Local conditions separating expansion from collapse in spherically symmetric models with anisotropic pressures

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We investigate spherically symmetric spacetimes with an anisotropic fluid and discuss the existence and stability of a separating shell dividing expanding and collapsing regions. We resort to a $3 + 1$ splitting and obtain gauge invariant conditions relating intrinsic spacetime quantities to properties of the matter source. We find that the separating shell is defined by a generalization of the Tolman-Oppenheimer-Volkoff equilibrium condition. The latter establishes a balance between the pressure gradients, both isotropic and anisotropic, and the strength of the fields induced by the Misner-Sharp mass inside the separating shell and by the pressure fluxes. This defines a local equilibrium condition, but conveys also a nonlocal character given the definition of the Misner-Sharp mass. By the same token, it is also a generalized thermodynamical equation of state as usually interpreted for the perfect fluid case, which now has the novel feature of involving both the isotropic and the anisotropic stresses. We have cast the governing equations in terms of local, gauge invariant quantities that are revealing of the role played by the anisotropic pressures and inhomogeneous electric part of the Weyl tensor. We analyze a particular solution with dust and radiation that provides an illustration of our conditions. In addition, our gauge invariant formalism not only encompasses the cracking process from Herrera and co-workers but also reveals transparently the interplay and importance of the shear and of the anisotropic stresses.

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I. INTRODUCTION

The universe close to us is inhomogeneous, exhibiting structures at different scales that are the result of the nonlinear collapse of overdensities, and below certain scales these structures seem to be immune to the overall expansion of the universe. On the other hand, this picture reveals two different dynamical behaviors that we wish to describe by a global general relativistic solution. This solution must exhibit expansion on the large scales, and infall at smaller scales, eventually producing bound structures. It is the understanding of the interplay between collapsing and expanding regions within the theory of general relativity (GR) that we aim to address here. This issue is connected to the general problem of assessing the influence of global physics into local physics [1,2], as well as to the approach to nonperturbative backreaction through model building [3,4]. Another related problem is that of recollapsing [5–9].

In a previous paper [10], we have obtained local conditions for perfect fluid solutions to collapse within an otherwise cosmologically expanding background (also in [11,12]). We have characterized the locally defined separating shells between the collapsing and the expanding regions.

In the present paper we wish to deepen our understanding of the problem under consideration by overcoming the limits placed by the consideration of a perfect fluid. While such a description of the matter content is justified when one deals with an equilibrium configuration, the consideration of nonequilibrium states requires a more general viewpoint, where anisotropic stresses are present [13]. Indeed, in models exhibiting anisotropies and/or inhomogeneities, the consideration of a fluid with isotropic pressures is very much a restriction to the model (often of a simplifying nature). The most likely situation seems to be that the fluid inherits the geometrical features of the model and is of an anisotropic type. So one should envisage different directional behaviors, which is precisely what

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should be expected both from collapsing or expanding regions within spherically symmetric models, since the radial and transverse directions behave differently.

In the present work we investigate spherically symmetric spacetimes with an anisotropic fluid, but no heat fluxes since we want to concentrate on the role of the stresses, shear, and intrinsic curvature regarding the problem under consideration. We leave the role of heat fluxes for a subsequent work. As in our previous works [10,14], we resort to a $3 + 1$ splitting. We use a nonperturbative approach that relies on the use of the formalism that has been developed in a remarkable series of papers by Lasky and Lun using Generalized Painlevé-Gullstrand (GPG) coordinates [15–18]. The GPG coordinates are helpful to set up a single metric describing both collapse and expanding regions. This approach allows us to evade having to deal with the spacetime matching problem (an illustration of matching two different spacetime metrics can be found in [19]). We must stress, though, that this choice of coordinate is not the only one achieving this goal, and we use it as a matter of convenience and consistency with our previous work. We assess the existence and stability of a separating shell dividing expanding and collapsing regions, in a gauge invariant way. The local conditions that we find generalize our previous results and relate intrinsic spacetime quantities to quantities characterizing the matter source. This happens through a generalization of the Tolman-Oppenheimer-Volkoff equilibrium condition, which, itself, is a generalization of the corresponding isotropic generalized TOV condition found in [10]. Our condition establishes a relation between the pressure gradients, both isotropic and anisotropic, and the strength of the fields induced by the Misner-Sharp mass inside the separating shell and by the pressure fluxes. This defines a local equilibrium condition but conveys also a nonlocal character given the definitions of the Misner-Sharp mass, and of the energy function E [see definition in Eq. (2.1) below]. By the same token, it is also a generalized thermodynamical equation of state as usually interpreted for the perfect fluid case, which now has the novel feature of involving both the isotropic and the anisotropic stress.

