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Constraint algebra in loop quantum gravity reloaded. I. Toy model of a U(1)^3 gauge theory

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We analyze the issue of anomaly-free representations of the constraint algebra in loop quantum gravity (LQG) in the context of a diffeomorphism-invariant U(1)^3 theory in three spacetime dimensions. We construct a Hamiltonian constraint operator whose commutator matches with a quantization of the classical Poisson bracket involving structure functions. Our quantization scheme is based on a geometric interpretation of the Hamiltonian constraint as a generator of phase space-dependent diffeomorphisms. The resulting Hamiltonian constraint at finite triangulation has a conceptual similarity with the $\bar{\mu}$ scheme in loop quantum cosmology and highly intricate action on the spin-network states of the theory. We construct a subspace of non-normalizable states (distributions) on which the continuum Hamiltonian constraint is defined which leads to an anomaly-free representation of the Poisson bracket of two Hamiltonian constraints in loop quantized framework. Our work, along with the work done in [C. Tomlin and M. Varadarajan, Phys. Rev. D **87**, 044039 (2013)], suggests a new approach to the construction of anomaly-free quantum dynamics in Euclidean LQG.

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I. INTRODUCTION

Loop quantum gravity (LQG) started out as an approach to nonperturbative quantization of the gravitational field using a classical canonical formulation of gravity as a starting point [1,2]. The (spatial) diffeomorphism invariance of the theory guaranteed a pretty kinematical framework with tight analytic control rarely seen in four-dimensional quantum field theories [3]. The initial attempts at a formulation of the dynamics (via the implementation of the Hamiltonian constraint) were very promising. In the mid-nineties Thiemann proposed a quantization of the Euclidean as well as the Lorentzian Hamiltonian constraint in a series of remarkable papers titled ''Quantum Spin Dynamics''[4–6]. Thiemann's Hamiltonian constraint had some rather intriguing properties, like UV finiteness; however, despite this initial promise, the Hamiltonian constraint program in canonical LQG has reached a strange impasse. Several issues still remain open and in our opinion it is hard to argue against the assertion that there is no satisfactory definition of Hamiltonian constraint. (For some very interesting recent progress in this direction we refer the reader to [7,8].) This impasse has in turn led to new avenues to analyze the dynamics of LQG; e.g., the master constraint program [9,10], covariant spin foam models (for reviews, see [11,12]), and deparametrized dust models [13,14].

There are two primary reasons for the above assertion. On one hand, there is no unique definition of the Hamiltonian constraint; quantization of the Hamiltonian constraint in LQG involves (just as for any composite operator in any quantum field theory) an intermediate choice of regularization. This regularization amounts to choosing a family of loops, edges, surfaces, and certain discrete representation-theoretic labels from which a regularized Hamiltonian constraint operator is built. There are an infinite number of choices for each of the regulating structures involved, and in principle each such choice can give rise to a distinct operator that is well defined on the kinematical Hilbert space $\mathcal{H}_{\rm kin}$ of LQG. This would not be a problem if the continuum limit of the regularized constraint were independent of the regulating structures involved. However, this is not the case, and even the continuum Hamiltonian has an infinite-dimensional parameter worth of ambiguity.

As the Hamiltonian constraint in canonical gravity is a generator of the so-called Dirac algebra, *a priori* one might expect that, as we require the quantum Hamiltonian constraint to be anomaly free (in the sense that there exists a representation of the Dirac algebra in quantum theory), there will be certain nontrivial restrictions on the quantization choices mentioned above. However as it turns out, this is not quite true: Anomaly freedom is only achieved in LQG (in the language of gauge theories) partially on shell. That is, one requires both the left- and the right-hand sides of the commutator relations to trivialize on states that are solutions to (at least some of) the constraints. This condition places relatively few restrictions on the available (and, even in the continuum limit, distinguishable) quantization choices. Even more worrisome, there are some very concrete signs that this on-shell trivialization might extend to states which are not on shell, thus indicating some serious problems with the definition of the quantum constraint.

In this paper, we analyze the issue in a simple three-dimensional diffeomorphism-invariant gauge theory with Abelian gauge group U(1)^3; to the best of our knowledge, this model was first conceived by Smolin in four spacetime dimensions [15]. Our aim is to quantize the Hamiltonian constraint of the theory (in the loop formulation) such that it has a chance to generate an anomaly-free Dirac algebra. In the next section, we outline what we regard as problems with Thiemann's Hamiltonian constraint in more detail and outline the work done in this paper.

II. MOTIVATION AND OUTLINE

A. The issue

In this section we explain the problems with Thiemann's Hamiltonian constraint that we referred to in the Introduction. The underlying issues are rather involved and we may not be able to do justice to different points of view which exist in the literature. We refer the reader to [16–18] for more details.

Traditionally, the continuum Hamiltonian constraint operator is a densely defined operator on $\mathcal{H}_{\mathrm{kin}}$. This is accomplished by placing a rather unusual topology [referred to as the Uniform-Rovelli-Smolin (URS) topology] on the space of operators in which convergence of the one-parameter family of finite-triangulation Hamiltonian constraint operators turns out to be an operator on $\mathcal{H}_{\mathrm{kin}}$. Roughly speaking, this topology is such that limit points of any two operator sequences (indexed by finite triangulation T) $\hat{O}_{1T}$ and $\hat{O}_{2T}$ which are such that

$$\hat{O}_{1T} - \hat{O}_{2T} = (\hat{U}(\phi) - 1)\hat{O}_{3T} \tag{2.1}$$

for some diffeomorphism $\phi \in \mathrm{Diff}(\Sigma)$ (where Σ is the spatial manifold, and $\hat{U}$ denotes the usual unitary representation of $\mathrm{Diff}(\Sigma)$ on $\mathcal{H}_{\mathrm{kin}}$), and for some $\hat{O}_{3T}$, are identified; that is, in the URS topology, $\lim_{T\to\infty}(\hat{O}_{1T} - \hat{O}_{2T}) = 0$.

In this topology the commutator of two (continuum) Hamiltonian constraints $[\hat{H}[N], \hat{H}[M]]$ on $\mathcal{H}_{\mathrm{kin}}$ vanishes (here N and M are scalar lapse smearing functions). As shown in [6], there exists a quantization (at finite triangulation) of the right-hand side of the Poisson bracket relation (see Sec. III)

$$\{H[N], H[M]\} = V[\vec{\omega}], \tag{2.2}$$

[where $V[\vec{\omega}]$ denotes the vector constraint smeared with the phase space-dependent vector field $\omega^a = q^{ab}(M\partial_b N - N\partial_b M)$] which is of the form $(\hat{U}(\phi) - 1)\hat{O}_{3T}$, whence its continuum limit in the URS topology is zero. Thus the quantization of both the left- and right-hand sides of (2.2) vanish. Although this was originally taken to be a sign of internal consistency and anomaly freedom, a closer look at the structural aspects of the computation was performed in a series of remarkable papers by Gambini *et al.* [19,20]. They looked at the convergence of the finite-triangulation commutator sequence not on $\mathcal{H}_{\mathrm{kin}}$, but on a certain distributional space (which is an extension of the space of diffeomorphism-invariant distributions) referred to as a habitat. Since the habitat consists of (distributional) states which are not diffeomorphism invariant, *a priori* neither the quantization of the left-hand side (LHS) or the right-hand side (RHS) of (2.2) is expected to be a trivial operator. However, it turned out that $\widehat{\{H[N], H[M]\}}$ is the zero operator on the habitat, and there exists a quantization of $V[\vec{\omega}]$ at finite triangulation on $\mathcal{H}_{\mathrm{kin}}$ whose continuum limit on the habitat is trivial. However, this vanishing of the quantization of both the RHS and LHS of (2.2) on diffeomorphism noninvariant states is rather unsatisfactory. Perhaps more worrisome, the reasons for the vanishing of the LHS and RHS are entirely different. The most transparent way to see this is given in [20], where the authors argue that, if instead of working with density weight one constraints, one works with higher density constraints, but keeps the quantization choices essentially the same, then on their habitat, the LHS will continue to vanish, but the RHS will not, whence suggesting the presence of an anomaly in the whole scheme (we come back to this point in more detail below).

B. Our goal

Success of the canonical LQG program is defined by the following. Starting with the (unique) diffeomorphism-covariant representation of the holonomy-flux algebra, does there exist a vector space $\mathcal{V}$ (whose elements are linear combinations of spin network states) that is a representation space for the Dirac algebra, in the sense that

$$\begin{aligned} \hat{U}(\phi_1)\hat{U}(\phi_2)\Psi &= \hat{U}(\phi_1 \circ \phi_2)\Psi, \\ [\hat{H}[N], \hat{H}[M]]\Psi &= \hat{V}[\vec{\omega}]\Psi, \\ \hat{U}(\phi^{-1})\hat{H}[N]\hat{U}(\phi)\Psi &= \hat{H}[\phi^* N]\Psi, \end{aligned} \tag{2.3}$$

$\forall\, \Psi \in \mathcal{V}$, where

(i) ϕ is a spatial diffeomorphism (usually taken to be in the semianalytic category) and $\hat{U}(\phi)$ is a representation of the diffeomorphism group on $\mathcal{V}$ induced via its unitary representation on $\mathcal{H}_{\mathrm{kin}}$ and[1]

(ii) $\hat{H}[N]$ is a continuum quantum Hamiltonian constraint operator obtained as a limit point of a net of finite-triangulation operators defined on $\mathcal{H}_{\mathrm{kin}}$?

We refer to (2.3), and in particular to the second equation in (2.3), as the off-shell closure condition for $\hat{H}[N]$ [22].

Once a quantization of the Hamiltonian constraint is found which meets the above criteria, its kernel (in $\mathcal{V}$) is expected to give rise to the physical Hilbert space of the theory. This is a rather ambitious aim, and it is instructive to accomplish it in models which are, on one hand, diffeomorphism-invariant field theories with a gauge algebra being the Dirac algebra, and on the other hand, are simpler and more tractable than gravity. Models where this

[1] In light of work done in [21], one could ask for a genuine representation of Dirac algebra involving the diffeomorphism constraint operator $\hat{V}[\vec{N}]$ instead of $\hat{U}(\phi)$, but we do not attempt this here.

aim has been accomplished include a two-dimensional parametrized field theory (PFT) [23,24], and the Husain-Kuchař (HK) model[2] [21]. However, the main reasons that the constraint algebra in these models could be represented without anomalies are the following:

(1) Two dimensions are special, in the sense that the Dirac algebra in two-dimensional PFT is a true Lie algebra which is isomorphic to the Witt algebra $[\mathrm{diff}(S^1) \oplus \mathrm{diff}(S^1)]$. In the HK model, the only constraint (apart from the Gauss constraint) is the spatial diffeomorphism constraint, whose algebra is isomorphic to the Lie algebra of (spatial) vector fields.

(2) The Poisson action of constraint functionals on classical fields in these models has a clear geometric interpretation which provided key insights into the possible quantization choices.

Neither of the above are true in the case of canonical gravity, and so as appealing as the results obtained in the PFT and the HK model are, we need to see if the lessons learned there can be applied to models which are more closely related to canonical gravity. In this paper we propose just such a model. It is a diffeomorphism-invariant (in fact topological) field theory in three dimensions which can be thought of as a weak coupling limit of three-dimensional Euclidean gravity.[3] Though the usual weak-coupling limit in Euclidean gravity amounts to switching off self-interactions, in this case it amounts to switching the gauge group from SU(2) to $\mathrm{U}(1)^3$. The canonical formulation of the theory corresponds to the phase space of a $\mathrm{U}(1)^3$ Yang-Mills theory in $2+1$ dimensions with Hamiltonian, diffeomorphism, and Gauss constraints. On the (Gauss) gauge-invariant sector of the phase space, the remaining constraints (Hamiltonian and diffeomorphism) generate the Dirac algebra.

Our aim in this paper is to loop quantize this system such that the (continuum) Hamiltonian constraint satisfies the second equation in (2.3).[4,5] However, blindly looking for a possible quantization of the Hamiltonian constraint which will lead to off-shell closure is a hopeless task, so we draw upon the lessons learned in [21,23] to achieve our goal. In order to familiarize the reader with these lessons, we briefly recall them below.

1. Determination of the correct density weight

As explained rather beautifully in [19], if one chooses to work with the density one Hamiltonian constraint, then no matter what domain one chooses to take the limit of finite-triangulation Hamiltonian constraint on, the resulting operator will always have a vanishing commutator with itself (as long as limits of the finite-triangulation commutator are well defined). In particular, the commutator of two density one constraints can never give rise to an operator which could resemble a quantization of the RHS. The reason for this is rather simple. Consider the Hamiltonian constraint in D spatial ($D \geq 2$) dimensions:

$$H[N] = \int_\Sigma \mathrm{d}^D x N^{(1-k)}(x) H^{(k)}(x), \tag{2.4}$$

where the superscripts indicate the density weights of the various fields with $k \in \mathbb{R}$: The smearing function N is a scalar density of weight $(1-k)$ while the local Hamiltonian density $H^{(k)}(x) \sim q^{(k-2)/2} FEE$ has density weight k.[6] In LQG, the quantization of $H[N]$ proceeds by first approximating the integral by a Riemann sum over simplices (which constitute a triangulation T of Σ) and then approximating each term in the sum by a function of appropriate holonomies and fluxes. Typically in LQG the "fineness" of the triangulation is measured by a parameter δ (usually associated with the coordinate volume of a simplex $\triangle$ in T), and a simple dimensional analysis shows that, when one uses regulating structures of size δ, the Hamiltonian approximant at triangulation fineness $T(\delta)$ reads[7]

$$\begin{aligned} H_{T(\delta)}[N] &= \sum_{\triangle \in T(\delta)} \delta^D \cdot \delta^{-2D(k-2)/2} \cdot \delta^{-2} \cdot \delta^{-(D-1)} \\ &\quad \cdot \delta^{-(D-1)} \cdot N^{(1-k)}(v(\triangle)) O_\triangle(v(\triangle)) \\ &= \sum_{\triangle \in T(\delta)} \delta^{D(1-k)} N^{(1-k)}(v(\triangle)) O_\triangle(v(\triangle)), \end{aligned} \tag{2.5}$$

where $v(\triangle)$ is a point in the simplex $\triangle$, and $O(v(\triangle))$ is a function of holonomies and fluxes, constructed out of loops and (hyper)surfaces associated to $\triangle$. Quantization choices involved in the definition of $O_\triangle(v(\triangle))$ are such that it has no explicit dependence on δ. Thus for any d, if the lapse N is a scalar with density weight zero ($k=1$), then $H_{T(\delta)}[N]$ has no explicit dependence on δ. Along with a special property of Thiemann's quantization [that the Hamiltonian constraint operator does not act on the vertices (of a spin network state) that it creates], this ensures

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[2] The HK model is essentially canonical gravity without the Hamiltonian constraint.

[3] We are indebted to Miguel Campiglia for pointing this out to us.

[4] The remaining relations which are related to the diffeomorphism covariance properties of the Hamiltonian constraint will be analyzed in [25].

[5] In the terminology of [22] we are aiming towards a quantization of the Hamiltonian constraint which satisfies the off-shell closure condition.

[6] F denotes the curvature of a connection one-form A, and E a vector density of weight 1 conjugate to A. The metric determinant $q = q[E]$ is a function of E; its specific form depends on the number of spatial dimensions and the internal symmetry group. See Sec. III A below for precise definitions.

[7] It is straightforward to verify that $q^{D-1} \equiv (\det q)^{D-1} \sim P_{2D}(E)$, where P_{2D} is some homogeneous polynomial of degree $2D$ (E^a_i need not be expressible as a square matrix; this is the case in $2+1$ dimensions), and E^a_i gets smeared over $(D-1)$-dimensional hypersurfaces associated with $T(\delta)$, hence $q^\alpha \sim \delta^{-2D\alpha}$. The (leading order) regularized F is proportional to δ^{-2}, and the coordinate measure to δ^D.

that the commutator of two density one operators vanish. Note that as there are no explicit factors of δ left in the definition of $\hat{H}_{T(\delta)}[N]$, and so also in the commutator, a quantization of the RHS can never arise, as it involves derivatives of the lapse functions, which themselves require at least one factor of δ^{-1}. This observation led the authors of [20] to conclude that one must quantize higher density constraints in order to have explicit factors of δ^{-1} which in the continuum limit could give rise to terms like $(N\partial_a M - M\partial_a N)$. The lesson we draw from these arguments is that even though density one constraints can be quantized on $\mathcal{H}_{\rm kin}$, if we are interested in seeking an anomaly-free representation of the constraint algebra, then one needs to work with higher density constraints (for an interesting counterpoint to this argument, see [24]). It was also argued in [20] that, if one chose to work with higher density constraints such that one has enough factors of δ^{-1} to obtain a nontrivial quantization of $V[\vec{\omega}]$ on the habitat, then the LHS will continue to vanish unless the Hamiltonian constraint acts nontrivially on the vertices that it creates.

In [23] these observations were taken seriously and applied to two-dimensional PFT. It was shown that one could obtain an anomaly-free representation of the Dirac algebra if one quantized density two constraints. The key point was to work with an appropriate density weight such that at finite triangulation, the Hamiltonian constraint, when written in terms of holonomies and fluxes, should have precisely one explicit factor of δ^{-1}. In this case, the commutator will have a factor of $(\delta\delta')^{-1}$, and this has the correct dimensionality to yield a quantization of $V[\vec{\omega}]$ in the continuum limit.[8]

In the model considered in this paper, where the classical Hamiltonian constraint is

$$H[N] = \int_\Sigma d^2x N q^{-\alpha/2} \epsilon^{ijk} F^i_{ab} E^a_j E^b_k(x), \tag{2.6}$$

in order to quantize the constraint such that, at finite triangulation, it has an explicit factor of δ^{-1}, we need to choose $\alpha = \frac{1}{2}$, so that N needs to be a scalar density of weight $-\frac{1}{2}$.

2. What should the constraint operators do?

As we mentioned above, one of the aspects which distinguishes two-dimensional PFT and the HK model from canonical gravity (or the model studied in this paper) is that the Poisson action of the constraints on phase space has a transparent geometric interpretation. In the first case, the Hamiltonian constraint, being a generator of the Witt algebra, is intricately linked to spatial diffeomorphisms on S^1. In the HK model, as the only constraint is the diffeomorphism constraint, its Poisson action on (A, E) is nothing but the Lie derivative by the shift field. These interpretations were key inputs in pinning down the quantization choices for these constraints. The connection between the geometric interpretation and quantization can be encoded in the following schematic equation. Given a spin network state Ψ, let the corresponding classical cylindrical function be denoted $\Psi(A)$. Then

$$\hat{O}_{T(\delta)}[V]\Psi \equiv \frac{1}{\delta}[(\text{Finite action generated by } O[V], \text{ parametrized by } \delta) - 1]\Psi(A). \tag{2.7}$$

For example, in the case of the HK model, the quantization choices made in [21] to construct the diffeomorphism constraint operator were such that this operator at finite triangulation equalled $D[\vec{N}] = \frac{1}{\delta} \times (\hat{U}(\phi^{\vec{N}}_\delta) - 1)$.

If we were to follow this route to find out what quantization choices are to be made to construct $\hat{H}[N]$, we are forced to look for a geometric interpretation of Poisson action of the Hamiltonian constraint. As we show in Sec. VI, there indeed does exist such an interpretation[9]:

$$X_{H[N]}A^i_a \approx \epsilon^{ijk}\mathcal{L}_{q^{-1/4}N\vec{E}_j}A^k_a \cong \epsilon^{ijk}\frac{(\phi_\delta^{q^{-1/4}N\vec{E}_j})^* - 1}{\delta}A^k_a, \tag{2.8}$$

where $\approx$ refers to equality modulo Gauß law. Thus the change in, say, A^1 under the action of the Hamiltonian vector field of $H[N]$ equals the Lie derivative of A^3 with respect to the vector field $q^{-1/4}N\vec{E}_2$ minus the Lie derivative of A^2 with respect to the vector field $q^{-1/4}N\vec{E}_3$. The second approximation is a discrete approximant to the Hamiltonian vector field. We will seek a quantization of $H[N]$ at finite triangulation which mimics this action on spin-network states.

3. Where should the continuum limit be taken?

As the finite-triangulation Hamiltonian constraint has an explicit factor of δ^{-1}, it cannot admit a continuum limit

[8]One factor of δ^{-1} can, in the continuum limit, give one derivative, and hence can yield terms like $N\partial_a M - M\partial_a N$, and the other factor is precisely the factor one needs to obtain a quantum diffeomorphism constraint, which we expect to be linked with Lie derivatives.

[9]Rather remarkably this interpretation also holds for the SU(2) case, and is likely to be important in extending this program to Euclidean quantum gravity.

on $\mathcal{H}_{\rm kin}$ (in any operator topology), and so the obvious questions are, is there any admissible topology on the space of operators, and are there any subspaces of the space of distributions on which the continuum limit can be taken? Once again the case of two-dimensional PFT and the HK model provide important clues. One needs to build spaces (or habitats as termed in [19]) by studying the specific deformations of a spin network that the finite triangulation constraints generate. The topology on the space of operators can then come by looking at seminorms defined by (generalized) matrix elements of operators between these habitat states and states in $\mathcal{H}_{\rm kin}$. *A priori*, there can be infinitely many habitats which can function as a home for the quantum constraints. In this paper, we consider the simplest possible habitat on which the continuum limit of the Hamiltonian constraint exists and on which the off-shell closure relation is to be checked. We do not know if this habitat is physically interesting in the sense that it is a representation space for Dirac observables. However, as our modest aim in this paper is to see if there is an anomaly-free representation of the Hamiltonian constraint on *some* space, we leave a detailed analysis of the construction of "physically interesting" habitats for future research.

In the remainder of this section, we outline how we implement the above lessons in our model and arrive at a quantum Hamiltonian constraint which satisfies the off-shell closure condition.

C. Outline

1. The idea of the quantum shift and the role of inverse volume

As we want to quantize the Hamiltonian constraint at finite triangulation such that it mimics the action in (2.8) on charge network states,[10] we need to define the quantum counterparts of the classical vector fields $q^{-1/4}NE^a_i$. This is where the loop representation throws its first surprise. Although classically the triad fields are smooth, quantum mechanically they turn out to be operator-valued distributions. More precisely, given any charge network state $|c\rangle$, the graph of this charge network is the "locus of discontinuity" of the $\hat{q}^{-1/4}N\hat{E}^a_i$ operator. Nonetheless, as we show below, one can quantize $q^{-1/4}NE^a_i$ in such a way that each charge network is an eigenstate, with its spectrum belonging to $T_v\Sigma$ (where $v \in \Sigma$ is some vertex of the graph underlying c). Given any charge network state $|c\rangle$ based on the graph γ, the expectation value of $\hat{q}^{-1/4}N\hat{E}^a_i(v)$ is nonvanishing only if $v \in V(\gamma)$; the resulting "vectors" at the vertices of γ will be referred to as the *quantum shift* associated to that state.

[10]We refer to the spin network states of the $U(1)^3$ theory as charge networks [26], as the edge labels in this case are $U(1)$ charges.

2. Image of the regularized Hamiltonian constraint and the birth of extraordinary vertices

Given a charge network state $|c\rangle$, the action of the regularized Hamiltonian constraint operator produces two generic effects:

(i) The change in the edge labels is state dependent.

(ii) A new degenerate vertex (by degenerate we mean that the inverse volume operator acting at that vertex vanishes) is created whose location depends on the quantum shift $\langle \hat{E}^a_i(v)\rangle$.

We call such vertices extraordinary vertices. The restriction of a charge network in the neighborhood of an extraordinary vertex has certain invariant properties that we enumerate below, and these help us isolate all the charge network states which lie in the image of the regularized Hamiltonian constraint operator.

3. Geometric interpretation of the Poisson action of the product of Hamiltonian constraints

As the extraordinary vertices which are created by the Hamiltonian constraint are degenerate, naively one would expect the (regularized) Hamiltonian constraint to act trivially at such vertices. However, this result relies upon specific quantization choices and would lead to an anomaly in the constraint algebra. We cure this problem by once again taking a cue from a classical computation.

The classical discrete approximant to the action of the Hamiltonian vector field of $H[N]$ on a cylindrical function $f_c(A)$ creates a linear combination of cylindrical functions with exact analogs of the extraordinary vertices mentioned above. These vertices are located along the integral curves of a triad-dependent vector field, so the action of a second Hamiltonian constraint on a cylindrical function containing such vertices moves them (as such an action will have a nontrivial effect on integral curves of phase space-dependent vector fields). This observation helps us in *modifying* the action of the Hamiltonian constraint on extraordinary vertices, by making use of the quantization ambiguities that are available to us due to the structure of the loop representation.

4. Proposal for the habitat

Finally by studying the precise nature of extraordinary vertices (that is, the deformations in a charge network that the Hamiltonian constraint operator creates), we propose a definition of a habitat. It is a subspace of distributions, where each distribution is a linear combination of an infinite number of a specific class of charge networks with coefficients being dependent on the vertex set of the charge network. The underlying idea of this habitat is precisely the same as that proposed in [19] (though the habitat itself is completely different) whence we call it a Lewandowski-Marolf-Inspired (LMI) habitat.

As we show in Appendix D, on this habitat, the regularized Hamiltonian constraint admits a continuum limit.

5. Quantization of the RHS and the off-shell closure condition

We finally demonstrate that there exists a quantization of the RHS, $\hat{V}[\vec{\omega}]$ on the LMI habitat such that the continuum limit of regularized commutators between the two Hamiltonian constraints equals $\hat{V}[\vec{\omega}]$. This is our main result. We finally end with conclusions and highlight the open issues and some of the unsatisfactory aspects of our construction. Some of these issues will be analyzed in the sequel [25].

III. CLASSICAL THEORY

In this section we describe the constrained Hamiltonian system that we aim to quantize in this paper. The algebra of constraints that we eventually arrive at can be obtained simply by replacing the internal gauge group SU(2) of general relativity in connection variables with a direct product of three commuting copies of U(1), but there is another way, due to Smolin [15], in which one takes Newton's gravitational constant $G_{\mathrm{N}} \to 0$ at the level of the action, and analyzes the resulting canonical theory. We follow this second route.

Our starting point is the Palatini action for general relativity in three dimensions:

$$S_{\mathrm{P}}[e, \omega] = \frac{1}{16\pi G_{\mathrm{N}}} \int_M \mathrm{d}^3x \eta^{\mu\nu\rho} \Omega^i_{\mu\nu} e_{\rho i}, \tag{3.1}$$

where the basic variables are a (dimensionless) cotriad e^i_μ, and a connection ω^i_μ (with dimensions of inverse length) with curvature $\Omega^i_{\mu\nu}$. G_{N} has units of inverse momentum (in $c = 1$ units), and the Planck length is defined as $l_{\mathrm{P}} = \hbar G_{\mathrm{N}}$ (where $\hbar$ as usual has units of angular momentum). Here $\mu, \nu, \ldots = 0, 1, 2$ are spacetime indexes (while below we will use $a, b, \ldots = 1, 2$ as spatial indexes), and $i, j, \ldots$ label the generators of a group G in whose Lie algebra both e_μ and ω_μ take values. $\eta^{\mu\nu\rho}$ is the (e- and A-independent) Levi-Civita tensor density of weight $+1$ on the manifold M. We take the manifold M to be topologically $\Sigma \times \mathbb{R}$ with Σ a closed two-dimensional Riemann surface. For $G = \mathrm{SU}(2)$, the action is equivalent to that of Euclidean-signature general relativity, while $G = \mathrm{SU}(1, 1)$ corresponds to Lorentzian general relativity. In the SU(2) case,

$$\Omega^i_{\mu\nu} = 2\partial_{[\mu}\omega^i_{\nu]} + \epsilon^{ijk}\omega^j_\mu \omega^k_\nu, \tag{3.2}$$

and setting $A^i_\mu := (8\pi G_{\mathrm{N}})^{-1}\omega^i_\mu$ [which has units of momentum (or mass) per length], one can rewrite the action as

$$S_{\mathrm{P}}[e, \omega] = \frac{1}{2}\int_M \mathrm{d}^3x \eta^{\mu\nu\rho}(2\partial_{[\mu}A^i_{\nu]} + 8\pi G_{\mathrm{N}}\epsilon^{ijk}A^j_\mu A^k_\nu)e_{\rho i}. \tag{3.3}$$

In the limit $G_{\mathrm{N}} \to 0$, we obtain the following action:

$$S[e, A] = \frac{1}{2}\int_M \mathrm{d}^3x \eta^{\mu\nu\rho} F^i_{\mu\nu} e_{\rho i}, \quad F^i_{\mu\nu} := 2\partial_{[\mu}A^i_{\nu]}, \tag{3.4}$$

in which the SU(2) gauge symmetry

$$\omega_\mu \to g\omega_\mu g^{-1} - \frac{1}{8\pi G_{\mathrm{N}}}(\partial_\mu g)g^{-1} \tag{3.5}$$

has become a $\mathrm{U}(1)^3$ gauge symmetry:

$$A^i_\mu \to A^i_\mu - \partial_\mu \theta^i. \tag{3.6}$$

A. Constraints

In this section we essentially follow [27]. Canonical analysis of the theory defined by S reveals $E^a_i := \eta^{ab} e^i_b$ as the momentum conjugate to A^i_a, where η^{ab} is the Levi-Civita density on Σ, a symplectic structure given by

$$\{A^i_a(x), E^b_j(y)\} = \delta^b_a \delta^i_j \delta^{(2)}(x, y), \tag{3.7}$$

and first class constraints

$$G[\Lambda] := \int \mathrm{d}^2x \Lambda^i \partial_a E^a_i, \qquad F[N] := \frac{1}{2}\int \mathrm{d}^2x N_i \eta^{ab} F^i_{ab}, \tag{3.8}$$

where Λ^i, N_i are Lagrange multipliers. $G[\Lambda]$ constitutes three U(1) Gauss constraints, and $F[N]$ is referred to as the curvature constraint.

