Are quantization rules for horizon areas universal?

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Doubts have been expressed on the universality of holographic/string-inspired quantization rules for the horizon areas of stationary black holes, or the products of their radii, already in 4-dimensional general relativity. Realistic black holes are not stationary but time-dependent. We produce three examples of 4-dimensional general-relativistic spacetimes containing dynamical black holes for at least part of the time, and we show that the quantization rules (even counting virtual horizons) cannot hold, except possibly at isolated instants of time, and do not seem to be universal.

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I. INTRODUCTION

Recently, there has been some excitement in the research community working on the holographic principle and stringy/supergravity black holes following the observation that the products of Killing horizon areas for certain multihorizon black holes are independent of the black hole mass and depend only on the quantized charges (supergravity and extradimensional black holes with angular momentum and electric and magnetic charges were considered) [1–9]. Older results on black holes far from extremality [1,2,10] induce one to take into account both outer and inner black hole horizons when studying the quantization of black hole entropies and horizon areas. Expressions for products of the horizon areas of black holes in four and higher dimensions have been hypothesized or suggested [3–6] and then questioned in more recent work [9].

This literature is inspired by the holographic principle and string theories (although the results are not, strictly speaking, derived from string theories), and it stems from the underlying idea that quantized products of areas depending on combinations of integers must carry the signature of some specific microphysics. This feature would not be too surprising if the area A of a horizon is related to its entropy S through the famous Bekenstein–Hawking formula $S = A/4$ (in units in which $c = \hbar = 1$) and corresponds to a statistical mechanics based on microscopic models counting microstates determined by quantum gravity (see, e.g., Ref. [11]). When there are outer $(+)$ and inner $(-)$ horizons, the quantization rules recurrent in the literature are

$$A_{\pm} = 8\pi l_{\rm pl}^2\left(\sqrt{N_1} \pm \sqrt{N_2}\right), \qquad N_1, N_2 \in \mathbb{N}, \tag{1}$$

or

$$A_+ A_- = (8\pi l_{\rm pl}^2)^2 N, \qquad N \in \mathbb{N}, \tag{2}$$

where $l_{\rm pl}$ is the Planck length [1,2,10]. $N_{1,2}$ are integers for supersymmetric extremal black holes but are related to the numbers of branes, antibranes, and strings in less simple situations [12]. A weaker rule states that the product of horizon areas is independent of the black hole mass and depends only on the quantized charges. Rules of the type (2) are found for Einstein–Maxwell black holes in 5 and 6 dimensions, asymptotically flat [1,4–6,8,10], or asymptotically de Sitter or anti-de Sitter, and also for black holes in $D = 3$ and $D \geq 6$ dimensions [9], and it seems to apply also to black rings and black strings in higher dimensions [6] (asymptotically de Sitter and anti-de Sitter black holes in general relativity and other theories of gravity, in various dimensions, are discussed in Ref. [9]).

A word of caution against the temptation of regarding these rules as universal for all types of black holes endowed with multiple horizons [3] has been voiced by Visser [13,14]. Visser considered black holes in 4-dimensional general relativity and found that, in these situations, products of areas do not give mass-independent quantities, nor are they related in a simple way to integers. Rather, it is quadratic combinations of the various horizon radii (with the dimensions of an area, which can be referred to as "generalized areas") which generate mass-independent quantities and are, presumably, the best candidates to be quantized [13,14], although no evidence has been presented thus far that these generalized areas have any special physical significance. Moreover, it is essential to include in these algebraic combinations also cosmological and virtual horizons in addition to the black hole horizons [14]. Virtual horizons correspond to negative or imaginary roots of the equation locating the horizons (which, in nonasymptotically flat solutions of the Einstein equations, provides also cosmological horizons).

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The quantization rules break down also for general Myers–Perry black holes in dimension $D \geq 6$ and for Kerr–anti-de Sitter black holes with $D \geq 4$ unless the virtual horizons are included in the picture [15].

In this paper we point out a fact which induces even more caution in discussing the products of horizon areas. The horizons considered in the literature are Killing (and event) horizons. Realistic black holes are not stationary if nothing else because they emit Hawking radiation and the backreaction due to this effect changes their masses which become time-dependent, together with their horizon radii and areas. For astrophysical black holes, the effect is completely negligible, but the same cannot be said for quantum black holes. Therefore, a timelike Killing vector will not be present, and in realistic situations one should consider not Killing and event horizons but other kinds of horizons. Dynamical horizons have received much attention in quantum gravity [16]; at present it seems that apparent horizons (AHs; see Ref. [17] for reviews) are the best and most versatile candidates for the notion of time-dependent "horizon," and it is claimed that thermodynamical laws can be associated with AHs [18]. In any case, AHs are used as proxies for event horizons in studies of gravitational collapse in numerical relativity [19]. AHs coincide with event horizons in stationary situations, but, in dynamical situations, they are spacelike or even timelike. In the following we consider dynamical situations, and we focus on AHs.