In addition, this approach has allowed us to express the Einstein field equations as a dynamical system involving scalar invariants and local quantities. This formulation reveals the fundamental roles of combinations of expansion with shear and two sets combining the electric Weyl with anisotropic stress scalars that are discussed in their flow evolution, relation to curvature and impact on shear evolution.

To illustrate our results we analyze a particular solution with dust and radiation. Such a solution stems from the work of Sussman and Pavón [20], where, not withstanding the generality of their initial formalism, they analyzed only the thermodynamic aspects of the spatially flat spherical solution. We find the conditions characterizing the matter and radiation content to fulfill the existence of a separating shell. In turn we also obtain the nonflat elliptic solutions.

On a different context, Herrera and co-workers [21] have studied small anisotropic perturbations around spherically symmetric homogeneous fluids in equilibrium. They concluded that this may lead to instabilities that result in the "cracking" of boundary surfaces of compact objects in astrophysics. We recover their results within our gauge invariant formalism, which not only confirms the important role of the shear and of the anisotropic stresses but also reveals transparently their interplay and how they trigger the cracking process.

Following is an outline of the paper: in Sec. II the GPG formalism of Lasky and Lun and the $3 + 1$ splitting is revised. We also define gauge invariant kinematical quantities. In Sec. III we discuss the existence of a shell separating collapse from expansion and give general dynamical conditions. In Sec. IV we present illustrations with a dust plus radiation solution and with the relation between the separating shell and cracking. Section V gives a discussion of our results.

We shall use $\kappa^2 = 8\pi G$, $c = 1$ and the following index convention: greek indices $\alpha, \beta, \ldots = 1, 2, 3$ while latin indices $a, b, \ldots = 0, 1, 2, 3$.

II. 3 + 1 SPLITTING AND GAUGE INVARIANTS KINEMATICAL QUANTITIES

We set the basic equations in generalized Painlevé-Gullstrand coordinates following the formalism developed by Lasky and Lun (LL) [16,17], while adapting their derivations for our standpoint, which is concerned with the collapse within an underlying overall expansion, rather than with collapse on its own.

A. Metric and ADM splitting

We assume that the flow of the fluid is characterized by the timelike, normalized vector[1] $u_a := -\alpha\nabla_a t = [-\alpha, 0, 0, 0]$ $(u_a u^a = -1)$, defined with its lapse $N = \alpha$ and its radial shift vector $N^\mu = (\beta, 0, 0)$, and an evolution of the spatially curved 3-metric ${}^3g_{\mu\nu} = \mathrm{diag}(\frac{1}{1+E}, r^2, r^2\sin^2\theta)$. Consequently, we write the spherically symmetric line element as

$$ds^2 = -\alpha(t,R)^2 dt^2 + \frac{1}{1+E(t,R)}(\beta(t,R)dt + dR)^2 + r(t,R)^2 d\Omega^2, \qquad (2.1)$$

[1]In this paper we will adopt some notations differing from previous conventions adopted in our works [10,14] and those providing their technical framework [16–18]: the previous flow vector n^a, shear scalar a, tangential eigenvalues of the traceless 3-Ricci q and of the electric Weyl tensor Σ are now noted, respectively, u^a, σ, η and Ξ.