Considering $F[N]$, when the 2-metric $q_{ab} := e^i_a e^j_b \delta_{ij}$ has nonzero determinant $\det q \equiv q$, one may perform an invertible phase space-dependent transformation on the Lagrange multipliers N_i and arrive at an alternative set of constraints that more closely resemble those that arise in $3+1$ dimensions (for Euclidean signature and Barbero-Immirzi parameter equal to 1). Namely, one may define a vector field N^a and a scalar density N of weight $-\frac{1}{2}$ such that

$$N^i = N^a \eta_{ab} E^b_i + N q^{-1/4} E^i, \tag{3.9}$$

where $E^i := \frac{1}{2}\epsilon^{ijk}\eta_{ab}E^a_j E^b_k$ is the called degeneracy vector in [27], which satisfies $E^i E^i = q$ and $E^i E^a_i = 0$. With this decomposition of N^i, the single curvature constraint can be written as the sum of two constraints

$$F[N] = V[\vec{N}] + H[N], \tag{3.10}$$

where

$$\begin{aligned} V[\vec{N}] &:= \int \mathrm{d}^2x N^a F^i_{ab} E^b_i, \\ H[N] &:= \frac{1}{2}\int \mathrm{d}^2x N q^{-1/4} \epsilon^{ijk} F^i_{ab} E^a_j E^b_k. \end{aligned} \tag{3.11}$$

By subtracting a multiple of the Gauß constraint from $V[\vec{N}]$ one obtains the generator of diffeomorphisms:

$$D[\vec{N}] := V[\vec{N}] - G[A\cdot\vec{N}] = \int d^2x E^a_i \mathfrak{L}_{\vec{N}} A^i_a$$
$$= -\int d^2x A^i_a \mathfrak{L}_{\vec{N}} E^a_i. \tag{3.12}$$

The Poisson algebra of constraints $G[\Lambda]$, $D[\vec{N}]$, $H[N]$ is first class:

$$\{G[\Lambda], G[\Lambda']\} = \{G[\Lambda], H[N]\} = 0, \tag{3.13}$$

$$\{D[\vec{N}], G[\Lambda]\} = G[\mathfrak{L}_{\vec{N}}\Lambda], \tag{3.14}$$

$$\{D[\vec{N}], D[\vec{N}']\} = D[\mathfrak{L}_{\vec{N}}\vec{N}'], \tag{3.15}$$

$$\{D[\vec{N}], H[N]\} = H[\mathfrak{L}_{\vec{N}}N], \tag{3.16}$$

$$\{H[N], H[M]\} = D[\vec{\omega}] + G[A\cdot\vec{\omega}] = V[\vec{\omega}],\quad \omega^a := q^{-1/2}E^a_i E^b_i(M\partial_b N - N\partial_b M). \tag{3.17}$$

IV. QUANTUM KINEMATICS

Here we briefly review the Hilbert space on which the basic kinematical operators, the holonomies and fluxes, are defined. It is in complete analogy with the SU(2) case so we direct the reader to [16] for further details. The kinematical Hilbert space can be defined by specifying a complete orthonormal basis, as follows. We refer to the basis states as charge networks, which, like the SU(2) spin networks, are specified by a graph with representation labels. These states are written as

$$|c\rangle = \bigotimes_{i=1}^{3}\bigotimes_{I=1}^{N} h_{e_I}^{n^i_{e_I}}. \tag{4.1}$$

c denotes the compound label $c = \{\gamma, \{\vec{n}_I\}_{I=1}^N\}$ where γ is a finite, piecewise-analytic graph embedded in Σ consisting of N oriented analytic edges e_I meeting at vertices v (technically, since the usual Gelfand-Neimark-Segal construction would provide states based on three distinct graphs γ_i, $i = 1, 2, 3$, where the charge labels on each γ_i are all nonzero, we consider the graph γ to be the finest possible graph associated with the union $\cup_i\gamma_i$). Each edge e_I is colored by a triplet $n^i_{e_I}$ [$i = 1, 2, 3$ labeling the different U(1) copies] of integers, which we denote in vector notation by $\vec{n}_I = (n^1_{e_I}, n^2_{e_I}, n^3_{e_I})$. By splitting edges in their interior at "trivial vertices" (points at which γ remains analytic), we arrange that at each nontrivial vertex all edges are outgoing by reversing the orientation of appropriate segments; if a segment's orientation is reversed by this procedure, the corresponding charges undergo a change of sign. On the right side of (4.1),

$$h^i_{e_I}[A] := e^{i\kappa\int_{e_I} A^i_a dx^a} \tag{4.2}$$

is the holonomy of the $U(1)_i$ connection A^i along the oriented edge e_I in the fundamental representation ($n^i_{e_I} = 1$), and $h_{e_I}^{n^i_{e_I}}[A]$ is the holonomy in the $n^i_{e_I}$ representation (the factor of κ has the same units as G_N and is needed to make the exponent dimensionless). $|c\rangle$ will be gauge invariant with respect to $U(1)^3$ gauge transformations only if it is gauge invariant with respect to each $U(1)_i$ separately, and $|c\rangle$ is $U(1)_i$ gauge invariant if, at each nontrivial vertex v, the sum of the charges $\sum_I n^i_{e_I}$ on (outgoing) edges e_I at v vanishes. The set of all gauge-invariant charge networks provides a complete orthonormal basis [with respect to the Ashtekar-Lewandowski measure built from the normalized U(1) Haar measure] for the kinematical $U(1)^3$ gauge-invariant Hilbert space $\mathcal{H}_{\rm kin}$.

In the connection representation, holonomies act on charge network functions $c(A)$ by multiplication, and the densitized triads as

$$\hat{E}^a_i(x)c(A) = i\hbar\{E^a_i(x), c(A)\} = -i\hbar\frac{\delta c(A)}{\delta A^i_a(x)}$$
$$= \kappa\hbar\sum_I\left(\int_0^1 dt_I\,\delta^{(2)}(x, e_I(t_I))\dot{e}^a_I(t_I)\right)n^i_{e_I}c(A), \tag{4.3}$$

where each edge e_I is parametrized by $t_I \in [0, 1]$, and $\dot{e}^a_I$ is its tangent. Given a one-dimensional oriented surface L, parametrized by $s \in [0, 1]$ with tangent $\dot{L}^a$ one can define a flux operator

$$\hat{E}_i(L) := \int_0^1 ds\,\eta_{ab}\hat{E}^a_i(L(s))\dot{L}^b(s). \tag{4.4}$$

Its action on a holonomy functional based on an edge which emanates from L is given by

$$\hat{E}_i(L)h_e^{n^i}[A] = \kappa\hbar\int_0^1 ds\,\eta_{ab}\dot{L}^b(s)\times\int_0^1 dt\,\delta^{(2)}(L(s), e(t))\dot{e}^a(t)n^i h_e^{n^i}[A]$$
$$= \frac{1}{2}\kappa\hbar\epsilon(L, e)n^i h_e^{n^i}[A], \tag{4.5}$$

where $\epsilon(L, e) = \pm1, 0$ is the relative orientation L and e. The factor of $\frac{1}{2}$ appears because we have assumed that e has an end point on L and evaluated one of the δ functions at the boundary of integration. If e has an end point on the boundary of L, then an additional factor of $\frac{1}{2}$ appears. $\hat{E}_i(L)$ extends by Leibniz's rule to charge networks, and we observe that it is diagonal in the charge network basis.

V. THE ACTION OF THIEMANN'S HAMILTONIAN CONSTRAINT

In this section we describe Thiemann's seminal construction of the Euclidean Hamiltonian constraint in

LQG. As we are working in $2+1$ dimensions, we will summarize the construction as given in [27], and will restrict ourselves to the $U(1)^3$ case.[11] We will only focus on the salient features of his construction which are most relevant to us. This will help us bring out the contrast between quantization choices we make and the choices made in [27].

Given a graph γ with a vertex set $V(\gamma)$, Thiemann's construction involves a choice of the following ingredients:

(1) A one parameter family of triangulations $T(\gamma)$ adapted to the graph γ.
(2) An approximation of the classical smeared Hamiltonian constraint by a suitable Riemann sum over the simplices of $T(\gamma)$ such that, in the limit of shrinking triangulation, one recovers the continuum expression.
(3) Associated to each simplex in the triangulation which contributes to the Riemann sum, the choice of a loop (to approximate the curvature) and a choice of a collection of edge segments (to approximate the inverse metric determinant).
(4) The approximant to the continuum curvature is a holonomy around the prechosen (based) loop in a representation which Thiemann selects to be the fundamental representation (or at least this representation is chosen to be fixed once and for all and is not considered to be state dependent).

Choice of Triangulation: Given a vertex $v \in V(\gamma)$, let there be n (outgoing) edges emanating from v, $\{e_1, \ldots, e_n\}$. Let us assume that these edges are such that (e_i, e_{i+1}) are right oriented $\forall\, i \in \{1, \ldots, n\}$ with $n+1 := 1$.[12] Now assign to each pair (e_i, e_{i+1}) a two simplex $\triangle_i(v)|_{i=1,\ldots,n}$ which has one of the vertices as v and whose boundary is traversed by two segments s, s' in e, e', respectively, and an (analytic) arc $a_{s,s'}$ between end points of s and s'. Note that Thiemann's choice of triangulation is such that the Riemann sum which approximates $H[N]$ is given by

$$H_{T(\gamma)}[N] = \sum_{v\in V(\gamma)} \frac{4}{n} \sum_{i=1}^{n} \mathcal{F}_{\triangle_i(v)}(A, E, N) + \text{Sum over simplices which do not contain vertices of}\,\gamma. \tag{5.1}$$

Here $\mathcal{F}_{\triangle_i(v)}(A, E)$ is a suitable approximant to $\int_{\triangle_i(v)} d^3x N \sqrt{q} \epsilon^{ijk} F^i_{ab} E^a_j E^b_k - k$ written in terms of various holonomies and fluxes.[13]

This nice split of the Riemann sum means that upon quantization, the sum over simplices which do not contain vertices of γ gives the zero operator. Thus

$$\hat{H}_{T(\gamma)}[N] = \sum_{v\in V(\gamma)} \frac{4}{n} \sum_{i=1}^{n} N(v) \widehat{\mathcal{F}_{\triangle_i(v)}(A, E)}, \tag{5.2}$$

where $\widehat{\mathcal{F}_{\triangle_i(v)}(A, E)}$ is a composite operator built out of holonomy operators $\hat{h}_{\partial(\triangle_i(v))}$, $\hat{h}_{s_i}$, $\hat{h}_{s_{i+1}}$ and the volume operator $\hat{V}(v)$ at v. Schematically, it looks like

$$\widehat{\mathcal{F}_{\triangle_i(v)}(A, E)} = \frac{1}{\delta^m} O(\hat{h}_{\partial(\triangle_i(v))} \hat{h}_{s_i} \hat{h}_{s_{i+1}}, \hat{V}(v)), \tag{5.3}$$

where the parameter δ is such that the coordinate area of $\triangle_i(v)$ is $O(\delta^2)$ and m depends on what density weight constraint one chooses.[14]

We emphasize three of the four features mentioned at the beginning of this section once again, as it will help us illustrate the key difference between Thiemann's regularization and the one we adopt in this paper.

(1) All the holonomies in the construction are typically in the fundamental (or at least a state-independent) representation.
(2) The action of this (finite-triangulation) Hamiltonian constraint is on a state based on γ results in the addition of two new vertices and one new edge. The locations of these vertices and edges are independent of the colorings of the state, and only depend on the graph.
(3) The Hamiltonian constraint has a trivial action on the newly created vertices.[15]

[11] Although Thiemann has defined a quantum Hamiltonian constraint in $2+1$ as well as $3+1$ dimensions for the SU(2) theory, his constructs can be trivially generalized to any compact group, and in particular $U(1)^3$.

[12] The notion of orientation is given in (Definition 4.1 of) [27]. Roughly speaking, it means that, given a pair of edges e, e', if upon starting at e and moving counterclockwise one encounters e' before encountering the analytic extension of e, then the ordered pair (e, e') is said to be right oriented.

[13] Usually k is chosen to be 1, but at least as far as the finite-triangulation operator is concerned, one can be more general. The density weight of the lapse depends on k.

[14] Thiemann's quantization is specific to the density one constraint. This is because it is only for density weight one that one can take the continuum limit of $\hat{H}_{T(\gamma)}[N]$ on $\mathcal{H}_{\rm kin}$. If one were to work with higher density constraints (whose continuum limit will not be a well-defined operator on $\mathcal{H}_{\rm kin}$ but on some distributional space), the germ of Thiemann's construction would essentially go through except for extra factors of $\frac{1}{\delta^m}$ floating around.

[15] This statement is less obvious in two dimensions than in three; however, in [27] it is ensured by constraining the tangent space structure of the graph at these vertices.

In Sec. VII D, we will see how the quantization of the Hamiltonian constraint performed in this paper differs in these three aspects from Thiemann's construction.

VI. A CLASSICAL COMPUTATION

In this section we exhibit a classical computation which motivates our proposal for the action of the quantum Hamiltonian constraint operator at finite triangulation. We work in this section with the density two Hamiltonian constraint, i.e., with no power of the metric determinant q appearing (and the lapse is a scalar density with weight -1). This simplifies the calculation considerably, and allows for a geometric interpretation of the action of the constraint. Moreover, in the $\mathrm{U}(1)^3$ quantum theory, $\hat{q}$ is just a multiple of the identity on charge networks (and hence any power of it is also), so we expect this simplified calculation to capture the most important ingredients of the classical theory that we want to retain in quantization.

We are interested in the action of the density two Hamiltonian constraint

$$H[N] = \frac{1}{2}\int \mathrm{d}^2x N\epsilon^{ijk}E^a_i E^b_j F^k_{ab} \tag{6.1}$$

on cylindrical functions. First observe that the action of the corresponding Hamiltonian vector field on the connection can be written as

$$\begin{aligned} X_{H[N]}A^i_a(x) &:= \{H[N], A^i_a(x)\} = -\epsilon^{ijk}NE^b_jF^k_{ab}(x) \\ &= \epsilon^{ijk}\mathcal{L}_{V_j}A^k_a(x) - \epsilon^{ijk}\partial_a(NE^b_jA^k_b(x)). \end{aligned} \tag{6.2}$$

Here $V^a_i := NE^a_i$ is a phase space-dependent vector field (of density weight zero) for each value of i. The second term can be seen as the result of a $\mathrm{U}(1)$ gauge transformation, and since we will work with a basis of states (the charge networks) that are invariant under $\mathrm{U}(1)^3$, this term will not contribute to the analysis. We see in (6.2) that the action results in a linear combination of (phase space-dependent) infinitesimal "diffeomorphisms." Of course these are not diffeomorphisms in the usual sense since the $\mathrm{U}(1)^3$ indexes get reshuffled.

Consider now the action of $X_{H[N]}$ on a holonomy functional associated with an edge e,

$$\begin{aligned} &h^{\vec{n}_e}_e \equiv h^{n^1_e}_e h^{n^2_e}_e h^{n^3_e}_e : \mathcal{A} \to \mathrm{U}(1)^3, \\ &A = (A^1, A^2, A^3) \mapsto h^{n^1_e}_e[A^1]h^{n^2_e}_e[A^2]h^{n^3_e}_e[A^3], \end{aligned} \tag{6.3}$$

where $\mathcal{A}$ is the space of smooth $\mathrm{U}(1)^3$ connections. We suppose that the vector fields V^a_i have support only in some

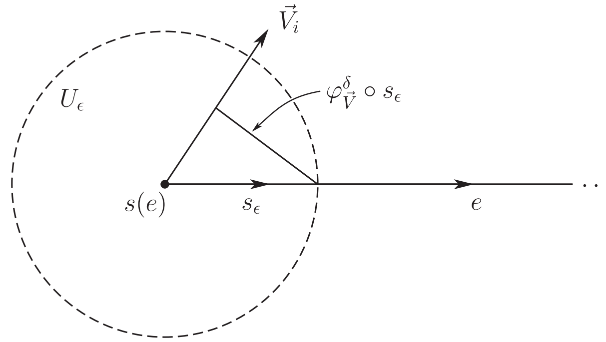

FIG. 1. The action of a diffeomorphism generated by the vector field $\vec{V}_i$ (which is compactly supported in U_ϵ) on a holonomy functional.

ϵ-neighborhood U_ϵ of $s(e)$, the source of e (as mentioned above, in the quantum theory, which features a nontrivial power of $\hat{q}$, this is the only relevant situation, in fact with ϵ as small as one pleases, since $\hat{q}$ acts only at the vertices of charge networks). Using (6.2), we find (discarding the terms coming from the total derivative)

$$\begin{aligned} X_{H[N]}h^{\vec{n}_e}_e[A] = \kappa\Big(\mathrm{i}n^2_e\int_e \mathcal{L}_{V_3}A^1 - \mathrm{i}n^3_e\int_e \mathcal{L}_{V_2}A^1 + \text{cyclic}\Big) \\ \times h^{n^1_e}_e[A^1]h^{n^2_e}_e[A^2]h^{n^3_e}_e[A^3]. \end{aligned} \tag{6.4}$$

We approximate the Lie derivatives by

$$\mathcal{L}_V A = \frac{1}{\delta}((\varphi^\delta_V)^*A - A) + O(\delta), \tag{6.5}$$

where φ^δ_V is a one-parameter family (parametrized by δ) of finite transformations generated by the vector field V, $(\varphi^\delta_V)^*$ being the pullback map. Since $\mathrm{supp}(V) = U_\epsilon$, we have

$$\begin{aligned} \mathrm{i}\kappa n_e\int_e \mathcal{L}_V A &= \mathrm{i}n_e\int_{s_\epsilon}\mathcal{L}_V A \\ &= \frac{1}{\delta}\,\mathrm{i}\kappa n_e\int_{s_\epsilon}((\varphi^\delta_V)^*A - A) + O(\delta) \\ &= \frac{1}{\delta}\kappa\Big(\mathrm{i}n_e\int_{\varphi^\delta_V\circ s_\epsilon} A - \mathrm{i}n_e\int_{s_\epsilon} A\Big) + O(\delta) \\ &= \frac{1}{\delta}(h^{n_e}_{\varphi^\delta_V\circ s_\epsilon}[A] - h^{n_e}_{s_\epsilon}[A]) + O(\delta), \end{aligned} \tag{6.6}$$

where $s_\epsilon = e\cap U_\epsilon$ is a small segment of e lying in U_ϵ [see Fig. 1]. Substituting in (6.4), we obtain

$$X_{H[N]}h_e^{\vec{n}_e}[A] = \frac{1}{\delta}((h^{n_e^2}_{\varphi^\delta_{V_3}\circ s_\epsilon}[A^1] - h^{n_e^2}_{s_\epsilon}[A^1]) - (h^{n_e^3}_{\varphi^\delta_{V_2}\circ s_\epsilon}[A^1] - h^{n_e^3}_{s_\epsilon}[A^1]) + \text{cyclic})h_e^{n_e^1}[A^1]h_e^{n_e^2}[A^2]h_e^{n_e^3}[A^3] + O(\delta)$$
$$= \frac{1}{\delta}(h^{n_e^2}_{\varphi^\delta_{V_3}\circ s_\epsilon}[A^1]h^{-n_e^2}_{s_\epsilon}[A^1] - 1)h^{n_e^2}_{s_\epsilon}[A^1]h_e^{n_e^1}[A^1]h_e^{n_e^2}[A^2]h_e^{n_e^3}[A^3]$$
$$- \frac{1}{\delta}(h^{n_e^3}_{\varphi^\delta_{V_2}\circ s_\epsilon}[A^1]h^{-n_e^3}_{s_\epsilon}[A^1] - 1)h^{n_e^3}_{s_\epsilon}[A^1]h_e^{n_e^1}[A^1]h_e^{n_e^2}[A^2]h_e^{n_e^3}[A^3] + \text{cyclic} + O(\delta). \tag{6.7}$$

Approximating $h^{n_e^i}_{s_\epsilon}[A^1]$ outside the parentheses as $1 + O(\epsilon)$, we have finally

$$X_{H[N]}h_e^{\vec{n}_e}[A] = \frac{1}{\delta}(h^{n_e^2}_{\varphi^\delta_{V_3}\circ s_\epsilon}[A^1]h^{-n_e^2}_{s_\epsilon}[A^1] - 1)h_e^{n_e^1}[A^1]h_e^{n_e^2}[A^2]h_e^{n_e^3}[A^3]$$
$$- \frac{1}{\delta}(h^{n_e^3}_{\varphi^\delta_{V_2}\circ s_\epsilon}[A^1]h^{-n_e^3}_{s_\epsilon}[A^1] - 1)h_e^{n_e^1}[A^1]h_e^{n_e^2}[A^2]h_e^{n_e^3}[A^3] + \text{cyclic} + O(\epsilon, \delta). \tag{6.8}$$

We can extend this calculation to charge networks. Consider a charge network c based on a graph containing an N-valent vertex $v \in \text{supp}(V_i)$ [with $\text{supp}(V_i)$ an ϵ-neighborhood of v] and suppose no other vertex of c lies in $\text{supp}(V_i)$. Then a simple Leibniz rule application of (6.8) yields

$$X_{H[N]}c(A) = c(A)\frac{1}{\delta}\sum_{I=1}^{N}[(h^{n^2_{e_I}}_{\varphi^\delta_{V_3}\circ s^I_\epsilon}[A^1]h^{-n^2_{e_I}}_{s^I_\epsilon}[A^1] - 1) - (h^{n^3_{e_I}}_{\varphi^\delta_{V_2}\circ s^I_\epsilon}[A^1]h^{-n^3_{e_I}}_{s^I_\epsilon}[A^1] - 1)] + \text{cyclic} + O(\epsilon, \delta). \tag{6.9}$$

We can rewrite this result in terms of a product over I by noting that, given some ϵ-dependent quantities $f_I(\epsilon) = 1 + \epsilon g_I$ (short holonomies being an example),

$$\sum_I (f_I(\epsilon) - 1) = \prod_I f_I(\epsilon) - 1 + O(\epsilon^2). \tag{6.10}$$

Using (6.10) and (6.9) becomes

$$X_{H[N]}c(A) = c(A)\frac{1}{\delta}\left[\prod_{I=1}^{N} h^{n^2_{e_I}}_{\varphi^\delta_{V_3}\circ s^I_\epsilon}[A^1]h^{-n^2_{e_I}}_{s^I_\epsilon}[A^1] - \prod_{I=1}^{N} h^{n^3_{e_I}}_{\varphi^\delta_{V_2}\circ s^I_\epsilon}[A^1]h^{-n^3_{e_I}}_{s^I_\epsilon}[A^1]\right] + \text{cyclic} + O(\epsilon, \delta). \tag{6.11}$$

It is easy to check that if $c(A)$ is gauge invariant at v then the $X_{H[N]}c(A)$ derived in (6.11) will be gauge invariant as well.

We can now restate our goal: Our aim is to quantize the Hamiltonian constraint (3.11) at finite triangulation in such a way that its action on charge networks gives the linear combination in (6.11) [up to factors coming from the quantization of the nontrivial power of q appearing in (3.11)]. Of course (6.11) is not the only approximant one can obtain starting with the geometric action of $X_{H[N]}$. The justification of the choices we have made lies in the fact that the off-shell closure condition is satisfied.

VII. DEFINITION OF $\hat{H}[N]$

The classical Hamiltonian constraint (3.11) is written in terms of the local connection and densitized triad fields, but neither of these objects is a well-defined operator in our quantum theory, so we cannot immediately write down an operator $\hat{H}[N]$ corresponding to (3.11). The strategy [16] is to first derive a classical approximant $H_T[N]$ to $H[N]$, where $H_T[N]$ is written solely in terms of holonomies and fluxes, and then quantize it as an operator on $\mathcal{H}_{\text{kin}}$. Since there are an uncountably infinite number of different ways that the connection and triad can be approximated using holonomies and fluxes (generally leading to inequivalent quantum operators), we tune our choices to the end we seek, which is to mimic the classical action found above. There are many choices to be made, and in the following subsections, we motivate and specify each.

A. Choice of triangulation

All subsequent regularization choices are based first on a one parameter family of *triangulations* $T(\delta)$ of Σ, by which we mean, for a fixed value of δ, a tesselation or cover of Σ by subsets $\triangle \subset \Sigma$.[16] We proceed to spell out what is required of $T(\delta)$.

We fix once and for all a volume form ω on Σ used to assign areas in subsequent constructions. Then, the argument δ of $T(\delta)$ is a parameter which roughly measures the (square root of the) area of each $\triangle \in T(\delta)$. All we require of $T(\delta)$ is that the Riemann sum $\sum_{\triangle\in T(\delta)} H_\triangle[N]$ converge

[16]We use the term triangulation rather loosely here. It does not mean a triangulation in the sense of algebraic topology where an n-dimensional triangulation of a space implies covering the space with n-dimensional simplices which themselves intersect only in lower dimensional simplices.

to $H[N]$ as $\delta \to 0$ (the triangulation becoming infinitely fine), where $H_{\triangle}[N]$ is some approximant to $H[N]$ in $\triangle$. The allowed values of δ, and hence the class of admissible triangulations, will be fixed by the charge network state c that $\hat{H}_T$ will act on. We emphasize that any one-parameter family $T(\delta)$ satisfying this requirement is an admissible family of triangulations, in the sense that the Riemann sum correctly approximates the classical Hamiltonian constraint, and we use this freedom to choose $T(\delta)$ which are "adapted" to charge networks, in a way we describe below.

Let c be a charge network with an underlying graph γ. We construct a one parameter family of triangulations $T(\delta, \gamma)$ adapted to the graph γ, satisfying the following criteria:

(i) The coordinate areas $|\triangle| := \int_{\triangle} \omega$ of all $\triangle \in T(\gamma, \delta)$ containing a vertex of γ satisfy

$$|\triangle| = \delta_1 \delta_2 = \delta^2, \tag{7.1}$$

where $\delta_1 \gg \delta \gg \delta_2$. That is, the plaquettes are chosen to be "long rectangles" oriented along the edges of γ in some local coordinates.[17]

(ii) We tailor $T(\delta, \gamma)$ so that each N-valent vertex v in the vertex set $V(\gamma)$ of γ is in the interior of precisely N plaquettes $\{\triangle_v^I\}_{I=1}^N$; moreover, we require that $\triangle_v^I$ is aligned along the edge e_I emanating from v as shown in Fig. 2. This requirement ensures that the overlap of any plaquettes is a region of area $\sim\delta_2^2$, and hence the contribution to the Riemann sum of these regions will be subleading in δ in the sense that it will vanish in the continuum limit.

(iii) The triangulation of the space $\Sigma - \bigcup_{v\in V(\gamma)} \bigcup_I \triangle_v^I$ is only subject to the requirements that no plaquettes overlap (except in their boundaries), and that their areas scale with δ^2.

The existence of such $T(\delta)$, in which the contribution to a Riemann sum from overlapping cuboids vanishes in the continuum limit, was shown in [21] for three dimensions. We assume here a precisely similar construct in two dimensions.

B. Riemann sum

Given an admissible triangulation $T(\delta, \gamma)$ adapted to γ, our next task is to construct an approximant $H_{\triangle}[N]$ to $H[N]$ in each $\triangle \in T(\delta, \gamma)$. First we expand the classical

[17]Given a vertex v, fix an open neighborhood U_v and a coordinate system $\{x_v\}$ around v. For sufficiently small δ, the edges meeting at v are analytic in the open ball $B_{2\sqrt{\delta}}(v) \subset U_v$ of radius $2\sqrt{\delta}$ centered at v, and hence, for sufficiently small δ, are "almost straight lines" in the coordinate system $\{x_v\}$ restricted to $B_{2\sqrt{\delta}}(v)$. Then for a given edge e_I emanating from v, construct a plaquette $\triangle_v^I$ as a long rectangle in $\{x_v\}$ along the direction tangent to e at v, with length $\sqrt{\delta}$ and width $\delta^{3/2}$, which overlaps the beginning point v of e [see Fig. 2 and requirement (ii)].

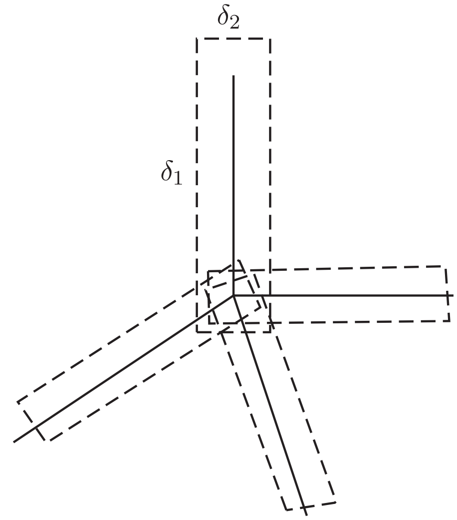

FIG. 2. Portion of an admissible triangulation adapted to a graph near a vertex. Each plaquette $\triangle_v^I$ containing the vertex overlaps the others in an area $\sim\delta_2^2$.