II. TOY MODELS FOR DYNAMICAL BLACK HOLES

Here we consider three toy models of *dynamical* black holes, which are implemented by embedding them in a Friedmann–Lemaître–Robertson–Walker (FLRW) cosmological "background" (we use quotation marks because, due to the nonlinearity of the Einstein equations, one cannot split the metric into a background and a deviation from it in a covariant way). In the first model, a McVittie spacetime, there are a black hole, a cosmological, and a virtual horizon. In the second model, a generalized McVittie solution of the Einstein equations, the "McVittie no-accretion condition" is relaxed to allow accretion of energy, and then we have either two real horizons (a black hole and a cosmological horizon) or two virtual horizons. The third model consists of an electrically charged (but nonaccreting) generalization of the McVittie spacetime. In this case there is a charge to quantize, but the behavior of the horizons is the same as in the uncharged case. Our main point is that, in dynamical situations, even if combinations of AH radii which are mass-independent exist, they depend continuously on time and cannot be expressed as combinations of integers.

A. McVittie spacetime

The McVittie metric [20] describes a black hole embedded in a FLRW universe, which is a truly dynamical spacetime.[1] Limiting ourselves, for simplicity, to a spatially flat FLRW background, the line element can thus be written in the form [21]

$$ds^2 = -\left[1 - \frac{2m}{R} - H^2(t)R^2\right]dt^2 + \frac{dR^2}{1-\frac{2m}{R}} - \frac{2H(t)R}{\sqrt{1-\frac{2m}{R}}}dtdR + R^2 d\Omega_{(2)}^2, \quad (3)$$

where m is a constant related to the mass of the central inhomogeneity, $d\Omega_{(2)}^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the metric on the unit 2-sphere, R is the areal radius, $H(t) \equiv \dot{a}(t)/a(t)$ is the Hubble parameter, $a(t)$ is the scale factor of the FLRW background, and an overdot denotes differentiation with respect to the comoving time t. The locally static Schwarzschild–de Sitter–Kottler spacetime corresponds to $a(t) = \exp(\sqrt{\Lambda/3}t)$ and $H = \sqrt{\Lambda/3}$ (where $\Lambda > 0$ is the cosmological constant) and is a special case of the McVittie metric which can be obtained using a simple transformation of the time coordinate [22]. Assuming a perfect fluid stress-energy tensor, the Einstein equations provide the energy density $\rho(t)$ and pressure $P(t, R)$ of the background fluid. Again for simplicity, let us restrict ourselves to a cosmic fluid which reduces to dust (equation of state parameter $w \equiv P/\rho = 0$) at spatial infinity; then

$$\rho(t) = \frac{3}{8\pi}H^2(t), \quad (4)$$

$$P(t, R) = \rho(t)\left(\frac{1}{\sqrt{1-\frac{2m}{R}}} - 1\right). \quad (5)$$

The inverse metric is

$$(g^{\mu\nu}) = \begin{pmatrix} \frac{-1}{1-2m/R} & \frac{-HR}{\sqrt{1-2m/R}} & 0 & 0 \\ \frac{-HR}{\sqrt{1-2m/R}} & \left(1-\frac{2m}{R}-H^2R^2\right) & 0 & 0 \\ 0 & 0 & \frac{1}{R^2} & 0 \\ 0 & 0 & 0 & \frac{1}{R^2\sin^2\theta} \end{pmatrix}. \quad (6)$$

For any spherically symmetric metric written in terms of the areal radius R, the AHs are located by solving the equation $\nabla^c R \nabla_c R = 0$ or $g^{RR} = 0$ [23]. For the Schwarzschild–de Sitter–Kottler spacetime, which is a special case of McVittie, this equation coincides with the horizon condition reported in Ref. [14] but, in the general

[1] The special case of a de Sitter background admits a timelike Killing vector and is locally static in the region between the black hole and the de Sitter cosmological horizons.

