

**Exact mapping between tensor and most general scalar power spectra**Jose Beltrán Jimenez,<sup>\*</sup> Marcello Musso,<sup>†</sup> and Christophe Ringeval<sup>‡</sup>*Centre for Cosmology, Particle Physics and Phenomenology, Institute of Mathematics and Physics, Louvain University,  
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We prove an exact relation between the tensor and the scalar primordial power spectra generated during inflation. Such a mapping considerably simplifies the derivation of any power spectra as they can be obtained from the study of the tensor modes only, which are much easier to solve. As an illustration, starting from the second order slow-roll tensor power spectrum, we derive in a few lines the next-to-next-to-leading order power spectrum of the comoving curvature perturbation in generalized single field inflation with a varying speed of sound.

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**I. INTRODUCTION**

Cosmic inflation is currently considered to be the standard lore to explain the origin of the cosmic microwave background (CMB) anisotropies and the large scale structures of our Universe. In addition to solving the so-called “problems” of the standard Friedmann-Lemaître-Robertson-Walker model, inflation makes definite predictions for the cosmological perturbations in the earliest times of the Universe’s history [1–4]. At linear order, it predicts an almost scale invariant power spectrum for the comoving curvature perturbation  $\zeta$  in complete agreement with the spectral index measured in the most recent CMB data [5–10]. Confronting the predictions of inflationary models with increasingly more accurate cosmological data has pushed forward various theoretical developments. Among them, the search for non-Gaussianities has triggered interest in the calculation of higher  $n$ -point functions for  $\zeta$  which are expected to trace any departures from a single slow-rolling field [11–16]. This is particularly timely as the Planck satellite has severely constrained the amount of possible non-Gaussianities in the CMB data [17,18] while dramatically increasing the accuracy in the measurement of the scalar spectral index. Reference [10] reports  $n_S = 0.9603 \pm 0.0073$  using Planck temperature data complemented with WMAP polarization [5]. Both of these results suggest that admissible inflationary models cannot be too far from the slow-roll single field inflation paradigm. Within this landscape of models, the shape of the primordial power spectra is an observable of choice to discriminate between various scenarios. In this respect several complementary approaches have been proposed. Given a model of inflation, it is always possible to exactly evaluate the power spectrum using numerical methods,<sup>1</sup> eventually complemented with Bayesian model comparison to determine how well they suit in any cosmological data set [19–22].

A second method, which is the one we will be interested in, consists in parametrizing the power spectra of a broad class of models using the slow-roll expansion [23–28]. The modern approach consists in defining an infinite hierarchy of so-called Hubble flow functions [26,29] (also simply referred to as the slow-roll parameters)

$$\epsilon_{i+1} \equiv \frac{d \ln |\epsilon_i|}{dN}, \quad \epsilon_1 \equiv -\frac{d \ln H}{dN}, \quad (1)$$

where  $H$  is the Hubble parameter and  $N \equiv \ln a$  with  $a$  the scale factor during inflation. By definition, the expansion of the Universe is accelerated if  $\epsilon_1 < 1$  and the slow-roll approximation relies on the extra-assumptions that  $\epsilon_1 \ll 1$  and all the  $\epsilon_i$  are of the same order of magnitude that we denote by  $\mathcal{O}(\epsilon)$ . Provided this is verified, one can consistently solve order by order the evolution equations for the cosmological perturbations. After some appropriate field redefinitions, the equations for the two polarization degrees of freedom  $h_\lambda$  of the tensor modes and for the scalar comoving curvature perturbation  $\zeta$  can all be written in Fourier space in terms of a Mukhanov-Sasaki variable  $v$  verifying [30]

$$v'' + \left(k^2 - \frac{z''}{z}\right)v = 0, \quad (2)$$

where a “prime” denotes differentiation with respect to an appropriate time variable  $\tau$ , and  $z$  is a suitable function of  $\tau$ . For instance, in canonical single field inflation  $\tau = \eta$  is the standard conformal time ( $d\eta = dt/a$ ); for the tensor modes  $v(k, \eta) \equiv h_\lambda(k, \eta)z(\eta)$  with  $z(\eta) \equiv a(\eta)$  [1], while for the scalar mode  $v(k, \eta) \equiv \sqrt{2}\zeta(k, \eta)z(\eta)$  with  $z(\eta) \equiv a(\eta)\sqrt{\epsilon_1(\eta)}$  [31].