Hamiltonian $H[N]$ into terms labeled by the curvature's U(1) index:

$$H[N] = \frac{1}{2}\int d^2x N q^{-1/4}(\epsilon^{1jk}F_{ab}^1 E_j^a E_k^b + \epsilon^{2jk}F_{ab}^2 E_j^a E_k^b + \epsilon^{3jk}F_{ab}^3 E_j^a E_k^b) := \sum_{i=1}^{3} H^{(i)}[N]. \tag{7.2}$$

Let us focus on $H^{(1)}$; $H^{(2)}$ and $H^{(3)}$ can be obtained by cyclic permutations of the $\mathrm{U}(1)_i$ indexes. For an admissible $T(\delta, \gamma)$, the following expression converges to $H^{(1)}[N]$ as $\delta \to 0$:

$$H^{(1)}_{T(\delta,\gamma)}[N] = \frac{1}{2}\sum_{\triangle\in T(\delta,\gamma)} |\triangle| N(v_\triangle) q^{-1/4}(v_\triangle)\epsilon^{1jk}F_{ab}^1(v_\triangle) \times E_j^a(v_\triangle)E_k^b(v_\triangle), \tag{7.3}$$

where $v_\triangle$ is a point in $\triangle$, which we specify after splitting the sum in the following way: The sum over $\triangle$ is split into those $\triangle_v$ that contain a vertex $v \in V(\gamma)$, and those $\bar{\triangle}$ that do not:

$$2H^{(1)}_{T(\delta,\gamma)}[N] = \sum_{\triangle_v | v\in V(\gamma)} |\triangle_v| N q^{-1/4}\epsilon^{1jk}F_{ab}^1 E_j^a E_k^b(v) + \sum_{\bar{\triangle}} |\bar{\triangle}| N q^{-1/4}\epsilon^{1jk}F_{ab}^1 E_j^a E_k^b(v_{\bar{\triangle}}), \tag{7.4}$$

and in the first sum, $v_\triangle$ is chosen to be v, the vertex contained in $\triangle$, while in the second sum, $v_{\bar{\triangle}}$ is a basepoint of $\bar{\triangle}$, chosen once and for all. When we quantize (7.4) as an operator acting on charge networks based on the graph γ with $\hat{q}^{-1/4}$ acting rightmost, the latter sum will not contribute, since as shown in Appendix A, $\hat{q}^{-1/4}$ acts nontrivially only at charge network vertices.

Next we need to approximate the various local fields in the first sum of (7.4) by holonomies and fluxes. This consists of choosing surfaces over which the triads are smeared, holonomies around loops that feature in the curvature approximant, and holonomies along short paths that feature in the inverse metric determinant approximant. We take a cue from the treatment of the diffeomorphism constraint [21], and tailor the curvature approximant to the underlying state. In this case however, there is no fixed shift vector field that one can use to define a small loop. To stand in its place, we introduce a key ingredient in our construction, the quantum shift.

C. The quantum shift

Note that classically $Nq^{-1/4}E^a_j$ is a vector field (of density weight zero) for each j. The rough idea is to quantize this operator on charge networks and use its eigenvalues as shift vector components, which then feed into the definition of the small loop used to approximate the curvature. It turns out that in the $\mathrm{U}(1)^3$ theory, our quantization of $Nq^{-1/4}E^a_j$ yields an operator diagonal in the charge network basis, so we define the (regularized) quantum shift components by

$$V^a_j(x)|_\epsilon := \langle c|N\hat{E}^a_j(x)|_\epsilon \hat{q}_\epsilon^{-1/4}(x)|c\rangle, \tag{7.5}$$

where $\hat{E}^a_j|_\epsilon$ and $\hat{q}_\epsilon^{-1/4}$ denote some ϵ-regularized $\hat{E}^a_j$ and $\hat{q}^{-1/4}$, which we construct below. As suggested by the notation, we will quantize E^a_j and $q^{-1/4}$ separately.

As demonstrated in more detail in Appendix A, the regulated operator $\hat{q}_\epsilon^{-1/4}$ we employ is proportional to $(\kappa\hbar)^{-1}$, as well as the small parameter ϵ used to construct the classical identity that is quantized to define $\hat{q}_\epsilon^{-1/4}$. We leave these factors explicit, and write the eigenvalues as

$$\hat{q}_\epsilon^{-1/4}(v)|c\rangle := \frac{\epsilon}{\kappa\hbar}\lambda(\vec{n}^c_v)|c\rangle, \tag{7.6}$$

where $v \in V(c)$ (otherwise the right-hand side is zero), and the $\lambda(\vec{n}^c_v)$ are dimensionless numbers depending on relations amongst the tangents of the edges emanating from v, their charges, as well as additional regularization choices.

As for $\hat{E}^a_j|_\epsilon$, we require some extra structure: At each vertex $v \in c$, we fix, once and for all, an ϵ'-neighborhood $U_{\epsilon'}(\gamma, v)$ with a coordinate chart $\{x_v\}$ with origin at v, and a coordinate ball $B_x(v, \epsilon) \subset U_{\epsilon'}(\gamma, v)$ of radius ϵ centered at v (ϵ, ϵ' are independent parameters). Using this structure, we regularize the δ function appearing in the action of $\hat{E}^a_j$, resulting in a regularized operator $\hat{E}^a_j|_\epsilon$ which acts as

$$\hat{E}^a_j|_\epsilon(v)|c\rangle := \kappa\hbar\sum_{e_I\cap v}\left(\int_0^1 \mathrm{d}t\frac{\chi_{B_x(v,\epsilon)}(e_I(t))}{\pi\epsilon^2}\dot{e}^a_I(t)\right)n^j_I|c\rangle, \tag{7.7}$$

where χ_S is the characteristic function on S. This evaluates to

$$\begin{aligned}\hat{E}^a_j|_\epsilon(v)|c\rangle &= \frac{\kappa\hbar}{\pi\epsilon^2}\sum_{e_I\cap v}\left(\int_{e_I\cap B_x(v,\epsilon)}\mathrm{d}e^a_I\right)n^j_I|c\rangle \\ &= \frac{\kappa\hbar}{\pi\epsilon}\sum_{e_I\cap v}\hat{e}^a_I n^j_I|c\rangle + O(1),\end{aligned} \tag{7.8}$$

where $\hat{e}^a_I$ is a unit vector in $\{x_v\}$ which passes through $e_I \cap B_x(v, \epsilon)$.[18] Thus we have

$$\begin{aligned}\hat{V}^a_j(v)|_\epsilon|c\rangle &= N(v)\hat{E}^a_j|_\epsilon(v)\hat{q}_\epsilon^{-1/4}(v)|c\rangle \\ &= \frac{1}{\pi}N(v)\lambda(\vec{n}^c_v)\sum_{e_I\cap v}\hat{e}^a_I n^j_I|c\rangle := V^a_j(v,\epsilon,c)|c\rangle.\end{aligned} \tag{7.9}$$

This is a heavily coordinate-dependent construction (note that it also depends on choices made in the classical identity used in the construction of $\hat{q}_\epsilon^{-1/4}$). We place a bound $\delta_0 := \delta_0(c, \epsilon)$ on the parameter δ [associated with $T(\delta, \gamma)$] by requiring that for all $\delta \le \delta_0$, the end point of the arc δE^a_j is in the ball $U_\epsilon(\gamma, v)$.

D. The Hamiltonian constraint operator at finite triangulation

In this subsection we lay out our proposal for the Hamiltonian constraint operator at finite triangulation. Let us order $\hat{H}^{(1)}_{T(\delta,\gamma)}[N]$ in the following way:

$$\begin{aligned}2\hat{H}^{(1)}_{T(\gamma,\delta)}[N] = &\sum_{v\in V(\gamma)}\sum_{\triangle_v|v\in V(\gamma)}|\triangle_v|\epsilon^{1jk}(\widehat{F^1_{ab}E^b_k})_\delta\hat{V}^a_j(v_x)|_\delta \\ &+ \sum_{\bar{\triangle}}|\bar{\triangle}|\epsilon^{1jk}(\widehat{F^1_{ab}E^b_k})_\delta\hat{V}^a_j(v_{\bar{\triangle}})|_\delta.\end{aligned} \tag{7.10}$$

Since $\hat{q}_\delta^{-1/4}$ vanishes everywhere except at the vertices of γ, the second sum gives no contribution, leaving

$$\begin{aligned}2\hat{H}^{(1)}_{T(\gamma,\delta)}[N]|c\rangle = &\sum_{v\in V(\gamma)}\sum_{\triangle_v|v\in V(\gamma)}|\triangle_v|(V^a_2(v_x,\delta,c) \\ &\times(\widehat{F^1_{ab}E^b_3}(\triangle_v))_\delta - V^a_3(v_x,\delta,c) \\ &\times(\widehat{F^1_{ab}E^b_2}(\triangle_v))_\delta)|c\rangle.\end{aligned} \tag{7.11}$$

We now use the eigenvalues V^a_i to specify the loops used to define the curvature operator. Specifically, at a given vertex v, we associate one loop with each edge emanating from v. $T(\gamma, \delta)$ is chosen such that for an N-valent vertex v, there are N plaquettes $\{\triangle^I_v\}^N_{I=1}$ containing v, and with each a loop is associated. We now construct these loops.

For a given edge e_I, one segment of the loop is formed by a coordinate-length δ segment of e_I itself, and another

[18] The $O(1)$ term is subleading in $\frac{1}{\epsilon}$.

by a segment of length $\delta|E^a_i|$ in the direction of V^a_i. Note here that

$$|E^a_i| := \frac{|V^a_i(v_x, \delta, c)|}{|N(v_x)\lambda(\vec{n}^c_v)|} \tag{7.12}$$

is the norm of the quantum shift eigenvalue, apart from the inverse volume eigenvalue and the value of the lapse; that is, we do not use the entire quantum shift for the loop specification [if $N(v_x)\lambda(\vec{n}^c_v) = 0$, then the quantum shift is zero, and the Hamiltonian operator is defined to act trivially]. First we describe the generic case, where the end point of the arc δE^a_i does not lie on γ, and later describe the special case when this end point lies on γ.

The final segment is an arc connecting the ends of these two segments which is tangent to the edges of γ at its end points (this is a consequence of the fact that the quantum shift direction determined by E^a_i is tangential to each edge at the arc position, and it ensures that the operator $\hat{q}^{-1/4}$, and hence the Hamiltonian, acts trivially at these newly created trivalent vertices). We postpone specifying further properties of these arcs, as they are irrelevant, except that they do not create any spurious new intersections, and that their areas satisfy a property spelled out below. Let us denote the full loops by $\beta^v_{i,I}$. By convention, they are oriented such that the segment which overlaps e_I is ingoing at the vertex. Note that the segment δE^a_i is shared by all $\beta^v_{i,I}$ (as I varies). Now consider the following classical approximant:

$$(V^a_i F^j_{ab} E^b_k(\triangle^I_v))_\delta = N(v_x)\frac{(h^j_{\beta^v_{i,I}})^{n^k_{e_I}} - 1}{\mathrm{i}\kappa n^k_{e_I}|\beta^v_{i,I}|}\frac{E_k(L_I)}{|L_I|}q(v_x)^{-1/4}_\delta. \tag{7.13}$$

Here $(h^j_{\beta^v_{i,I}})^{n^k_{e_I}}$ is the $\mathrm{U}(1)_j$ holonomy around the loop $\beta^v_{i,I}$ in the $n^k_{e_I}$ crepresentation, $|\beta^v_{i,I}|$ is the coordinate area of $\beta^v_{i,I}$, and L_I is a flux surface transverse to e_I of area $|L_I|$. This converges to $V^a_i F^j_{ab} E^b_k(v)$ classically as $|\beta^v_{i,I}|, |L_I| \to 0$. Making this replacement as an operator in $\hat{H}^{(1)}_{T(\gamma,\delta)}[N]$, we obtain

$$\begin{aligned} 2\hat{H}^{(1)}_{T(\gamma,\delta)}[N]|c\rangle &= \sum_{v\in V(\gamma)} N(v_x)\lambda(\vec{n}^c_v)\sum_{\triangle_v|v\in V(\gamma)}|\triangle_v|\left(\frac{(h^1_{\beta^v_{2,I}})^{n^3_{e_I}}-1}{\mathrm{i}\kappa n^3_{e_I}|\beta^v_{2,I}|}\frac{\hat{E}_3(L_I)}{|L_I|} - \frac{(h^1_{\beta^v_{3,I}})^{n^2_{e_I}}-1}{\mathrm{i}\kappa n^2_{e_I}|\beta^v_{3,I}|}\frac{\hat{E}_2(L_I)}{|L_I|}\right)|c\rangle \\ &= \frac{\hbar}{\mathrm{i}}\sum_{v\in V(\gamma)} N(v_x)\lambda(\vec{n}^c_v)\sum_I \frac{|\triangle_I|}{|L_I|}\left(\frac{(h^1_{\beta^v_{2,I}})^{n^3_{e_I}}-1}{|\beta^v_{2,I}|} - \frac{(h^1_{\beta_{3,I}})^{n^2_{e_I}}-1}{|\beta^v_{3,I}|}\right)|c\rangle, \end{aligned} \tag{7.14}$$

where the sum over I extends over the valence of the vertex v and we have chosen flux surfaces L_I such that $\epsilon(L_I, e_I) = +1$. The charges $n^2_{e_I}$, $n^3_{e_I}$ are chosen to be those coloring the edge e_I of c. If either $n^2_{e_I}$, $n^3_{e_I}$ is zero, then we choose the holonomy to be in the fundamental representation. We have the freedom of tuning the loop, flux, and plaquette areas so as to arrive at an overall factor of δ^{-1}:

$$2\hat{H}^{(1)}_{T(\gamma,\delta)}[N]|c\rangle = \frac{\hbar}{\mathrm{i}}\sum_{v\in V(\gamma)} N(v_x)\lambda(\vec{n}^c_v)\frac{1}{\delta}\sum_I(((h^1_{\beta^v_{2,I}})^{n^3_{e_I}}-1) - ((h^1_{\beta^v_{3,I}})^{n^2_{e_I}}-1))|c\rangle. \tag{7.15}$$

We may again pass to the product form (discarding terms which vanish classically as $\delta \to 0$)[19]

$$\begin{aligned} \hat{H}^{(1)}_{T(\gamma,\delta)}[N]|c\rangle &= \frac{\hbar}{2\mathrm{i}}\sum_{v\in V(\gamma)} N(v_x)\lambda(\vec{n}^c_v)\left(\frac{\prod_I(h^1_{\beta^v_{2,I}})^{n^3_{e_I}}-1}{\delta} - \frac{\prod_I(h^1_{\beta^v_{3,I}})^{n^2_{e_I}}-1}{\delta}\right)|c\rangle \\ &=: \frac{\hbar}{2\mathrm{i}\delta}\sum_{v\in V(\gamma)} N(v_x)\lambda(\vec{n}^c_v)(|c_1\cup\alpha^\delta_v(\langle\hat{E}_2\rangle, n^3), c_2, c_3\rangle - |c_1\cup\alpha^\delta_v(\langle\hat{E}_3\rangle, n^2), c_2, c_3\rangle). \end{aligned} \tag{7.16}$$

We have introduced the following notation:

$$|c\rangle \equiv |c_1, c_2, c_3\rangle, \qquad |c_1\cup\alpha^\delta_v(\langle\hat{E}_2\rangle, n^3), c_2, c_3\rangle := \prod_I (h^1_{\beta^v_{2,I}})^{n^3_{e_I}}|c\rangle, \tag{7.17}$$

[19]The reader may wonder about the physical motivation in switching to the product form. Although the anomaly freedom of the continuum Hamiltonian constraint achieved in this paper is intricately tied to the structure of the operator and in particular to this product form, we believe that an off-shell closure condition could be satisfied even without switching to the product form, if the key ideas developed here are followed closely. However the real reason to pass to the product form lies in keeping an eye on the SU(2) theory, which is our main goal. In that case, only the product rule will ensure that the newly created vertex will be nondegenerate and the second Hamiltonian constraint will have a nontrivial action on it.

suggestive of the fact that only $c_{i=1}$ has been altered by the action of $\hat{H}^{(i=1)}_{T(\gamma,\delta)}$, and this deformation is performed near the vertex v, is of "size" δ, and depends on the (vector-valued) eigenvalue $\langle \hat{E}_2 \rangle$ (in the state c) as well as the values of the $n^{i=3}$ charges of c. If c is not charged in the $i = 2$ or $i = 3$ factors of U(1), then $\hat{H}^{(1)}_{T(\delta,\gamma)}$ annihilates $|c\rangle$, with similar statements for $\hat{H}^{(2)}_{T(\gamma,\delta)}$ and $\hat{H}^{(3)}_{T(\gamma,\delta)}$. To see this, note that if c is not charged in $\mathrm{U}(1)_2$ for example, then $n^2_{e_I} = 0$, and hence $\beta^v_{2,I}$ collapses to a retraced curve (so that the corresponding holonomy equals 1), and $\hat{E}_2$ annihilates $|c\rangle$. The term $|c_1 \cup \alpha^\delta_v(\langle \hat{E}_2 \rangle, n^3), c_2, c_3\rangle$ produced by $\hat{H}^{(1)}_{T(\gamma,\delta)}[N]$ is depicted in Fig. 3.

In the case that the end point of δE^a_i lies on γ, the construction proceeds just as above; however, we observe that the quantum shift actually points along some edge in this case, and hence the resulting state has all edge tangents parallel or antiparallel at this point, as shown in Fig. 4.

At each N-valent vertex v, $\hat{H}^{(i)}_{T(\gamma,\delta)}[N]$ acts by attaching at most N loops $\beta^v_{j,I}$ charged in $U(1)_i$ only with charge $n^k_{e_I}$, the charge on the edge that $\beta^v_{j,I}$ partially overlaps. Our construction is such that at most two loops do not intersect any other edges except the ones they overlap, and remaining

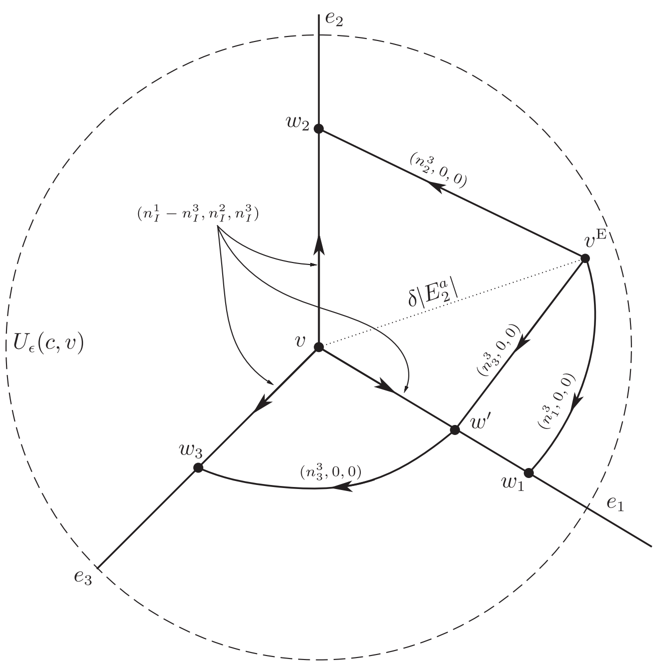

FIG. 3. The state $|c_1 \cup \alpha^\delta_v(\langle \hat{E}_2 \rangle, n^3), c_2, c_3\rangle$ as produced by the action of $\hat{H}^{(1)}_{T(\gamma,\delta)}[N]$ in the generic case where v^{E} does not lie on γ. Each segment e_I now leaving v has had its $\mathrm{U}(1)_1$ charge shifted by $-n^3_I$, and the segments which leave the extraordinary vertex v^{E} are only charged in $\mathrm{U}(1)_1$. The dotted segment $\delta|E^a_2|$ shared by all $\beta^v_{2,I}$ is totally uncharged as a result of gauge invariance. The trivalent vertices w_I are such that all edge tangents are here parallel or antiparallel, and the four-valent vertex w' is such that there are two pairs of edges which are analytic extensions of each other.

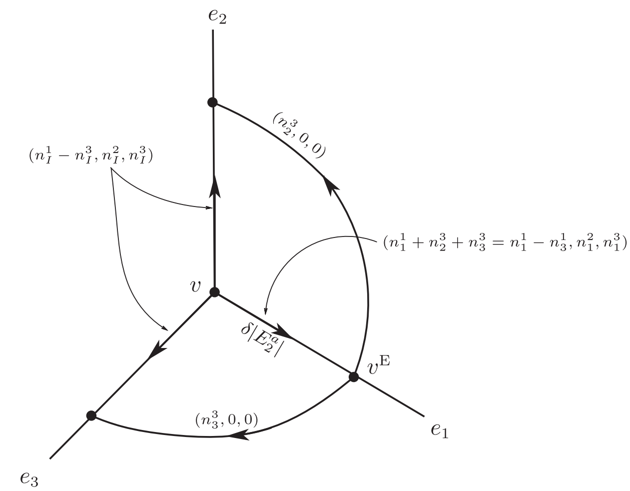

FIG. 4. The state $|c_1 \cup \alpha^\delta_v(\langle \hat{E}_2 \rangle, n^3), c_2, c_3\rangle$ as produced by the action of $\hat{H}^{(1)}_{T(\gamma,\delta)}[N]$ in the special case where v^{E} lies on γ.

loops will have nontrivial intersections with the edges apart from the ones they overlap (this trivial observation, unavoidable in two dimensions, will be important later).

Recall that all the attached loops have precisely one common segment which is given by the straight line δE^a_i. Gauge invariance ensures that this segment is [as part of the resulting state $|c_1 \cup \alpha^\delta_v(\langle \hat{E}_2 \rangle, n^3), c_2, c_3\rangle$ for instance] uncharged, and thus its end point (the beginning point being v) will be an $N_v \leq N$-valent vertex charged only in $\mathrm{U}(1)_i$. Whence the action of $\hat{H}^{(1)}_{T(\gamma,\delta)}[N]$ at a charge network vertex creates precisely two charge network states, each of which have an additional N_v-valent vertex, and precisely N_v additional trivalent vertices. We will refer to the N_v-valent vertex created by the action of $H^{(1)}_{T(\gamma,\delta)}[N]$ as an extraordinary vertex v^{E}. Note that this extraordinary vertex can lie off the original graph or be in the interior of one of the edges of the original graph depending on the quantum shift. By construction, v^{E} lies in the interior of $U_\epsilon(\gamma, v)$. In order to specify the action of the Hamiltonian constraint on arbitrary charge networks we need a classification scheme given in the following section.

To summarize, the action of a Hamiltonian constraint at finite triangulation creates three kinds of vertices. The extraordinary vertices, whose location depends on the quantum shift, a set of trivalent vertices which by construction are such that $\hat{q}^{-1/4}$ vanishes at such vertices, and the four-valent vertices which have very specific charge configurations and analyticity properties. As we will see later, these trivalent and four-valent vertices will play no role in proving the off-shell closure condition, and hence we will refer to them as irrelevant vertices.

E. Classification of extraordinary vertices

As we saw above, the action of the Hamiltonian constraint $\hat{H}^{(i)}_{T(\gamma,\delta)}[N]$ on a charge network state $|c\rangle$ results in the creation of what we called extraordinary (EO) vertices. In this section, we analyze the structure of these vertices in more detail. Our aim is to argue that, given a charge-network state $|\bar{c}\rangle$ with its vertex set $V(\bar{c})$, we can uniquely determine

(a) which of the vertices are EO;
(b) if v^{E} is EO, then there exists a *unique* charge network state $|c\rangle$ such that the action of $\hat{H}^{(i)}_{T(\gamma,\delta)}(v)$ on $|c\rangle$ [for a unique v in $V(c_1 \cup c_2 \cup c_3)$ and a particular value of i] results in $|\bar{c}\rangle$ with v^{E} [where $\hat{H}^{(i)}_{T(\gamma,\delta)}(v)$ is defined via $\hat{H}^{(i)}_{T(\gamma,\delta)}[N]|c\rangle = \sum_{v\in V(\gamma(c))} N(v)\hat{H}^{(i)}_{T(\gamma,\delta)}(v)|c\rangle$].

We will first give a classification scheme, which helps us isolate EO vertices inside any charge network $\bar{c}$ unambiguously. We then show that the removal of an EO vertex v^{E} along with all the edges incident on it and appropriate shifts in charges on the remaining edges of the graph results in c with a vertex v such that the action of $\hat{H}^{(i)}_{T(\gamma,\delta)}(v)$ for some δ and i results in $\bar{c}$.

As we saw earlier, if we act on a state $|c\rangle$ by $\hat{H}^{(i)}_{T(\gamma,\delta)}[N]$, EO vertices are end points of a straight line arc determined by quantum shift vectors. Generically these vertices will lie off the graph $\gamma(c)$; however, there can be states in which the EO vertices will lie on some edge which was already present in the original graph. We will distinguish these two types of vertices and call them type A and type B vertices, respectively.

Given a charge-network c with a vertex v^{E}, we give a minimal set of independent conditions which, if satisfied, determine that v^{E} is an EO vertex. The conditions characterizing type A vertices are given below. The set of conditions characterizing an EO vertex of type B are given in Appendix B. We caution the reader that the conditions as listed here are rather technical and not too illuminating. The most efficient way to understand them is to consult Fig. 3 simultaneously.

Let $\bar{c} =: (\bar{c}_1, \bar{c}_2, \bar{c}_3)$ be a charge network with $\gamma(\bar{c})$ the coarsest graph associated underlying it. Let v^{E} be a vertex of $\gamma(\bar{c})$. We will call v^{E} an EO vertex of type (A, $M \in \{1,2,3\}, j \in \{1,2,3\}$) or type (B, $M \in \{1,2,3\}$, $j \in \{1,2,3\}$) if and only if it satisfies the set of conditions A or B, respectively.

Remark on notation: Sometimes we will indicate the type of EO vertices only by omitting one or two of the labels. For example, when the analysis only depends on the fact that the EO vertex is type ($M = 1, j = 2$), we will omit the label A/B.

1. Set A

(1) All the edges beginning at v^{E} are charged in the Mth copy [this is the result of the action of $\hat{H}^{(i=M)}$].
(2) If the valence of v^{E} is N_v,[20] then we will denote the N_v vertices which are the end points of the N_v edges beginning at v^{E} by the set $\mathcal{S}_{v^{\mathrm{E}}} := \{v^{\mathrm{E}}_{(1)}, \ldots, v^{\mathrm{E}}_{(N_v)}\}$. The valence of all these vertices is bounded between 3 and 4.
   (a) At most two vertices in $\mathcal{S}_{v^{\mathrm{E}}}$ are trivalent.
(3) The trivalent vertices are such that the edges which are not incident on v^{E} are analytic extensions of each other and the four-valent vertices are such that two of the edges which are not incident on v^{E} are analytic extensions of each other, and the fourth edge is the analytic extension of the edge which is incident at v^{E}.
   (a) Any four-valent vertex defined in (3) is such that, if the four edges (e_1, e_2, e_3, e_4) incident on it are such that $e_1 \circ e_2$ is entire analytic and $e_3 \circ e_4$ is entire analytic then $\vec{n}_{e_1} = \vec{n}_{e_2}$, $\vec{n}_{e_3} = \vec{n}_{e_4}$.
(4) Let $e_{v^{\mathrm{E}}}$ be an edge beginning at v^{E} which ends in a four-valent vertex $f(e_{v^{\mathrm{E}}})$. By (3), there exists an edge $e'_{v^{\mathrm{E}}}$ beginning at $f(e_{v^{\mathrm{E}}})$ such that $e_{v^{\mathrm{E}}} \circ e'_{v^{\mathrm{E}}} =: \tilde{e}_{v^{\mathrm{E}}}$ is the analytic extension of $e_{v^{\mathrm{E}}}$ in $E(\bar{c})$ (the edge set of $\bar{c}$) beginning at v^{E}. The final vertex $f(\tilde{e}_{v^{\mathrm{E}}})$ of $\tilde{e}_{v^{\mathrm{E}}}$ is always trivalent. Thus, restricting attention to analytic extensions of each of the edges beginning at v^{E}, all such edges end in trivalent vertices, and all of these trivalent vertices are such that the remaining two edges incident on them are analytic extensions of each other (see Fig. 3). The set of these N_v three-valent vertices associated to v^{E} will be denoted $\bar{\mathcal{S}}_{v^{\mathrm{E}}} := \{\bar{v}^{\mathrm{E}}_1, \ldots, \bar{v}^{\mathrm{E}}_{N_v}\}$.[21]
   (a) All three edges incident on any element in $\bar{\mathcal{S}}_{v^{\mathrm{E}}}$ have parallel (or anitparallel) tangents.
(5) Let us denote these [maximally analytic inside $E(\gamma)$] edges beginning at v^{E} by $\{\tilde{e}^1_{v^{\mathrm{E}}}, \ldots \tilde{e}^{N_v}_{v^{\mathrm{E}}}\}$. Without loss of generality, consider the case when all the edges incident at v^{E} are charged in $\mathrm{U}(1)_1$ [in this case we say that v^{E} is of type (A, $M = 1$, $j \in \{2,3\}$)]. Let the charges on these edges be $\{(n_{\tilde{e}^1_{v^{\mathrm{E}}}}, 0, 0), \ldots, (n_{\tilde{e}^{N_v}_{v^{\mathrm{E}}}}, 0, 0)\}$. If $\tilde{e}^k_{v^{\mathrm{E}}}$ ($k \in \{1,\ldots,N_v\}$) ends in a trivalent vertex $f(\tilde{e}^k_{v^{\mathrm{E}}})$ and if the charges on the remaining two (analytically related) edges $e^{k\prime}_{v^{\mathrm{E}}}, e^{k\prime\prime}_{v^{\mathrm{E}}}$ incident at $f(\tilde{e}^k_{v^{\mathrm{E}}})$ are $(n^1_{e^{k\prime}_{v^{\mathrm{E}}}}, n^2_{e^{k\prime}_{v^{\mathrm{E}}}}, n^3_{e^{k\prime}_{v^{\mathrm{E}}}})$ and $(n^1_{e^{k\prime\prime}_{v^{\mathrm{E}}}}, n^2_{e^{k\prime\prime}_{v^{\mathrm{E}}}} = n^2_{e^{k\prime}_{v^{\mathrm{E}}}}, n^3_{e^{k\prime\prime}_{v^{\mathrm{E}}}} = n^3_{e^{k\prime}_{v^{\mathrm{E}}}})$, then either
   (a) $n^1_{\tilde{e}^k_{v^{\mathrm{E}}}} = n^2_{e^{k\prime}_{v^{\mathrm{E}}}}$, or
   (b) $n^1_{\tilde{e}^k_{v^{\mathrm{E}}}} = n^3_{e^{k\prime}_{v^{\mathrm{E}}}}$.
(6) Now consider the set $\bar{\mathcal{S}}_{v^{\mathrm{E}}}$. Recall that each element in this set is a trivalent vertex. Consider a vertex $f(\tilde{e}_{v^{\mathrm{E}}})$ whose three incident edges are $\tilde{e}_{v^{\mathrm{E}}}$, $e'_{v^{\mathrm{E}}}$, and $e''_{v^{\mathrm{E}}}$. Recall that $e'_{v^{\mathrm{E}}}, e''_{v^{\mathrm{E}}}$ are analytic continuations of each other. Depending on whether $n_{\tilde{e}_{v^{\mathrm{E}}}} \lessgtr 0$, choose

[20] The subscript v in N_v may seem out of place; however, when we list all the conditions in A, its relevance will become clear.