which adopts the GPG coordinates of Ref. [17] ($d\Omega^2 := d\theta^2 + \sin^2\theta d\phi^2$). Notice that the areal radius r differs from the R coordinate and is used here for convenience to handle particular cases.[2]

Performing a ADM $3+1$ splitting [16–18,22], we use the projection operators along and orthogonal to the flow,

$$N^a_b := -u^a u_b, \qquad h^{ab} := g^{ab} + u^a u^b, \tag{2.2}$$

where h^{ab} is the 3-metric on the surface S_3 normal to the flow. Those projectors are also used for covariant derivatives. Along the flow, the proper time (also known as convective) derivative of any tensor $X^{ab}{}_{cd}$ is

$$\dot{X}^{ab}{}_{cd} := u^e X^{ab}{}_{cd;e}, \tag{2.3}$$

and in the orthogonal 3-surface, each component is projected with h (the overbar denotes the covariant derivative and tensor full orthogonal projection)

$$X^{ab}{}_{cd;\bar{e}} := h^a{}_f h^b{}_g h^i{}_c h^j{}_d h^k{}_e X^{fg}{}_{ij;k}. \tag{2.4}$$

Then the covariant derivative of the flow, from its projections, is defined as

$$u_{a;b} = N_b{}^c u_{a;c} + u_{a;\bar{b}} = -u_b \dot{u}_a + \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab}, \tag{2.5}$$

where the trace of the projection is the expansion of the flow, $\Theta = u^a{}_{;\bar{a}} = u^a{}_{;a}$, the rate of shear σ_{ab} is its symmetric trace-free part and its skew-symmetric part is the vorticity ω_{ab}.

Ahead, we will find it useful to use a "mixed" approach with the kinematic variables of the covariant $1+3$ formalism of Ehlers and Ellis [23,24] and some ADM variables like ϵ, η and the gradients of the lapse function α. However, for clarity, the correspondence between those ADM variables and the $1+3$ kinematic variables is written in Appendix B.

On the other hand, we consider an energy-momentum tensor

$$T^{ab} = \rho u^a u^b + P h^{ab} + \Pi^{ab}, \tag{2.6}$$

where ρ is the energy density, P is the pressure and Π^{ab} is the anisotropic stress tensor. $\Pi^{ab} u_b = 0$ and $\Pi^a{}_a = 0$, i.e., the anisotropic stress Π^{ab} is orthogonal to u^a and traceless.

The spherical symmetry implies that all the quantities $X_{\alpha\beta} = h_\alpha{}^a h_\beta{}^a X_{ab}$ share the same spatial eigendirections characterized by the traceless 3-tensor $P^\alpha{}_\beta = \mathrm{diag}[-2, 1, 1]$, such that

$$X_{\alpha\beta} = w(t, R) P_{\alpha\beta}. \tag{2.7}$$

Given a unit vector n^a, in the direction orthogonal to the flow u^a, $n^a u_a = 0$ and $n^a n_a = 1$, so $n^a = \sqrt{1+E}\delta^a_R$, and one can define, in four dimensions, the traceless projector as $P_{ab} = h_{ab} - h^c_c n_a n_b$, admitting from the bulk all the properties of the 3-projector in the hypersurfaces. Because of spherical symmetry, we then can decompose any spatial two-tensor into its trace and traceless parts: $X_{ab} = h^c_a h^d_b X_{cd} = X\frac{h_{ab}}{3} + \xi P_{ab}$, with its trace being $X = X^a_a$ and its traceless tangential eigenvalue being ξ. Therefore, we have the following decompositions for traceless quantities:

(i) For the anisotropic stress,

$$\Pi_{ab} = \Pi(t, R) P_{ab}. \tag{2.8}$$

(ii) For the shear tensor[3] (traceless extrinsic curvature),

$$\sigma_{ab} = \sigma(t, R) P_{ab}. \tag{2.9}$$

(iii) For the trace-free, three-dimensional Riemann tensor, which measures the departures from constant spatial curvature,

$${}^{(3)}R_{\alpha\beta} - \frac{1}{3}{}^{(3)}g_{\alpha\beta}{}^{(3)}R = \eta(t, R) P_{\alpha\beta}. \tag{2.10}$$