It is well known that this equation remains the same for any single field model with the most generic quadratic action, such as K-inflation [32–38]. In that case,  $\tau$  is a rescaled conformal time defined by  $d\tau = c_s(\eta)d\eta$ , where  $c_s$  stands for the “sound speed” associated with the scalar perturbations. The Mukhanov variable still reads  $v(k, \tau) \equiv \sqrt{2}\zeta(k, \tau)z(\tau)$ , with now  $z(\tau) \equiv a(\tau)\sqrt{\epsilon_1(\tau)}/c_s(\tau)$ .

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The slow-roll approximation allows us to consistently solve Eq. (2), at a given order of approximation. For standard single field models, the first calculation of this kind was done in Ref. [1] for the tensor modes and in Refs. [23,39] for the scalar modes. The next-to-leading order corrections were first derived in Ref. [24]. The scalar mode solution was then rederived and extended using the more general Green function method in Ref. [40]. These results have been also recovered for both scalar and tensor modes by using the Wentzel-Kramers-Brillouin (WKB) approximation in Ref. [41] and the uniform approximation in Refs. [42,43]. Next-to-next-to-leading order corrections for the scalar spectral index were first derived in Ref. [40] while the expanded power spectrum at second order has been explicitly derived in Ref. [26] for the scalars and in Ref. [27] for the tensor modes, still using the Green function method. These results have been recovered with an improved WKB approximation in Refs. [44,45]. As we will need its expression later on, the second order tensor power spectrum obtained in Ref. [27] reads

$$\begin{aligned} \mathcal{P}_h = & \frac{2H_*^2}{\pi^2 M_{\text{p}}^2} \left\{ 1 - 2(1+C)\epsilon_{1*} + \left( \frac{\pi^2}{2} - 3 + 2C + 2C^2 \right) \epsilon_{1*}^2 \right. \\ & + \left( \frac{\pi^2}{12} - 2 - 2C - C^2 \right) \epsilon_{1*} \epsilon_{2*} + [-2\epsilon_{1*} + (2+4C)\epsilon_{1*}^2 \\ & - 2(1+C)\epsilon_{1*} \epsilon_{2*}] \ln \left( \frac{k}{k_*} \right) \\ & \left. + (2\epsilon_{1*}^2 - \epsilon_{1*} \epsilon_{2*}) \ln^2 \left( \frac{k}{k_*} \right) \right\}, \quad (3) \end{aligned}$$

where  $M_{\text{p}}^2 = 8\pi G$  is the reduced Planck mass and  $C$  is a constant equal to  $C = \gamma + \ln 2 - 2 \simeq -0.729637$ . For the sake of clarity, let us emphasize that this expression simply comes from the integration of Eq. (2) by keeping all *functions* of order less than or equal to  $\mathcal{O}(\epsilon^2)$ , and dropping the higher order ones. Moreover, the power spectrum has been expanded around a unique pivot scale,  $k_*$ , such that all “star” quantities are evaluated at the time  $\eta_*$  defined by  $k_* \eta_* = -1$ . This last step is necessary in order to make explicit the dependence on the wave number  $k$ . Let us notice that the spectral index  $n_{\text{T}} = \text{d} \ln \mathcal{P}_h / \text{d} \ln k|_{k_*}$  and the running  $\alpha_{\text{T}} \equiv \text{d}^2 \ln \mathcal{P}_h / \text{d} (\ln k)^2|_{k_*}$  are immediately obtained by expanding the logarithm of Eq. (3).<sup>2</sup>

As one can check in the references mentioned earlier, the scalar mode calculations are usually much more involved than those for the tensors. The technical difficulties in solving the scalar equations are exacerbated when one wants to apply these techniques to models for which the

<sup>2</sup>At second order, it is also crucial to specify the pivot scale definition. In particular, the pivot of Refs. [27,40] is set at  $k = aH$  instead of  $k_* \eta_* = -1$  here. This is the reason why some numerical coefficients in Eq. (3) are different in front of the second order terms.

perturbations propagate with a varying speed of sound  $c_s(\eta)$ . A first attempt at next-to-leading order was performed in Refs. [46–48] but their results were implicitly assuming a constant  $c_s$  leading to some missing terms in the first order corrections to the power spectrum amplitude. The first consistent calculation for the scalar modes at next-to-leading order in K-inflation was presented in Ref. [34] with the Green functions and in Ref. [49] by means of the uniform approximation. There is also a nontrivial dependence in  $c_s$  arising in the tensor power spectrum due to the pivot shift between scalars and tensors. This has been first discussed and derived in Refs. [49,50]. Finally, next-to-next-to-leading order corrections have only been derived very recently in Ref. [51] within the uniform approximation only.

