PHYSICAL REVIEW D **88,** 044014 (2013)

Area products for stationary black hole horizons

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(Received 19 June 2012; published 7 August 2013)

Area products for multihorizon stationary black holes often have intriguing properties, and are often (though not always) independent of the mass of the black hole itself (depending only on various charges, angular momenta, and moduli). Such products are often formulated in terms of the areas of inner (Cauchy) horizons and outer (event) horizons, and sometimes include the effects of unphysical "virtual" horizons. But the conjectured mass independence sometimes fails. Specifically, for the Schwarzschild–de Sitter [Kottler] black hole in (3 + 1) dimensions it is shown by explicit exact calculation that the product of event horizon area and cosmological horizon area is *not* mass independent. (Including the effect of the third "virtual" horizon does not improve the situation.) Similarly, in the Reissner-Nordstrom–anti-de Sitter black hole in (3 + 1) dimensions the product of the inner (Cauchy) horizon area and event horizon area is calculated (perturbatively), and is shown to be *not* mass independent. That is, the mass independence of the product of physical horizon areas is *not* generic. In spherical symmetry, whenever the quasilocal mass $m(r)$ is a Laurent polynomial in aerial radius, $r = \sqrt{A/4\pi}$, there are significantly more complicated mass-independent quantities, the elementary symmetric polynomials built up from the complete set of horizon radii (physical *and* virtual). Sometimes it is possible to eliminate the unphysical virtual horizons, constructing combinations of physical horizon areas that are mass independent, but they tend to be considerably more complicated than the simple products and related constructions currently being mooted in the literature.

DOI: 10.1103/PhysRevD.88.044014 PACS numbers: 04.70.−s, 04.70.Bw, 04.70.Dy, 04.20.Jb

I. INTRODUCTION

There has recently been some considerable ongoing interest in the products of horizon areas for various types of stationary black holes. Some of this interest has arisen specifically within the general relativity community [1–4], while for somewhat different reasons interest has also arisen from within the string community [5–8]. In some cases the product of horizon areas is in fact independent of the mass of the black hole.

For instance, based on classical general relativistic techniques it is known that both for standard (3 + 1) dimensional Kerr-Newman, and even for (3 + 1) dimensional Kerr-Newman black holes distorted by the presence of arbitrary stationary axisymmetric matter, the product of the inner (Cauchy) horizon area and outer (event) horizon areas is [1–4]

$$A_C A_E = (8\pi)^2 \left[J^2 + \frac{Q^4}{4} \right]. \tag{1}$$

The underlying physics here is that due to stationarity there can be no matter present between the inner and outer horizons (where the radial direction is timelike) [9]. The region between inner and outer horizons is then stationary, axisymmetric, and electro-vac; this is not quite enough to be able to apply the black hole uniqueness theorems, but it appears that enough of the flavor of uniqueness survives to guarantee that the area product is not only independent of the mass of the black hole, but more remarkably is independent of the way the static axisymmetric matter external to the black hole (and distorting its gravitational field away from exact Kerr-Newman) is distributed. (Note that in the special case of extremal horizons, such conditions are enough to guarantee that the near horizon geometry uniquely corresponds to extreme Kerr–Newman [10,11].) These results are closely related to mass-independent inequalities for the area of generic dynamical axisymmetric apparent horizons, holding in particular for the outer Killing horizon in stationary axisymmetric black holes with surrounding matter [12–22]:

$$A_E \geq 8\pi \sqrt{J^2 + \frac{Q^4}{4}}. \tag{2}$$

Apart from the standard (3 + 1) dimensional Kerr-Newman spacetime, there are also many multidimensional string-inspired black hole configurations for which similar formulas hold [5–8]. More boldly, there are also conjectures to the effect that this product of areas is sometimes quantized. That is, in the supersymmetric extremal limit one often finds

$$A_C A_E = (8\pi)^2 L_P^4 N \quad \text{with} \quad N \in \mathbb{N}. \tag{3}$$

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For specific discussion of potential pitfalls for such a conjecture see [23,24]. A safer statement is that when one moves away from extremality and supersymmetry then quite often the product of areas is discretized in terms of the Planck area and fine structure constant with

$$A_C A_E = (8\pi)^2 L_P^4 \left\{ \ell(\ell+1) + \frac{\alpha^2 q^2}{4} \right\}, \tag{4}$$

or some natural generalization thereof [23]. Here $\ell \in \mathbb{N}$ and $q \in \mathbb{Z}$.