(iv) For the trace-free Hessian of the lapse function,

$$\frac{1}{\alpha}\left(D_\gamma D_\mu - \frac{1}{3}{}^{(3)}g_{\gamma\mu} D^\beta D_\beta\right)\alpha = \epsilon(t, R) P_{\gamma\mu}, \tag{2.11}$$

where D_μ is the covariant derivative on the hypersurface (note $D_\mu = h^a_\mu \nabla_a$ with the covariant derivative written ∇_a).

(v) For the electric part of the Weyl tensor, we have

$$E_{ab} = \Xi(t, R) P_{ab}. \tag{2.12}$$

There is no magnetic part of the Weyl tensor due to the spherical symmetry [25], and thus the models fall into the class that has been dubbed "silent" universes [26,27]. Another consequence of the spherical symmetry is that the flow is irrotational, $\omega_{ab} = 0$.

B. The Einstein field equations

It is well known that the ADM approach separates the ten Einstein field equations (EFE) into four constraints on the hypersurfaces and six evolution equations. Spherical symmetry reduces them to $2+2$.

[2]As discussed below, it becomes possible to restrict to $R = r$. However, we will restrain here from doing this identification to maintain generality in the following equations.

[3]Note that σ here is more general than that used e.g. in [22], as it allows for negative values.

The EFE can then be written as a set of propagation equations[4]: the trace and tracefree[5] orthogonal contractions of the EFE, the double orthogonal contracted and flow projected Bianchi identity (once contracted with the tracefree orthogonal projector)[6] read

$$-2\dot{\Theta} = \frac{^3R}{2} + \Theta^2 + 9\sigma^2 - \frac{2}{\alpha}D^\mu D_\mu \alpha + 3\kappa^2 P - 3\Lambda, \tag{2.13}$$

$$\dot{\sigma} = -\sigma\Theta + \epsilon - \eta + \kappa^2\Pi \tag{2.14}$$

$$\dot{\Xi} = -\frac{\kappa^2}{2}\dot{\Pi} - \frac{\kappa^2}{2}(\rho + P - 2\Pi)\sigma - \left(3\Xi + \frac{\kappa^2}{2}\Pi\right)\left(\frac{\Theta}{3} + \sigma\right). \tag{2.15}$$

They are accompanied with spacelike constraints: the gauge invariant radial balance, which proceeds from the cross projection of the EFE, and the tidal forces, obtained from the double orthogonal contracted, acceleration projected Bianchi identity (again, once with the tracefree orthogonal projector),[7] yield

$$\left(\frac{\Theta}{3} + \sigma\right)' = -3\sigma\frac{r'}{r}, \tag{2.16}$$

$$\frac{4\pi}{3}(\rho + 3\Pi)' = -\Xi' - 3\left(\Xi + \frac{\kappa^2}{2}\Pi\right)\frac{r'}{r}. \tag{2.17}$$

Finally, the Hamiltonian constraint reads, in the presence of a cosmological constant,

$$^{(3)}R + \frac{2}{3}\Theta^2 - 6\sigma^2 = 2\kappa^2\rho + 2\Lambda. \tag{2.18}$$

From the twice-contracted Bianchi identities, we also derive, along and orthogonal to the flow,

$$\dot{\rho} = -\Theta(\rho + P) - 6\Pi\sigma \tag{2.19}$$

$$0 = (D^k + \dot{u}^k)(\Pi_{ik} + h_{ik}P) + [\rho - (P - 2\Pi)]\dot{u}_i - u_i[\Theta P + 6\Pi\sigma]. \tag{2.20}$$

[4]For the scalar equations presented here, the Lie and convective derivatives coincide as $\mathcal{L}_u X = u^a X_{,a} = \frac{1}{\alpha}(\partial_t X - \beta\partial_R X)$. The Lie derivative is used in [17,18]. We use the notations $u^a X_{,a} = \dot{X}$ and $\partial_R X = X'$. We use convective derivatives instead of Lie derivatives for consistency with the 1 + 3 formalism.