[21] Note that $\mathcal{S}_{v^{\mathrm{E}}} \cap \bar{\mathcal{S}}_{v^{\mathrm{E}}}$ = 3-valent vertices in $\mathcal{S}_{v^{\mathrm{E}}}$.

the one of the two edges $e'_{v^{\rm E}}$, $e''_{v^{\rm E}}$ which has lesser or greater charge in the first copy than the other edge. Consider the set of all such chosen edges for each vertex in $\bar{\mathcal{S}}_{v^{\rm E}}$. We refer to this set as $\mathcal{T}_{v^{\rm E}}$.

(a) If all these edges meet in a vertex v which is such that, if the number of edges incident on v is greater than N_v and if the charges $\{\tilde{e}^k_{v^{\rm E}}\}_{k=1,\ldots,N_v}$ are the $\mathrm{U}(1)_j$ charges on the edges in $\mathcal{T}_{v^{\rm E}}$, then the $\mathrm{U}(1)_j$ charge on edges incident at v which are not in $\mathcal{T}_{v^{\rm E}}$ is zero. As shown in Appendix D, v, if it exists, is unique.

(7) Finally consider the graph $\gamma := \gamma(\bar{c}) - \{\tilde{e}^1_{v^{\rm E}}, \ldots, \tilde{e}^{N_v}_{v^{\rm E}}\}$ and a charge-network c based on γ obtained by deleting $\{\tilde{e}^1_{v^{\rm E}}, \ldots, \tilde{e}^{N_v}_{v^{\rm E}}\}$ along with the charges on them, and also deleting exactly the same amount of charge from the edges in $\mathcal{T}_{v^{\rm E}}$. Note that by construction, v belongs to γ. Now consider $U_\epsilon(\gamma, v)$. The final and key feature of an EO vertex $v^{\rm E}$ is $v^{\rm E} \in U_\epsilon(\gamma, v)$ and $v^{\rm E}$ is the end point of the "straight line curve" $\delta\langle\hat{E}^a_j\rangle_c$ for *some* δ, where $j = 2$ if in (6) condition (a) is satisfied, and $j = 3$ if in (5), (b) is satisfied. For example, $|c_1 \cup \alpha^\delta_v(\langle\hat{E}_2\rangle, n^3), c_2, c_3\rangle$ has an EO vertex of type $M = 1$, $j = 2$.

It is easy to see that the conditions listed above are independent of each other, as one could easily conceive of a charge-network state which satisfies all but one of the conditions. If all the conditions given in Set A above, or Set B in the Appendix B, are satisfied, then we call the pair $(v, v^{\rm E})$ extraordinary. For the benefit of the reader we emphasize once again that the type of extraordinariness of $v^{\rm E}$ is labeled by the triple (A/B, $M \in \{1, 2, 3\}$, $j \in \{1, 2, 3\}$). For example, $M = 1$ when all the edges incident at $v^{\rm E}$ are only charged under $\mathrm{U}(1)_1$ and $j \in \{2, 3\}$ if these charges equal the charges in $\mathrm{U}(1)_j$ on edges in $\mathcal{T}_{v^{\rm E}}$.

We now prove a lemma which shows that EO vertices are always associated to the action of some $\hat{H}^{(i)}_{T(\gamma,\delta)}[N]$. This will imply that any charge network which has an EO vertex is always in the image of $\hat{H}^{(i)}_{T(\gamma,\delta)}[N]$ for some i, δ, N.

Claim: Let $\cup_{K\in\{\rm A,B\}} \cup_{j\in\{2,3\}} \{\bar{c}^j_K\} = \mathcal{C}$ be the set of charge network states such that $(v, v^{\rm E})$ is an EO pair for each charge network in this set and $v^{\rm E}$ is an EO vertex of type $(K, M = 1, j)$.[22] Let c be a charge network obtained by performing the surgery described in condition (7) above.[23] Also let N be a lapse function such that it has support in a neighborhood of v (which, as we saw above, belongs to both c and $\bar{c}^j_K$ for each K and j). If the vertex v is nondegenerate (i.e., $\langle\hat{q}^{-1/4}\rangle_c \neq 0$), and if

$$\hat{H}^{(1)}_{T(\gamma,\delta)}[N]|c\rangle = \frac{1}{\delta} N(v)\langle\hat{q}^{-1/4}\rangle_c(v)[\alpha|c'\rangle + \beta|c''\rangle], \tag{7.18}$$

where α, $\beta \in \{\pm 1\}$, then

(a) both c', c'' belong to the set $\mathcal{C}$;

(b) conversely, given any $\bar{c}$ that is obtained from c (containing a nondegenerate vertex v which is not EO) by adding an EO vertex $v^{\rm E}$ of type $(K, 1, j)$ for some K, j, then $\bar{c}$ is always one of the two charge networks one gets by letting $\hat{H}^{(1)}_{T(\gamma,\delta)}(v)$ act on $|c\rangle$ for some δ.

Proof.—(a) follows by construction. That is, it is straightforward to verify that both c' and c'' satisfy all the conditions listed in Set A or Set B [see Eq. (7.16)].

For (b), consider a charge network $\bar{c}$ obtained by adding an EO vertex $v^{\rm E}$ of type $(K = \mathrm{A}, M = 1, j = 2)$ to c such that $(v, v^{\rm E})$ is the EO pair (other types of EO vertices can be treated similarly). Let the N_v-valent segments beginning at $v^{\rm E}$ and terminating at the N_v trivalent vertices $\{v_{e_1}, \ldots, v_{e_{N_v}}\}$ be denoted by $\{s_{e_1}, \ldots, s_{e_{N_v}}\}$. As the vertex is of type 1, all of these segments are charged in $\mathrm{U}(1)_1$. Let the vertex $v^{\rm E}$ be along the straight line $\delta_0\langle\hat{E}^a_2(v)\rangle_c$ (in the coordinate system that we have fixed once and for all).

Now consider the Hamiltonian constraint operator $\hat{H}^{(i)}_{T(\delta_0)}(v)$ at a given vertex $v \in V(c)$ as constructed out of products of holonomies around loops, described above. Each loop is constructed out of a segment along an edge of $\gamma(c)$, the straight line arc given by the quantum shift, and an arc which joins $v^{\rm E}$ with one of the vertices in $\mathcal{S}_{v^{\rm E}}$. We need N_v such arcs and upon choosing them to be $(s_{e_1}, \ldots, s_{e_{N_v}})$, respectively,[24] $\hat{H}^{(i)}_{T(\delta_0)}(v)|c\rangle$ will result in a linear combination of states, one of which will be $\bar{c}$. This completes the proof.

2. Weakly extraordinary vertices

One highly unpleasant feature of EO vertices is their background dependence. As we require such vertices to lie in the coordinate neighborhood $U_\epsilon(v, \gamma)$ of v, the property that we termed extraordinariness is not a diffeomorphism-covariant notion. That is, if $v^{\rm E}$ is EO with respect to $v \in V(c)$, then it does not imply that $\phi(v^{\rm E})$ is EO with respect to $\phi(v) \in V(\phi \cdot c)$. With this drawback in mind, we introduce a generalization of extraordinary vertices in this section. As we will see later, this generalization will play an important role when we construct a habitat.

Let $\bar{c}$ be a charge network with an EO pair $(v, v^{\rm E})$, where $v^{\rm E}$ is an EO vertex of type $M = 1$, say. Let ϕ be a semi-analytic diffeomorphism of Σ such that $\phi \cdot c = c$ and

[22] We are restricting attention to the $M = 1$ case in this lemma. The proof is exactly analogous for $M \in \{2, 3\}$.

[23] It is easy to see that under this surgery, one ends up with the same charge network c no matter which $\bar{c}^j_K \in \mathcal{C}$ one starts with.

[24] This can always be done as there is enough freedom in choosing the loops underlying the holonomies out of which the Hamiltonian constraint is built [respecting the area constraints mentioned below Eq. (7.14)].

consider the state $\phi \cdot \bar{c}$ such that $(\phi(v), \phi(v^{\rm E}))$ is not an EO pair. In this case we refer to the image of $v^{\rm E}$ in this state as a weakly extraordinary (WEO) vertex. Notice that as ϕ keeps c invariant, $\phi \cdot \bar{c}$ has the same "topological structure" as $\bar{c}$. In particular, such diffeomorphisms cannot change N_v (defined in the previous section). This implies the following:

Vertices which have all the properties stated above except property (7) in Set A or Set B are weakly extraordinary vertices.

We would like to emphasize that the real motivation behind introducing WEO vertices will become clear in [25] where we will analyze the issue of diffeomorphism covariance of the Hamiltonian constraint.

F. Action of a "second" Hamiltonian

1. Hints from the classical theory

As observed in [20] and explained in the introduction, one of the reasons Thiemann's quantum Hamiltonian constraint can never produce a nontrivial commutator (even if one worked with higher density constraints) is due to the fact that it has trivial action on the vertices that it creates. At first sight, it seems like we have run into the same problem. As we saw above, the EO vertices created by $\hat{H}^{(i)}_{T(\delta)}[N]$ are degenerate, whence the action of the Hamiltonian constraint on a state containing an EO pair will act trivially at the EO vertex. It then seems plausible that an analysis similar to the one done in [20] would lead to a trivial continuum commutator. However, following a simple observation in the classical theory tells us how this triviality could be circumvented. The computation done in Sec. VI (which motivated our quantization choices in the construction of $\hat{H}^{(i)}_{T(\delta)}[N]$) demonstrated how the (classical) action of a Hamiltonian constraint could be understood in terms of spatial diffeomorphisms generated by triad fields. The Poisson action of two successive Hamiltonian constraints involves terms which in turn act on these triad fields. More precisely, the triad field E_i has a nonvanishing Poisson bracket with $H^{(i)}[N]$ and is given by (in the density two case)

$$\begin{aligned}\{H^{(i)}[N], E^a_i(x)\} &= -\epsilon^{ijk} E^a_j \partial_a (N E^b_k)(x) \\ &\approx -\epsilon^{ijk}(E^a_j E^b_k \partial_a N)(x) - (N E^a_j \partial_a E^b_k)(x), \qquad (7.19)\end{aligned}$$

where we have used the Gauss constraint $\partial_a E^a_i = 0$. As the Hamiltonian vector field action of $H[N]$ is approximated by a transformation involving a triad-dependent diffeomorphism, as in (6.8), we would like the second Hamiltonian constraint to act nontrivially on an EO vertex via its action on the generator of this diffeomorphism. In essence, this is the action that is captured by the extra term in the Hamiltonian constraint (described below) when it acts on EO vertices. More precisely, the extra term in $\hat{H}^{(i)}_{T(\delta)}[N]$ will induce an action on the EO vertex which will mirror the first term in (7.19) only.[25,26]

2. The action of $\hat{H}_{T(\delta)}[N]$ on EO pairs

Based on the classical insight of the previous section, we modify the definition of $\hat{H}^{(i)}_{T(\delta)}[N]$ such that on any $|c\rangle$ not containing an EO pair, it is still given by (7.16). However, if $|c\rangle$ contains an EO pair, then $\hat{H}^{(i)}_{T(\delta)}[N]$ contains an additional term constructed to mimic (7.19) as it modifies the quantum shift. This term utilizes a dichotomy present between the classical theory and loop quantized quantum field theories, which arises due to the underlying representation of the holonomy-flux algebra.

Consider an edge e and a transversal (codimension one) surface L^δ_e which intersects e in some interior point and whose coordinate length scales with δ. Classically, quadratic functions of fluxes like $E_i(L^\delta_e)E_j(L^\delta_e)$ are higher order in δ than $E_i(L^\delta_e)$, but in the quantum theory,

$$\begin{aligned}\frac{1}{\hbar}\hat{E}^i(L^\delta_e) h^{\vec{n}_e}_e(A) &= n^i_e h^{\vec{n}_e}_e(A), \\ \frac{1}{\hbar^2}\hat{E}_i(L^\delta_e)\hat{E}_j(L^\delta_e) h^{\vec{n}_e}_e(A) &= n^i_e n^j_e h^{\vec{n}_e}_e(A).\end{aligned} \qquad (7.20)$$

Thus, owing to the peculiarity of the holonomy-flux representation, spectra of flux operators do not carry the memory of coordinate area of the underlying surfaces. We interpret this dichotomy as a quantization ambiguity, and it is this ambiguity which we will use to modify $\hat{H}^{(i)}_{T(\delta)}[N]$.

In order to explain the most important nontriviality of the modification, we will first work with density two constraints.[27] Finally we will switch to the density $\frac{5}{4}$ constraint by choosing a particular operator ordering when including $\hat{q}^{-1/4}$. Let us first compute $\hat{H}^{(1)}_{T(\delta')}[M]\hat{H}^{(2)}_{T(\delta)}[N]|c\rangle$, with the constraint operators given in (7.16). Let $v \in V(c)$ be the only vertex which lies inside the support of N and M. Then, suppressing all the factors of $\hbar$,

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[25]This information suffices to obtain an anomaly-free commutator of Hamiltonian constraints, as will be shown in Sec. VII F 2.

[26]It is possible to find a discrete approximant to $X_{H[M]}X_{H[N]}f_c(A)$ which illustrates this point rather clearly. Such a computation will produce terms involving $f_{\phi^\delta_{\vec{V}}\cdot c}(A)$ where $\vec{V}$ is a triad-dependent vector field of the type given in (7.19). However, this computation is rather involved and as our primary motivation for considering such classical computations is merely as guiding tools to make quantization choices, we do not reproduce it here.

[27]Density two constraints, when quantized, should have at finite triangulation an overall factor of $(\delta\epsilon)^{-1}$, where ϵ comes from the regularization of quantum shift. This ϵ is removed when one switches to density $\frac{5}{4}$ constraint since the quantization of $q^{-1/4}$ involves an overall factor of ϵ. Whence we will suppress the factor of $\frac{1}{\epsilon}$ in the density two case, as it is not relevant in the final result.

$$\begin{aligned}
&\hat{H}^{(1)}_{T(\delta')}[M]\hat{H}^{(2)}_{T(\delta)}[N]|c_1, c_2, c_3\rangle \\
&= \frac{1}{\delta}N(v)\hat{H}^{(1)}_{T(\delta')}[M][|c_1, c_2 \cup \alpha_v^{\delta}(\langle \hat{E}_3\rangle_{c_3}, n_{c_1}), c_3\rangle - |c_1, c_2 \cup \alpha_v^{\delta}(\langle \hat{E}_1\rangle_{c_1}, n_{c_3}), c_3\rangle] \\
&= \frac{1}{\delta\delta'}N(v)M(v)[(|c_1 \cup \alpha_v^{\delta'}(\langle \hat{E}_2\rangle_{c_2\cup\alpha_v^{\delta}(\langle \hat{E}_3\rangle)}, n_{c_3}), c_2 \cup \alpha_v^{\delta}(\langle \hat{E}_3\rangle_{c_3}, n_{c_1}), c_3\rangle \\
&\quad - |c_1 \cup \alpha_v^{\delta'}(\langle \hat{E}_3\rangle_{c_3}, n_{c_2\cup\alpha_v^{\delta}(\langle \hat{E}_3\rangle)}), c_2 \cup \alpha_v^{\delta}(\langle \hat{E}_3\rangle_{c_3}, n_{c_1}), c_3\rangle) \\
&\quad - (|c_1 \cup \alpha_v^{\delta'}(\langle \hat{E}_2\rangle_{c_2\cup\alpha_v^{\delta}(\langle \hat{E}_3\rangle)}, n_{c_3}), c_2 \cup \alpha_v^{\delta}(\langle \hat{E}_1\rangle_{c_1}, n_{c_3}), c_3\rangle \\
&\quad - |c_1 \cup \alpha_v^{\delta'}(\langle \hat{E}_3\rangle_{c_3}, n_{c_2\cup\alpha_v^{\delta}(\langle \hat{E}_3\rangle)}), c_2 \cup \alpha_v^{\delta}(\langle \hat{E}_1\rangle_{c_1}, n_{c_3}), c_3\rangle)].
\end{aligned} \tag{7.21}$$

Here n_{c_1} denotes the n^1 charges on the subset of $E(c)$ at v; we suppress the superscript. Let $e \in E(c)$. Given a point $v' \in \mathrm{Int}(e)$ in the interior of the edge, let $L_e^{v'}(\delta)$ be a surface of codimension one (so $L_e^{v'}$ is just a segment which intersects e transversely) whose coordinate length is of the order $\delta' = O(\delta^2)$.[28] Consider a state $|c_1, c_2', c_3\rangle$ which has an EO pair (v, v^{E}) with v^{E} an EO vertex of type $(M = 2, j = 1)$. This state is of the type

$$|c_1, c_2', c_3\rangle = |c_1, c_2 \cup \alpha_v^{\delta_0}(\langle \hat{E}_1\rangle_{c_1}, n_{c_3}), c_3\rangle \tag{7.22}$$

for some fixed δ_0. Our proposal for the action of $\hat{H}^{(1)}_{T(\delta)}[M]$ on $|c_1, c_2 \cup \alpha_v(\langle \hat{E}_1\rangle_{c_1}, n_{c_3}), c_3\rangle$ is as follows (since δ_0 is fixed, we suppress it for the clarity of presentation):

$$\begin{aligned}
&\hat{H}^{(1)}_{T(\delta)}[M]|c_1, c_2 \cup \alpha_v(\langle \hat{E}_1\rangle_{c_1}, n_{c_3}), c_3\rangle \\
&= \sum_{v\in V(c)} M(v)\hat{H}^{(1)}_{T(\delta')}(v)|c_1, c_2 \cup \alpha_v(\langle \hat{E}_1\rangle_{c_1}, n_{c_3}), c_3\rangle \\
&\quad + \delta_{\mathrm{supp}(M), v^{\mathrm{E}}}\epsilon\Bigg(\frac{1}{\delta}\sum_{e\in E(\gamma)} \langle \hat{E}_2(L_e^{v'}(\delta'))\rangle_{c_2}(M(v + \delta\dot{e}(0)) - M(v))|c_1, c_2 \cup \alpha_v(\langle \hat{E}_3\rangle_{c_3}, n_{c_3}), c_3\rangle \\
&\quad - \frac{1}{\delta}\sum_{e\in E(\gamma)} \langle \hat{E}_3(L_e^{v'}(\delta'))\rangle_{c_3}(M(v + \delta\dot{e}(0)) - M(v))|c_1, c_2 \cup \alpha_v(\langle \hat{E}_2\rangle_{c_2}, n_{c_3}), c_3\rangle\Bigg),
\end{aligned} \tag{7.23}$$

where

(a) the first term is the unmodified action coming from (7.16);
(b) the second term is the proposed modification which is designed to capture the displacement of EO vertex as motivated from (7.19);
(c) ϵ is a numerical coefficient which we will choose to be 1 and as we will see later, with this value, the off-shell closure condition is satisfied; and
(d) $\delta_{\mathrm{supp}(M), v^{\mathrm{E}}} = 1$ if v^{E} lies inside the support of M, and is zero otherwise.

We need to show that

(1) there exists an operator which when acting on $|c\rangle$ accomplishes (7.23); and
(2) the continuum limit of the classical function which is quantized to this operator should yield the familiar classical Hamiltonian.

We proceed by defining an operator which yields (7.23) and then argue that it differs from the unmodified operator by terms subleading in δ, thus showing that it has the correct classical continuum limit. Once again we assume that v^{E} is inside the support of the lapse M. If it lies outside the support then the modification is absent by definition.

[28]As we have a length scale in the theory $\kappa\hbar$, one could use it to define $\delta' = \frac{\delta^2}{\kappa\hbar}$.

$$\hat{H}^{(1)}_{T(\delta)}[M]|c_1, c_2 \cup \alpha_v(\langle\hat{E}_1\rangle_{c_1}, n_{c_3}), c_3\rangle$$
$$= \frac{1}{\delta}M(v)\Big[(\hat{h}^{(1)}_{\alpha_v^\delta(\langle\hat{E}_2\rangle_{c_2\cup\alpha_v(\langle\hat{E}_1\rangle_{c_1},n_{c_3})},n_{c_3})}) - (\hat{h}^{(1)}_{\alpha_v^\delta(\langle\hat{E}_3\rangle_{c_3},n_{c_2}-n_{c_3})})\Big]|c_1, c_2 \cup \alpha_v(\langle\hat{E}_1\rangle_{c_1}, n_{c_3}), c_3\rangle$$
$$+ \frac{1}{\kappa\hbar\delta}\Big(\sum_{e\in E(\gamma)}\langle\hat{E}_2(L_e^{v'}(\delta'))\rangle_{c_2}(M(v+\delta\dot{e}(0)) - M(v))\hat{h}^{(2)}_{\alpha_v(\langle\hat{E}_3\rangle_{c_3},n_{c_3})}(\hat{h}^{(2)}_{\alpha_v(\langle\hat{E}_1\rangle_{c_1},n_{c_3})})^{-1}$$
$$- \sum_{e\in E(\gamma)}\langle\hat{E}_3(L_e^{v'}(\delta'))\rangle_{c_3}(M(v+\delta\dot{e}(0)) - M(v))\hat{h}^{(2)}_{\alpha_v(\langle\hat{E}_2\rangle_{c_2},n_{c_3})}(\hat{h}^{(2)}_{\alpha_v(\langle\hat{E}_1\rangle_{c_1},n_{c_3})})^{-1}\Big)|c_1, c_2 \cup \alpha_v(\langle\hat{E}_1\rangle_{c_1}, n_{c_3}), c_3\rangle. \quad (7.24)$$

The extra terms in the last two lines of (7.24) are subleading in δ as compared to the first (unmodified) term. This can be seen as follows. The unmodified operator $\hat{H}^{(1)}_{T(\delta)}(v)$ is an operator of the form $\frac{1}{\delta}[\hat{h}_{\alpha(\delta)} - \hat{h}^{-1}_{\alpha(\delta)}]$ and hence $O(\delta)$. The second and third terms are of the form $\frac{1}{\delta}\hat{E}_i(L_e(\delta'))(M(v+\delta) - M(v))$ and hence to leading order in δ they are $O(\delta') = O(\delta^2)$.

These finite-triangulation operators, due to the structure of the extra terms, are nonlocal in the sense that they can never be perceived as (quantum counterparts of) the discretization of a classical local functional. A similar feature was observed in the correction to the fundamental LQG curvature operator, that was defined in [21] and led to an anomaly-free quantization of the diffeomorphism constraint. Nonetheless, as we will see later, the continuum limit of the Hamiltonian constraint operator will be local in the sense that it will be expressed in terms of local differential operators.

This then is our proposal for the density two $\hat{H}^{(1)}_{T(\delta)}[N]$ when it acts on a state containing an EO pair (v, v^{E}) with v^{E} being an EO vertex of type $(K \in \{\mathrm{A}, \mathrm{B}\}, M = 2, j = 1)$ [i.e., it is either a type A or type B vertex with all incident edges charged only in $\mathrm{U}(1)_2$, located at a position determined by the $j = 1$ quantum shift, with the charge magnitudes coming from the $\mathrm{U}(1)_3$ labels on edges incident at vertices in $\bar{\mathcal{S}}_{v^{\mathrm{E}}}$]. Other cases can be considered similarly. We now modify our results appropriately for the realistic case of density $\frac{5}{4}$ constraint. As we remarked earlier, this amounts to choosing a particular operator ordering for $\hat{q}^{-1/4}$, which is a scalar multiple of the identity operator on any charge network state. The ordering we choose is given by

$$\hat{H}^{(1)}_{T(\delta)}[M]|c_1, c_2 \cup \alpha_v(\langle\hat{E}_1\rangle_{c_1}, n_{c_3}), c_3\rangle$$
$$= \frac{1}{\delta}M(v)\langle\hat{q}(v)^{-1/4}\rangle_c\Big[(\hat{h}^{(1)}_{\alpha_v^\delta(\langle\hat{E}_2\rangle_{c_2\cup\alpha_v(\langle\hat{E}_1\rangle_{c_1},n_{c_3})},n_{c_3})}) - (\hat{h}^{(1)}_{\alpha_v^\delta(\langle\hat{E}_3\rangle_{c_3},(n_{c_2}-n_{c_3})})\Big]|c_1, c_2 \cup \alpha_v(\langle\hat{E}_1\rangle_{c_1}, n_{c_3}), c_3\rangle$$
$$+ \frac{1}{\kappa\hbar\delta}\Big(\sum_{e\in E(\gamma)}\langle\hat{E}_2(L_e^{v'}(\delta'))\rangle_{c_2}(M(v+\delta\dot{e}(0)) - M(v))\hat{h}^{(2)}_{\alpha_v(\langle\hat{E}_3\rangle_{c_3},n_{c_3})})\hat{q}(v)^{-1/4}(\hat{h}^{(2)}_{\alpha_v(\langle\hat{E}_1\rangle_{c_1},n_{c_3})})^{-1}$$
$$- \sum_{e\in E(\gamma)}\langle\hat{E}_3(L_e^{v'}(\delta'))\rangle_{c_3}(M(v+\delta\dot{e}(0)) - M(v))\hat{h}^{(2)}_{\alpha_v(\langle\hat{E}_2\rangle_{c_2},n_{c_3})}\hat{q}(v)^{-1/4}(\hat{h}^{(2)}_{\alpha_v(\langle\hat{E}_1\rangle_{c_1},n_{c_3})})^{-1}\Big)|c_1, c_2 \cup \alpha_v(\langle\hat{E}_1\rangle_{c_1}, n_{c_3}), c_3\rangle. \quad (7.25)$$

Thus we finally have a definition of the Hamiltonian constraint operator on an arbitrary charge network state. If the charge-network contains an EO pair then the constraint operator has an additional piece which is nonlocal and can be thought of as having a nontrivial action on the EO pair rather then acting on a single isolated vertex. The complete implications of having an operator which at finite triangulation not only changes charge network in the neighborhood of a single vertex but also changes it in the neighborhood of a subgraph are not clear to us.

VIII. LMI HABITAT

In the previous section we completed the construction of the Hamiltonian constraint operator at finite triangulation, which is densely defined on $\mathcal{H}_{\mathrm{kin}}$. As is well known, due to the higher density weight of the operator, it will not have a continuum limit (in any operator topology that we know of) which is well defined on $\mathcal{H}_{\mathrm{kin}}$. In this section we construct an arena which we call the Lewandowski-Marolf-inspired habitat, on which the net of finite-triangulation operators admits a continuum limit. We will come back to the issue of operator topology later in the section. First we engineer a habitat taking a cue from Lewandowski and Marolf's seminal construction [19].

