^{\mu\nu,\rho\sigma}h_{\rho\sigma}\right), \quad \text{(C22)}$$

and therefore $S_{\text{gauge-inv}}$ is the same as the nonlocal action $S_{\rm nonloc}$ given in Eq. (4.1). We therefore have the identities

$$\begin{aligned} S_{\rm nonloc} &\equiv \int d^{d+1}x\frac{1}{2}h_{\mu\nu}\left(1 - \frac{m^2}{\Box}\right)\mathcal{E}^{\mu\nu,\rho\sigma}h_{\rho\sigma} \\ &= \int d^{d+1}x\left[\frac{1}{2}\hat{h}_{\mu\nu}\mathcal{E}^{\mu\nu,\rho\sigma}\hat{h}_{\rho\sigma} - \frac{m^2}{2}(\hat{h}_{\mu\nu}\hat{h}^{\mu\nu} - \hat{h}^2)\right] \\ &= \int d^{d+1}x\left[\frac{1}{2}\hat{h}_{\mu\nu}(\Box - m^2)\hat{h}^{\mu\nu} - \frac{1}{2}\hat{h}(\Box - m^2)\hat{h}\right], \end{aligned} \quad \text{(C23)}$$

where in the last line we used Eq. (B2) to simplify $\hat{h}_{\mu\nu}\mathcal{E}^{\mu\nu,\rho\sigma}\hat{h}_{\rho\sigma}$. Thus, $S_{\rm nonloc}$ is a *local* functional of $\hat{h}_{\mu\nu}$, and the nonlocality of $S_{\rm nonloc}$ as a functional of the metric perturbation $h_{\mu\nu}$ is now hidden in the nonlocal relation between $\hat{h}_{\mu\nu}$ and $h_{\mu\nu}$.

In the action (C23) the terms $\hat{h}_{\mu\nu}(\Box - m^2)\hat{h}^{\mu\nu}$ and $\hat{h}(\Box - m^2)\hat{h}$ are not independent, since $\hat{h} = \eta^{\mu\nu}\hat{h}_{\mu\nu}$. We can however decouple them using Eq. (C19) to write

$$\hat{h}_{\mu\nu} = h^{\rm TT}_{\mu\nu} + \frac{1}{d}\left(\eta_{\mu\nu} - \frac{1}{\Box}\partial_\mu\partial_\nu\right)\hat{h}, \tag{C24}$$

and we can use $h^{\rm TT}_{\mu\nu}$ and $\hat{h}$ as independent fields. Then the action (C28) becomes

$$S_{\rm nonloc} = \frac{1}{2}\int d^{d+1}x\left[h^{\rm TT}_{\mu\nu}(\Box - m^2)h^{\rm TT\mu\nu} - \frac{d-1}{d}\hat{h}(\Box - m^2)\hat{h}\right]. \tag{C25}$$

We see that $h^{\rm TT}_{\mu\nu}$ has a healthy kinetic term. In contrast, for $d > 1$, in a theory governed by $S_{\rm nonloc}$ the scalar $\hat{h}$ is a ghost, since its kinetic term has the wrong sign. Equation (C25) confirms the analysis made in Sec. IV: the action $S_{\rm nonloc}$ describes 6 degrees of freedom, out of which 5 correspond to the helicities 0, ± 1, and ± 2 of a massive spin-2 particle, while the sixth degree of freedom is a Lorentz scalar, and we further see that it is a ghost. Taking the $m = 0$ limit and comparing with Eq. (B7) we see that in this limit $\hat{h}$ reduces to the nonradiative field s. However, from the discussion in Sec. IV it follows that in the massive case $\hat{h}$ is truly dynamical, while we saw that in the massless case s is a nonradiative degree of freedom, despite its KG action. Similarly, in the massive case $h^{\rm TT}_{\mu\nu}$ describes 5 dynamical degrees of freedom.

Consider now FP massive gravity. According to Eq. (3.15), we must then add to $S_{\rm nonloc}$ the term

$$-2m^2\int d^4x N\frac{1}{\Box}\partial_\mu\partial_\nu(h^{\mu\nu} - \eta^{\mu\nu}h). \tag{C26}$$

Observe, from Eq. (B6), that

$$\hat{h} = -\frac{1}{\Box}\partial_\mu\partial_\nu(h^{\mu\nu} - \eta^{\mu\nu}h). \tag{C27}$$

Therefore the FP action (3.15) can be rewritten as

$$S_{\rm FP} + S_{\rm int} = \int d^{d+1}x\left[\frac{1}{2}h^{\rm TT}_{\mu\nu}(\Box - m^2)h^{\rm TT\mu\nu} - \frac{d-1}{2d}\hat{h}(\Box - m^2)\hat{h} + 2m^2N\hat{h}\right] + \frac{\kappa}{2}\int d^{d+1}x\left[h^{\rm TT}_{\mu\nu}T^{\rm TT\mu\nu} + \frac{1}{d}\hat{h}T\right]. \tag{C28}$$

We also wrote Eq. (3.15) in a generic space-time dimension and we used the fact that, because of $\partial_\mu T^{\mu\nu} = 0$, upon integration by parts $h_{\mu\nu}T^{\mu\nu} = \hat{h}_{\mu\nu}T^{\mu\nu} = h^{\rm TT}_{\mu\nu}T^{\rm TT\mu\nu} + (1/d)\hat{h}T$.[20] We see that the Lagrange multiplier N imposes the constraint $\hat{h} = 0$ and kills the ghost. Thus, the action (C28) is equivalent to

$$S_{\rm FP} + S_{\rm int} = \int d^{d+1}x\left[\frac{1}{2}h^{\rm TT}_{\mu\nu}(\Box - m^2)h^{\rm TT\mu\nu} + \frac{\kappa}{2}h^{\rm TT}_{\mu\nu}T^{\rm TT\mu\nu}\right]. \tag{C29}$$

Equation (C29) provides a gauge-invariant description of the 5 physical degrees of freedom of the massive graviton of FP theory. The price to be paid for explicit gauge invariance is of course nonlocality, which is now hidden in the relation between the gauge-invariant field $h^{\rm TT}_{\mu\nu}$ and the metric perturbation $h_{\mu\nu}$, given by Eqs. (B6) and (C19). It is also interesting to observe that, under an infinitesimal conformal transformation of the metric $g_{\mu\nu} \to e^{-2\theta}g_{\mu\nu}$, we have $h_{\mu\nu} \to h_{\mu\nu} - 2\theta\eta_{\mu\nu}$ and

$$\hat{h}_{\mu\nu} \to \hat{h}_{\mu\nu} - 2(\eta_{\mu\nu} - \Box^{-1}\partial_\mu\partial_\nu)\theta. \tag{C30}$$

Plugging this into Eq. (C19) we find that $h^{\rm TT}_{\mu\nu}$ is invariant. Thus, the reduction from the 10 degrees of freedom of $h_{\mu\nu}$ to the 5 of $h^{\rm TT}_{\mu\nu}$ can be understood as a consequence of diff invariance (which eliminates 4 degrees of freedom) plus an "accidental" conformal invariance of the linearized theory.

[20] Assuming that $T_{\mu\nu}$ has compact support, the integrations by parts involving nonlocal terms are well defined.

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