But how generic are such mass-independence results? For instance, to what extent do they survive introduction of a cosmological constant? It is already known that the area inequalities behave in a more complicated manner once a cosmological constant is introduced [15,19]. Herein we address this issue in an elementary way by straightforwardly exhibiting several simple spherically symmetric (3 + 1) dimensional examples where, due precisely to a nonzero cosmological constant, the product of physical horizon areas is explicitly *not* mass independent. We shall explicitly consider the Schwarzschild–(anti)-de Sitter and Reissner-Nordström–(anti)-de Sitter spacetimes, before considering general lessons we can extract for generic static spherically symmetric spacetimes. The fact that asymptotically anti-de Sitter black holes often fail to have mass-independent area products is perhaps of most interest to the string community, indicating that more complicated functions of horizon area might be of interest.

There will typically be *some* (sometimes several) more complicated functions of physical horizon areas that are mass independent, but generically these functions are nowhere near as straightforward as a simple product of areas. As we shall soon see, obtaining mass-independent functions of horizon areas in spherical symmetry is intimately related to the quasilocal mass $m(r)$ being a Laurent polynomial of the areal radius r defined by $A(r) = 4\pi r^2$. (Because of spherical symmetry the quasilocal mass is always guaranteed to be well defined, and so is a sufficiently general tool for the current article. Any attempt at moving to axisymmetry would require slightly more subtle tools; the norm of the horizon-generating Killing vector is an appropriate quantity to consider.) The relevant mass-independent area-related functions are constructed in terms of the elementary symmetric polynomials built up from the radii of the various horizons (both physical *and* virtual). Sometimes one can eliminate the virtual horizons to obtain more complicated mass-independent qualities depending only on the physical horizons.

II. FRAMEWORK

Based only on symmetry one can without any loss of generality write any static spherically symmetric spacetime in the form [25]

$$ds^2 = -\exp\{2\Phi(r)\}\left(1 - \frac{2m(r)}{r}\right)dt^2 + \frac{dr^2}{1 - 2m(r)/r} + r^2\{d\theta^2 + \sin^2\theta d\varphi^2\}. \tag{5}$$

Here $m(r)$ denotes the quasilocal mass [26,27], and $\Phi(r)$ is the anomalous redshift [25]. The Killing horizons are then found by solving

$$\Delta(r) \equiv 1 - \frac{2m(r)}{r} = 0. \tag{6}$$

Once we have extracted the various roots of this equation, the individual horizon areas are immediate.

III. SCHWARZSCHILD–DE SITTER BLACK HOLES

For Schwarzschild–de Sitter (Kottler) black holes the Killing horizons are found by solving the equation

$$\Delta(r) = 1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2 = 0. \tag{7}$$

This is equivalent to solving the cubic

$$r^3 - 3r/\Lambda + 6m/\Lambda = 0. \tag{8}$$

For $\Lambda > 0$ it is convenient to set $\Lambda = 1/a^2$, where a is (asymptotically) the spatial radius of curvature. Then

$$r^3 - 3ra^2 + 6ma^2 = 0. \tag{9}$$

The three *exact* roots for this cubic are (see the Appendix)

$$r = 2a\sin\left(\frac{1}{3}\sin^{-1}\left(\frac{3m}{a}\right) + \epsilon\frac{2\pi}{3}\right); \quad \epsilon \in \{0, \pm 1\}. \tag{10}$$

A. Killing horizons

The two physical roots are the event horizon at

$$r_E = 2a\sin\left(\frac{1}{3}\sin^{-1}\left(\frac{3m}{a}\right)\right) = 2m + \frac{8m^3}{3a^3} + \mathcal{O}\left(\frac{m^5}{a^4}\right), \tag{11}$$

and the cosmological horizon at

$$r_\Lambda = 2a\sin\left(\frac{2\pi}{3} + \frac{1}{3}\sin^{-1}\left(\frac{3m}{a}\right)\right) = \sqrt{3}a - m + \mathcal{O}\left(\frac{m^2}{a}\right). \tag{12}$$