As we have just summarized, all the integration techniques performed so far have been independently applied to either the tensor or the scalar modes, at a given order, and for a given class of single field models. In this work, we derive an exact mapping between all the power spectra by noticing that even though the integration methods are different, they all start from the same functional form, namely Eq. (2). In the next section, we explicitly prove the existence of such a transformation by introducing some generalized flow functions, very similar to the usual ones of Eq. (1). Then we apply our method to the Green function integration approach and derive, for the first time and in a few lines, the power spectrum of the curvature perturbation at second order for K-inflation.

## II. GENERALIZED FLOW FUNCTIONS

From Eq. (2), one can reinterpret the function  $z(\tau)$  as being a generalized scale factor from which one could define a generalized  $e$ -fold number  $\tilde{N} \equiv \ln z$  and a generalized Hubble parameter  $\tilde{H}$  with its conformal analogue  $\tilde{\mathcal{H}} \equiv z\tilde{H}$  such that

$$\tilde{\mathcal{H}}(\tau) \equiv \frac{z'}{z} = \frac{\text{d}\tilde{N}}{\text{d}\tau}. \quad (4)$$

As a result, one can construct an infinite hierarchy of generalized flow functions  $\alpha_i$ , exactly as in Eq. (1), but based on this rescaled Hubble parameter:

$$\alpha_{i+1} \equiv \frac{\text{d} \ln |\alpha_i|}{\text{d}\tilde{N}}, \quad \alpha_1 \equiv -\frac{\text{d} \ln \tilde{H}}{\text{d}\tilde{N}}. \quad (5)$$

In terms of these generalized quantities, we know that, up to an overall normalization accounting for the different quantum initial conditions for scalars versus tensors (such as the number of polarization states), the power spectrum of any quantity at second order using the Green function method must be given by Eq. (3) with the replacement

$$\epsilon_{i*} \rightarrow \alpha_{i*}, \quad H_* \rightarrow \tilde{H}_*. \quad (6)$$

Even though this might seem a trivial remark, as it stems from well-known field redefinitions in the quadratic action for the perturbations, up to our knowledge this property has never been used before to actually solve the equations of motion and compute the relevant observables with the accuracy achieved in this paper. These generalized quantities are the only ones that we can measure by detecting the amplitude and spectral index of the scalar perturbations alone. From a purely effective point of view, it has been noticed that the interplay of  $\epsilon_1$  and  $c_s$  can lead to an exactly scale invariant power spectrum also for finite values of  $\epsilon_1$ , at the cost of breaking the scale invariance of the tensor modes [52] and increasing the amount of non-Gaussianity [38]. Even more radically, scale invariant scalar perturbations can also be obtained in noninflationary backgrounds if one relies on other mechanisms to solve the horizon and flatness problem [53,54].

Assuming that  $c_s$  is a free function, from Eqs. (4) and (5) we have

$$\begin{aligned}\tilde{N} &= N + \frac{1}{2}(\ln \epsilon_1 - \ln c_s), \\ \tilde{H} &= \frac{H}{\sqrt{\epsilon_1 c_s}} \left(1 + \frac{\epsilon_2 + \delta_1}{2}\right),\end{aligned}\quad (7)$$

where we have defined the usual sound flow hierarchy [49]

$$\delta_{i+1} \equiv \frac{d \ln |\delta_i|}{dN}, \quad \delta_1 \equiv -\frac{d \ln c_s}{dN}. \quad (8)$$

It is important to notice that Eq. (7) contains exact functional relations between the usual Hubble and sound flow functions and the generalized ones. As such, they can be used in *any* approximation schemes or expansions. They are also complete as they fix by recurrence the mapping of the full hierarchy. For instance, using  $dN/d\tilde{N} = [1 + (\epsilon_2 + \delta_1)/2]^{-1}$  the exact functional relations for the first two generalized flow functions  $\alpha_1$  and  $\alpha_2$  are