case, the Hubble parameter is time-dependent instead of constant. This cubic equation,

$$R^3 - \frac{R}{H^2(t)} + \frac{2m}{H^2(t)} = 0, \tag{7}$$

has three solutions which, under conditions specified below, correspond to a time-dependent black hole AH with (proper) radius $R_{BH}(t)$, a cosmological AH with radius $R_C(t)$, and a virtual AH with negative radius $R_V(t)$. The three roots are

$$R_{BH} = \frac{2H^{-1}}{\sqrt{3}} \sin\psi, \tag{8}$$

$$R_C = -R_V = H^{-1}\left(\cos\psi - \frac{1}{\sqrt{3}}\sin\psi\right), \tag{9}$$

with $\psi(t)$ given by $\sin(3\psi) = 3\sqrt{3}mH(t)$. Here m and H are both necessarily positive (we only consider expanding universes), and R_V defines the negative root. As discussed in Ref. [24], the condition for the black hole and cosmological AHs to exist is $0 < \sin(3\psi) < 1$, which corresponds to $mH(t) < 1/(3\sqrt{3})$ (and $mH(t) > 0$, which is always satisfied). Unlike the Schwarzschild–de Sitter–Kottler case where the Hubble parameter is a constant, this inequality will only be satisfied at certain times during the cosmological expansion and will be violated at other times. The threshold between these two regimes is the time at which $mH(t_*) = 1/(3\sqrt{3})$ [for a dust-dominated background with $H(t) = 2/(3t)$, this critical time is $t_* = 2\sqrt{3}m$]. At early times $t < t_*$, it is $m > \frac{1}{3\sqrt{3}H(t)}$, and both $R_{BH}(t)$ and $R_C(t)$ are complex and therefore unphysical. In this case all the AHs are virtual. At the critical time $t = t_*$, it is $m = \frac{1}{3\sqrt{3}H(t)}$, and the AHs with radii $R_{BH}(t_*)$ and $R_C(t_*)$ coincide at a real, physical location. There are then a single real AH at $\frac{1}{\sqrt{3}H(t)}$ and one virtual AH. At "late" times $t > t_*$, it is $m < \frac{1}{3\sqrt{3}H(t)}$, and both $R_{BH}(t)$ and $R_C(t)$ are real and, therefore, physical—there are two real and one virtual AHs. The dynamics of the black hole and cosmological AH radii as functions of comoving time are pictured in Fig. 1.

The phenomenology of AHs appearing and annihilating in pairs appears to be rather common for black holes embedded in cosmological backgrounds, in both general relativity and alternative theories of gravity [25–27]. The physical reason why a pair of AHs suddenly appears in the McVittie spacetime (3) is discussed in Ref. [24]. The same phenomenology of Fig. 1 is found for generalized McVittie metrics [28] and in Lemaître–Tolman–Bondi spacetimes ([29]; see also Ref. [30]) describing black holes embedded in (spatially flat) FLRW universes.[2]

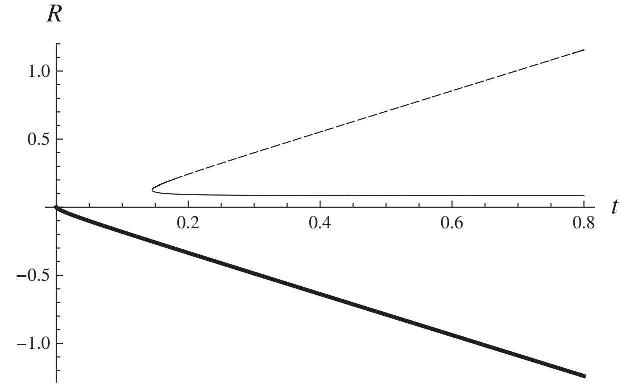

FIG. 1. The proper radii of the AHs of a dust-dominated McVittie metric vs time. The negative radius represents the virtual horizon. At a critical time a cosmological AH (dashed curve) appears together with a black hole AH (solid curve), the former expanding and the latter shrinking.