There is a third (unphysical and purely formal) "virtual" horizon which is located at negative r:

$$r_V = -r_\Lambda - r_E. \tag{13}$$

Note that the product of physical horizon areas, $A_E \times A_\Lambda$, has no nice quantization features. Nor does it have any nice "independence of mass" features. Indeed

$$A_E \times A_\Lambda = (16\pi a^2)^2 \sin^2\left(\frac{2\pi}{3} + \frac{1}{3}\sin^{-1}\left(\frac{3m}{a}\right)\right) \times \sin^2\left(\frac{1}{3}\sin^{-1}\left(\frac{3m}{a}\right)\right) \quad (14)$$

$$= (8\pi a^2)^2 \left[\cos\left(\frac{2\pi}{3} + \frac{2}{3}\sin^{-1}\left(\frac{3m}{a}\right)\right) - \frac{1}{2}\right]^2 \quad (15)$$

$$= (8\pi)^2\{3m^2a^2 - 2\sqrt{3}m^3a + 6m^4 + \mathcal{O}(m^5/a)\}. \quad (16)$$

If one restricts attention to the two physical horizons at the two physical roots of the cubic, then in terms of area products this is the best one can do. If one includes the effect of the virtual horizon r_V, as advocated in Ref. [5], then we have the exact results

$$r_V r_E r_\Lambda = -6ma^2; \qquad A_V A_E A_\Lambda = (4\pi)^3 36 m^2 a^4. \quad (17)$$

These are, however, explicitly mass-dependent quantities.

B. Mass independence

In counterpoint, note that there *is* an exact mass-independent quantity arising from a *quadratic* sum over all three roots of the cubic. Namely,

$$\sum_{i>j} r_i r_j = -3a^2. \quad (18)$$

That is

$$r_V\{r_E + r_\Lambda\} + r_E r_\Lambda = -3a^2. \quad (19)$$

We can eliminate the virtual radius and rewrite this as

$$\{r_\Lambda + r_E\}^2 - r_\Lambda r_E = 3a^2, \quad (20)$$

and so

$$r_\Lambda^2 + r_E^2 + r_\Lambda r_E = 3a^2. \quad (21)$$

If one prefers to work in terms of areas one has

$$A_\Lambda + A_E + \sqrt{A_\Lambda A_E} = 12\pi a^2. \quad (22)$$

So there is certainly *some* function of physical horizon areas that is mass independent, but the function that exhibits mass independence is nowhere near as straightforward as a simple product of horizon areas.

IV. SCHWARZSCHILD–ANTI-DE SITTER BLACK HOLES

Consider the Schwarzschild–anti-de Sitter black hole. Now set $\Lambda = -1/|a|^2$. We determine the Killing horizons via the polynomial

$$r^3 + 3r|a|^2 - 6m|a|^2 = 0. \quad (23)$$

There is now only one physical root, only one physical horizon (an event horizon), located at

$$r_E = 2|a| \sinh\left(\frac{1}{3}\sinh^{-1}\left(\frac{3m}{|a|}\right)\right). \quad (24)$$

To make this fully explicit, in terms of Cardano's formulas one can rewrite this as

$$r_E = |a|\left\{\left[\sqrt{1+\frac{9m^2}{|a|^2}}+\frac{3m}{|a|}\right]^{1/3} - \left[\sqrt{1+\frac{9m^2}{|a|^2}}-\frac{3m}{|a|}\right]^{1/3}\right\}. \quad (25)$$

There are now two purely formal and unphysical virtual horizons, at complex conjugate values of r_V and r_V^*. There is an exact result that

$$r_V r_V^* r_E = 6m|a|^2; \qquad A_V A_V^* A_E = (4\pi)^3 36 m^2 a^4. \quad (26)$$

This is again explicitly mass dependent. The mass-independent quantity constructed from the event horizon and two virtual horizons is now

$$\sum_{i>j} r_i r_j = 3|a|^2. \quad (27)$$

That is

$$r_E\{r_V + r_V^*\} + r_V r_V^* = 3|a|^2. \quad (28)$$

We can simplify this a little by noting that

$$r_E + r_V + r_V^* = 0, \quad (29)$$

so that

$$r_E^2 = r_V r_V^* - 3|a|^2; \qquad A_E = |A_V| - 12\pi|a|^2. \quad (30)$$

This is at least formally mass independent—but since $|A_V|$ is not directly observable (and not calculable except by explicitly solving the mass-dependent cubic), the result is not particularly useful.