[5]Note the sign differences in front of the Lie/convective derivatives terms compared with [16,17]; otherwise, the Raychaudhuri equation restricted to the Friedmann-Lemaître-Robertson-Walker (FLRW) case does not get the usual sign for $\dot{H}$.

[6]In other terms, contracted with $h^b_d u^a P^c_e$, the Bianchi identities yield the Weyl evolution.

[7]Or more precisely, the Bianchi identities contracted with $h^c_e \dot{u}^b P^a_d$, so they yield the Weyl constraint.

The latter equation gives the heat fluxes evolution [17], which we set to zero here, since we are restricting our analysis to the case where these fluxes are absent. We thus have

$$0 = -(\rho + P - 2\Pi)\frac{\alpha'}{\alpha} - (P - 2\Pi)' + 6\Pi\frac{r'}{r}, \tag{2.21}$$

which we write to facilitate the connection between Eq. (2.20) and its formulation in terms of fluid and metric elements. The inspection of the system of equations (2.13)–(2.21) thus tells us that the anisotropic stress shows up in all but Eqs. (2.13), (2.16), and (2.18). This reveals the importance of the anisotropic pressures in explicitly contributing to the evolution of the shear, the electric part of the Weyl tensor and the lapse function α [21,28–33].

It is worth noticing at this point that we have included the cosmological constant Λ for the sake of completeness. However, none of the results that follow will depend on its presence. Indeed, we can, without loss of generality, make $\Lambda = 0$, or alternatively absorb it into ρ and P.

Introducing the Misner-Sharp mass [34] and following [17],

$$M' = \frac{\kappa^2}{2}\rho r^2 r', \tag{2.22}$$

it is possible to derive[8]

$$(\dot{r})^2 = \frac{2M}{r} + (1 + E)(r')^2 - 1 + \frac{1}{3}\Lambda r^2 \tag{2.25}$$

and

$$-\ddot{r} = \frac{M}{r^2} + \frac{\kappa^2}{2}(P - 2\Pi)r - (1 + E)\frac{\alpha'}{\alpha}r' - \frac{1}{3}\Lambda r. \tag{2.26}$$

This allows us to extend the generalization of the TOV function made in [10] to the case where anisotropic stresses are present,

$$\mathrm{gTOV} = -\ddot{r}. \tag{2.27}$$

Since, in the absence of heat fluxes we have

$$-\frac{\alpha'}{\alpha} = \frac{1}{(\rho + P - 2\Pi)}\left[(P - 2\Pi)' - 6\Pi\frac{r'}{r}\right], \tag{2.28}$$

then Eqs. (2.26) and (2.27) become

[8]By analogy with the perfect fluid case, it is also possible to derive

$$r'\dot{E} = 2(1 + E)\left[-\dot{r}' - \frac{\beta'}{\alpha}r'\right], \tag{2.23}$$

$$\dot{M} = -\frac{\kappa^2}{2}r^2(P - 2\Pi)\dot{r}. \tag{2.24}$$

$$\begin{aligned} \mathrm{gTOV} &= -\ddot{r} \\ &= \frac{M}{r^2} + \frac{\kappa^2}{2}(P - 2\Pi)r \\ &\quad + \frac{(1+E)r'}{(\rho + P - 2\Pi)}\left[(P - 2\Pi)' - 6\Pi\frac{r'}{r}\right] - \frac{1}{3}\Lambda r. \end{aligned} \tag{2.29}$$

This tells us that, when going from the isotropic perfect fluid to the case of an anisotropic content in the above equations, we have to replace P by $P - 2\Pi$ and introduce an extra term related to the anisotropic stresses.