We want to build our habitat in such a way that not only does it admit some sort of continuum limit of the Hamiltonian constraint, but that it admits a representation of the entire Dirac algebra. We build our habitat keeping this requirement in mind. Starting with a charge network c which has no monocolored vertex, construct a set $[c_1, c_2, c_3]_{(i)}$ as follows:

$$[c]_{(i=1)} = [c_1, c_2, c_3]_{(i=1)} = \{c\} \cup \bigcup_{c_1'} \{(c_1', c_2, c_3)\}, \tag{8.1}$$

where (c_1', c_2, c_3) has at least one additional weakly extraordinary (WEO) vertex as compared to c_1. $[c]_{(2)}$ and $[c]_{(3)}$ are defined similarly. Now we consider the following type of elements of Cyl*:

$$\Psi^{f^{(i)}}_{[c]_{(i)}} = \sum_{(c_1', c_2, c_3) \in [c]_{(i)}} f^{(i)}(\bar{V}(c_1' \cup c_2 \cup c_3)) \langle c_1', c_2, c_3|, \tag{8.2}$$

where

(i) $f^{(i)}$, $i = 1, 2, 3$ are smooth functions on $\Sigma^{|V(c)|}$;

(ii) $\bar{V}(c_1' \cup c_2 \cup c_3)$ is defined as follows: Let $V(c_1' \cup c_2 \cup c_3) = \{v_0, v_0^{\rm WE}, \ldots, v_K, v_K^{\rm WE}, v_{K+1}, \ldots, v_N\}$, where $\{v_0^{\rm WE}, \ldots, v_K^{\rm WE}\}$ are WEO vertices of $c_1' \cup c_2 \cup c_3$ associated to $\{v_0, \ldots, v_K\} \subset V(c)$, respectively. Then

$$\bar{V}(c_1' \cup c_2 \cup c_3) := \{v_0^{\rm WE}, \ldots, v_K^{\rm WE}, v_{K+1}, \ldots, v_M\}. \tag{8.3}$$

Note that by construction $|\bar{V}(c_1' \cup c_2 \cup c_3)| = |V(c)|$ so that $f^{(i)}$ are functions on $\Sigma^{|V(c)|}$.

We define $\mathcal{V}_{\rm LMI}$ as a subspace of Cyl* spanned by distributions of the type $\Psi^{f^{(i)}}_{[c]_{(i)}}$.

We will now show that $\hat{H}^{(i)}_{T(\delta)}[N]$ admits a continuum limit as a linear operator from $\mathcal{V}_{\rm LMI} \to$ Cyl*. The topology on the space of operators in which we consider the continuum limit is defined via the following family of seminorms: Given any pair $(\Psi, |c\rangle) \in \mathcal{V}_{\rm LMI} \times \mathcal{H}_{\rm kin}$, we say that $\hat{H}^{(i)}[N]'$ is a continuum limit of $\hat{H}^{(i)}_{T(\delta)}[N]$ if for $\epsilon > 0$, $\exists\, \delta_0 = \delta_0(\Psi, c, N)$ such that

$$|(\hat{H}^{(i)}[N]'\Psi)|c\rangle - \Psi(\hat{H}^{(i)}_{T(\delta)}[N]|c\rangle)| < \epsilon \tag{8.4}$$

$\forall\, \delta < \delta_0$ (we will generally decorate operators acting on elements of Cyl* with a prime). It turns out that the continuum Hamiltonian constraint does not preserve the LMI habitat; rather

$$\hat{H}^{(i)}[N]' \colon \mathcal{V}_{\rm LMI} \to {\rm Cyl}^*. \tag{8.5}$$

This happens because when acting on a state, say $\Psi^{f^{(1)}}_{[c]_{(1)}} \in \mathcal{V}_{\rm LMI}$, the resulting states are still infinite linear combinations of (duals of) charge network states, with amplitudes being functions of vertices. However, in contrast to $f^{(1)}$ which is smooth, coefficients of the charge networks in these linear combinations will be discontinuous functions.

A. The continuum limit

Consider $\Psi^{f^{(1)}}_{[c]_{(1)}}$, where $f^{(1)} \colon \Sigma^{|V(c)|} \to \mathbb{R}$. The action of the continuum Hamiltonian constraint $\hat{H}^{(1)}[N]' + \hat{H}^{(2)}[N]' + \hat{H}^{(3)}[N]'$ on such states can be deduced from Eqs. (8.8), (8.10), and (8.14) that are given below. Derivations of these results can be found in Appendix D.

We first consider the action of $\hat{H}^{(1)}[N]'$ on $\Psi^{f^1}_{[c_1,c_2,c_3]_{(1)}}$:

$$\hat{H}^{(1)}[N]'\Psi^{f^{(1)}}_{[c]_{(1)}} = \sum_{v \in V(c)} \Big[\Psi^{\bar{f}^{(1)(1)}_v}_{[c]_{(1)}} - \Psi^{\bar{\bar{f}}^{(1)(1)}_v}_{[c]_{(1)}}\Big], \tag{8.6}$$

where[29] $\bar{f}^{(1)(1)}_v$ is given by (see below for $\bar{\bar{f}}^{(1)(1)}_v$)

$$\bar{f}^{(1)(1)}_v(v_1, \ldots, v_{|V(c)|}) = f^{(1)}(v_1, \ldots, v_{|V(c)|}) \tag{8.7}$$

if the following hold:

(1) $\{v_1,\ldots,v_{|V(c)|}\} \neq V(c_1' \cup c_2 \cup c_3)$ for any $(c_1', c_2, c_3) \in [c]_{(1)}$, or

(2) $\{v_1,\ldots,v_{|V(c)|}\} = V(c_1' \cup c_2 \cup c_3)$ for some $(c_1', c_2, c_3) \in [c]_{(1)}$ but $v \notin \{v_1, \ldots, v_{|V(c)|}\}$.

In the case that the complements of (1) and (2) hold, we have

$$\bar{f}^{(1)(1)}_v(v_1, \ldots, v_{|V(c)|}) = N(v)\lambda(\vec{n}^c_v)\langle \hat{E}^a_2(v)\rangle_{c_2} \frac{\partial}{\partial v^a} f^{(1)}(v_1, \ldots, v_{|V(c)|}). \tag{8.8}$$

$\bar{\bar{f}}^{(1)(1)}_v$ is defined analogously, except that $\langle \hat{E}^a_2(v)\rangle_{c_2}$ in (8.8) is replaced by $\langle \hat{E}^a_3(v)\rangle_{c_3}$.

We now consider the action of $\hat{H}^{(2)}[N]'$ on $\Psi^{f^{(1)}}_{[c_1,c_2,c_3]_{(1)}}$:

$$\hat{H}^{(2)}[N]'\Psi^{f^{(1)}}_{[c]_{(1)}} = \sum_{v \in V(c)} [\Psi^{\bar{f}^{(1)(2)}_v}_{[c]_{(1)}} - \Psi^{\bar{\bar{f}}^{(1)(2)}_v}_{[c]_{(1)}}], \tag{8.9}$$

where $\bar{f}^{(1)(2)}_v$ and $\bar{\bar{f}}^{(1)(2)}_v$ are defined as follows: Let $\{v_1, \ldots, v_{|V(c)|}\} = V(c_1' \cup c_2 \cup c_3)$ such that

(1) $(c_1' \cup c_2 \cup c_3) \in [c]_{(1)}$;

(2) $v \in \{v_1, \ldots, v_{|V(c)|}\}$ such that $v \in V(c)$ and there is an EO vertex $v^{\rm E}$ associated to v of type $(M=1, j=2)$ which lies inside the support of N. In this case

$$\bar{f}^{(1)(2)}_v(\bar{V}(c_1' \cup c_2 \cup c_3)) = \Big[\sum_{e \in E(c)|b(e)=v} \langle \hat{E}_1(L_e)\rangle_{c_1} \dot{e}^a(0)\partial_a N(v)\Big] f^{(1)}(\bar{V}(c_1' \cup c_2 \cup c_3)), \tag{8.10a}$$

$$\bar{\bar{f}}^{(1)(2)}_v(\bar{V}(c_1' \cup c_2 \cup c_3)) = \Big[\sum_{e \in E(c)|b(e)=v} \langle \hat{E}_3(L_e)\rangle_{c_3} \dot{e}^a(0)\partial_a N(v)\Big] f^{(1)}(\bar{V}(c_1' \cup c_2 \cup c_3)), \tag{8.10b}$$

where L_e is as defined in Eq. (D20).

[29]To avoid notational clutter, we do not explicitly indicate the dependence of $\bar{\bar{f}}^{(1)(1)}_v$ etc. on N, (c_1, c_2, c_3).

In the case where the complement of the two conditions (1) and (2) hold, we have

$$\bar{f}_v^{(1)(2)}(v_1, \ldots, v_{|V(c)|}) = f^{(1)}(v_1, \ldots, v_{|V(c)|}), \tag{8.11}$$

$$\bar{\bar{f}}_v^{(1)(2)}(v_1, \ldots, v_{|V(c)|}) = f^{(1)}(v_1, \ldots, v_{|V(c)|}). \tag{8.12}$$

The action of $\hat{H}^{(3)}[N]'$ on $\Psi^{f^{(1)}}_{[c]_{(1)}}$ can be written in analogy with (8.9):

$$\hat{H}^{(3)}[N]'\Psi^{f^{(1)}}_{[c]_{(1)}} = \sum_{v\in V(c)} \Big[\Psi^{\bar{f}_v^{(1)(3)}}_{[c]_{(1)}} - \Psi^{\bar{\bar{f}}_v^{(1)(3)}}_{[c]_{(1)}}\Big], \tag{8.13}$$

where $\bar{f}_v^{(1)(3)}$ and $\bar{\bar{f}}_v^{(1)(3)}$ are defined as follows: Let $\{v_1, \ldots, v_{|V(c)|}\} = V(c_1' \cup c_2 \cup c_3)$ such that

(1) $(c_1' \cup c_2 \cup c_3) \in [c]_{(1)}$;

(2) $v \in \{v_1, \ldots, v_{|V(c)|}\}$ such that $v \in V(c)$ and there is an EO vertex v^{E} associated to v of type $(M = 1, j = 3)$. In this case

$$\bar{f}_v^{(1)(3)}(\bar{V}(c_1' \cup c_2 \cup c_3)) = \Big[\sum_{e\in E(c)|b(e)=v} \langle \hat{E}_2(L_e)\rangle_{c_2} \dot{e}^a(0)\partial_a N(v)\Big] f^{(1)}(\bar{V}(c_1' \cup c_2 \cup c_3)), \tag{8.14a}$$

$$\bar{\bar{f}}_v^{(1)(3)}(\bar{V}(c_1' \cup c_2 \cup c_3)) = \Big[\sum_{e\in E(c)|b(e)=v} \langle \hat{E}_1(L_e)\rangle_{c_1} \dot{e}^a(0)\partial_a N(v)\Big] f^{(1)}(\bar{V}(c_1' \cup c_2 \cup c_3)). \tag{8.14b}$$

As before, if the set $\{v_1, \ldots, v_{|V(c)|}\}$ does not satisfy conditions (1) or (2), then the two functions $\bar{f}_v^{(1)(3)}$ and $\bar{\bar{f}}_v^{(1)(3)}$ take the same value as $f^{(1)}$.

The definitions of $\bar{f}_v^{(1)(i)}|_{i=1,2,3}$ make it rather clear that the continuum Hamiltonian constraint does not preserve the LMI habitat. These functions have a discontinuity as soon as one of their arguments is the vertex v. This discontinuity is due to the discontinuous nature of the quantum shift vector, which is in turn tied to the choice of representation we are forced upon in LQG.

B. The action of the Hamiltonian constraint on irrelevant vertices

Before we compute the continuum limit of the commutator of two (regularized) Hamiltonian constraints, we make two observations which vastly simplify the structure of the computation (and indeed, without which, $\hat{H}[N]'$ will not satisfy the off-shell closure condition). These observations are related to the action of a Hamiltonian constraint on a charge network state which lies in the image of $\hat{H}_T^{(i)}[N]$ for some $i \in \{1, 2, 3\}$, N, and T.

The action of such a finite triangulation Hamiltonian constraint on a charge network which has no EO vertex, creates a set of vertices which we termed irrelevant vertices (the name finds its justification in this section). When a finite-triangulation Hamiltonian acts on a trivalent irrelevant vertex, it vanishes (as all such trivalent vertices are in the kernel of a $\hat{q}^{-1/4}$ operator). Whence these vertices are irrelevant as far as the action of a second Hamiltonian on such a charge network is concerned. This is not quite true for the four-valent irrelevant vertices.[30] However, we now argue that the continuum limit of the action of a finite-triangulation Hamiltonian constraint on a four-valent irrelevant vertex is trivial. This feature is tied to the choice of our habitat [or more precisely to the definition of $[c]_{(i)}$].

Let $(\tilde{c}_1, c_2, c_3)$ be a charge-network with an EO vertex v_0^{E}, which for the sake of concreteness we consider to be of type $(M = 1, j = 2)$. That is,

$$(\tilde{c}_1, c_2, c_3) = (c_1 \cup \alpha_{v_0}^{\delta_0}(\langle\hat{E}_2(v_0)\rangle_{c_2}, n_{c_3}), c_2, c_3), \tag{8.15}$$

where c does not have any EO vertices and where v_0^{E} is associated with v_0. There is a set of irrelevant vertices in $V(\tilde{c}_1 \cup c_2 \cup c_3)$, and let us denote this set by $\{v_1^{v_0}, \ldots, v_k^{v_0}\}$. Let us consider one of them, say $v_1^{v_0}$ and let the four edges incident on v_1 be $e^1_{v_1^{v_0}}, \ldots, e^4_{v_1^{v_0}}$ such that $(e^1_{v_1^{v_0}}, e^3_{v_1^{v_0}})$ and $(e^2_{v_1^{v_0}}, e^4_{v_1^{v_0}})$ are analytic pairs. Let us assume that $(e^1_{v_1^{v_0}}, e^3_{v_1^{v_0}})$ are charged only under $\mathrm{U}(1)_1$, whence a simple computation shows that $\hat{H}^{(1)}_{T(\delta')}(v_1^{v_0})$ acting on $|\tilde{c}_1, c_2, c_3\rangle$ vanishes. However this is not true for $\hat{H}^{(2)}_{T(\delta')}(v_1^{v_0})$ or $\hat{H}^{(3)}_{T(\delta')}(v_1^{v_0})$. Their action will produce EO pairs $(v_1^{v_0}, (v_1^{v_0})^{\mathrm{E}})$ which are of type $(M = 2)$ or type $(M = 3)$. Thus the action of the Hamiltonian constraint on $|\tilde{c}_1, c_2, c_3\rangle$ produces a state which has a vertex v_0^{E} charged in $\mathrm{U}(1)_1$ and a vertex $(v_1^{v_0})^{\mathrm{E}}$ charged in $\mathrm{U}(1)_2$. As there exists no set

[30] It is important to note that our entire construction, when generalized to three dimensions, would generically be free of such four-valent vertices.

$[c']_{(i)}$ in which there is ever a charge network with two monocolored vertices charged in different $U(1)_i$, we have

$$\Psi^{f^{(i)}}_{[c']}(\hat{H}^{(i)}_{T(\delta)}[N]|\tilde{c}_1, c_2, c_3\rangle) = 0 \tag{8.16}$$

$\forall\ \delta > 0$ and $\forall\ i$. Hence we will ignore the action of the Hamiltonian constraint on irrelevant vertices in what follows.

IX. COMMUTATOR OF TWO HAMILTONIAN CONSTRAINTS

In this section we embark upon the key computation performed in this paper. We argue that the quantum Hamiltonian constraint that we have obtained above has the right basic ingredients to achieve an anomaly-free representation of the Dirac algebra. We will show that the commutator between two Hamiltonian constraints is, in a precise sense, *a* quantization of the right-hand side of the corresponding classical Poisson bracket.

Let us first describe what it is that we want to show. Recall that

$$\{H[M], H[N]\} = V[\vec{\omega}],\\ \omega^a := q^{-1/2}E^a_i E^b_i(N\partial_b M - M\partial_b N). \tag{9.1}$$

Our aim is to show that the above equality holds at the quantum level. That is, schematically we want to prove that

$$[\hat{H}[M], \hat{H}[N]] = i\hbar\hat{V}[\vec{\omega}]. \tag{9.2}$$

Our strategy will be the following. As the continuum Hamiltonian constraint does not preserve the habitat $\mathcal{V}_{\mathrm{LMI}}$, but maps it elements into elements of Cyl*, the commutator of two continuum Hamiltonians does not make sense on $\mathcal{V}_{\mathrm{LMI}}$. However, things are not as bad as they look. Let us assume for a moment an ideal scenario where we had a habitat $\mathcal{V}_{\mathrm{grand}}$ on which any product of a finite number of continuum Hamiltonian constraints is a well-defined operator. Then $\forall\ \Psi \in \mathcal{V}_{\mathrm{grand}}$, we would have

$$\begin{aligned}\sum_{i,j}([\hat{H}^{(i)}[M]', \hat{H}^{(j)}[N]']\Psi)|c_1, c_2, c_3\rangle &= \sum_{i,j}((\hat{H}^{(i)}[M]'\hat{H}^{(j)}[N]' - (M \leftrightarrow N))\Psi)|c_1, c_2, c_3\rangle\\ &= \sum_{i,j}\lim_{\delta\to 0}\lim_{\delta'\to 0}\Psi(\hat{H}^{(j)}_{T(\delta')}[N]\hat{H}^{(i)}_{T(\delta)}[M]\\ &\quad - (M \leftrightarrow N)))|c_1, c_2, c_3\rangle.\end{aligned} \tag{9.3}$$

As we show below, the right-hand side of this equation is well defined and constitutes a definition of the *continuum commutator*:

$$\begin{aligned}([\hat{H}[M], \hat{H}[N]]'\Psi)|c\rangle &:= \sum_{i,j}\lim_{\delta\to 0}\lim_{\delta'\to 0}\Psi((\hat{H}^{(j)}_{T(\delta')}[N]\hat{H}^{(i)}_{T(\delta)}[M]\\ &\quad - (M \leftrightarrow N))|c\rangle).\end{aligned} \tag{9.4}$$

In light of (9.3), the equality in (9.2) amounts to proving that

$$\begin{aligned}\sum_{i,j}\lim_{\delta\to 0}\lim_{\delta'\to 0}\Psi(\hat{H}^{(j)}_{T(\delta')}[N]\hat{H}^{(i)}_{T(\delta)}[M] - (M \leftrightarrow N))|c\rangle\\ = \lim_{\delta''\to 0}(-i\hbar)\Psi(\hat{V}_{T(\delta'')}[\vec{\omega}]|c\rangle)\end{aligned} \tag{9.5}$$

$\forall\ \Psi \in \mathcal{V}_{\mathrm{LMI}}$, $\forall\ (c_1, c_2, c_3)$.

The minus sign on the RHS of (9.5) may seem surprising but it arises due to the argument given in Appendix E.

The strategy used proving (9.5) will be as follows:

(1) The first step in obtaining the continuum commutator on $\mathcal{V}_{\mathrm{LMI}}$ is computing $\sum_{i,j}[\hat{H}^{(i)}_{T(\delta')}[N]\hat{H}^{(j)}_{T(\delta)}[M] - (M \leftrightarrow N)]|c'\rangle$ for any charge network state $|c'\rangle$. As we show in Sec. IX A, the regularized commutator vanishes $\forall\ \delta,\ \delta'$ if $\mathrm{supp}(N) \cap \mathrm{supp}(M) = \varnothing$. For the case when $\mathrm{supp}(N) \cap \mathrm{supp}(M) \neq \varnothing$, the computation is slightly more involved and details are provided in Appendix F.

(2) In Section IX B we use the results of Appendix F to derive the continuum limit of the right-hand side of Eq. (9.5).

(3) In Sec. IX C, we define $\hat{V}_{T(\delta,\delta')}[\vec{\omega}]$ such that $\hat{V}_{T(\delta,\delta')}[\vec{\omega}]|c'\rangle$ precisely equals the relevant terms[31] in

$$\sum_{i,j}[\hat{H}^{(i)}_{T(\delta')}[N]\hat{H}^{(j)}_{T(\delta)}[M] - (M \leftrightarrow N)]|c'\rangle. \tag{9.6}$$

This will finally lead us to our main result.

A. Analyzing the case when supp(N) ∩ supp(M) = ∅

Consider a state $|c'\rangle$ such that the only vertices in $V(c')$ which lie in the support of N and M, respectively are, v_N and v_M. Then for any i, j, we have schematically

$$\begin{aligned}&[\hat{H}^{(i)}_{T(\delta')}[N]\hat{H}^{(j)}_{T(\delta)}[M] - (M \leftrightarrow N)]|c'\rangle\\ &= \frac{1}{\delta\delta'}[\hat{O}^{(i)}_{T(\delta')}(v_N)\hat{O}^{(j)}_{T(\delta)}(v_M)(\mathcal{F}[N; v_N)\mathcal{G}[M; v_M)\\ &\quad - (M \leftrightarrow N))]|c'\rangle,\end{aligned} \tag{9.7}$$

where $\mathcal{F}[N; v_N)$ and $\mathcal{G}[M; v_M)$ are in general local functionals of N and M, evaluated at v_N and v_M, respectively.[32] Locality of these functionals implies that

[31] By relevant terms we mean those states which do not vanish once they are "dotted" with a state in the habitat.

[32] The precise form of these functionals depends on the nature of $|c'\rangle$, but what is important for our purposes is that they can at most involve the first derivatives of the lapses evaluated at v_N or v_M.

$$[\hat{H}^{(i)}_{T(\delta')}[N]\hat{H}^{(j)}_{T(\delta)}[M] - (M \leftrightarrow N)]|c'\rangle = 0 \tag{9.8}$$

$\forall\ \delta, \delta'$ and $\forall\ |c'\rangle \in \mathcal{H}_{\rm kin}$. Whence for any state in $\mathcal{V}_{\rm LMI}$ we have

$$\lim_{\delta\to 0}\lim_{\delta'\to 0} \Psi^{f^{(k)}}_{[c]_{(k)}}([\hat{H}^{(i)}_{T(\delta')}[N]\hat{H}^{(j)}_{T(\delta)}[M] - (M \leftrightarrow N)]|c'\rangle) = 0. \tag{9.9}$$

This implies that as long as N, M have nonintersecting supports,

$$[\hat{H}[M], \hat{H}[N]]\Psi^{f^{(k)}}_{[c]_{(k)}} = 0. \tag{9.10}$$

B. Analyzing the case when supp(N) ∩ supp(M) ≠ ∅

Let $\Psi^{f^{(i)}}_{[c]_{(i)}}$ be such that there exists a single vertex in $V(c)$ which falls inside $\text{supp}(N) \cap \text{supp}(M)$. The case where more than one element of the vertex set $V(c)$ falls in the overlap region is a straightforward generalization of the analysis given here. In this case, in order to evaluate

$$\Psi^{f^{(i)}}_{[c]_{(i)}}\left(\sum_{i,j}[\hat{H}^{(i)}_{T(\delta')}[N]\hat{H}^{(j)}_{T(\delta)}[M] - (M \leftrightarrow N)]|c'\rangle\right), \tag{9.11}$$

it suffices to consider only those states $|c'\rangle$ which have only one vertex in the support of $V(c)$, so we first consider a state $|c'\rangle$ with precisely one vertex $v_0 \in V(c') \cap \text{supp}(N) \cap \text{supp}(M)$. As shown in Appendix F, we have the following:

$$\sum_{i\neq j}[\hat{H}^{(i)}_{T(\delta')}[N]\hat{H}^{(j)}_{T(\delta)}[M] - (M \leftrightarrow N)]|c'_1, c'_2, c'_3\rangle = |\psi^{\delta,\delta'}_1(v_0, c', [M,N])\rangle + |\psi^{\delta,\delta'}_2(v_0, c', [M,N])\rangle + |\psi^{\delta,\delta'}_3(v_0, c', [M,N])\rangle, \tag{9.12}$$

where the $|\psi^{\delta,\delta'}_i(v_0, c', [M,N])\rangle$ are given in Appendix F in (F9)–(F11), respectively.

Claim.—$\lim_{\delta,\delta'\to 0}\Psi^{f^{(i)}}_{[c]_{(i)}}(|\psi^{\delta,\delta'}_j(v_0, c', [N,M])\rangle) = 0$ $\forall\ i \neq j, c, c'$.

Proof.—The proof is straightforward, since $|\psi^{\delta,\delta'}_j(v_0, c', [N,M])\rangle$ is a linear combination of four states, and each contains (with respect to c') one EO state of type j. Thus clearly these states are orthogonal to all states in $[c]_{(i)}$ for any c as long as $i \neq j$. This completes the proof.

Then

$$\lim_{\delta,\delta'\to 0}\Psi^{f^{(i)}}_{[c]_{(i)}}\left(\sum_{j\neq k}[\hat{H}^{(j)}_{T(\delta')}[N]\hat{H}^{(k)}[M] - (M \leftrightarrow N)]|c'\rangle\right) = \lim_{\delta,\delta'\to 0}\Psi^{f^{(i)}}_{[c]_{(i)}}(|\psi^{\delta,\delta'}_i(v_0, c', [M,N])\rangle). \tag{9.13}$$

Without loss of generality, we consider the $i = 1$ case.

Lemma.—$\forall\ c', c, N, M$,

$$\lim_{\delta,\delta'\to 0}\Psi^{f^{(1)}}_{[c]_{(1)}}(|\psi^{\delta,\delta'}_1(v_0, c', [M,N])\rangle) = (\Psi^{f^{(1)(3,1)}_{v_0}[M,N]}_{[c]_{(1)}} - \Psi^{f^{(1)(1,3)}_{v_0}[M,N]}_{[c]_{(1)}} - \Psi^{f^{(1)(1,2)}_{v_0}[M,N]}_{[c]_{(1)}} + \Psi^{f^{(1)(2,1)}_{v_0}[M,N]}_{[c]_{(1)}})|c'\rangle, \tag{9.14}$$

where $\Psi^{f^{(1)(i,j)}_{v_0}[M,N]}_{[c]_{(1)}} \notin \mathcal{V}_{\rm LMI}$ are distributions with vertex functions $f^{(1)(i,j)}_{v_0}[M,N]$: $\Sigma^{|V(c)|} \to \mathbb{R}$ defined as

$$f^{(1)(i,j)}_{v_0}[M,N](v_1, \ldots, v_{|V(c)|}) = f^{(1)}(v_1, \ldots, v_{|V(c)|}) \tag{9.15}$$

if the one of the following holds:

(1) $\{v_1, \ldots, v_{|V(c)|}\} \neq V(c'_1 \cup c_2 \cup c_3)$ for any $(c'_1, c_2, c_3) \in [c]_{(1)}$, or

(2) $\{v_1, \ldots, v_{|V(c)|}\} = V(c'_1 \cup c_2 \cup c_3)$ for some $(c'_1, c_2, c_3) \in [c]_{(1)}$ but $v_0 \notin \{v_1, \ldots, v_{|V(c)|}\}$.

In the case that the complements of (1) and (2) hold, we have

$$f^{(1)(i,j)}_{v_0}[M,N](v_1, \ldots, v_{|V(c)|}) = \frac{1}{4}\left(\frac{\hbar}{\mathrm{i}}\right)^2 \lambda^2(\vec{n}^c_v)\sigma(1;i,j)\left(\sum_{e\in E(c)}\langle \hat{E}_i(L(e))\rangle \dot{e}^a(0) \times (M(v_0)\partial_a N(v_0) - N(v_0)\partial_a M(v_0))\right)\langle \hat{E}^b_j(v_0)\rangle \times \frac{\partial}{\partial v^a} f^{(1)}(v_1, \ldots, v_{|V(c)|}), \tag{9.16}$$

where

$$\sigma(1;i,j) = \begin{cases} 0, & i = j \\ 0, & \text{at least one of } i \text{ or } j \neq 1 \\ +1, & i \neq j = 1 \\ -1, & i = 1 \neq j \end{cases}. \tag{9.17}$$

The proof is exactly analogous to the proof for the continuum limit of $\hat{H}^{(1)}_{T(\delta)}[N]$ on $\mathcal{V}_{\rm LMI}$, hence we do not give further details here.

Thus finally, in the topology that we have put on the space of (finite-triangulation) operators, the continuum limit of the commutator is as follows:

$$[\hat{H}[M], \hat{H}[N]]'\left(\Psi^{f^{(1)}}_{[c]_{(1)}} + \Psi^{f^{(2)}}_{[c]_{(2)}} + \Psi^{f^{(3)}}_{[c]_{(3)}}\right) = \sum_{i=1}^{3}\sum_{j\neq i}\left(\Psi^{f^{(i)(j,i)}_{v_0}[M,N]}_{[c]_{(i)}} - \Psi^{f^{(i)(i,j)}_{v_0}[M,N]}_{[c]_{(i)}}\right). \tag{9.18}$$

$f^{(i)(j,i)}_{v_0}[M,N]$ and $f^{(i)(i,j)}_{v_0}[M,N]$ are defined above in (9.16) for $i = 1$. For $i = 2, 3$ they are defined similarly. We remind the reader that our analysis has been restricted to $\mathcal{K} := V(c) \cap \text{supp}(N) \cap \text{supp}(M) = \{v_0\}$. The most general case is when this set contains more than one element and in this case Eq. (9.18) generalizes to

$$[\hat{H}[M], \hat{H}[N]]'(\Psi^{f^{(1)}}_{[c]_{(1)}} + \Psi^{f^{(2)}}_{[c]_{(2)}} + \Psi^{f^{(3)}}_{[c]_{(3)}}) = \sum_{v\in\mathcal{K}} \sum_{i=1}^{3} \sum_{j\neq i} (\Psi^{f_v^{(i)(j,i)}[M,N]}_{[c]_{(i)}} - \Psi^{f_v^{(i)(i,j)}[M,N]}_{[c]_{(i)}}). \tag{9.19}$$

Next we quantize $V[\vec{\omega}]$ on $\mathcal{V}_{\mathrm{LMI}}$ and show that (9.5) is satisfied.