V. REISSNER-NORDSTRÖM–DE SITTER BLACK HOLES

The situation improves somewhat for Reissner-Nordström–de Sitter black holes. To locate the Killing horizons we need to find the roots of

$$\Delta(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{1}{3}\Lambda r^2 = 0. \quad (31)$$

Again setting $\Lambda = 1/a^2$, we now rearrange this to obtain the quartic

$$r^4 - 3r^2a^2 + 6mra^2 - 3Q^2a^2 = 0. \quad (32)$$

Taking $\Lambda \to 0$ (corresponding to $a \to \infty$) gives the standard Reissner-Nordström geometry. Also, $Q \to 0$ gives the Schwarzschild–de Sitter (Kottler) solution previously considered. Let us now write the quartic as

$$r^4 - 3a^2\{r^2 - 2mr + Q^2\} = 0, \quad (33)$$

and reformulate this as

$$r^4 - 3a^2(r - r_+)(r - r_-) = 0. \tag{34}$$

Here $r_\pm$ are the locations where the horizons would be in the limit where the cosmological constant is switched off ($\Lambda \to 0$, that is, $a \to \infty$). For simplicity we shall take $|Q| \le m$, so that the $r_\pm$ are guaranteed real. (There is no real point to considering the subcase where $r_\pm$ are complex.)

A. Approximate results

While we know on general principles that the quartic appearing above has an exact solution, it can be more advantageous to perturbatively extract approximate solutions. First, rearrange the quartic to yield the exact equation

$$r = r_\pm + \frac{r^4}{3a^2(r - r_\mp)}. \tag{35}$$

We shall now solve this equation perturbatively.

1. Event and Cauchy horizons

To a first approximation, for the event horizon we have

$$r_E \approx r_+ + \frac{r_+^4}{3a^2(r_+ - r_-)} = r_+\left\{1 + \frac{r_+^3}{3a^2(r_+ - r_-)}\right\}. \tag{36}$$

For the inner (Cauchy) horizon we have

$$r_C \approx r_- - \frac{r_-^4}{3a^2(r_+ - r_-)} = r_-\left\{1 - \frac{r_-^3}{3a^2(r_+ - r_-)}\right\}. \tag{37}$$

Consequently,

$$r_E r_C \approx r_+ r_-\left\{1 + \frac{r_+^3 - r_-^3}{3a^2(r_+ - r_-)}\right\}, \tag{38}$$

and so

$$r_E r_C \approx r_+ r_-\left\{1 + \frac{r_+^2 + r_+ r_- + r_-^2}{3a^2}\right\}. \tag{39}$$

But in terms of the mass and charge we know

$$r_\pm = m \pm \sqrt{m^2 - Q^2}, \tag{40}$$

whence

$$r_+ r_- = Q^2, \tag{41}$$

and

$$r_\pm^2 = 2m^2 - Q^2 \pm 2m\sqrt{m^2 - Q^2}, \tag{42}$$

so

$$r_+^2 + r_+ r_- + r_-^2 = 4m^2 - Q^2. \tag{43}$$

This implies

$$r_E r_C \approx Q^2\left\{1 + \frac{4m^2 - Q^2}{3a^2}\right\}, \tag{44}$$

which can also be written as

$$r_E r_C = Q^2\{1 + \frac{1}{3}\Lambda(4m^2 - Q^2) + \mathcal{O}(\Lambda^2)\}. \tag{45}$$

Therefore

$$A_E A_C = 16\pi^2 Q^4\{1 + \frac{2}{3}\Lambda(4m^2 - Q^2) + \mathcal{O}(\Lambda^2)\}, \tag{46}$$

which is certainly *not* mass independent.

For completeness we also note

$$r_E + r_C \approx 2m + \frac{r_+^4 - r_-^4}{3a^2(r_+ - r_-)}, \tag{47}$$

which again is explicitly mass dependent.