$$\begin{aligned}\alpha_1 &= \frac{1}{(1 + \frac{\epsilon_2 + \delta_1}{2})^2} \left( \epsilon_1 + \frac{1}{2} \epsilon_2 - \frac{1}{2} \delta_1 + \frac{1}{2} \epsilon_1 \epsilon_2 + \frac{1}{4} \epsilon_2^2 - \frac{1}{2} \epsilon_2 \epsilon_3 + \frac{1}{2} \epsilon_1 \delta_1 - \frac{1}{2} \delta_1 \delta_2 - \frac{1}{4} \delta_1^2 \right), \\ \alpha_2 &= \frac{1}{1 + \frac{\epsilon_2 + \delta_1}{2}} \left( -\frac{\epsilon_2 \epsilon_3 + \delta_1 \delta_2}{1 + \frac{\epsilon_2 + \delta_1}{2}} \right. \\ &\quad \left. + \frac{2\epsilon_1 \epsilon_2 + \epsilon_2 \epsilon_3 - \delta_1 \delta_2 + \epsilon_1 \epsilon_2^2 + \epsilon_1 \epsilon_2 \epsilon_3 + \epsilon_2^2 \epsilon_3 - \epsilon_2 \epsilon_3^2 - \epsilon_2 \epsilon_3 \epsilon_4 + \epsilon_1 \epsilon_2 \delta_1 + \epsilon_1 \delta_1 \delta_2 - \delta_1 \delta_2^2 - \delta_1 \delta_2 \delta_3 - \delta_1^2 \delta_2}{2\epsilon_1 + \epsilon_2 - \delta_1 + \epsilon_1 \epsilon_2 + \frac{1}{2} \epsilon_2^2 - \epsilon_2 \epsilon_3 + \epsilon_1 \delta_1 - \delta_1 \delta_2 - \frac{1}{2} \delta_1^2} \right).\end{aligned}\quad (9)$$

Standard single field inflation is recovered by plugging  $c_s = 1$  and  $\delta_i = 0$  in Eqs. (7) and (9). In the following, motivated by the Planck results, we adopt a conservative approach by assuming  $c_s$  is a free but slowly varying function such that  $\delta_i \sim \mathcal{O}(\epsilon)$ .

### III. POWER SPECTRA WITH VARYING SPEED OF SOUND

The above mapping can now be applied to straightforwardly derive the second order power spectrum for the comoving curvature perturbation in generalized single field models with varying speed of sound. Plugging Eqs. (7) and (9) into Eq. (3) and Taylor expanding everything at second order in the  $\epsilon_i$  and  $\delta_i$  parameters yields the desired scalar spectrum. One should also not forget to divide the result by the well-known factor 16 which accounts for the different normalization of the scalar action with respect to the one for tensors, arising from the number of graviton polarization states [31]. One finally gets

$$\begin{aligned}\mathcal{P}_\zeta &= \frac{H_\triangleright^2}{8\pi^2 M_{\text{Pl}}^2 \epsilon_{1\triangleright} c_{s\triangleright}} \left\{ 1 - 2(1+C)\epsilon_{1\triangleright} - C\epsilon_{2\triangleright} + (2+C)\delta_{1\triangleright} + \left(\frac{\pi^2}{2} - 3 + 2C + 2C^2\right)\epsilon_{1\triangleright}^2 \right. \\ &\quad + \left(\frac{7\pi^2}{12} - 6 - C + C^2\right)\epsilon_{1\triangleright}\epsilon_{2\triangleright} + \left(\frac{\pi^2}{8} - 1 + \frac{C^2}{2}\right)\epsilon_{2\triangleright}^2 + \left(\frac{\pi^2}{24} - \frac{C^2}{2}\right)\epsilon_{2\triangleright}\epsilon_{3\triangleright} + \left(\frac{\pi^2}{8} + n_\triangleright + C + \frac{C^2}{2}\right)\delta_{1\triangleright}^2 \\ &\quad + \left(-\frac{\pi^2}{24} + 2 + 2C + \frac{C^2}{2}\right)\delta_{1\triangleright}\delta_{2\triangleright} + \left(-\frac{\pi^2}{2} + p_\triangleright - 3C - 2C^2\right)\delta_{1\triangleright}\epsilon_{1\triangleright} + \left(-\frac{\pi^2}{4} + q_\triangleright - C - C^2\right)\delta_{1\triangleright}\epsilon_{2\triangleright} \\ &\quad + \left[-2\epsilon_{1\triangleright} - \epsilon_{2\triangleright} + \delta_{1\triangleright} + (2+4C)\epsilon_{1\triangleright}^2 + (-1+2C)\epsilon_{1\triangleright}\epsilon_{2\triangleright} + C\epsilon_{2\triangleright}^2 - C\epsilon_{2\triangleright}\epsilon_{3\triangleright} + (1+C)\delta_{1\triangleright}^2 \right. \\ &\quad \left. + (2+C)\delta_{1\triangleright}\delta_{2\triangleright} - (3+4C)\delta_{1\triangleright}\epsilon_{1\triangleright} - (1+2C)\delta_{1\triangleright}\epsilon_{2\triangleright} \right] \ln\left(\frac{k}{k_\triangleright}\right) + \left[2\epsilon_{1\triangleright}^2 + \epsilon_{1\triangleright}\epsilon_{2\triangleright} + \frac{1}{2}\epsilon_{2\triangleright}^2 - \frac{1}{2}\epsilon_{2\triangleright}\epsilon_{3\triangleright} \right. \\ &\quad \left. + \frac{1}{2}\delta_{1\triangleright}^2 + \frac{1}{2}\delta_{1\triangleright}\delta_{2\triangleright} - 2\delta_{1\triangleright}\epsilon_{1\triangleright} - \delta_{1\triangleright}\epsilon_{2\triangleright} \right] \ln^2\left(\frac{k}{k_\triangleright}\right)\},\end{aligned}\quad (10)$$