III. GENERAL CONDITIONS DEFINING A SHELL SEPARATING EXPANSION FROM COLLAPSE

By analogy with the perfect fluid case [10], we now derive the necessary local conditions for the existence of a separating shell, denoted $\star$. First, we require a stationarity condition, analogous to the Newtonian and classic Lemaître-Tolman-Bondi (LTB) turnaround condition discussed in [35],

$$(\dot{r}_\star)^2 = \frac{2M_\star}{r_\star} + (1 + E_\star)(r'_\star)^2 - 1 + \frac{\Lambda}{3}r_\star^2 = 0, \tag{3.1}$$

and second, we need an equilibrium condition to be satisfied on the shell,

$$\begin{aligned} -\ddot{r}_\star &= \frac{M_\star}{r_\star^2} + \frac{\kappa^2}{2}(P_\star - 2\Pi_\star)r_\star - (1 + E_\star)\frac{\alpha'_\star}{\alpha_\star}r'_\star - \frac{\Lambda}{3}r_\star \\ &= 0, \end{aligned} \tag{3.2}$$

where the subscript $\star$ denotes evaluation on the shell $r = r_\star$. Indeed, from Eq. (2.29), the $\mathrm{gTOV}_\star = 0$ equation of state for the stationarity of the separating shell becomes now

$$\begin{aligned} &-\frac{1}{(\rho + P - 2\Pi)_\star}\left[(P - 2\Pi)' - 6\Pi\frac{r'}{r}\right]_\star \\ &\quad = \left[\frac{\frac{M}{r^2} + \left[\frac{\kappa^2}{2}(P - 2\Pi) - \frac{1}{3}\Lambda\right]r}{1 - 2\frac{M}{r} - \frac{1}{3}\Lambda r^2}\right]_\star r'_\star. \end{aligned} \tag{3.3}$$

Thus the existence of a spherical shell separating an expanding outer region from an inner region collapsing in the direction of the center of symmetry, depends essentially on two conditions.[9] The former (3.1) amounts to the vanishing of the kinetic energy of the shell, and establishes the precise balance between the analogues of the total and potential energies at the separating shell. The latter condition (3.2), combined with the former (3.1), is the generalization of the TOV equation for the present case, and is necessary for the equilibrium of the shell. There are noticeable differences with respect to the original problem in the form of the TOV equation [36,37]. The isotropic pressure gradient P' is replaced by $(P - 2\Pi)'$, the gravitational mass $\rho + P$ is consistently traded into $(\rho + P - 2\Pi)$, and there is a new additional term, $-6\Pi\frac{r'}{r}$, involving the anisotropic stress Π and hence reflecting its additional contribution to the balance of pressures and forces per unit mass. It is worth stressing that our result does not rely on the assumption of a static equilibrium of the spherical distribution of matter, and consequently does not assume that all the internal spherical shells are constrained to satisfy the TOV equation. Here the generalized TOV equation is just satisfied at the separating shell. On the neighboring shells it will not be satisfied, and these shells will either be collapsing or expanding since they are not in equilibrium.[10] Moreover, the generalized TOV function depends on the spatial 3-curvature in a more general way than the original TOV function.

It goes without saying that, from the conditions (3.1) and (3.2), it is straightforward to realize that the absence of pressure gradients between the neighboring shells prevents the existence of a separating shell in the spatially homogeneous FLRW models.

Since we have

$$\left(\frac{\Theta}{3} + \sigma\right) = \frac{\dot{r}}{r} \tag{3.4}$$

$$\left(\frac{\Theta}{3} + \sigma\right)^{\cdot} + \left(\frac{\Theta}{3} + \sigma\right)^2 = \frac{\ddot{r}}{r}, \tag{3.5}$$

we see that the turning point condition (3.1) does not imply necessarily the vanishing of the expansion nor of the shear, but it rather means that these quantities should satisfy $\Theta_\star = -3\tilde{\sigma}_\star$ at the separating shell $r = r_\star$ as

$$\Theta_\star + 3\sigma_\star = 0 \tag{3.6}$$

$$\left(\frac{\Theta}{3} + \sigma\right)^{\cdot}_\star = 0. \tag{3.7}$$

If one of Θ or a were to vanish at this locus, we would then have the other quantity vanishing as well. This limit case corresponds to the total staticity of the separating shell.