2. Cosmological horizon

From the exact result

$$r^2 = 3a^2\frac{(r - r_+)(r - r_-)}{r^2}, \tag{48}$$

we have, as a zero order approximation,

$$r_\Lambda \approx \sqrt{3}a. \tag{49}$$

Therefore as a first order approximation

$$r_\Lambda \approx \sqrt{3}a\sqrt{\frac{(\sqrt{3}a - r_+)(\sqrt{3}a - r_-)}{3a^2}} \tag{50}$$

$$\approx \sqrt{3}a\left\{1 - \frac{r_+ + r_-}{2\sqrt{3}a}\right\} \tag{51}$$

$$= \sqrt{3}a\left\{1 - \frac{m}{\sqrt{3}a}\right\}. \tag{52}$$

So for the cosmological horizon

$$r_\Lambda \approx \sqrt{3}a - m. \tag{53}$$

Oddly enough the location of the cosmological horizon is to this order independent of the charge Q, but it does definitely depend on the mass m.

3. Virtual horizon

Finally, from the exact quartic, we know there is a (unphysical) virtual horizon at negative r:

$$r_V = -\{r_E + r_C + r_\Lambda\}. \tag{54}$$

So to a first approximation

$$r_V \approx -\sqrt{3}a - m. \tag{55}$$

B. Exact results

What quantities might actually be independent of m? From the exact quartic we know

$$r_V r_E r_C r_\Lambda = -3Q^2 a^2, \tag{56}$$

implying, in terms of physical horizons, that the quantity

$$\{r_E + r_C + r_\Lambda\} r_E r_C r_\Lambda = 3Q^2 a^2 \tag{57}$$

is strictly independent of m. But this looks nothing like the product of event and Cauchy horizon areas $A_+ A_-$. Perhaps more promising is the exact condition

$$\sum_{i>j} r_i r_j = -3a^2. \tag{58}$$

That is

$$r_V\{r_E + r_C + r_\Lambda\} + r_E\{r_C + r_\Lambda\} + r_C r_\Lambda = -3a^2, \tag{59}$$

whence

$$\{r_E + r_C + r_\Lambda\}^2 - r_E\{r_C + r_\Lambda\} - r_C r_\Lambda = 3a^2, \tag{60}$$

so that

$$r_E^2 + r_C^2 + r_\Lambda^2 + r_E r_C + r_C r_\Lambda + r_\Lambda r_E = 3a^2. \tag{61}$$

We can furthermore eliminate explicit (though not implicit) occurrence of the cosmological constant by dividing these two exact results to get

$$\frac{\{r_E + r_C + r_\Lambda\} r_E r_C r_\Lambda}{r_E^2 + r_C^2 + r_\Lambda^2 + r_E r_C + r_C r_\Lambda + r_\Lambda r_E} = Q^2. \tag{62}$$

This is certainly mass independent, but is a rather complicated function of physical horizon radii. As $a \to \infty$ (that is $\Lambda \to 0$, so $r_\Lambda \to \infty$) one recovers the usual Reissner-Nordström result

$$\lim_{a\to\infty} r_E r_C = Q^2. \tag{63}$$

If one insists on working with areas then we have the exact result that $4\pi Q^2$ is equal to

$$\frac{\{\sqrt{A_E} + \sqrt{A_C} + \sqrt{A_\Lambda}\}\sqrt{A_E A_C A_\Lambda}}{A_E^2 + A_C^2 + A_\Lambda^2 + \sqrt{A_E A_C} + \sqrt{A_C A_\Lambda} + \sqrt{A_\Lambda A_E}}. \tag{64}$$

Again, there is certainly *some* function of physical horizon areas that is mass independent (and in this particular case, even free of explicit cosmological constant dependence), but it is nowhere near as straightforward as a simple product of horizon areas.

VI. REISSNER-NORDSTRÖM–ANTI-DE SITTER BLACK HOLES

Set $\Lambda = -1/|a|^2$. The relevant quartic becomes

$$r^4 + 3r^2|a|^2 - 6mr|a|^2 + 3Q^2|a|^2 = 0. \tag{65}$$

There are now two complex conjugate (utterly formal and unphysical) virtual horizons $r_V^\pm$, and two physical horizons: an event horizon r_E and an inner (Cauchy) horizon r_C. Because there are only two physical horizons, this particular situation is closest in spirit to the standard Reissner-Nordström spacetime.