where the three constants  $n_{\triangleright}$ ,  $p_{\triangleright}$  and  $q_{\triangleright}$  read

$$n_{\triangleright} = 0, \quad p_{\triangleright} = 2, \quad q_{\triangleright} = 2. \quad (11)$$

The new index “ $\triangleright$ ” is different from “ $*$ ” of Eq. (3). Indeed, it is important to understand that the mapping method automatically induces a transformation on the pivot definition. Starting from the tensor mode pivot of Eq. (3) defined at  $k_*\eta_* = -1$ , we get the scalar pivot defined in the same way but with the transformed quantities, i.e. at  $k_{\triangleright}\tau_{\triangleright} = -1$ . As a result, Eq. (10) is expressed in terms of quantities evaluated at the time  $\eta_{\triangleright}$  such that

$$k_{\triangleright} \int_{\eta_{\triangleright}}^0 c_s(\eta) d\eta = -1. \quad (12)$$

Within standard single field models, i.e. those having  $c_s(\eta) = 1$ , there is no difference between the two pivots and  $\eta_{\triangleright} = \eta_*$  (at the same observable pivot mode  $k_{\triangleright} = k_*$ ).

The expression of Eq. (10) has never been derived before, but we can make some cross-checks with other approximation methods. First of all, setting  $c_{s\triangleright} = 1$  and  $\delta_{\triangleright} = 0$ , we recover exactly the same expression as in Refs. [27,40], once the pivot has been switched from  $k = aH$  to ours, i.e.  $k_{\triangleright}\eta_{\triangleright} = -1$  for  $c_{s\triangleright} = 1$  (see the discussion above). In the general case, Ref. [55] claims to have performed such a second order expansion using Green functions but their derivation does not include the pivot expansion and it is assumed that  $c_s \simeq 1$ , which makes it hardly comparable with our result. On the other hand, as for the tensor modes, the spectral index  $n_s - 1$  and the running  $\alpha_s$  at second order can be immediately read out from the logarithm of Eq. (10). These quantities have already been derived in the literature, as for instance in Refs. [34,49], using the trick described in Ref. [28], which allows us to derive spectral index and running at second order from the power spectrum at first order. Our results match both expressions, and we do not repeat them here. However, applying the method of Ref. [28] to our results gives now the spectral index and running at third order (see the Appendix). Finally, Ref. [51] has recently derived the very same power spectrum, at the pivot scale  $k_{\diamond}\eta_{\diamond}c_{s\diamond} = -1$ , by using the uniform approximation to

directly solve the scalar equation of motion. Up to the well-known differences between the Green function method and the WKB/uniform approximations, Eq. (10) is compatible with this reference after one has performed the change of pivot described below. A last check we have performed is to apply the mapping technique within the uniform approximation scheme. Starting from the tensor power spectrum at second order given in Ref. [51], using Eqs. (7) and (9), we have reproduced the second order power spectrum in the uniform approximation derived in that reference.