A. The relation in dynamics between nonlocal and local conditions

Although the conditions (3.1) and (3.3) that characterize the separating shell hold locally, at $r = r_\star$, they involve

[9] We emphasize again that Λ is written here for the sake of generality but is not required for the conditions to hold.

[10] We won't consider here the possible case where the inner shells move outwards and the outer shells move inwards, so that shell crossing occurs. Here we are just interested in characterizing the converse situation where the inner and outer shells depart. The occurrence of shell crossing in inhomogeneous models with anisotropic pressures is discussed in [38].

nonlocal quantities, namely M and E. Indeed, from the construction of M, Eq. (2.22), we see that the profile of the distribution of matter inside the separating shell is taken into account.

It is however possible to find local conditions involving local, rather than nonlocal quantities and this is addressed in what follows.

Given Eq. (3.4) it is possible to relate the condition (2.25) to the Hamiltonian constraint (2.18) that generalizes the Friedmann equation. With that purpose, we recast the latter (also known as the Gauß-Codazzi equation, obtained from $\frac{1}{3}u^a u^b G_{ab}$) as

$$\left(\frac{\Theta}{3}+\sigma\right)^2 = \frac{\kappa^2}{3}\rho - \frac{{}^{(3)}R}{6} + \frac{\Lambda}{3} + 2\sigma\left(\frac{\Theta}{3}+\sigma\right), \tag{3.8}$$

so that we conclude that

$$\frac{2M}{r^3} + (1+E)\left(\frac{r'}{r}\right)^2 - \frac{1}{r^2} = \frac{\kappa^2}{3}\rho - \frac{{}^{(3)}R}{6} + 2\sigma\left(\frac{\Theta}{3}+\sigma\right). \tag{3.9}$$

In parallel, we also wish to clarify the relation between the gTOV function, expressed with the gauge invariant of Eq. (3.4),

$$\mathrm{gTOV} = -r\left(\left(\frac{\Theta}{3}+\sigma\right)^{\cdot} + \left(\frac{\Theta}{3}+\sigma\right)^2\right), \tag{3.10}$$

and the "generalized" Raychaudhuri equation, obtained from contracting the Ricci identity with the combination of projectors $-\frac{1}{6}(2h^{ac}+P^{ac})u^b$,

$$\left(\frac{\Theta}{3}+\sigma\right)^{\cdot} + \left(\frac{\Theta}{3}+\sigma\right)^2 = \epsilon + \frac{1}{3\alpha}D^k D_k \alpha - \frac{\kappa^2}{6}(\rho+3P) - \left(\Xi - \frac{\kappa^2}{2}\Pi\right) + \frac{\Lambda}{3}. \tag{3.11}$$

It is interesting to relate ${}^{(3)}R$ to E from its metric expression (A5)

$$\frac{{}^{(3)}R}{2} = -(1+E)\left(\frac{r'}{r}\right)^2 + \frac{1}{r^2} - 2\frac{\sqrt{1+E}}{r}(\sqrt{1+E}r')'. \tag{3.12}$$

We see that the separating conditions (3.1) and (3.3) now translate into [from Eq. (3.8)]

$$\frac{{}^{(3)}R_\star}{2} = \kappa^2\rho_\star + \Lambda, \tag{3.13}$$

and [from Eq. (3.11)]

$$-\epsilon_\star - \frac{1}{3\alpha_\star}D^k D_k \alpha_\star = -\frac{\kappa^2}{6}(\rho+3P)_\star - \left(\Xi - \frac{\kappa^2}{2}\Pi\right)_\star + \frac{\Lambda}{3}. \tag{3.14}$$

The former of these equations reveals that the stationarity condition requires ${}^{(3)}R > 0$, when $\rho, \Lambda > 0$.[11] It no longer explicitly involves the Misner-Sharp mass $M_\star$, but just the local energy density $\rho_\star$. The latter condition emerges from the generalized Raychaudhuri equation (3.11) and, besides involving local quantities defined at $\star$ as well, it reveals that the important role of the pressure gradient of Eq. (3.3) is now translated by the Hessian trace and traceless tangential eigenvalue on the left-hand side of Eq. (3.14).