C. Quantization of $V[\vec{\omega}]$

Recall that

$$\{H[M], H[N]\} = V[\vec{\omega}] = \int_{\Sigma} \mathrm{d}^2x q^{-1/2} E^a_i E^b_i (N\partial_b M - M\partial_b N) F^j_{ac} E^c_j. \tag{9.20}$$

Before quantizing this classical functional, let us rewrite it as

$$\begin{aligned} V[\vec{\omega}] &= \int_{\Sigma} \mathrm{d}^2x (N\partial_a M - M\partial_a N)\bigg(\sum_{i\neq 1}(E^a_i E^b_i)F^1_{bc}E^c_1 + \sum_{i\neq 2}(E^a_i E^b_i)F^2_{bc}E^c_2 + \sum_{i\neq 3}(E^a_i E^b_i)F^3_{bc}E^c_3\bigg)q^{-1/2} \\ &\equiv V^1([N,M]) + V^2([N,M]) + V^3([N,M]), \end{aligned} \tag{9.21}$$

where

$$\begin{aligned} V^1([N,M]) &:= \int_{\Sigma} \mathrm{d}^2x (N\partial_a M - M\partial_a N)\sum_{i\neq 1}(E^a_i E^b_i)F^1_{bc}E^c_1 q^{-1/2} \\ &= \int_{\Sigma} \mathrm{d}^2x (N\partial_a M - M\partial_a N)(E^a_2 E^c_1 F^1_{bc}E^b_2 + E^a_3 E^c_1 F^1_{bc} E^b_3)q^{-1/2} \\ &= \int_{\Sigma} \mathrm{d}^2x (N\partial_a M - M\partial_a N)((E^a_2 E^c_1 - E^a_1 E^c_2)F^1_{bc}E^b_2 + (E^a_3 E^c_1 - E^a_1 E^c_3)F^1_{bc}E^b_3)q^{-1/2}, \end{aligned} \tag{9.22}$$

where we have subtracted classically trivial terms like $E^a_1 E^c_2 F^1_{bc} E^b_2$ which will give a nontrivial contribution in the quantum theory (these terms upon quantization are higher order in $\hbar$ whence there is no contradiction). $V^2([N,M])$ and $V^3([N,M])$ are defined similarly, and involve terms containing F^2 and F^3, respectively. We will quantize $V[\vec{\omega}]$ as the sum of the quantizations of these functionals.

1. *Quantization of* $V^1([N,M])$

Before presenting the quantization of $V^1([N,M])$ in detail, we explain the underlying idea. Spiritually the quantization is similar to the quantization of $\hat{H}^{(i)}_{T(\delta)}[N]$, but there are some differences. Consider a graph γ and a triangulation $T(\gamma,\delta)$ adapted to γ such that every vertex $v\in V(\gamma)$, whose valence is N, is contained in N "rectangles" whose area is δ^2 with the area of overlap region being $O(\delta^3)$.

From (9.22) we see that $V^1([N,M])$ is the sum of two terms. Let us focus on one of them which involves $F^1_{bc}E^b_2$:

$$\begin{aligned} V^1([N,M]) &= V^{1,2}([N,M]) + V^{1,3}([N,M]) \\ &:= \int_{\Sigma} \mathrm{d}^2x (N\partial_a M - M\partial_a N)((E^a_2 E^c_1 - E^a_1 E^c_2)F^1_{bc}E^b_2)q^{-1/2} + \text{term containing } F^1_{bc}E^b_3. \end{aligned} \tag{9.23}$$

We will quantize $V^{1,2}([N,M])$ as a product of elementary operators in following rough sense:

$$\begin{aligned} &E^a_{i=1,2}(N\partial_a M - M\partial_a N) \to \text{flux}, \\ &E^c_{j=1,2} q^{-1/4}|_\epsilon \to \text{quantum shift} =: V^c_{j=1,2}(x), \\ &E^c_{j=1,2} q^{-1/4}(x)|_\epsilon F^1_{bc}E^b_2 \to \text{holonomy around a loop generated by } V^c_{j=1,2} \\ &\qquad \text{and charged with eigenvalue of flux associated to } E^b_2, \\ &q_\epsilon^{-1/4} \to \text{quantize separately.} \end{aligned}$$

The regularized flux quantization of $E^a_{i=1,2}(N\partial_a M - M\partial_a N)$ is defined as follows:

$$\hat{E}^a_{i=1,2}(N\partial_a M - M\partial_a N)(v)|_{\delta,\delta'}|c\rangle := \sum_{e\in E(\gamma)|b(e)=v} \frac{\hat{E}_2(L^\delta_e)(N(v)(M(v+\delta'\dot{e}(0)) - M(v)) - (N\leftrightarrow M))}{|L^\delta_e|\delta'}. \quad (9.24)$$

Now we follow essentially the same strategy used in quantizing $\hat{H}^{(i)}_{T(\delta)}[N]$. However, there is one key technical difference in the construction of the quantum shift as compared to the quantum shift used in $\hat{H}^{(i)}_{T(\delta)}[N]$. Recall that the definition of the quantum shift $\hat{V}^c_j(v)|_\epsilon$ at a given point v depended on the regularization of $\hat{E}^a_j(v)$ and $q_\epsilon^{-1/4}(v)$ separately. The regularization required $B(v,\epsilon)$, which was used as a smearing object for regularizing the distribution $\hat{E}^a_j(v)$ and which also went into the construction of $q_\epsilon^{-1/4}(v)$ (as shown in Appendix A). These regularizations gave rise to

$$\hat{E}^a_j(v) = \frac{1}{\epsilon}\hat{O}^a_{1j}(v), \qquad q_\epsilon^{-1/4}(v) = \epsilon\hat{O}_2(v,\epsilon), \quad (9.25)$$

where $\hat{O}_1$, $\hat{O}_2$ are densely defined operators on $\mathcal{H}_{\rm kin}$, in contrast to being operator-valued distributions, and $\hat{O}_2$ implicitly depends on ϵ. This construction implied that the quantum shift $\hat{V}^a_j(v) = \hat{O}^a_{1j}(v)\hat{O}_2(v,\epsilon)$ was (explicitly) independent of ϵ. However, for defining the quantum shift in $\hat{V}([N,M])$ we use a different regularization, where the ball $B(v,\epsilon)$ used for smearing $\hat{E}^a_j$ is four times as large as the ball used in constructing $q_\epsilon^{-1/4}$. This implies that

$$\hat{V}^a_j(v) = \frac{1}{4}\hat{O}^a_{1j}(v)\hat{O}_2(v,\epsilon). \quad (9.26)$$

The $\frac{1}{4}$ factor will account for the overall $\frac{1}{4}$ factor that we obtained on the LHS.

We are now ready to put all the pieces together. Given a graph γ and a triangulation $T(\gamma,\delta)$ adapted to γ, a quantization of $V^{1,2}_{T(\delta,\delta')}([N,M])$ is given by

$$\begin{aligned}\hat{V}^{1,2}_{T(\delta,\delta')}([N,M])|c'\rangle = -\frac{\hbar^2}{4i}\Bigg[\Bigg(&\sum_{v\in V(\gamma)}\sum_{\triangle_v|v\in V(\gamma)}|\triangle_v|\sum_{e\in E(\gamma)|b(e)=v}\frac{\hat{E}_2(L^\delta_e)(N(v)(M(v+\delta'\dot{e}(0)) - M(v)) - (N\leftrightarrow M))}{|L^\delta_e|\delta'}\\ &\times\sum_{e'\in E(\gamma)|b(e')=v}\frac{1}{|L^\epsilon_{e'}|\delta^2}((\hat{h}^1_{\alpha(e',\langle\hat{E}_1(v)\rangle)})^{n^2_{e'}} - 1)\hat{q}_\epsilon^{-1/4}(v)\hat{O}_2(v,\epsilon)\Bigg)\\ &-\Bigg(\sum_{v\in V(\gamma)}\sum_{\triangle_v|v\in V(\gamma)}|\triangle_v|\sum_{e\in E(\gamma)|b(e)=v}\frac{\hat{E}_1(L^\delta_e)(N(v)(M(v+\delta'\dot{e}(0)) - M(v)) - (N\leftrightarrow M))}{|L^\delta_e|\delta'}\\ &\times\sum_{e'\in E(\gamma)|b(e')=v}\frac{1}{|L^\epsilon_{e'}|\delta^2}((\hat{h}^1_{\alpha(e',\langle\hat{E}_2(v)\rangle)})^{n^2_{e'}} - 1)\hat{q}_\epsilon^{-1/4}(v)\hat{O}_2(v,\epsilon)\Bigg)\Bigg]|c'\rangle, \end{aligned} \quad (9.27)$$

where we have the following.

(1) L^δ_e is a (codimension one) surface transversal to e, intersecting in a point which is in the coordinate neighborhood of v and the length of L^δ_e is δ.

(2) $\alpha(e',\langle\hat{E}_1(v)\rangle)$ is the loop starting at v, spanned by a straight-line arc along the $\hat{E}^a_1(v)$, and a segment along e' such that the area of the loop is δ^2.

(3) The factor of $|L^\epsilon_{e'}|$ in the denominator comes from the fact that the quantization of $F^i_{ab}E^b_j$ (see Sec. VII D) along an edge e' by "charging" the loop along with holonomy of A^i is defined by the eigenvalue of flux $E_j(L^\epsilon_{e'})$, which requires dividing the resulting operator by $|L^\epsilon_{e'}|$. Note that on choosing $|L^\epsilon_{e'}| = \epsilon$, this factor cancels with the factor of ϵ present in the quantization of $\hat{q}^{-1/4}$.

(4) The factor of $\frac{1}{4}$ in front is due to the quantum shift being $\frac{1}{4}\hat{V}^a$ and the factor of (-1) is due to the classical object being $E^c_i F^1_{bc} E^b_2 = -E^b_i F^1_{bc} E^c_2$, the latter of which we actually quantize.

(5) The two factors of $\hbar$ come from the quantum shift $\langle\hat{E}^a_i\rangle$, and the flux eigenvalue under which the holonomy around the loop is charged (this is the same convention we used when quantizing the Hamiltonian constraint at finite triangulation).[33]

(6) The factor of i^{-1} comes from expressing the curvature in terms of a loop holonomy.

We now follow the same steps that we followed in Sec. VII D, and replace the sum over holonomies by a product. The resulting final operator at finite triangulation is

[33]Note that our convention is always that $\langle\hat{E}^a_i\rangle$ is without a factor of $\hbar$.

$$\hat{V}^{1,2}_{T(\delta,\delta')}([N,M])|c'\rangle = \frac{i\hbar^2}{4\delta\delta'}\Bigg[\Bigg(\sum_{v\in V(\gamma)}\sum_{e\in E(\gamma)|b(e)=v}\hat{E}_2(L_e^\delta)(N(v)(M(v+\delta'\dot{e}(0))-M(v)) - (N\leftrightarrow M))\Bigg[\prod_{e'\in E(\gamma)|b(e')=v}\hat{h}^{n^2_{e'}}_{\alpha(e',\langle\hat{E}^a_1(v)\rangle)}-1\Bigg]\hat{q}_\epsilon^{-1/4}(v)\hat{O}_2(v)\Bigg) - \Bigg(\sum_{v\in V(\gamma)}\sum_{e\in E(\gamma)|b(e)=v}\hat{E}_1(L_e^\delta)(N(v)(M(v+\delta'\dot{e}(0))-M(v)) - (N\leftrightarrow M))\Bigg[\prod_{e'\in E(\gamma)|b(e')=v}\hat{h}^{n^2_{e'}}_{\alpha(e',\langle\hat{E}^a_2(v)\rangle)}-1\Bigg]\hat{q}_\epsilon^{-1/4}(v)\hat{O}_2(v)\Bigg)\Bigg]|c'\rangle, \tag{9.28}$$

which finally yields the following linear combination of charge network states:

$$\hat{V}^{1,2}_{T(\delta,\delta')}([N,M])|c'\rangle = \frac{i\hbar^3}{4\delta\delta'}\frac{1}{\hbar^2}\Bigg[\Bigg(\sum_{v\in V(\gamma)}\lambda^2(\vec{n}^c_v)\sum_{e\in E(\gamma)|b(e)=v}\langle E_2(L_e^\delta)\rangle(N(v)(M(v+\delta'\dot{e}(0))-M(v)) - (N\leftrightarrow M))[|c'_1\cup\alpha^\delta_v(\langle\hat{E}_1(v)\rangle,n^2),c'_2,c'_3\rangle-|c'\rangle]\Bigg) - \Bigg(\sum_{v\in V(\gamma)}\lambda^2(\vec{n}^c_v)\sum_{e\in E(\gamma)|b(e)=v}\langle\hat{E}_1(L_e^\delta)\rangle(N(v)(M(v+\delta'\dot{e}(0))-M(v)) - (N\leftrightarrow M))[|c'_1\cup\alpha^\delta_v(\langle\hat{E}_2(v)\rangle,n^2),c'_2,c'_3\rangle-|c'\rangle]\Bigg)\Bigg]. \tag{9.29}$$

$\hat{V}^{1,3}_{T(\delta),\delta'}([N,M])$ can be defined analogously. Thus finally we have that

$$\hat{V}^{1}_{T(\delta,\delta')}([N,M])|c'\rangle = -\frac{\hbar}{i}\frac{1}{4\delta\delta'}\sum_{v\in V(\gamma)}N(v)\lambda^2(\vec{n}^c_v)\sum_{e\in E(c')|b(e)=v}(M(v+\delta'\dot{e}(0)) - M(v))[(\langle\hat{E}_3(L_e)\rangle(|c'_1\cup\alpha^\delta_v(\langle\hat{E}_1\rangle,n_{c'_3}),c'_2,c'_3\rangle-|c'\rangle)-\langle\hat{E}_1(L_e)\rangle(|c'_1\cup\alpha^\delta_v(\langle\hat{E}_3\rangle,n_{c'_3}),c'_2,c'_3\rangle-|c'\rangle)) - (\langle\hat{E}_1(L_e)\rangle(|c'_1\cup\alpha^\delta_v(\langle\hat{E}_2\rangle,n_{c'_2}),c'_2,c'_3\rangle-|c'\rangle)-\langle\hat{E}_2(L_e)\rangle(|c'_1\cup\alpha^\delta_v(\langle\hat{E}_2\rangle,n_{c'_2}),c'_2,c'_3\rangle-|c'\rangle))] - (N\leftrightarrow M). \tag{9.30}$$

Without loss of generality let us assume that the only vertex in $V(c')$ which is contained in the supports of both N, M, is v_0, so that then[34]

$$\hat{V}^{1}_{T(\delta),\delta'}([N,M])|c'\rangle = -\frac{\hbar}{i}\frac{1}{4\delta\delta'}N(v_0)\lambda^2(\vec{n}^c_v)\sum_{e\in E(c')|b(e)=v_0}(M(v_0+\delta'\dot{e}(0)) - M(v_0))[(\langle\hat{E}_3(L_e)\rangle(|c'_1\cup\alpha^\delta_{v_0}(\langle\hat{E}_1\rangle,n_{c'_3}),c'_2,c'_3\rangle-|c'\rangle)-\langle\hat{E}_1(L_e)\rangle(|c'_1\cup\alpha^\delta_{v_0}(\langle\hat{E}_3\rangle,n_{c'_3}),c'_2,c'_3\rangle-|c'\rangle)) - (\langle\hat{E}_1(L_e)\rangle(|c'_1\cup\alpha^\delta_{v_0}(\langle\hat{E}_2\rangle,n_{c'_2}),c'_2,c'_3\rangle-|c'\rangle)-\langle\hat{E}_2(L_e)\rangle(|c'_1\cup\alpha^\delta_{v_0}(\langle\hat{E}_2\rangle,n_{c'_2}),c'_2,c'_3\rangle-|c'\rangle))] - (N\leftrightarrow M). \tag{9.31}$$

Now notice that [using (F9) and (9.31)],

$$(-i\hbar)\hat{V}^1_{T(\delta,\delta')}([N,M])|c'\rangle = |\psi^{\delta,\delta'}_1(v_0,c',[M,N])\rangle. \tag{9.32}$$

The remaining $\hat{V}^{i,j}_{T(\delta,\delta')}([N,M])$ operators are defined analogously. The sum of all these operators constitutes a quantization of $V[q^{-1}[N,M]]$ on $\mathcal{H}_{\rm kin}$:

$$(-i\hbar)\hat{V}_{T(\delta,\delta')}([N,M])|c'\rangle = (-i\hbar)\sum_i\hat{V}^i_{T(\delta,\delta')}([N,M])|c'\rangle = \sum_i|\psi^{\delta,\delta'}_i(v_0,c',[M,N])\rangle. \tag{9.33}$$

We are now ready to state our main result:

[34]If N and M have support containing different vertices of the underlying state then, it is easy to see that the operator vanishes at finite triangulation $\forall\ \delta'$ whence its continuum limit will vanish on $\mathcal{V}_{\rm LMI}$. In this case, the equality of the RHS and LHS of (9.5) follows.

Theorem.—$[\hat{H}[M]', \quad \hat{H}[N]']\Psi = (-i\hbar)\hat{V}([N, M])'\Psi$, $\forall\ \Psi \in \mathcal{V}_{\mathrm{LMI}}$.

Proof.—For any $\Psi^{f^{(i)}}_{[c]_{(i)}}$, the LHS is given by (9.18) and it is a result of the continuum limit of the net given in (9.13). As the RHS is the continuum limit of the net given in (9.33), which is precisely the same net as in (9.13), the result follows.

Thus there exists a quantization of $\{H[M], H[N]\}$ as an operator from $\mathcal{V}_{\mathrm{LMI}}$ to Cyl* (as a limit point of a net of finite-triangulation operators in a particular topology which is similar to the weak *-topology) which equals a quantization of $V[q^{-1}[N, M]]$ as an operator from $\mathcal{V}_{\mathrm{LMI}}$ to Cyl*. This, in our opinion, demonstrates that the quantization of the Hamiltonian constraint we have proposed in this paper has the right structural properties such that it can give rise to a faithful representation of Dirac algebra.

X. CONCLUSIONS

A satisfactory definition of quantum dynamics in canonical LQG is still missing. Even in the Euclidean sector of the theory, the progress is rather fragmented and is mainly achieved in a variety of minisuperspace models. To the best of our knowledge the only midisuperspace models where a completely satisfactory definition of the quantum constraints in generally covariant loop quantized field theories is known are, in fact, nongravitational theories, namely, two-dimensional PFT and the HK model. However, these models miss perhaps the most interesting aspect of the constraint algebra of canonical gravity: That it is a Lie algebroid instead of being a Lie algebra [28]; that is, the fact that the Poisson bracket of two Hamiltonian constraints involves phase space-dependent structure functions. In this paper, we proposed a toy model which has the same Dirac algebra as Euclidean three-dimensional canonical gravity but here, in a certain sense, nonlinear aspects of gravity are absent. We focused on a part of the constraint algebra and finally derived a quantization of continuum Hamiltonian constraint, which has the potential to give rise to a faithful representation of the Dirac algebra. To the best of our knowledge, our work along with [29] is a first attempt towards a quantum realization of off-shell closure of the Dirac algebra within the LQG framework.

As the theory is a topological Abelian gauge theory, one might perceive this model as being too simplistic. However this is not quite true. As our main focus has been on understanding the off-shell closure condition in the quantum theory (and as the theory is topological only on shell), we work in a genuine field-theoretic context. It is quite straightforward to generalize our results to (3 + 1)-dimensional $U(1)^3$ theory (which is precisely the model studied in [29]) and, in fact, some of the technical annoyance that we face in two dimensions (e.g., the presence of irrelevant vertices) can be evaded in three spatial dimensions. On the other hand, in 2 + 1 dimensions the physical spectrum of this topological gauge theory is well understood [as states supported on the moduli space of flat $U(1)^3$ connections], and a complete set of Dirac observables is known. Hence one could investigate the consistency of the quantum theory defined here by investigating issues associated to the kernel of constraints, and the representation of quantum Dirac observables.

We now recap the most salient aspects of our constructions before highlighting the key open issues and some of the unsatisfactory aspects.

The usual construction of composite operators in LQG is via some classical polynomial function of holonomies and fluxes. However, as we were motivated to look for a quantization of Hamiltonian constraint which mimicked a certain discrete approximant of the classical geometric action involving phase space-dependent diffeomorphisms, our quantization choices involved quantizing $F^i_{ab}E^b_j$ as a holonomy operator (with the holonomy being in a state-dependent representation) and quantizing the remaining triad E^a_k (or more precisely $Nq^{-1/4}E^a_k$) as a quantum shift which generated the loop underlying the holonomy associated to $F^i_{ab}E^b_j$. These choices have a spiritual similarity to the "$\bar{\mu}$-scheme" which led to a physically viable quantization of the Hamiltonian constraint in LQC [30].[35]

The Hamiltonian constraint at finite triangulation $\hat{H}_{T(\delta)}[N]$ created very specific types of vertices that we called extraordinary (EO) vertices. As the underlying gauge group is $U(1)^3$, these vertices are *always* degenerate (i.e., they are in the kernel of the inverse volume operator). This aspect of the construction bares some similarity to Thiemann's Hamiltonian, in which the newly created vertices are also degenerate.[36] Whence at first sight it seemed as if one faces the same problem that Thiemann's Hamiltonian does, in that the action of a second successive Hamiltonian will have no nontrivial action at EO vertices and the objections raised in [20] will remain true in our case. However, this expected triviality overlooked a key fact about the EO vertices: Their "location" with respect to the original charge network is state dependent, which is in turn due to the fact that these EO vertices are created along straight-line arcs of the quantum shift. This fact along with a classical Poisson bracket computation suggested a plausible modification of Hamiltonian constraint when it acted on EO vertices. The modification was such that its precise interpretation was a "nonlocal" action (i.e., these actions were generated by operators which involved holonomies around finite loops) not only on the EO vertex

[35]However notice that the analogy is only superficial. In the $\bar{\mu}$ scheme, the triad dependence underlying the holonomy operator does not come from the $q^{-1/2}E \wedge E$ term. We thank Martin Bojowald for pointing this out to us.

[36]We should note here that the similarity is only restricted to $U(1)^3$. The superficial extension of our analysis to SU(2) suggests that the EO vertices created in that case will not be degenerate.

but on a subgraph containing the EO pair. However, the detailed understanding of such terms in the context of the underlying continuum classical theory is not clear to us and should be investigated further [31].

The action of the finite-triangulation Hamiltonian constraint $\hat{H}_{T(\delta)}[N]$ on a charge network $|c\rangle$ is remarkably different from the action considered so far in LQG, and it is worth summarizing its three main features:

(1) The deformations of the graph underlying c are state-dependent (as dictated by the quantum shift).
(2) The $\mathrm{U}(1)^3$ edge labels in c change, with the change itself being state dependent.
(3) A certain special class of degenerate vertices moves under the action of $\hat{H}_{T(\delta)}[N]$.

All three of these conditions hint at a rather rich structure of quantum dynamics in the model that could be of interest to discrete approaches inspired by canonical loop quantum gravity.

We then constructed a habitat which was designed in such a way that the continuum limit of the finite-triangulation Hamiltonian constraint could be taken in the topology induced by a family of seminorms. The continuum limit Hamiltonian constraint $\hat{H}[N]'$ does not preserve this habitat and can only be interpreted as a linear operator from $\mathcal{V}_{\mathrm{LMI}}$ to Cyl^*. Although this implies that the commutator of $\hat{H}[N]'$ with itself is ill defined, it turns out that the limit of finite-triangulation commutators is still well defined on $\mathcal{V}_{\mathrm{LMI}}$ and is nonvanishing. We finally showed that there exists a quantization of the RHS, which is not quantized as an ordinary diffeomorphism with triad-dependent shift, but requires a specific operator ordering. This quantization matches with the continuum quantum commutator which is the LHS of the off-shell closure relation.

We now come to the open issues and certain related unsatisfactory aspects of our work. Our entire construction is based upon decomposing the Hamiltonian constraint into three pieces involving F^1, F^2, and F^3, respectively. Although each of these pieces is gauge invariant in the present model, this is not true in the case of SU(2). Thus more careful analysis is needed to extend our proposal to a quantization of the Hamiltonian constraint in the SU(2) case.

The second issue lies in the choice of the habitat. Our experience of how to construct habitats on which higher density operators in LQG admit a continuum limit is rather limited. After the seminal work done in [19], where a habitat was constructed in which Thiemann's (regularized) Hamiltonian constraint admitted a continuum limit (once again in a seminorm topology induced by the habitat states and states in $\mathcal{H}_{\mathrm{kin}}$), the only places where habitats have been utilized have been in [21,23]. In these two examples habitats even turned out to be physically appropriate homes for the quantum constraints, as the kernel (which was known via other methods) is a subspace of those habitats. However, in our case, the nature of the regularized constraints makes it rather difficult to construct a suitable habitat on which not only do the regularized constraints admit a continuum limit, but also that all the details of the regularized constraint operators remain intact when we take continuum limit (for example, the change in edge labels induced by the action of $\hat{H}_{T(\delta)}[N]$ go amiss when we consider the dual action on the habitat). It is important to note that we have constructed a habitat only with two goals in mind:

(1) $\hat{H}^i_{T(\delta)}[N]$ admits a continuum limit; and
(2) $[\hat{H}_T[N], \hat{H}_{T'}[M]]$ admits a continuum limit.

$\mathcal{V}_{\mathrm{LMI}}$ need not be a physically relevant habitat since
(a) $H[N]'$ does not preserve $\mathcal{V}_{\mathrm{LMI}}$.
(b) We do not know if the states in the moduli space of flat $\mathrm{U}(1)^3$ connections are included in $\mathcal{V}_{\mathrm{LMI}}$.[37]
(c) The classical theory is a completely integrable system, but we do not know if the habitat admits a representation of quantum observables and if there a precise sense in which these observables commute with the quantum constraints.

Detailed investigations of all these three issues could be key in constructing physically interesting habitats.

There is an alternate viewpoint one could adhere to. Let us assume that we can extend our constructions to three spatial dimensions, and appropriately density weighted constraints that satisfy the off-shell closure condition on some habitat. As far as the $\mathrm{U}(1)^3$ theory is concerned, since the inverse volume operator is just a multiple of the identity operator on charge network states, the higher density weighted operator induces an unambiguous definition of a density one operator. It is quite plausible that in the Uniform-Rovelli-Smolin topology (or some suitable generalization thereof), this density one operator converges to a densely defined operator on $\mathcal{H}_{\mathrm{kin}}$. Then the requirement of off-shell closure would only be used to select, out of an infinitude of possible choices, a density one quantum Hamiltonian constraint, and one could just choose to work on $\mathcal{H}_{\mathrm{kin}}$.[38]

A faithful representation of the Dirac algebra entails not only the second equation in Eq. (2.3), but all three of them. In particular, the construction of a finite-triangulation Hamiltonian constraint operator involved certain background structure, which survived in the continuum limit. The definition of the quantum shift involved a certain regularization scheme and it is far from clear if this scheme is diffeomorphism covariant (or if it could be rendered so). The notion of extraordinary vertices, which are required to lie inside certain balls around nondegenerate vertices is

[37]Naively speaking, the habitat states with constant vertex functions can indeed be thought of as states with support only on flat connections. However this issue has not been investigated in detail.

[38]We do not believe these ideas can work in two spatial dimensions, due to the presence of irrelevant vertices which would be absent in three dimensions.

certainly a noncovariant notion, as one can map an EO vertex with respect to some charge network c into a WEO vertex with respect to the same charge network via some diffeomorphism. Hence in light of the noncovariant structures which have gone into the construction of the (continuum) Hamiltonian, it is far from clear if the third equation in (2.3) is satisfied. We will come back to these issues in [25].

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APPENDIX A: VOLUME AND INVERSE VOLUME OPERATORS

In this appendix we discuss the construction of a $U(1)^3$ volume operator, as well as an operator corresponding to $q^{-1/4}$, which are used in the main body of the paper. We closely follow Thiemann [27].