To be complete, we would like to express the power spectrum at the more widespread pivot  $\eta_{\diamond}$  defined by

$$k_{\diamond}\eta_{\diamond}c_{s\diamond} = -1. \quad (13)$$

Changing from one pivot to the other is a straightforward, but lengthy, calculation that requires performing slow-roll expansions for all terms of Eq. (10). Details on such a transformation can be found in Ref. [49] and we simply here report the result. One gets exactly the same expression as Eq. (10) but with three different numerical coefficients for  $n$ ,  $p$  and  $q$  given by

$$n_{\diamond} = -1, \quad p_{\diamond} = 4, \quad q_{\diamond} = 3. \quad (14)$$

These numbers are only involved in the overall amplitude, i.e. not in front of any  $k$ -dependent terms and therefore this change of pivot does not affect the spectral index and running at second order (it does at third order). With this new pivot, we have checked that the numerical values of all multiplying coefficients are within a few percent to those given by the uniform approximation of Ref. [51], even though they are defined from different combinations of irrational numbers and stem from a completely different approach to solving the equations of motion.

From the data analysis point of view, one should simultaneously use both the scalar and tensor power spectra. In particular, this allows us to measure, or bound, the tensor-to-scalar ratio. In order to get meaningful results it is however crucial to evaluate them at the same pivot. From Eq. (3), moving the pivot from  $k_*\eta_* = -1$  to  $k_{\diamond}\eta_{\diamond}c_{s\diamond} = -1$ , one gets

$$\begin{aligned} \mathcal{P}_h = & \frac{2H_{\diamond}^2}{\pi^2 M_{\text{p}}^2} \left\{ 1 - 2(1 + C - \ln c_{s\diamond})\epsilon_{1\diamond} + \left[ \frac{\pi^2}{2} - 3 + 2C + 2C^2 - (2 + 4C)\ln c_{s\diamond} + 2\ln^2 c_{s\diamond} \right] \epsilon_{1\diamond}^2 \right. \\ & + \left[ \frac{\pi^2}{12} - 2 - 2C - C^2 + 2(1 + C)\ln c_{s\diamond} - \ln^2 c_{s\diamond} \right] \epsilon_{1\diamond}\epsilon_{2\diamond} + [-2\epsilon_{1\diamond} + (2 + 4C - 4\ln c_{s\diamond})\epsilon_{1\diamond}^2 \\ & \left. - 2(1 + C - \ln c_{s\diamond})\epsilon_{1\diamond}\epsilon_{2\diamond} \right] \ln\left(\frac{k}{k_{\diamond}}\right) + (2\epsilon_{1\diamond}^2 - \epsilon_{1\diamond}\epsilon_{2\diamond})\ln^2\left(\frac{k}{k_{\diamond}}\right) \Big\}, \end{aligned} \quad (15)$$

which now explicitly depends on  $c_{s\diamond}$ .

#### IV. CONCLUSION

We have derived a simple transformation that is summarized by Eqs. (5)–(7) which allows us to map the tensor mode perturbations into the scalar ones. This transformation being exact, it can be used at any order of a flow expansion and within any approximation schemes to integrate the mode equation. We have illustrated its usefulness by deriving for the first time the second order power spectrum for the comoving curvature perturbation using the Green function method and for generalized single field inflation models having a varying speed of sound.

It is important to stress that since  $\mathcal{O}(\epsilon) = \mathcal{O}(n_s - 1)$ , taking the Planck results quoted in the introduction, one has  $\mathcal{O}(\epsilon^2) \simeq 10^{-3}$ . This number is of comparable amplitude with the measurement accuracy of the spectral index and shows that the Planck data are already sensitive to the second order corrections. From a Bayesian data analysis point of view, it means that even if the second order terms cannot yet be measured, they should be included in the data analysis and marginalized over to allow for a robust determination of the  $\epsilon_i$  at first order. Let us also mention that these terms, and eventually the third order ones, will be crucial in the context of 21-cm cosmology [56–59]. At last, and as it is discussed in Ref. [60], direct detection of primordial gravitational waves requires higher order

corrections to be included in the tensor power spectrum because the observable wave numbers are quite different from the ones the CMB is sensitive to. For all these reasons, we give in the Appendix the third order expression of the spectral index, its running and the running of the running for both the tensor and scalar primordial power spectra.

Finally, as the calculations involving the tensor modes are far easier than those involving the scalars, our approach opens the feasibility window for higher order expansions. In principle, our mapping can also be used directly for the perturbed variables and this could also simplify the derivation of higher  $n$ -point functions involved in the calculations of non-Gaussianities.