The Misner–Sharp–Hernandez mass $M_{\rm MSH}$ [31] of a sphere of areal radius R (which is defined for spherically symmetric spacetimes) is [24]

$$M_{\rm MSH} = m + \frac{4\pi G}{3}\rho R^3 \tag{10}$$

and coincides with the Hawking–Hayward quasilocal mass [32] in spherical symmetry. It is interpreted as the contribution of the black hole mass m (which is constant because of the "McVittie condition" $G^0_1 = 0$, which implies $T^0_1 = 0$ for the stress-energy tensor of the cosmic fluid and forbids accretion of the latter onto the black hole) and a contribution due to the energy of the cosmic fluid inside the sphere. Searching for generalized areas which are independent of the black hole mass, Visser's discussion for the Schwarzschild–de Sitter–Kottler black hole can be repeated almost without changes. Including the virtual horizon in the count, it is straightforward to see that the quantities

$$R_V(R_{\rm BH} + R_C) + R_{\rm BH}R_C = -\frac{1}{H^2(t)} \tag{11}$$

and

$$(R_{\rm BH} + R_C)^2 - R_{\rm BH}R_C = \frac{1}{H^2(t)} \tag{12}$$

are independent of the black hole mass m. This situation can be regarded as a special case of Visser's discussion [14] computing mass-independent combinations of AH radii whenever the Misner–Sharp–Hernandez mass is a Laurent polynomial of the areal radius R. This is clearly the case of the McVittie metric; see Eq. (10). In the present case, the physical mass contained in a sphere is actually given by the Misner–Sharp–Hernandez notion, but the cosmic fluid here serves the only purpose of generating a cosmological background to make the central black hole dynamical, and it seems that the relevant mass to consider when mass-independent quantities such as Eqs. (11) and (12) are

[2]In the first case, both decelerating and accelerating FLRW background universes are considered while in the second case, by necessity, only a dust-dominated background is considered.

searched for is the black hole contribution m, not the total $M_{\rm MSH}$. In any case, the AH radii identify different spheres and correspond to different Misner–Sharp–Hernandez masses $M^{(i)}_{\rm MSH} = 2R^{(i)}_{\rm AH}$ [from Eq. (10)]. Here we stick to m.

Following Ref. [14], we have included the virtual horizon to obtain the mass-independent quantity (11). Now, when the AH radii change with time, the combinations (11) and (12) are not constant but depend on time: therefore, if they are expressed by combinations of integers at an initial time, they will not be combinations of integers immediately afterward. They could only be a combination of integers at times forming a set of zero measure in any time interval.

B. Generalized (accreting) McVittie spacetime

The McVittie solution of the Einstein equations can be generalized to allow for the possibility of radial energy flow onto the central inhomogeneity [28,33]. Among the class of spherically symmetric solutions of the Einstein equations thus obtained, there is a late-time attractor for expanding background universes, which is given by the line element [34]

$$ds^2 = -\frac{(1-\frac{m}{2r})^2}{(1+\frac{m}{2r})^2}dt^2 + a^2(t)\left(1+\frac{m}{2r}\right)^4(dr^2 + r^2 d\Omega^2_{(2)}) \tag{13}$$

in isotropic coordinates, where $a(t)$ is the scale factor of the background FLRW universe and m is a constant. In terms of the areal radius $R(t,r) = a(t)r(1+\frac{m}{2r})^2$, the AHs of this solution of the Einstein equations corresponding to an expanding FLRW background universe are [34]

$$R_{\rm BH} = \frac{1-\sqrt{1-8MH}}{2H}, \tag{14}$$

$$R_{\rm CH} = \frac{1+\sqrt{1-8MH}}{2H}, \tag{15}$$

where $M(t) = ma(t)$ and $H \equiv \dot{a}/a$ is the Hubble parameter of the background FLRW universe [34]. The time evolution of the radii of the apparent horizons is reported in Fig. 2 for a dust-dominated FLRW universe with scale factor $a(t) = a_0 t^{2/3}$. These AHs are real when $8MH \leq 1$ and virtual when $8MH > 1$ (in which case we label them R_{V1} and R_{V_2}), which happens before a critical time t_*.

The products of the horizon radii prescribed in Ref. [14] reduce to

$$R_{\rm CH}R_{\rm BH} = R_{V_1}R_{V_2} = \frac{2M}{H(t)} = 3ma_0 t^{5/3}, \tag{16}$$

which is time-dependent, for both cases in which the AHs are real or virtual. Following the same reasoning as in the previous section, we conclude that it cannot be expressed as a combination of integers. The Misner–Sharp–Hernandez/Hawking–Hayward mass of the black hole (when the latter exists) is [34]