1. Volume operator

In [27], an operator-valued distribution corresponding to the degeneracy vector $E^i = \frac{1}{2}\epsilon^{ijk}\eta_{ab}E^a_j E^b_k$ is constructed when the gauge group is SU(2). In the case of $U(1)^3$, the construction proceeds analogously and leads to the operator action

$$\hat{E}^i(x)|c\rangle = \frac{1}{8}(\kappa\hbar)^2 \sum_{v\in V(c)} \delta^{(2)}(x, v) \times \sum_{e_I\cap e_{I'}=\{v\}} \epsilon(e_I, e_{I'})\epsilon^{ijk} n^j_I n^k_{I'}|c\rangle, \quad \text{(A1)}$$

where $V(c)$ is the vertex set of c, and

$$\epsilon(e, e') := \frac{\eta_{ab}\dot{e}^a(0)\dot{e}'^b(0)}{|\eta_{ab}\dot{e}^a(0)\dot{e}'^b(0)|} = \pm 1, 0. \quad \text{(A2)}$$

The additional factor of $\frac{1}{4}$ comes from evaluating the δ functions at end points of integration over t, t' (we have arranged all edges as outgoing at vertices). Classically $q = E^i E^i$, but the presence of the δ function requires an additional regularization as an intermediate step. One can show that the regularized $\hat{q}$ is the square of an essentially self-adjoint operator, and it is positive semidefinite, so its square root is well defined as an operator-valued distribution:

$$\sqrt{\hat{q}(x)}|c\rangle = \frac{1}{8}(\kappa\hbar)^2 \sum_{v\in V(c)} \delta^{(2)}(x, v) \times \sqrt{\left(\sum_{e_I\cap e_{I'}=\{v\}} \epsilon(e_I, e_{I'})\epsilon^{ijk} n^j_I n^k_{I'}\right)^2}|c\rangle. \quad \text{(A3)}$$

We can use this to define a regular operator corresponding to the volume $V(R)$ of a region $R \subset \Sigma$:

$$\hat{V}(R)|c\rangle := \int_R d^2x\sqrt{\hat{q}(x)}|c\rangle = \frac{1}{8}(\kappa\hbar)^2 \times \sum_{v\in V(c)\cap R} \sqrt{\left(\sum_{e_I\cap e_{I'}=\{v\}} \epsilon(e_I, e_{I'})\epsilon^{ijk} n^j_I n^k_{I'}\right)^2}|c\rangle. \quad \text{(A4)}$$

We observe that since $\hat{E}^i$ vanishes on states charged in only one copy of U(1), the volume also vanishes on these states. Moreover, due to the orientation factor $\epsilon(e_I, e_{I'})$, $\hat{V}$ vanishes at vertices at which there are not at least two edges with linearly independent tangents.

2. A $q^{-1/4}$ operator

In this subsection we derive a Thiemann-like classical identity for $q^{-1/4}$ which is then promoted to a regularized operator on $\mathcal{H}_{\text{kin}}$. One can imagine several variants on the following construction, but here we settle on one that satisfies two properties that we use in the main text:

(i) $\hat{q}^{-1/4}$ vanishes everywhere except at charge network vertices whose outgoing edges have linearly independent tangents.

(ii) $\hat{q}^{-1/4}$ vanishes at vertices, all whose incident edges are charged in a single $U(1)_i$.

We will refer to those charge network vertices at which $\hat{q}^{-1/4}$ is nonvanishing as nontrivial or nondegenerate, and those for which $\hat{q}^{-1/4}$ vanishes will be called trivial or degenerate.

We begin by noticing that classically, for $x \in R \subset \Sigma$,

$$\frac{\eta^{ab}\epsilon_{ijk}}{2(\frac{2}{3})^2(1-p)^2}\{A^j_a(x), V(R)^{\frac{2}{3}(1-p)}\} \times \{A^k_b(x), V(R)^{\frac{2}{3}(1-p)}\}V(R)^{\frac{2}{3}(1-p)} = V(R)^{-2p}E^i(x).$$

Now if $B(x, \epsilon)$ is a (coordinate) circular ball of coordinate radius ϵ centered at x, then

$$\lim_{\epsilon\to 0}\frac{V(x,\epsilon)}{\pi\epsilon^2}=\sqrt{q(x)},\quad\text{where}\quad V(x,\epsilon):=\int_{B(x,\epsilon)}\mathrm{d}^2x\sqrt{q(x)}. \tag{A5}$$

Therefore the Poisson bracket identity above allows us to write

$$q^{-5/8}E^i(x)=\lim_{\epsilon\to 0}8\epsilon^{5/2}\pi^{5/4}\eta^{ab}\epsilon_{ijk}\{A_a^j(x),V(x,\epsilon)^{\frac14}\}\times\{A_b^k(x),V(x,\epsilon)^{\frac14}\}V(x,\epsilon)^{\frac14}. \tag{A6}$$

We can form pure inverse powers of q by taking even powers of this identity. For instance, $q^{-1/4}=(q^{-5/8}E^i)^2$, which is the quantity we are interested in. Before such an identity can be quantized on $\mathcal{H}_{\mathrm{kin}}$, we must replace A by holonomies. Letting $h_a^i(x)$ be a coordinate length ϵ holonomy along an edge in the a direction which crosses or terminates at x, we have

$$q^{-5/8}E^i(x)=\lim_{\epsilon\to 0}8\epsilon^{5/2}\pi^{5/4}\eta^{ab}\epsilon_{ijk}\frac{(h_a^j(x))^{-1}}{\mathrm{i}\epsilon\kappa}\times\{h_a^j(x),V(x,\epsilon)^{\frac14}\}\frac{(h_b^k(x))^{-1}}{\mathrm{i}\epsilon\kappa}\times\{h_b^k(x),V(x,\epsilon)^{\frac14}\}V(x,\epsilon)^{\frac14}. \tag{A7}$$

Removing the $\epsilon\to 0$ limit, and making the replacements $V\to\hat{V}$ and $\{,\}\to(\mathrm{i}\hbar)^{-1}[,]$, we obtain a well-defined (ϵ-regularized) operator on $\mathcal{H}_{\mathrm{kin}}$. Squaring the resulting operator, we arrive at

$$\hat{q}_\epsilon^{-1/4}=\epsilon\frac{64\pi^{5/2}\eta^{ab}\eta^{cd}}{(\kappa\hbar)^4}\sum_{i,j}(h_a^i)^{-1}[h_a^i,\hat{V}^{\frac14}](h_b^j)^{-1}\times[h_b^j,\hat{V}^{\frac14}]\hat{V}^{\frac14}(h_c^i)^{-1}[h_c^i,\hat{V}^{\frac14}](h_d^j)^{-1}[h_d^j,\hat{V}^{\frac14}]\hat{V}^{\frac14}, \tag{A8}$$

where we have dropped the various arguments for notational clarity. We will choose the holonomies in (A8) based on the state $|c\rangle$ on which the operator acts. Specifically, given a vertex $v\in\gamma(c)$ which is at least bivalent with linearly independent tangents, we single out a pair of linearly independent edges to define the x and y coordinate axes of a coordinate system with origin at v. We let the holonomies of (A8) lie along these coordinate axes [so that they partially overlap the edges of $\gamma(c)$] and have end points (or beginning points) at v. Then in this coordinate system,

$$\hat{q}_\epsilon^{-1/4}=\epsilon\frac{64\pi^{5/2}\epsilon^{IJ}\epsilon^{KL}}{(\kappa\hbar)^4}\sum_{i,j}(h_I^i)^{-1}[h_I^i,\hat{V}^{\frac14}](h_J^j)^{-1}\times[h_J^j,\hat{V}^{\frac14}]\hat{V}^{\frac14}(h_K^i)^{-1}[h_K^i,\hat{V}^{\frac14}](h_L^j)^{-1}[h_L^j,\hat{V}^{\frac14}]\hat{V}^{\frac14} \tag{A9}$$

(with respect to another coordinate system, this expression will differ by an overall constant). If there are not two linearly independent directions defined by tangents of edges at v, then we pick some orthogonal direction by hand along which to lay holonomies; we will see shortly that in this case of "linear vertices," the operator has trivial action.

Since each action of the volume operator gives an eigenvalue proportional to $(\kappa\hbar)^2$, the eigenvalues of $\hat{q}_\epsilon^{-1/4}$ are proportional to $(\kappa\hbar)^{-1}$, and we separate this dimensionful dependence, as well as the ϵ dependence, from the dimensionless part of the eigenvalue, writing

$$\langle c|\hat{q}_\epsilon^{-1/4}(v)|c\rangle=:\frac{\epsilon}{\kappa\hbar}\lambda(\vec{n}_v^c) \tag{A10}$$

for the particular case where v is a nontrivial vertex of $|c\rangle$, and $\vec{n}_v^c$ denotes the collection of charge labels on the edges there. This operator is completely regular in its action on $\mathcal{H}_{\mathrm{kin}}$, so taking $\epsilon\to 0$ one obtains the zero operator. The strategy in the main text is to combine $\hat{q}_\epsilon^{-1/4}$ with a regularized $\hat{E}_i^a$ which behaves like ϵ^{-1} in the regulating parameter, and hence the combination remains regular as $\epsilon\to 0$. To see that the two properties (i) and (ii) are satisfied, note that $\hat{V}^{\frac14}$ acts rightmost, and hence annihilates volume-degenerate configurations.

Before concluding this section, we mention that the overall factor of ϵ is not the only possibility (nor is the overall constant); as noted in [19], one may choose the regions associated with each instance of $\hat{V}$ independently so as to obtain an arbitrary power of the regulating parameter (or some combination of several regulating parameters). Here and in the main text, we have merely followed an economical prescription, where all regulating parameters scale in the same way.

APPENDIX B: CHARACTERIZATION OF TYPE B EXTRAORDINARY VERTICES

In this section we classify the type B EO vertices. Recall that if, given a charge-network state $|c\rangle$, a quantum shift is such that the straight-line arc associated to it lies along one of the edges, then the corresponding vertex is called type B (depicted in Fig. 4). The conditions are a minor modification of the conditions characterizing type A vertices.

We will call an N_v-valent vertex v^{E} an extraordinary vertex of type (M,j,B) if and only if the following conditions are satisfied.

1. Set B

(1) Exactly two edges incident at v^{E} are analytic continuations of each other and these two edges are necessarily charged in more than one copy of $\mathrm{U}(1)^3$. The remaining N_v-2 edges are colored in only $\mathrm{U}(1)_M|_{M\in\{1,2,3\}}$. Let us refer to the two multicolored edges as $e_{v^{\mathrm{E}}}^{(1)}$, $e_{v^{\mathrm{E}}}^{(2)}$.

(a) Tangents to all the edges incident at $v^{\rm E}$ are parallel or anitparallel.

(2) Let us denote the $N_v - 2$ vertices which are the end points of the $N_v - 2$ edges beginning at $v^{\rm E}$ and are distinct from $e^{(1)}_{v^{\rm E}}$, $e^{(2)}_{v^{\rm E}}$ by the set $\mathcal{S}_{v^{\rm E}} := \{v^{\rm E}_{(1)}, \ldots, v^{\rm E}_{(N_v-2)}\}$.[39] The valence of all these vertices is bounded between three and four.

(a) At most two vertices in $\mathcal{S}_{v^{\rm E}}$ are trivalent.

(3) The trivalent vertices are such that the edges which are not incident at $v^{\rm E}$ are analytic extensions of each other and the four-valent vertices are such that two of the edges which are not incident at $v^{\rm E}$ are analytic extensions of each other and the fourth edge is the analytic extension of the edge which is incident at $v^{\rm E}$.

(a) Any four-valent vertex defined in (3) is such that if the four edges incident on it (e_1, e_2, e_3, e_4) are such that $e_1 \circ e_2$ is entire analytic and $e_3 \circ e_4$ is entire analytic, then $\vec{n}_{e_1} = \vec{n}_{e_2}$, $\vec{n}_{e_3} = \vec{n}_{e_4}$.

(4) Let $e_{v^{\rm E}}$ be an edge beginning at $v^{\rm E}$ which ends in a four-valent vertex $f(e_{v^{\rm E}})$. By (3), there exists an analytic extension $\tilde{e}_{v^{\rm E}}$ of $e_{v^{\rm E}}$ in $E(\bar{c})$ beginning at $v^{\rm E}$. The final vertex $f(\tilde{e}_{v^{\rm E}})$ of $\tilde{e}_{v^{\rm E}}$ is always trivalent. Thus restricting attention to analytic extensions of each of the edges beginning at $v^{\rm E}$, all such edges end in trivalent vertices, and all of these trivalent vertices are such that the remaining two edges incident on them are analytic extensions of each other. The set of these $N_v - 2$ trivalent vertices "associated" to $v^{\rm E}$ is $\bar{\mathcal{S}}_{v^{\rm E}} := \{\bar{v}^{\rm E}_{(1)}, \ldots, \bar{v}^{\rm E}_{(N_v-2)}\}$.[40]

(5) Let us denote these [maximally analytic inside $E(\gamma)$] edges beginning at $v^{\rm E}$ by $\{\tilde{e}^{(1)}_{v^{\rm E}}, \ldots, \tilde{e}^{N_v-2}_{v^{\rm E}}\}$. Without loss of generality, consider the case when all the edges incident on $v^{\rm E}$ except $e^{(1)}_{v^{\rm E}}$, $e^{(2)}_{v^{\rm E}}$ are charged in $\mathrm{U}(1)_1$.[41] Let the charges on these edges be $\{(n_{\tilde{e}^{(1)}_{v^{\rm E}}}, 0, 0), \ldots, (n_{\tilde{e}^{(N_v-2)}_{v^{\rm E}}}, 0, 0)\}$.

If $\tilde{e}^{(k)}_{v^{\rm E}}$ $(k \in \{1, \ldots, N_v - 2\})$ ends in a three-valent vertex $f(\tilde{e}^{(k)}_{v^{\rm E}})$ and if the charges on the remaining two (analytically related) edges $e^{(k)\prime}_{v^{\rm E}}$, $e^{(k)\prime\prime}_{v^{\rm E}}$ incident on $f(\tilde{e}^{(k)}_{v^{\rm E}})$ are $(n^{(1)}_{e^{(k)\prime}_{v^{\rm E}}}, n^{(2)}_{e^{(k)\prime}_{v^{\rm E}}}, n^{(3)}_{e^{(k)\prime}_{v^{\rm E}}})$ and $(n^{(1)}_{e^{(k)\prime\prime}_{v^{\rm E}}}, n^{(2)}_{e^{(k)\prime\prime}_{v^{\rm E}}} = n^{(2)}_{e^{(k)\prime}_{v^{\rm E}}}, n^{(3)}_{e^{(k)\prime\prime}_{v^{\rm E}}} = n^{(3)}_{e^{(k)\prime}_{v^{\rm E}}})$, then either

(a) $n^{(1)}_{\tilde{e}^{(k)}_{v^{\rm E}}} = n^{(2)}_{e^{(k)\prime}_{v^{\rm E}}}$ or

(b) $n^{(1)}_{\tilde{e}^{(k)}_{v^{\rm E}}} = n^{(3)}_{e^{(k)\prime}_{v^{\rm E}}}$.

(6) Now consider the set $\bar{\mathcal{S}}_{v^{\rm E}}$. Recall that each element in this set is a trivalent vertex. Consider the vertex $f(\tilde{e}_{v^{\rm E}})$ whose three incident edges are $\tilde{e}_{v^{\rm E}}$, $\tilde{e}'_{v^{\rm E}}$, and $\tilde{e}''_{v^{\rm E}}$, respectively. Recall that $\tilde{e}'_{v^{\rm E}}$, $\tilde{e}''_{v^{\rm E}}$ are analytic continuations of each other. Depending on whether $n^{(1)}_{\tilde{e}_{v^{\rm E}}} \lessgtr 0$, choose one out of the two edges, $\tilde{e}'_{v^{\rm E}}$, $\tilde{e}''_{v^{\rm E}}$ which has lesser or greater charge [depending on whether $n^{(1)}_{\tilde{e}_{v^{\rm E}}}$ is greater or less than zero] in $\mathrm{U}(1)_1$ than the other edge. Consider the set of all such chosen edges for each vertex in $\bar{\mathcal{S}}_{v^{\rm E}}$. We refer to this set as $\bar{\mathcal{T}}_{v^{\rm E}}$. Now consider $\sum_{\tilde{e}\in\bar{\mathcal{S}}_{v^{\rm E}}} n^{(1)}_{\tilde{e}_{v^{\rm E}}}$. Suppose this quantity is positive (negative). Then among the two edges $e^{(1)}_{v^{\rm E}}$ and $e^{(2)}_{v^{\rm E}}$ pick the edge whose charge in $\mathrm{U}(1)_1$ is lesser (greater) than the charge in $\mathrm{U}(1)_1$ of the other edge. Let us assume it is $e^{(1)}_{v^{\rm E}}$. Then

$$\mathcal{T}_{v^{\rm E}} := \bar{\mathcal{T}}_{v^{\rm E}} \cup \{e^{(1)}_{v^{\rm E}}\}. \tag{B1}$$

(a) All edges in $\mathcal{T}_{v^{\rm E}}$ meet at a vertex v which is such that if the number of edges incident at v is greater than N_v and if the charges $\{\tilde{e}^{v^{\rm E}}_k\}_{k=1,\ldots,N_v}$ are the $\mathrm{U}(1)_j$ charges on the edges in $\mathcal{T}_{v^{\rm E}}$, then the $(1)_j$ charge on the edges incident at v which are not in $\mathcal{T}_{v^{\rm E}}$ is zero. As shown in Appendix C, v, if it exists, is unique.

(7) Finally, consider the graph $\gamma := \gamma(\bar{c}) - \{\tilde{e}^{(1)}_{v^{\rm E}}, \ldots, \tilde{e}^{N_v-2}_{v^{\rm E}}\}$ and a charge-network c based on γ obtained by deleting $\{\tilde{e}^{(1)}_{v^{\rm E}}, \ldots, \tilde{e}^{N_v-2}_{v^{\rm E}}\}$ along with the charges on them, and also deleting exactly the same amount of charges from the edges in $\mathcal{T}_{v^{\rm E}}$. Note that by construction v belongs to γ. Now consider $U_\epsilon(\gamma, v)$. The final and key feature of the EO vertex $v^{\rm E}$ is $v^{\rm E} \in U_\epsilon(\gamma, v)$ and $v^{\rm E}$ is the end point of the "straight-line curve" $\delta\langle E^a_j\rangle$ for some δ, where $j = 2$ if in (6) condition (a) is satisfied, and $j = 3$ if in (5), (b) is satisfied.

We call the pair $(v, v^{\rm E})$ extraordinary with $v^{\rm E}$ a type B EO vertex.

APPENDIX C: UNIQUENESS OF v ASSOCIATED TO $v^{\rm E}$

Lemma.—Consider a charge-network $\bar{c}$ containing a vertex $v^{\rm E}$ of type $(M = 1, j = 2, K \in \{\mathrm{A}, \mathrm{B}\})$ satisfying conditions (1)–(7) as listed in Set A or Set B. Then the vertex v described in condition(s) (7) is unique.

Proof.—We will only prove the lemma for a type A EO vertex. The proof for the type B case is exactly analogous. Let us assume the contrary, i.e., there exist distinct $v, v' \in V(\bar{c})$ with respect to which $v^{\rm E}$ is EO. This implies that for all trivalent vertices in $\bar{\mathcal{S}}_{v^{\rm E}}$, two edges (which are analytic extensions of each other) begin, one edge (or its analytic

[39] This set satisfies exactly the same conditions that $\mathcal{S}_{v^{\rm E}}$ satisfies in the case of type A vertices.

[40] Note that $\mathcal{S}_{v^{\rm E}} \cap \bar{\mathcal{S}}_{v^{\rm E}} =$ trivalent vertices in $\mathcal{S}_{v^{\rm E}}$.

[41] In this case we will say that $v^{\rm E}$ is of type $(M = 1, j \in \{2, 3\}, \mathrm{B})$.

extension) ending at v and the other (or its analytic extension) ending at v'. Moreover all the edges incident at v and v', apart from those which begin at a vertex in $\bar{\mathcal{S}}_{v^{\mathrm{E}}}$, have zero charge in $\mathrm{U}(1)_2$. Now consider one such vertex $v_1 \in \bar{\mathcal{S}}_{v^{\mathrm{E}}}$. If the charge [in $\mathrm{U}(1)_1$] on edge e_{v^{E},v_1} (bounded by v^{E} and v_1) is positive, then depending on whether the $\mathrm{U}(1)_2$ charge on edge $e_{v_1,v}$ [which equals the $\mathrm{U}(1)_2$ charge on $e_{v_1,v'}$] is positive or negative, check on which of these two edges the $\mathrm{U}(1)_1$ charge is greater. This singles out one of v or v' with respect to which v^{E} is EO.

APPENDIX D: CONTINUUM LIMIT OF THE HAMILTONIAN CONSTRAINT OPERATOR

In this appendix we derive (8.6), (8.9), and (8.13). We do this by showing that for any N, $|\tilde{c}\rangle$, and $\Psi^{f^{(1)}}_{[c]_{(1)}}$,

$$(\hat{H}^{(i)}[N]'\Psi^{f^{(1)}}_{[c]_{(1)}})|\tilde{c}\rangle = \lim_{\delta\to 0}\Psi^{f^{(1)}}_{[c]_{(1)}}(\hat{H}^{(i)}_{T(\delta)}[N]|\tilde{c}\rangle) \quad \text{(D1)}$$

for $i = 1, 2, 3$. The left-hand side is given in (8.6), (8.9), and (8.13), respectively.

Proofs.—We want to show that for any $\tilde{c}$, N, and $\Psi^{f^{(i)}}_{[c]_{(i)}}$, the following holds:

$$\lim_{\delta\to 0}\Psi^{f^{(i)}}_{[c]_{(i)}}(\hat{H}^{(j)}_{T(\delta)}[N]|\tilde{c}\rangle) = \sum_{v\in V(c)}(\Psi^{\bar{f}^{(i,j)}_v}_{[c]_{(i)}} - \Psi^{\bar{\bar{f}}^{(i,j)}_v}_{[c]_{(i)}})|\tilde{c}\rangle \quad \text{(D2)}$$

for any $i, j \in \{1, 2, 3\}$, where $\Psi^{\bar{f}^{(i,j)}_v}_{[c]_{(i)}}$, $\Psi^{\bar{\bar{f}}^{(i,j)}_v}_{[c]_{(i)}}$ are in Cyl^*. The computation will be divided into several cases as follows.

Type A: The c on which $\Psi^{f^{(i)}}_{[c]_{(i)}}$ is based is such that all the EO vertices which can be created from this state necessary lie off $\gamma(c)$; i.e., all the EO vertices are type A.

Type B: Compliment of the type A case.

We will analyze only the type A case here as the complimentary case can be analyzed in a similar manner but requires more bookkeeping. The results proven here hold for both cases. We further divide the analysis of type A into several subcases.

1. Case (A,1): $i = j$

Without loss of generality, we take $i = j = 1$.

a. Case (A,1,a): $\tilde{c} = c$

Then

$$\begin{aligned}\lim_{\delta\to 0}\Psi^{f^{(1)}}_{[c]_{(1)}}(\hat{H}^{(1)}_{T(\delta)}[N]|c\rangle) &= \lim_{\delta\to 0}\frac{1}{\delta}\Psi^{f^{(1)}}_{[c]_{(1)}}\sum_{v\in V(c)}N(v)\lambda(\vec{n}^v_c)(|c_1\cup\alpha^\delta_v(\langle\hat{E}_2\rangle, n^3), c_2, c_3\rangle - |c_1\cup\alpha^\delta_v(\langle\hat{E}_3\rangle, n^2), c_2, c_3\rangle)\\ &= \lim_{\delta\to 0}\sum_{v\in V(c)}N(v)\lambda(\vec{n}^v_c)\frac{1}{\delta}(f^{(1)}_{[c]_{(1)}}(\bar{V}(c_1\cup\alpha^\delta_v(\langle\hat{E}_2\rangle, n^3)\cup c_2\cup c_3) - f^{(1)}_{[c]_{(1)}}(\bar{V}(c_1\cup\alpha^\delta_v(\langle\hat{E}_3\rangle, n^2)\cup c_2\cup c_3)))\\ &= \lim_{\delta\to 0}\sum_{v\in V(c)}N(v)\lambda(\vec{n}^v_c)\frac{1}{\delta}(f^{(1)}_{[c]_{(1)}}(\bar{V}(c_1\cup c_2\cup c_3\cup\{v^\delta_{\mathrm{E},(1,2)}\})) - f^{(1)}_{[c]_{(1)}}(\bar{V}(c_1\cup c_2\cup c_3\cup\{v^\delta_{\mathrm{E},(1,3)}\}))),\end{aligned} \quad \text{(D3)}$$

where $v^\delta_{\mathrm{E},(1,2)}$ is an EO vertex of type $(M = 1, j = 2)$ associated to v and which is at the "apex" of $\alpha^\delta_v(\langle\hat{E}_2\rangle, n^3)$. Similarly $v^\delta_{\mathrm{E},(1,3)}$ is the EO vertex that is associated to v and which is at the "apex" of $\alpha^\delta_v(\langle\hat{E}_3\rangle, n^2)$. We have used that

$$\bar{V}(c_1\cup\alpha^\delta_v(\langle\hat{E}_2\rangle, n^3)\cup c_2\cup c_3) = \bar{V}(c_1\cup c_2\cup c_3\cup\{v^\delta_{\mathrm{E},(1,2)}\}) \quad \text{(D4)}$$

as the irrelevant vertices do not appear in $\bar{V}(\tilde{c})$. We can now take the limit of the above matrix element and get

$$\lim_{\delta\to 0}\Psi^{f^{(1)}}_{[c]_{(1)}}(\hat{H}^{(1)}_{T(\delta)}[N]|c\rangle) = \sum_{v\in V(c)}N(v)\lambda(\vec{n}^v_c)\Big[\langle\hat{E}^a_2(v)\rangle\frac{\partial}{\partial v^a}f^1(V(c)) - \langle\hat{E}^a_3(v)\rangle\frac{\partial}{\partial v^a}f^1(V(c))\Big]. \quad \text{(D5)}$$

Let us now look at the RHS of (8.6):

$$\text{RHS of (8.6)} = \sum_{v\in V(c)}[\Psi^{\bar{f}^{(1)(1)}_v}_{[c]_{(1)}}(|c\rangle) - \Psi^{\bar{\bar{f}}^{(1)(1)}_v}_{[c]_{(1)}}(|c\rangle)] = \sum_{v\in V(c)}[\bar{f}^{(1)(1)}_v(V(c)) - \bar{\bar{f}}^{(1)(1)}_v(V(c))], \quad \text{(D6)}$$

which, using the definitions of $\bar{f}^{(1)(1)}_v$ and $\bar{\bar{f}}^{(1)(1)}_v$ given in Eq. (8.8), matches with (D5).

b. Case (A,1,b): $c \neq \tilde{c} \in [c]_{(i)}$

There are three separate subcases in (A,1,b): Let $v_0 \in V(\tilde{c}_1 \cup c_2 \cup c_3)$ and let $\mathrm{supp}(N) = B(v_0, \epsilon)$. Then we have the following.

Case (A,1,b,i): $v_0 \in V(\tilde{c}_1 \cup c_2 \cup c_3)$ is monocolored. Since v_0 is monocolored and $(\tilde{c}_1, c_2, c_3) \in [c]_{(1)}$, there exists $v_0' \in V(c)$ with respect to which v_0 is WEO. Then

$$\text{RHS of (8.6)} = \lim_{\delta\to 0} \Psi^{(1)}_{[c]_{(1)}}(\hat{H}^{(1)}_{T(\delta)}[N]|\tilde{c}_1, c_2, c_3\rangle) = 0. \tag{D7}$$

On the other hand,

$$\begin{aligned}\text{LHS of (8.6)} &= \sum_{v\in V(c)} (\Psi^{\bar{f}^{(1)}_v}_{[c]_{(1)}}(|\tilde{c}_1, c_2, c_3\rangle) - \Psi^{\bar{\bar{f}}^{(1)}_v}_{[c]_{(1)}}(|\tilde{c}_1, c_2, c_3\rangle))\\ &= \bar{f}^{(1)}_{v_0'}(\bar{V}(\tilde{c}_1 \cup c_2 \cup c_3)) - \bar{\bar{f}}^{(1)}_{v_0'}(\bar{V}(\tilde{c}_1 \cup c_2 \cup c_3))\\ &= \bar{f}^{(1)}_{v_0'}(V(c_1 \cup c_2 \cup c_3)/v_0' \cup \{v_0\}) - \bar{\bar{f}}^{(1)}_{v_0'}(V(c_1 \cup c_2 \cup c_3)/v_0' \cup \{v_0\})\\ &= f^{(1)}_{v_0'}(V(c_1 \cup c_2 \cup c_3)/v_0' \cup \{v_0\}) - f^{(1)}_{v_0'}(V(c_1 \cup c_2 \cup c_3)/v_0' \cup \{v_0\}) = 0.\end{aligned} \tag{D8}$$

Case (A,1,b,ii): $v_0 \in V(\tilde{c}_1 \cup c_2 \cup c_3) \cap V(c)$ such that v_0 has an associated WEO vertex $v_0' \in V(\tilde{c}_1 \cup c_2 \cup c_3)$. Let N be such that the support of N only includes v_0, and no other vertex of $V(c)$ lies inside the support of N. We prove a small lemma which will be useful while analyzing this case.

Lemma.—There exists no charge network in $[c]_{(1)}$ which corresponds to $(\tilde{c}_1 \cup \alpha^{\delta}_{v_0}(\langle\hat{E}_2\rangle/\langle\hat{E}_3\rangle, n_{c_3}/n_{c_2}), c_2, c_3)$.