A. Approximate results

To a first approximation, for the event horizon we have

$$r_E \approx r_+ - \frac{r_+^4}{3|a|^2(r_+ - r_-)} = r_+\left\{1 - \frac{r_+^3}{3|a|^2(r_+ - r_-)}\right\}. \tag{66}$$

For the inner (Cauchy) horizon we see

$$r_C \approx r_- + \frac{r_-^4}{3|a|^2(r_+ - r_-)} = r_-\left\{1 + \frac{r_-^3}{3|a|^2(r_+ - r_-)}\right\}. \tag{67}$$

Finally, for the two unphysical virtual horizons we obtain

$$r_V^\pm \approx \pm i\sqrt{3}|a| - m. \tag{68}$$

Then it is easy to compute

$$r_E r_C \approx r_+ r_-\left\{1 - \frac{r_+^3 - r_-^3}{3|a|^2(r_+ - r_-)}\right\}, \tag{69}$$

so that

$$r_E r_C \approx r_+ r_-\left\{1 - \frac{r_+^2 + r_+ r_- r_+ + r_-^2}{3|a|^2}\right\}, \tag{70}$$

and so

$$r_E r_C \approx Q^2\left\{1 - \frac{4m^2 - Q^2}{3|a|^2}\right\}. \tag{71}$$

Then (and I again emphasize that for $\Lambda < 0$ we are in an asymptotically AdS spacetime with no cosmological horizon, and we really only have these two physical horizons to deal with), we see

$$r_E r_C = Q^2\{1 - \frac{1}{3}|\Lambda|(4m^2 - Q^2) + \mathcal{O}(\Lambda^2)\}. \tag{72}$$

In fact this now implies that for *either* sign of the cosmological constant one has

$$r_E r_C = Q^2\{1 + \frac{1}{3}\Lambda(4m^2 - Q^2) + \mathcal{O}(\Lambda^2)\}. \tag{73}$$

Note this is very definitely *not* mass independent.

B. Exact results

Some exact results can again be obtained by computing various combinations of the roots of the quartic. Note that the key basic results obtained by picking off the various coefficients of the quartic are

$$r_V^+ + r_V^- + r_C + r_E = 0; \tag{74}$$

$$r_V^+ r_V^- + (r_V^+ + r_V^-)(r_C + r_E) = 3|a|^2; \tag{75}$$

$$r_V^+ r_V^-(r_C + r_E) + (r_V^+ + r_V^-) r_C r_E = -6m|a|^2; \tag{76}$$

and

$$r_V^+ r_V^- r_C r_E = 3Q^2|a|^2. \tag{77}$$

Therefore

$$(r_V^+ + r_V^-) = -(r_C + r_E); \qquad r_V^\pm = -\frac{1}{2}(r_C + r_E) \pm i\gamma, \tag{78}$$

and so

$$r_V^+ r_V^- = 3|a|^2 + (r_C + r_E)^2; \tag{79}$$

$$(r_V^+ r_V^- - r_C r_E)(r_C + r_E) = -6m|a|^2; \tag{80}$$

$$r_V^+ r_V^- = \frac{1}{4}(r_C + r_E)^2 + \gamma^2. \tag{81}$$

We can eliminate some of the unknowns in the above expressions but not all. In particular

$$\frac{r_V^+ r_V^- r_C r_E}{r_V^+ r_V^- + (r_V^+ + r_V^-)(r_C + r_E)} = Q^2, \tag{82}$$

so

$$\frac{[\frac{1}{4}(r_C + r_E)^2 + \gamma^2] r_C r_E}{\gamma^2 - \frac{3}{4}(r_C + r_E)^2} = Q^2. \tag{83}$$

Unfortunately, while the right-hand side depends only on the charge Q, the left-hand side contains the parameter γ, which is not directly accessible to physical observation. (Nor is it easy to calculate without explicitly solving the quartic.) Alternatively, one could also write

$$\{3|a|^2 + (r_C + r_E)^2\} r_C r_E = 3Q^2|a|^2. \tag{84}$$

Therefore

$$\left\{1 + \frac{1}{3}|\Lambda|(r_C + r_E)^2\right\} r_C r_E = Q^2. \tag{85}$$

This is at least m independent, and γ independent, but explicitly contains both Q and Λ. If we work in terms of areas

$$\left\{1 + \frac{1}{12\pi}|\Lambda|(\sqrt{A_C} + \sqrt{A_E})^2\right\}\sqrt{A_C A_E} = 4\pi Q^2. \tag{86}$$

Again, there is *some* function of the physical horizon areas that is mass independent, but it is nowhere near as straightforward as a simple product of horizon areas.