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#### APPENDIX: SPECTRAL INDEX AND RUNNINGS

In this appendix, for completeness, we give the spectral index, the running and the running of the running up to third order in slow-roll parameters for both scalar and tensor perturbations. The expressions for the scalar perturbations are

$$\begin{aligned}
n_s - 1 = & -(2\epsilon_{1\Diamond} + \epsilon_{2\Diamond} - \delta_{1\Diamond}) - 2\epsilon_{1\Diamond}^2 - (3 + 2C)\epsilon_{1\Diamond}\epsilon_{2\Diamond} - C\epsilon_{2\Diamond}\epsilon_{3\Diamond} + 3\delta_{1\Diamond}\epsilon_{1\Diamond} + \delta_{1\Diamond}\epsilon_{2\Diamond} - \delta_{1\Diamond}^2 + (2 + C)\delta_{1\Diamond}\delta_{2\Diamond} \\
& - 2\epsilon_{1\Diamond}^3 - (15 + 6C - \pi^2)\epsilon_{1\Diamond}^2\epsilon_{2\Diamond} + 5\delta_{1\Diamond}\epsilon_{1\Diamond}^2 - \left(7 + 3C + C^2 - \frac{7\pi^2}{12}\right)\epsilon_{1\Diamond}\epsilon_{2\Diamond}^2 \\
& - \left(6 + 4C + C^2 - \frac{7\pi^2}{12}\right)\epsilon_{1\Diamond}\epsilon_{2\Diamond}\epsilon_{3\Diamond} + \left(13 + 5C - \frac{\pi^2}{2}\right)\delta_{1\Diamond}\epsilon_{1\Diamond}\epsilon_{2\Diamond} - 4\delta_{1\Diamond}^2\epsilon_{1\Diamond} \\
& + \left(10 + 4C - \frac{\pi^2}{2}\right)\delta_{1\Diamond}\delta_{2\Diamond}\epsilon_{1\Diamond} - \left(2 - \frac{\pi^2}{4}\right)\epsilon_{2\Diamond}^2\epsilon_{3\Diamond} - \left(\frac{C^2}{2} - \frac{\pi^2}{24}\right)\epsilon_{2\Diamond}\epsilon_{3\Diamond}^2 + \left(3 + 2C - \frac{\pi^2}{4}\right)\delta_{1\Diamond}\epsilon_{2\Diamond}\epsilon_{3\Diamond} \\
& - \left(\frac{C^2}{2} - \frac{\pi^2}{24}\right)\epsilon_{2\Diamond}\epsilon_{3\Diamond}\epsilon_{4\Diamond} - \delta_{1\Diamond}^2\epsilon_{2\Diamond} + \left(3 + C - \frac{\pi^2}{4}\right)\delta_{1\Diamond}\delta_{2\Diamond}\epsilon_{2\Diamond} + \delta_{1\Diamond}^3 - \left(8 + 3C - \frac{\pi^2}{4}\right)\delta_{1\Diamond}^2\delta_{2\Diamond} \\
& + \left(2 + 2C + \frac{C^2}{2} - \frac{\pi^2}{24}\right)\delta_{1\Diamond}\delta_{2\Diamond}^2 + \left(2 + 2C + \frac{C^2}{2} - \frac{\pi^2}{24}\right)\delta_{1\Diamond}\delta_{2\Diamond}\delta_{3\Diamond}, \tag{A1}
\end{aligned}$$

$$\begin{aligned}
\alpha_s = & -2\epsilon_{1\Diamond}\epsilon_{2\Diamond} - \epsilon_{2\Diamond}\epsilon_{3\Diamond} + \delta_{1\Diamond}\delta_{2\Diamond} - 6\epsilon_{1\Diamond}^2\epsilon_{2\Diamond} - (3 + 2C)\epsilon_{1\Diamond}\epsilon_{2\Diamond}^2 - 2(2 + C)\epsilon_{1\Diamond}\epsilon_{2\Diamond}\epsilon_{3\Diamond} + 5\delta_{1\Diamond}\epsilon_{1\Diamond}\epsilon_{2\Diamond} \\
& + 4\delta_{1\Diamond}\delta_{2\Diamond}\epsilon_{1\Diamond} - C\epsilon_{2\Diamond}^2 - C\epsilon_{2\Diamond}\epsilon_{3\Diamond}\epsilon_{4\Diamond} + 2\delta_{1\Diamond}\epsilon_{2\Diamond}\epsilon_{3\Diamond} + \delta_{1\Diamond}\delta_{2\Diamond}\epsilon_{2\Diamond} - 3\delta_{1\Diamond}^2\delta_{2\Diamond} \\
& + (2 + C)\delta_{1\Diamond}\delta_{2\Diamond}(\delta_{2\Diamond} + \delta_{3\Diamond}), \tag{A2}
\end{aligned}$$