Proof.—$V(\tilde{c}_1 \cup \alpha^{\delta}_{v_0}(\langle\hat{E}_2\rangle/\langle\hat{E}_3\rangle, n_{c_3}/n_{c_2}) \cup c_2 \cup c_3)$ has a vertex which is not in $V(c)$ and is not WEO with respect to any vertex in $V(c)$. ■

Then

$$\begin{aligned}\lim_{\delta\to 0} \Psi^{f^{(1)}}_{[c]_{(1)}}(\hat{H}^{(1)}_{T(\delta)}[N]|\tilde{c}_1, c_2, c_3\rangle) &= \lim_{\delta\to 0} N(v_0)\lambda(\vec{n}^{v_0}_{(\tilde{c}_1,c_2,c_3)})\Psi^{f^{(1)}}_{[c]_{(1)}}(|\tilde{c}_1 \cup \alpha^{\delta}_{v}(\langle\hat{E}_2\rangle, n_{c_3}), c_2, c_3\rangle - |\tilde{c}_1 \cup \alpha^{\delta}_{v}(\langle\hat{E}_3\rangle, n_{c_2}), c_2, c_3\rangle)\\ &= \lim_{\delta\to 0} \Psi^{f^{(1)}}_{[c]_{(1)}} N(v_0)\lambda(\vec{n}^{v_0}_{c})(|\tilde{c}_1 \cup \alpha^{\delta}_{v}(\langle\hat{E}_2\rangle, n_{c_3}), c_2, c_3\rangle - |\tilde{c}_1 \cup \alpha^{\delta}_{v}(\langle\hat{E}_3\rangle, n_{c_2}), c_2, c_3\rangle),\end{aligned} \tag{D9}$$

where in the last line we have used $\lambda(\vec{n}^{v_0}_{(\tilde{c}_1,c_2,c_3)}) = \lambda(\vec{n}^{v_0}_{c})$.

Using the above lemma, we have

$$\Psi^{f^{(1)}}_{[c]_{(1)}}(|\tilde{c}_1 \cup \alpha^{\delta}_{v}(\langle\hat{E}_2\rangle, n_{c_3}), c_2, c_3\rangle) = \Psi^{f^{(1)}}_{[c]_{(1)}}(|\tilde{c}_1 \cup \alpha^{\delta}_{v}(\langle\hat{E}_3\rangle, n_{c_2}), c_2, c_3\rangle) = 0 \tag{D10}$$

$\forall\ \delta$. On the other hand, since N has support only in the neighborhood of $v_0 \in V(c)$,

$$(\Psi^{\bar{f}^{(1)}_{v_0}}_{[c]_{(1)}} - \Psi^{\bar{\bar{f}}^{(1)}_{v_0}}_{[c]_{(1)}})|\tilde{c}_1, c_2, c_3\rangle = (\bar{f}^{(1)}_{v_0}(\bar{V}(\tilde{c}_1 \cup c_2 \cup c_3)) - \bar{\bar{f}}^{(1)}_{v_0}(\bar{V}(\tilde{c}_1 \cup c_2 \cup c_3)). \tag{D11}$$

But since v_0 has a WEO vertex v_0' associated with it, the arguments of $\bar{f}^{(1)}_{v_0}$ and $\bar{\bar{f}}^{(1)}_{v_0}$ contain v_0' in place of v_0 and as both functions agree with $f^{(1)}$ when no argument is v_0, this also vanishes.

Case (A,1,b,iii): $v_0 \in V(\tilde{c}_1 \cup c_2 \cup c_3) \cap V(c)$ such that v_0 has an associated WEO vertex $v_0' \in V(\tilde{c}_1 \cup c_2 \cup c_3)$ which lies inside the support of N. The above argument goes through and both sides vanish.

Case (A,1,b,iv): $v_0 \in V(\tilde{c}_1 \cup c_2 \cup c_3) \cap V(c)$ such that v_0 has no associated WEO vertex. Let N be such that $\mathrm{supp}(N) \subset B(v_0, \epsilon)$ and $v \in V(c) \cap B(v_0, \epsilon) \Rightarrow v = v_0$. Then

$$\lim_{\delta\to 0} \Psi^{f^{(1)}}_{[c]_{(1)}}(\hat{H}^{(1)}_{T(\delta)}[N]|\tilde{c}_1, c_2, c_3\rangle) = \lim_{\delta\to 0} \Psi^{f^{(1)}}_{[c]_{(1)}}(|\tilde{c}_1 \cup \alpha^{\delta}_{v_0}(\langle\hat{E}_2\rangle, n^3), c_2, c_3\rangle - |\tilde{c}_1 \cup \alpha^{\delta}_{v_0}(\langle\hat{E}_3\rangle, n^2), c_2, c_3\rangle). \tag{D12}$$

Note that since $(\tilde{c}_1, c_2, c_3) \in [c]_{(1)}$, it is clear that $|\tilde{c}_1 \cup \alpha^{\delta}_{v_0}(\langle\hat{E}_2\rangle, n^3), c_2, c_3\rangle$ and $|\tilde{c}_1 \cup \alpha^{\delta}_{v_0}(\langle\hat{E}_3\rangle, n^2), c_2, c_3\rangle$ both belong to $[c]_{(1)}$. Whence, we have

$$\begin{aligned}\lim_{\delta\to 0}&\Psi^{f^{(1)}}_{[c]_{(1)}}(\hat{H}^{(1)}_{T(\delta)}[N]|\tilde{c}_1, c_2, c_3\rangle)\\ &=\lim_{\delta\to 0}\frac{1}{\delta}N(v_0)\lambda(\vec{n}^{v_0}_c)(f^{(1)}(\bar{V}(\tilde{c}_1\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_2\rangle, n^3)\cup c_2\cup c_3)) - f^{(1)}(\bar{V}(\tilde{c}_1\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_3\rangle, n^2)\cup c_2\cup c_3)))\\ &= N(v_0)\lambda(\vec{n}^{v_0}_c)\left[\langle\hat{E}^a_2\rangle\frac{\partial}{\partial v^a}f^{(1)}(V(c)) - \langle\hat{E}^a_3\rangle\frac{\partial}{\partial v^a}f^{(1)}(V(c))\right],\end{aligned} \tag{D13}$$

where in the first line we have used $\lambda(\vec{n}^{v_0}_{(\tilde{c}_1,c_2,c_3)}) = \lambda(\vec{n}^{v_0}_c)$ and in the second line we have used $\bar{V}(c) = V(c)$.

We now evaluate the RHS. Since N has support only in the neighborhood of the vertex v_0 in $V(c)$, the only nonvanishing contributions are through $\bar{f}_{v_0}$, $\bar{\bar{f}}_{v_0}$. $\bar{f}_{v_0}$: $\Sigma^{|V(c)|}\to\mathbb{R}$ such that

$$\bar{f}^{(1)}_{v_0}(v_1,\ldots,v_{|V(c)|}) = f^{(1)}(v_1,\ldots,v_{|V(c)|})\quad \text{if } \{v_1,\ldots,v_{|V(c)|}\}\neq V(c) \tag{D14}$$

and

$$\bar{f}^{(1)}_{v_0}(V(c)) = N(v_0)\lambda(\vec{n}^{v_0}_c)\langle\hat{E}_2\rangle(v_0)\frac{\partial}{\partial v^a_0}f^{(1)}(V(c)). \tag{D15}$$

Similarly, $\bar{\bar{f}}_{v_0}$: $\Sigma^{|V(c)|}\to\mathbb{R}$ such that

$$\bar{f}^{(1)}_{v_0}(v_1,\ldots,v_{|V(c)|}) = f^{(1)}(v_1,\ldots,v_{|V(c)|})\quad \text{if } (v_1,\ldots,v_{|V(c)|})\neq V(c) \tag{D16}$$

and

$$\bar{f}^{(1)}_{v_0}(V(c)) = N(v_0)\lambda(\vec{n}^{v_0}_c)\langle\hat{E}_2\rangle(v_0)\frac{\partial}{\partial v^a_0}f^{(1)}(V(c)). \tag{D17}$$

Case (A,1,c): $\tilde{c}\notin[c]_{(i)}$. It is rather straightforward to see that both sides vanish in this case.

2. Case (A,2): $i\neq j$

Let $i=1$, $j=2$. Other cases can be analyzed analogously. Thus our aim is to show that, given $\Psi^{f^{(1)}}_{[c]_{(1)}}$,

$$\lim_{\delta\to 0}\Psi^{f^{(1)}}_{[c]_{(1)}}(\hat{H}^{(2)}_{T(\delta)}[N]|\tilde{c}\rangle) = \sum_{v\in V(c)}(\Psi^{\bar{f}^{(1)(2)}_v}_{[c]_{(1)}} - \Psi^{\bar{\bar{f}}^{(1)(2)}_{v_0}}_{[c]_{(1)}})|\tilde{c}\rangle \tag{D18}$$

$\forall\, N$, $|\tilde{c}\rangle$, and where $\bar{f}^{(1)(2)}_v$ and $\bar{\bar{f}}^{(1)(2)}_{v_0}$ are defined in (8.9). As before we consider different cases, as follows.

a. Case (A,2,a): $\tilde{c}$ does not have an EO vertex of type $M=1$

In this case, $\forall\,\delta>0$, $\hat{H}^{(2)}_{T(\delta)}[N]$ will create states with EO vertices of type $M=2$ when acting on $|\tilde{c}\rangle$. However, as $[c]_{(1)}$ has no states with EO vertices of type $M=2$, LHS $=$ RHS $=0$.

b. Case (A,2,b): $\tilde{c}$ does have an EO vertex of type $M=1$, $j=2$

This condition implies that $\tilde{c} = (c'_1\cup\alpha^{\delta_0}_{v_0}(\langle\hat{E}_2\rangle_{\tilde{c}_2}, n_{\tilde{c}_3}), \tilde{c}_2, \tilde{c}_3)$ for some c'_1. Let us also assume that this vertex is inside the support of the lapse function N. Now both the LHS and RHS are nonzero if and only if $(c'_1, \tilde{c}_2, \tilde{c}_3) = c$. In this case we have [using the equation for the action of the Hamiltonian constraint on charge networks involving these specific types of EO vertices as given in (7.23)]

$$\begin{aligned}\text{LHS} &= \lim_{\delta\to 0}\Psi^{f^{(1)}}_{[c]_{(1)}}(\hat{H}^{(2)}_{T(\delta)}[N]|c_1\cup\alpha^{\delta_0}_{v_0}(\langle\hat{E}_2\rangle_{c_2}(v_0), n_{c_3}), c_2, c_3\rangle)\\ &= \lim_{\delta\to 0}\left(\frac{1}{\delta}\Psi^{f^{(1)}}_{[c]_{(1)}}\left[\sum_{e\in E(c)|b(e)=v_0}(\langle\hat{E}_3(L_e(\delta'))\rangle\dot{e}^a(0)(N(v_0+\delta\dot{e}(0)) - N(v_0))|c_1\cup\alpha^{\delta^0}_{v_0}(\langle\hat{E}_1\rangle(v_0), n^3), c_2, c_3\rangle\right]\right.\\ &\quad\left. - \frac{1}{\delta}\Psi^{f^{(1)}}_{[c]_{(1)}}\left[\sum_{e\in E(c)|b(e)=v_0}(\langle\hat{E}_1(L_e(\delta'))\rangle\dot{e}^a(0)(N(v_0+\delta\dot{e}(0)) - N(v_0))|c_1\cup\alpha^{\delta^0}_{v_0}(\langle\hat{E}_3\rangle(v_0), n^3), c_2, c_3\rangle\right]\right),\end{aligned} \tag{D19}$$

where $\delta' \to 0$ is faster than $\delta \to 0$. However note that the (net of) flux expectation values remains constant in the limit $\delta' \to 0$ (as they are simply equal to n^1 or n^3), whence $\langle \hat{E}_3(L_e(\delta'))\rangle$ is independent of δ' and we denote it simply as $\langle \hat{E}_3(L_e)\rangle$ where L_e could be any surface fixed once and for all. It is easy to see that the left-handside simplifies to

$$\mathrm{LHS} = \sum_{e\in E(c)|b(e)=v_0} \langle \hat{E}_3(L_e)\rangle \dot{e}^a(0)\partial_a N(v_0) f^{(1)}(\bar{V}(c_1 \cup \alpha_{v_0}^{\delta_0}(\langle \hat{E}_1\rangle(v_0), n^3) \cup c_2 \cup c_3) - \sum_{e\in E(c)|b(e)=v_0} \langle \hat{E}_1(L_e)\rangle \dot{e}^a(0)\partial_a N(v_0) f^{(1)}(\bar{V}(c_1 \cup \alpha_{v_0}^{\delta_0}(\langle \hat{E}_3(v_0)\rangle, n^3) \cup c_2 \cup c_3). \tag{D20}$$

But upon using (8.8) we see that this precisely equals the right-hand side.

APPENDIX E: ON THE MINUS SIGN IN EQ. (9.5)

Consider the ideal scenario in which the continuum Hamiltonian constraint $\hat{H}[N]'$ preserves $\mathcal{V}_{\mathrm{LMI}}$. In that case we would seek to prove that

$$[\hat{H}[N]', \hat{H}[M]']\Psi = \mathrm{i}\hbar \hat{V}[\vec{\omega}]'\Psi \tag{E1}$$

$\forall\, \Psi \in \mathcal{V}_{\mathrm{LMI}}$. This would imply that $\forall\, |c\rangle$,

$$\lim_{\delta,\delta'\to 0} (\Psi|(\hat{H}_{T(\delta')}[M]\hat{H}_{T(\delta)}[N] - (N \leftrightarrow M))|c\rangle = -\mathrm{i}\hbar \lim_{\delta\to 0} (\Psi|\hat{V}_{T(\delta)}[\vec{\omega}]|c\rangle. \tag{E2}$$

However notice that the left-hand side of the above equation is

$$([\hat{H}[N], \hat{H}[M]]'\Psi)|c\rangle$$

and the right-hand side is

$$(-\mathrm{i}\hbar)(\hat{V}[\vec{\omega}]'\Psi)|c\rangle.$$

Whence we are led to prove that

$$[\hat{H}[N], \hat{H}[M]]'\Psi = -\mathrm{i}\hbar \hat{V}[\vec{\omega}]\Psi. \tag{E3}$$

Technically we are seeking an anitrepresentation of the Hamiltonian constraint on $\mathcal{V}_{\mathrm{LMI}}$.

APPENDIX F: DETAILS OF THE COMMUTATOR COMPUTATION

In this appendix we derive Eq. (9.12). A key ingredient in this derivation is the fact that the action of the Hamiltonian constraint on irrelevant vertices is trivial. Given a charge-network state $|c'\rangle$, our first objective is to evaluate

$$\sum_{i,j} [\hat{H}^{(i)}_{T(\delta')}[N], \hat{H}^{(j)}_{T(\delta)}[M]]|c'\rangle. \tag{F1}$$

The nine terms in the commutator can be grouped in the following way:

$$\Big(\sum_i (\hat{H}^{(i)}_{T(\delta')}[N]\hat{H}^{(i)}_{T(\delta)}[M] - (N \leftrightarrow M)) + \sum_{i\neq j} (\hat{H}^{(i)}_{T(\delta')}[N]\hat{H}^{(j)}_{T(\delta)}[M] - (N \leftrightarrow M))\Big)|c'\rangle. \tag{F2}$$

It is easy to see that $[\hat{H}^{(i)}_{T(\delta')}[N], \hat{H}^{(i)}_{T(\delta)}[M]]$ do not contribute for any i. Let us consider the action of $[\hat{H}^{(1)}_{T(\delta')}[N], \hat{H}^{(1)}_{T(\delta)}[M]]$ on $|c'\rangle$. Recall that (in the case of a single vertex in the support of N)

$$\hat{H}^{(1)}_{T(\delta)}[N]|c'\rangle := N(v)\hat{H}^{(1)}_{T(\delta)}(v)|c'\rangle, \tag{F3}$$

and let

$$\hat{H}^{(1)}_{T(\delta)}(v)|c'\rangle = \sum_{j=1}^{2} |c'^{v}_{\delta j}\rangle. \tag{F4}$$

Then we have

$$\begin{aligned}
[\hat{H}^{(1)}_{T(\delta')}[N], \hat{H}^{(1)}_{T(\delta)}[M]]|c'\rangle &= \sum_{v\in V(c')} (\hat{H}^{(1)}_{T(\delta')}[N]M(v)\hat{H}^{(1)}_{T(\delta)}(v) - (N \leftrightarrow M))|c'\rangle \\
&= \sum_{j=1}^{2}\sum_{v\in V(c')}\sum_{v'\in V(c'^{v}_{\delta j})} (N(v')M(v) - M(v')N(v))\hat{H}^{(1)}_{T(\delta')}(v')\hat{H}^{(1)}_{T(\delta)}(v)|c'\rangle \\
&= \sum_{j=1}^{2}\sum_{v\in V(c')}\sum_{v'\in V(c')} (N(v')M(v) - M(v')N(v))\hat{H}^{(1)}_{T(\delta')}(v')\hat{H}^{(1)}_{T(\delta)}(v)|c'\rangle = 0.
\end{aligned} \tag{F5}$$

In the third line we have used the fact that action of $\hat{H}^{(1)}_{T(\delta)}(v)$ on EO vertices of type $M = 1$ is zero and that the action of $\hat{H}^{(1)}_{T(\delta')}$ on four-valent irrelevant vertices resulting from the action of $\hat{H}^{(1)}_{T(\delta)}$ is zero (due to the specific charge configuration on the incident edges). This shows that the first set of terms $\sum_i (\hat{H}^{(i)}_{T(\delta\prime)}[N]\hat{H}^{(i)}_{T(\delta)}[M] - (N \leftrightarrow M))$ does not contribute.

For the second set of terms with $\sum_{i\neq j}$, we first group them as follows:

$$\sum_{i\neq j}(\hat{H}^{(i)}_{T(\delta')}[N]\hat{H}^{(j)}_{T(\delta)}[M]-(N\leftrightarrow M))|c'\rangle$$
$$=[(\hat{H}^{(1)}_{T(\delta')}[N](\hat{H}^{(2)}_{T(\delta)}[M]+\hat{H}^{(3)}_{T(\delta)}[M])-(N\leftrightarrow M))+(\hat{H}^{(2)}_{T(\delta')}[N](\hat{H}^{(3)}_{T(\delta)}[M]+\hat{H}^{(1)}_{T(\delta)}[M])$$
$$-(N\leftrightarrow M))+(\hat{H}^{(3)}_{T(\delta')}[N](\hat{H}^{(1)}_{T(\delta)}[M]+\hat{H}^{(2)}_{T(\delta)}[M])-(N\leftrightarrow M))]|c'\rangle. \tag{F6}$$

Due to the antisymmetrization in the lapse functions, all the terms that are ultralocal (without derivatives) in the lapses vanish. Thus, the above equation simplifies to

$$\sum_{i\neq j}(\hat{H}^{(i)}_{T(\delta')}[N]\hat{H}^{(j)}_{T(\delta)}[M]-(N\leftrightarrow M))|c'\rangle$$
$$\approx\frac{1}{4}\left(\frac{\hbar}{\mathrm{i}}\right)^2\frac{1}{\delta\delta'}M(v_0)\lambda(\vec{n}^{c'}_{v_0})^2\sum_{e\in E(c')|b(e)=v_0}(N(v_0+\delta'\dot{e}(0))-N(v_0))[(\langle\hat{E}_3(L_e)\rangle|c'_1\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_1\rangle(v_0),n_{c'_3}),c'_2,c'_3\rangle$$
$$-\langle\hat{E}_1(L_e)\rangle|c'_1\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_3\rangle(v_0),n_{c'_3}),c'_2,c'_3\rangle)-(\langle\hat{E}_1(L_e)\rangle|c'_1\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_2\rangle(v_0),n_{c'_2}),c'_2,c'_3\rangle$$
$$-\langle\hat{E}_2(L_e)\rangle|c'_1\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_2\rangle(v_0),n_{c'_2}),c'_2,c'_3\rangle)+(\langle\hat{E}_1(L_e)\rangle|c'_1,c'_2\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_2\rangle(v_0),n_{c'_1}),c'_3\rangle$$
$$-\langle\hat{E}_2(L_e)\rangle|c'_1,c'_2\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_1\rangle(v_0),n_{c'_1}),c'_3\rangle)-(\langle\hat{E}_2(L_e)\rangle|c'_1,c'_2\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_3\rangle(v_0),n_{c'_3}),c'_3\rangle$$
$$-\langle\hat{E}_3(L_e)\rangle|c'_1,c'_2\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_2\rangle(v_0),n_{c'_3}),c'_3\rangle)+(\langle\hat{E}_2(L_e)\rangle|c'_1,c'_2,c'_3\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_3\rangle(v_0),n_{c'_3})\rangle$$
$$-\langle\hat{E}_3(L_e)\rangle|c'_1,c'_2,c'_3\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_2\rangle(v_0),n_{c'_3})\rangle)-(\langle\hat{E}_3(L_e)\rangle|c'_1,c'_2,c'_3\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_1\rangle(v_0),n_{c'_1})\rangle$$
$$-\langle\hat{E}_1(L_e)\rangle|c'_1,c'_2,c'_3\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_3\rangle(v_0),n_{c'_1})\rangle)]-(N\leftrightarrow M). \tag{F7}$$

Some remarks are in order: Without loss of generality we are assuming that δ is small enough such that all EO vertices that are created in the neighborhood of v_0 are in the support of both N and M. The weak equality $\approx$ indicates that we have thrown away all the terms resulting from the action of the second Hamiltonian on irrelevant vertices, as these terms will not contribute once we "dot" them with a habitat state. Henceforth we will understand that these additional terms have been thrown away and we will replace $\approx$ with exact equality. Due to reasons explained above (D19), we have omitted the δ' label from the surface L_e. The overall plus sign comes from the fact that we chose $\epsilon=1$ in (7.23).

Due to the underlying symmetry among the terms on the right-hand side, we can rewrite the above equation as

$$\sum_{i\neq j}(\hat{H}^{(i)}_{T(\delta')}[N]\hat{H}^{(j)}_{T(\delta)}[M]-(N\leftrightarrow M))|c'\rangle$$
$$=\frac{1}{4}\left(\frac{\hbar}{\mathrm{i}}\right)^2\frac{1}{\delta\delta'}M(v_0)\lambda(\vec{n}^{c'}_{v_0})^2\sum_{e\in E(c')|b(e)=v_0}(N(v_0+\delta'\dot{e}(0))-N(v_0))[(\langle\hat{E}_3(L_e)\rangle(|c'_1\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_1\rangle(v_0),n_{c'_3}),c'_2,c'_3\rangle-|c'\rangle)$$
$$-\langle\hat{E}_1(L_e)\rangle(|c'_1\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_3\rangle(v_0),n_{c'_3}),c'_2,c'_3\rangle-|c'\rangle))-(\langle\hat{E}_1(L_e)\rangle(|c'_1\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_2\rangle(v_0),n_{c'_2}),c'_2,c'_3\rangle-|c'\rangle)$$
$$-\langle\hat{E}_2(L_e)\rangle(|c'_1\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_2\rangle(v_0),n_{c'_2}),c'_2,c'_3\rangle-|c'\rangle))+(\langle\hat{E}_1(L_e)\rangle(|c'_1,c'_2\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_2\rangle(v_0),n_{c'_1}),c'_3\rangle-|c'\rangle)$$
$$-\langle\hat{E}_2(L_e)\rangle(|c'_1,c'_2\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_1\rangle(v_0),n_{c'_1}),c'_3\rangle-|c'\rangle))-(\langle\hat{E}_2(L_e)\rangle(|c'_1,c'_2\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_3\rangle(v_0),n_{c'_3}),c'_3\rangle-|c'\rangle)$$
$$-\langle\hat{E}_3(L_e)\rangle(|c'_1,c'_2\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_2\rangle(v_0),n_{c'_3}),c'_3\rangle-|c'\rangle))+(\langle\hat{E}_2(L_e)\rangle(|c'_1,c'_2,c'_3\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_3\rangle(v_0),n_{c'_3})\rangle-|c'\rangle)$$
$$-\langle\hat{E}_3(L_e)\rangle(|c'_1,c'_2,c'_3\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_2\rangle(v_0),n_{c'_3})\rangle-|c'\rangle))-(\langle\hat{E}_3(L_e)\rangle(|c'_1,c'_2,c'_3\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_1\rangle(v_0),n_{c'_1})\rangle-|c'\rangle)$$
$$-\langle\hat{E}_1(L_e)\rangle(|c'_1,c'_2,c'_3\cup\alpha^{\delta}_{v_0}(\langle\hat{E}_3\rangle(v_0),n_{c'_1})\rangle-|c'\rangle))]-(N\leftrightarrow M). \tag{F8}$$

We have added and subtracted $|c'\rangle$ to ensure that the commutator has a well-defined continuum limit on the LMI habitat.

We will divide the right-hand side of (F8) into three pieces. This will aide us in analyzing the continuum limit in a rather straightforward manner.

$$
\begin{aligned}
|\psi_1^{\delta,\delta'}(c',[M,N])\rangle := \frac{1}{4}\left(\frac{\hbar}{\mathrm{i}}\right)^2 \frac{1}{\delta\delta'} M(v_0)\lambda(\vec{n}_{v_0}^{c'})^2 \sum_{e\in E(c')|b(e)=v_0} &(N(v_0+\delta'\dot{e}(0)) \\
&- N(v_0))[(\langle\hat{E}_3(L_e)\rangle(|c_1'\cup\alpha_{v_0}^{\delta}(\langle\hat{E}_1\rangle(v_0),n_{c_3'}),c_2',c_3'\rangle - |c'\rangle) \\
&- \langle\hat{E}_1(L_e)\rangle(|c_1'\cup\alpha_{v_0}^{\delta}(\langle\hat{E}_3\rangle(v_0),n_{c_3'}),c_2',c_3'\rangle - |c'\rangle)) \\
&- (\langle\hat{E}_1(L_e)\rangle(|c_1'\cup\alpha_{v_0}^{\delta}(\langle\hat{E}_2\rangle(v_0),n_{c_2'}),c_2',c_3'\rangle - |c'\rangle) \\
&- \langle\hat{E}_2(L_e)\rangle(|c_1'\cup\alpha_{v_0}^{\delta}(\langle\hat{E}_2\rangle(v_0),n_{c_2'}),c_2',c_3'\rangle - |c'\rangle))] - (N\leftrightarrow M). \qquad \text{(F9)}
\end{aligned}
$$

$$
\begin{aligned}
|\psi_2^{\delta,\delta'}(c',[M,N])\rangle := \frac{1}{4}\left(\frac{\hbar}{\mathrm{i}}\right)^2 \frac{1}{\delta\delta'} M(v_0)\lambda(\vec{n}_{v_0}^{c'})^2 \sum_{e\in E(c')|b(e)=v_0} &(N(v_0+\delta'\dot{e}(0)) \\
&- N(v_0))[(\langle\hat{E}_1(L_e)\rangle(|c_1',c_2'\cup\alpha_{v_0}^{\delta}(\langle\hat{E}_2\rangle(v_0),n_{c_1'}),c_3'\rangle - |c'\rangle) \\
&- \langle\hat{E}_2(L_e)\rangle(|c_1',c_2'\cup\alpha_{v_0}^{\delta}(\langle\hat{E}_1\rangle(v_0),n_{c_1'}),c_3'\rangle - |c'\rangle)) \\
&- (\langle\hat{E}_2(L_e)\rangle(|c_1',c_2'\cup\alpha_{v_0}^{\delta}(\langle\hat{E}_3\rangle(v_0),n_{c_3'}),c_3'\rangle - |c'\rangle) \\
&- \langle\hat{E}_3(L_e)\rangle(|c_1',c_2'\cup\alpha_{v_0}^{\delta}(\langle\hat{E}_2\rangle(v_0),n_{c_3'}),c_3'\rangle - |c'\rangle))] - (N\leftrightarrow M). \qquad \text{(F10)}
\end{aligned}
$$

$$
\begin{aligned}
|\psi_3^{\delta,\delta'}(c',[M,N])\rangle := \frac{1}{4}\left(\frac{\hbar}{\mathrm{i}}\right)^2 \frac{1}{\delta\delta'} M(v_0)\lambda(\vec{n}_{v_0}^{c'})^2 \sum_{e\in E(c')|b(e)=v_0} &(N(v_0+\delta'\dot{e}(0)) \\
&- N(v_0))[(\langle\hat{E}_2(L_e)\rangle(|c_1',c_2',c_3'\cup\alpha_{v_0}^{\delta}(\langle\hat{E}_3\rangle(v_0),n_{c_3'})\rangle - |c'\rangle) \\
&- \langle\hat{E}_3(L_e)\rangle(|c_1',c_2',c_3'\cup\alpha_{v_0}^{\delta}(\langle\hat{E}_2\rangle(v_0),n_{c_3'})\rangle - |c'\rangle)) \\
&- (\langle\hat{E}_3(L_e)\rangle(|c_1',c_2',c_3'\cup\alpha_{v_0}^{\delta}(\langle\hat{E}_1\rangle(v_0),n_{c_1'})\rangle - |c'\rangle) \\
&- \langle\hat{E}_1(L_e)\rangle(|c_1',c_2',c_3'\cup\alpha_{v_0}^{\delta}(\langle\hat{E}_3\rangle(v_0),n_{c_1'})\rangle - |c'\rangle))] - (N\leftrightarrow M). \qquad \text{(F11)}
\end{aligned}
$$

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