VII. LAURENT POLYNOMIAL FOR THE QUASILOCAL MASS

Let us now try to put these specific results into a broader context. Suppose merely that the quasilocal mass $m(r)$ is some generic Laurent polynomial. Then without loss of generality $\Delta(r)$ is also a Laurent polynomial and can be written in the form

$$\Delta(r) = \Delta_* \frac{P(r)}{r^n}. \tag{87}$$

Here we have normalized the (ordinary) polynomial $P(r)$ so that its highest degree coefficient is unity, and its lowest degree coefficient (a constant term) is nonzero. The Killing horizons are located at the zeros r_i of the numerator $P(r)$. That is, we have

$$P(r) = \sum_{j=0}^{D-1} c_j r^j + r^D = \prod_{i=1}^{D}(r - r_i). \tag{88}$$

Furthermore, as is completely standard,

$$c_0 = (-1)^D \prod_{i=1}^{D} r_i; \quad c_1 = (-1)^{D-1}\sum_{j=1}^{D}\prod_{i=1, i\neq j}^{D} r_i; \; \ldots \tag{89}$$

$$\ldots \quad c_{D-2} = \sum_{i>j} r_i r_j; \qquad c_{D-1} = -\sum_{j=1}^{D} r_j. \tag{90}$$

In fact these coefficients are easily and explicitly calculable in terms of the elementary symmetric polynomials $e_i(\cdot)$ on D variables [28,29]:

$$c_{D-i} = (-1)^D e_i(r_1, r_2, \ldots, r_D). \tag{91}$$

We see that it is the coefficient c_{n-1} that leads to a $1/r$ falloff in $\Delta(r)$ at large r, and so it is this coefficient that is proportional to the mass of the black hole. (By construction $n \in \{1, \ldots, D\}$, otherwise the mass of the black hole will be zero.) All of the other coefficients (there are $D - 1$ of them),

$$c_i(r_1, r_2, \ldots, r_D)\colon 0 \le i \le D - 1; \quad i \neq n - 1, \tag{92}$$

will by construction be mass independent. That is, in terms of the elementary symmetric polynomials, all the quantities

$$e_i(r_1, r_2, \ldots, r_D)\colon 1 \le i \le D; \quad i \neq D - n + 1, \tag{93}$$

will be mass independent. In terms of horizon areas, $A_i = 4\pi r_i^2$, *all* $D - 1$ elementary symmetric polynomials

$$e_i\left(\sqrt{\frac{A_1}{4\pi}}, \sqrt{\frac{A_2}{4\pi}}, \ldots, \sqrt{\frac{A_D}{4\pi}}\right)\colon 1 \le i \le D, \tag{94}$$

for $i \neq D - n + 1$, will be mass independent. Of course not all the r_i need be physical (real and positive), so not all the A_i need be real. Since there are $D - 1$ of these mass-independent quantities, it might sometimes be possible to eliminate all the unphysical (virtual) horizons r_i, and reduce the situation to one of dealing with a smaller number of real mass-independent quantities determined solely in terms of physical horizon areas. With N virtual horizons one will generally have $D - N - 1$ mass-independent quantities constructible in terms of physical horizons. Whether or not this can successfully be achieved in practice depends very much on the precise details of the polynomial $P(r)$. For example, as we have seen in the previous sections:

(i) Schwarzschild–de Sitter spacetimes correspond to $D = 3$ and $N = 1$.
There are two mass-independent quantities [one trivial, Eq. (13); one nontrivial, Eq. (19)], but only one that depends solely on the physical horizons [Eq. (21) or equivalently (22)].

(ii) Schwarzschild–anti-de Sitter spacetimes correspond to $D = 3$ and $N = 2$.
There are two mass-independent quantities [one trivial, Eq. (29); one nontrivial, Eq. (30)], but none that depend solely on the physical horizon.

(iii) Reissner-Nordström–de Sitter spacetimes correspond to $D = 4$ and $N = 1$.
There are three mass-independent quantities [one trivial, Eq. (54); two nontrivial, Eqs. (56) and (59)], but only two that depend solely on the physical horizons [any two of Eqs. (57), (61), and (62)—or the equivalently (64)].