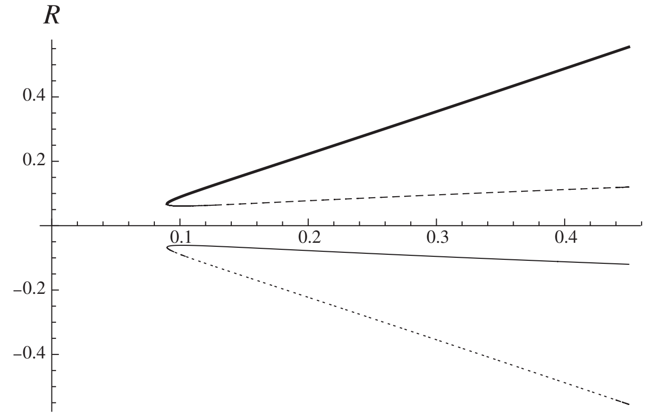

FIG. 2. The radii of the AHs of the generalized (accreting) McVittie spacetime vs time. There are always either two virtual AHs (with negative radii) or two real AHs with $R \geq 0$ (a cosmological AH, thick solid curve, and a black hole AH, dashed curve, both appearing at a critical time).

$$M_{\rm MSH} = \frac{R_{\rm BH}}{2} = \frac{1-\sqrt{1-8MH}}{4H}; \tag{17}$$

it cannot be split in any simple way into a contribution due to the central inhomogeneity and one due to the cosmic fluid inside the sphere of radius $R_{\rm BH}$. It is time-dependent due to the radial energy flow onto the black hole. The product (16) can be rewritten as

$$R_{\rm CH}R_{\rm BH} = R_{V_1}R_{V_2} = 2M_{\rm MSH}\left(\frac{1+\sqrt{1-8MH}}{2H}\right) = 2M_{\rm MSH}R_{\rm CH} \tag{18}$$

and depends on the physical black hole mass.

C. Electrically charged McVittie spacetime

Let us consider now the electrically charged, nonaccreting, generalization of the McVittie spacetime. This situation is physically more interesting because there is actually a charge which could be quantized. The electrically charged McVittie spacetime is described by the spherically symmetric line element [35]

$$ds^2 = -\frac{(1-\frac{m^2}{4a^2r^2})^2 + \frac{Q^2}{4a^2r^2}}{[(1+\frac{m}{2ar})^2 - \frac{Q^2}{4a^2r^2}]^2}dt^2 + a^2(t)\left[\left(1+\frac{m}{2ar}\right)^2 - \frac{Q^2}{4a^2r^2}\right]^2(dr^2 + r^2 d\Omega^2_{(2)}) \tag{19}$$

in isotropic coordinates, where $m = \text{const}$ is a mass parameter, Q is the electric charge of the central inhomogeneity, and $a(t)$ is the scale factor of the background FLRW universe, which is chosen here to be spatially flat.

The areal radius is clearly

$$R(t,r) = a(t)r\left[\left(1+\frac{m}{2ar}\right)^2 - \frac{Q^2}{4a^2r^2}\right] = ar + m + \frac{m^2}{4ar} - \frac{Q^2}{4ar}, \tag{20}$$

and the apparent horizons are located by the equation $\nabla^c R \nabla_c R = 0$. After a straightforward calculation, this equation becomes

$$H^2[(2ar+m)^2 - Q^2]^4 - 4(4a^2r^2 - m^2 + Q^2) = 0. \tag{21}$$

This form is not particularly useful for locating the apparent horizons because it is expressed in terms of the radial coordinate r; in order to turn it into a more useful expression involving only the proper (areal) radius R and the time t, one inverts Eq. (20) and obtains

$$r = \frac{1}{2a}\left(R - m + \sqrt{R^2 + Q^2 - 2mR}\right) \tag{22}$$

by choosing the positive sign of the square root. Using Eqs. (22) and (21) becomes

$$4H^2(t)R^4\left(R - m + \sqrt{R^2+Q^2-2mR}\right)^4 - \left(R - m + \sqrt{R^2+Q^2-2mR}\right)^2 + (m^2 - Q^2) = 0 \tag{23}$$

in terms of R and t. It is complicated to solve this transcendental equation or even to give explicit analytical criteria for the existence and number of its roots, but it is clear that, when solutions exist, they depend on time and generalized areas, and their products will also depend on time and will not be expressed as simple combinations of integers times a constant. For illustration, Eq. (23) is solved numerically for special values of the parameters using the scale factor $a(t) = a_0 t^{2/3}$ of a dust-dominated FLRW background, and the radii of the real apparent horizons are plotted in Fig. 3.