$$\beta_s = -2\epsilon_{1\Diamond}\epsilon_{2\Diamond}^2 - 2\epsilon_{1\Diamond}\epsilon_{2\Diamond}\epsilon_{3\Diamond} - \epsilon_{2\Diamond}\epsilon_{3\Diamond}^2 - \epsilon_{2\Diamond}\epsilon_{3\Diamond}\epsilon_{4\Diamond} + \delta_{1\Diamond}\delta_{2\Diamond}\delta_{3\Diamond} + \delta_{1\Diamond}\delta_{2\Diamond}^2. \tag{A3}$$

The spectral index of the tensor mode power spectrum at third order reads

$$\begin{aligned}
n_T = & -2\epsilon_{1\Diamond} - 2\epsilon_{1\Diamond}^2 - 2(1 + C - \ln c_{s\Diamond})\epsilon_{1\Diamond}\epsilon_{2\Diamond} - 2\epsilon_{1\Diamond}^3 - (14 + 6C - \pi^2 - 6 \ln c_{s\Diamond})\epsilon_{1\Diamond}^2\epsilon_{2\Diamond} \\
& - \left[2 + 2C + C^2 - \frac{\pi^2}{12} - 2(1 + C) \ln c_{s\Diamond} + \ln^2 c_{s\Diamond}\right]\epsilon_{1\Diamond}\epsilon_{2\Diamond}(\epsilon_{2\Diamond} + \epsilon_{3\Diamond}), \tag{A4}
\end{aligned}$$

while the running is given by

$$\alpha_T = -2\epsilon_{1\Diamond}\epsilon_{2\Diamond} - 6\epsilon_{1\Diamond}^2\epsilon_{2\Diamond} - 2(1 + C - \ln c_{s\Diamond})\epsilon_{1\Diamond}\epsilon_{2\Diamond}^2 - 2(1 + C - \ln c_{s\Diamond})\epsilon_{1\Diamond}\epsilon_{2\Diamond}\epsilon_{3\Diamond}, \quad (\text{A5})$$

and the running of the running is

$$\beta_T = -2\epsilon_{1\Diamond}\epsilon_{2\Diamond}(\epsilon_{2\Diamond} + \epsilon_{3\Diamond}). \quad (\text{A6})$$

Finally, the tensor-to-scalar ratio, up to third order, is given by

$$\begin{aligned} r = 16\epsilon_{1\Diamond}c_{s\Diamond} & \left\{ 1 - (2 + C)\delta_{1\Diamond} + C\epsilon_{2\Diamond} + 2\epsilon_{1\Diamond}\ln c_{s\Diamond} + \left(5 + 3C + \frac{C^2}{2} - \frac{\pi^2}{8}\right)\delta_{1\Diamond}^2 - \left(2 + 2C + \frac{C^2}{2} - \frac{\pi^2}{24}\right)\delta_{1\Diamond}\delta_{2\Diamond} \right. \\ & - \left(3 + 3C + C^2 - \frac{\pi^2}{4}\right)\delta_{1\Diamond}\epsilon_{2\Diamond} + \left(1 + \frac{C^2}{2} - \frac{\pi^2}{8}\right)\epsilon_{2\Diamond}^2 + \left(\frac{C^2}{2} - \frac{\pi^2}{24}\right)\epsilon_{2\Diamond}\epsilon_{3\Diamond} + 2(1 + \ln c_{s\Diamond})\ln c_{s\Diamond}\epsilon_{1\Diamond}^2 \\ & \left. - \left[8 + 3C - \frac{\pi^2}{2} + 2(2 + C)\ln c_{s\Diamond}\right]\delta_{1\Diamond}\epsilon_{1\Diamond} + \left[4 + C - \frac{\pi^2}{2} + 2(1 + 2C)\ln c_{s\Diamond} - \ln^2 c_{s\Diamond}\right]\epsilon_{1\Diamond}\epsilon_{2\Diamond} \right\}. \quad (\text{A7}) \end{aligned}$$

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