(iv) Reissner-Nordström–anti-de Sitter spacetimes correspond to $D = 4$ and $N = 2$.
There are three mass-independent quantities [one trivial, Eq. (74); two nontrivial, Eqs. (75) and (77)], but only one that depends solely on the physical horizons [any one of the equivalent Eqs. (84) and (85) or (86)].

But now we see that the key points of the preceding explicit discussion continue to hold in greater generality—whenever the quasilocal mass $m(r)$ is any generic Laurent polynomial. Generalizations to higher dimensional spacetimes with hyperspherical symmetry are immediate and straightforward. Generalizations to rotating black holes [30–34], and more complicated symmetries, are not quite as straightforward—but as long as the location of the horizons is determined by the roots of some Laurent polynomial we can expect similar results to hold. For instance, it is quite sufficient if, in terms of some natural r coordinate easily related to the horizon area, the norm of the horizon generating Killing vector is some entire function multiplied by a Laurent polynomial.

VIII. DISCUSSION

Generically, products of horizon areas may or may not be independent of the mass of the black hole. This depends on the precise form of the quasilocal mass, on whether one takes the product only over physical horizons, or whether one includes unphysical virtual horizons in the product. In spherical symmetry, as long as the quasilocal mass is a Laurent polynomial with $D = D_{\max} - D_{\min}$, there will be D horizons from which one can construct $D - 1$ mass-independent quantities in terms of the elementary symmetric polynomials built out of the horizon radii. If N of these horizons are "virtual" (negative or complex radius), then by algebraically eliminating the virtual horizons there will generally be $D - N - 1$ (quite complicated) mass-independent quantities constructible solely in terms of the physical horizon radii (and hence constructible in terms of the physical horizon areas). We have explicitly checked these results for validity by investigating the situation for Schwarzschild–(anti)-de Sitter and Reissner-Nordström–(anti)-de Sitter spacetimes.

As we have seen above, with regard to string-inspired area products the general situation is much more complicated than currently envisaged. The conjectured area quantization generally fails because certain parameters are not integers [23]. To quote the authors of [35], "we will refer to them as the numbers of branes, antibranes, and strings because (as will be seen) they reduce to those numbers in certain limits where these concepts are well defined." Furthermore, inspection of known exact solutions demonstrates that the conjectured mass independence often fails once a cosmological constant is added.

In contrast, for the general relativity inspired area bounds are not dependent on explicit exact solutions and at least partially survive the introduction of a cosmological constant [15,19]. There seems some hope of yet further progress along these lines. Similarly the Ansorg–Hennig area product theorems [1–3] are not dependent on explicit exact solutions—both the underlying framework and motivation is rather different—as are the required tools.

ACKNOWLEDGMENTS

This research was supported by the Marsden Fund, and by a James Cook Fellowship, both administered by the Royal Society of New Zealand.

APPENDIX: CUBIC POLYNOMIAL EQUATIONS

Consider a cubic polynomial equation in reduced form, with coefficients conveniently chosen to be

$$x^3 - 3p^2x + 2q = 0; \quad p > 0. \tag{A1}$$

Then the exact roots are given by a form of Viète's trigonometric solution

$$x = 2p \sin\left\{\frac{1}{3}\sin^{-1}\left[\frac{q}{p^3}\right] + \epsilon\frac{2\pi}{3}\right\}; \quad \epsilon \in \{0, \pm 1\}. \tag{A2}$$

If $|q| < p^3$ there are three real roots.

On the other hand, if we have

$$x^3 + 3p^2x - 2q = 0; \quad p > 0, \tag{A3}$$

then there is only one real root. It is given by a hyperbolic form of Viète's solution

$$x = 2p \sinh\left\{\frac{1}{3}\sinh^{-1}\left[\frac{q}{p^3}\right]\right\}. \tag{A4}$$

In terms of Cardano's formulas one can explicitly rewrite this as

$$x = p\left\{\left[\sqrt{1 + \frac{q^2}{p^6}} + \frac{q}{p^3}\right]^{1/3} - \left[\sqrt{1 + \frac{q^2}{p^6}} - \frac{q}{p^3}\right]^{1/3}\right\}. \tag{A5}$$

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