The case of a charged McVittie spacetime with $|Q| = m$ can be treated explicitly as the relevant expressions simplify considerably in this case. The areal radius is simply $R = ar + m \geq m$, and the equation locating the apparent horizons becomes

$$\frac{1}{16H^2} = R^4(R-m)^2 \tag{24}$$

or

$$S(R) \equiv R^2(R-m) = \frac{1}{4H} \tag{25}$$

in an expanding universe. The function $S(R)$ is a cubic with a local maximum of zero value at $R = 0$ and a local minimum (of value $-4m^3/27$) at $R = 2m/3$; in the

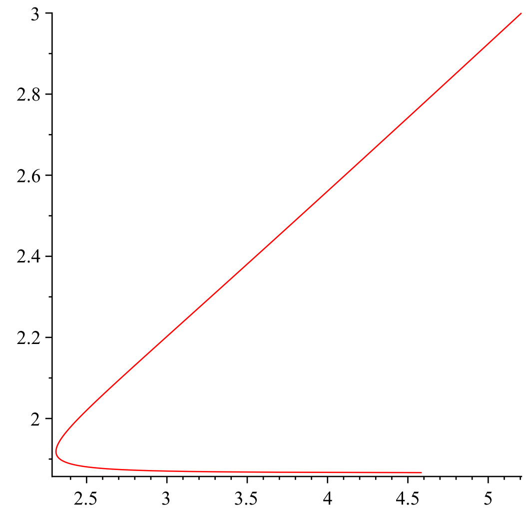

FIG. 3 (color online). The proper radii vs time of the real AHs of an electrically charged McVittie spacetime with a spatially flat, dust-dominated, FLRW background.

physical range $m \leq R < +\infty$, it is always increasing, starting from zero at $R = m$ and going to infinity as $R \to +\infty$. Therefore, for any $t > 0$ there is one and only one intersection between the graph of the function $S(R)$ and the horizontal line with ordinate value $1/4H = \frac{3t}{8} > 0$ [where, as usual, we assume $a(t) = a_0 t^{2/3}$ for a dust-dominated FLRW background], i.e., there is always one and only one apparent horizon with a radius which increases as the universe expands. (A detailed analysis of the apparent horizons of the charged McVittie spacetime, including the extremal case, will be reported elsewhere.)

III. CONCLUSIONS

The cosmological black holes reported here are just toy models for dynamical black hole horizons; the main point is that realistic black holes are time-dependent, not stationary. Therefore, far-reaching conclusions about the quantization of black hole horizon areas, or of quantities which are quadratic in the radii of Killing horizons (generalized areas), may be misleading and may not correspond to realistic, time-dependent, situations. It is interesting to probe the conjecture about mass independence and generalized area quantization using examples of time-varying black holes in 4-dimensional general relativity before approaching higher-dimensional black objects in supergravity or stringy objects. Exact solutions of the field equations of Einstein theory describing time-varying black holes are not easy to find, and we resort to the more well known cosmological black holes to provide examples of time-dependent black holes—the cosmological background is not conceptually essential here. In general, AHs depend on the spacetime foliation [36], but in the presence of spherical symmetry, to which we have restricted ourselves, this does not appear to be a significant problem. For the McVittie metric (as well as for its special

Schwarzschild–de Sitter–Kottler static case), there are generalized areas which are independent of the black hole mass. However, even if they can be expressed as $8\pi l_{\rm pl}^2$ times a combination of integers at some initial time, this expression changes as time goes by. The corresponding quantities for the generalized accreting McVittie and the electrically charged McVittie black holes are mass- and time-dependent.

Variations on the theme can be contemplated. If only the black hole and the cosmological AHs are retained and considered as physical, their area will be zero at all times $0 < t < t_*$; zero is an integer, alright, but this interpretation entails an entropy suddenly jumping from zero (describing a naked singularity in a FLRW background) to a value not reducible to a combination of integers and depending on the black hole mass. If the cosmological AH is excluded from the picture, then there remains only the black hole AH, the area of which is initially zero, then jumps to a positive value, and then decreases monotonically as time goes by (see Fig. 1). More complicated black holes with multiple AHs will lend themselves to the consideration of more possible combinations of the AH radii, but probably the most sensible way to proceed is to include all AH radii, even virtual ones, when searching for quantizable, mass-independent quantitites, as done in Ref. [14]. When realistic time-dependent horizons are considered, however, the connection between products of areas and combinations of integers becomes even more speculative, and perhaps it would be better to put it on a firmer ground or find out its limits of validity before assuming it as a postulate or a necessary accessory of the holographic principle. This conclusion reinforces that of Visser [13,14] that the black holes of 4-dimensional general relativity do not seem to reconcile with the usual quantization rules (1) and (2) and casts serious doubts on the universality of these expressions.

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