B. Nonlocality around the shell

It is possible to express the expansion scalar Θ in terms of the areal radius and its convective and radial derivatives,

$$\Theta = \left(\frac{\dot{r}'}{r'} + 2\frac{\dot{r}}{r}\right), \tag{3.15}$$

and from it to derive

$$(r^2\dot{r})' = \Theta r^2 r'. \tag{3.16}$$

This expression reveals that, in the inhomogeneous spherical models, the expansion scalar Θ is not just the logarithmic derivative of the spatial volume along the timelike flow, unlike what happens in the spatially homogeneous FLRW models. Indeed, we see that it rather contains the logarithmic Lie derivative along the flow of the areal radius r and of its radial gradient r'.

From (3.16), upon integration and choosing a fixed fiducial areal radius r_0 defined as $r_0 = r(t, R_0(t)) = cst$, we obtain

$$\dot{r} = \frac{1}{r^2}\int_{r_0}^{r}\Theta r^2 \mathrm{d}r + \frac{1}{r^2}[r^2\dot{r}]_{r_0}. \tag{3.17}$$

This result shows that the turning point condition at $r_\star$ yields

$$-[r^2\dot{r}]_{r_0} = \int_{r_0}^{r_\star}\Theta r^2 \mathrm{d}r. \tag{3.18}$$

The integral on the right-hand side vanishes if the initial parameter $[r^2\dot{r}]_{r_0}$ vanishes at some interior value $r_0 < r_\star$. This requires the vanishing of the expansion Θ at some intermediate value of r, $r_0 < \tilde{r} < r_\star$, since it has to change signs within the interval of integration (we assume that no shell crossing occurs in that range). Differentiating equation (3.17) with respect to the flow, we obtain

[11]Strictly speaking, when $\kappa^2\rho_\star + \Lambda > 0$.

$$\ddot{r} = -\frac{2\dot{r}}{r^3}\left(\int_{r_0}^{r}\Theta r^2 \mathrm{d}r + [r^2\dot{r}]_{r_0}\right) + \frac{1}{r^2}\left\{\left(\int_{r_0}^{r}\Theta r^2\mathrm{d}r\right)^{\cdot} + [r^2\dot{r}]^{\cdot}_{r_0}\right\} = \dot{r}\left\{\Theta - \frac{2}{r}\dot{r}\right\} + \frac{1}{r^2}\left\{\int_{r_0}^{r}\frac{\partial\Theta}{\partial\tau}r^2\mathrm{d}r + [r^2\dot{r}]^{\cdot}_{r_0}\right\} = -\mathrm{gTOV}, \quad (3.19)$$

where τ denotes proper time. This is the equation that generalizes the Eq. (3.27) of [10] and that corresponds to Eq. (21) of di Prisco *et al.* [39]. It corroborates once again their claim of a nonlocality of the radial acceleration. From Eq. (3.17) we realize that this nonlocality is inherent in the radial expansion, and is already present in the energy condition defining $r_\star$ Eqs. (3.1) and (3.6) and in our gTOV condition Eqs. (2.27), (2.26), and (3.2), since both implicate M which involves an integral between 0 and $r_\star$.

From the previous Eqs. (3.17) and (3.19), we see that at the separating shell we have

$$-[r^2\dot{r}]^{\cdot}_{r_0} = \int_{r_0}^{r_\star}\frac{\partial\Theta}{\partial\tau}r^2\mathrm{d}r, \quad (3.20)$$

which means that the integral on the right-hand side vanishes if the term $-[r^2\dot{r}]^{\cdot}_{r_0}$ vanishes at an interior value $r_0 < r_\star$. This shows that the vanishing of the proper time derivative of the expansion $\dot{\Theta}$ occurs then at some intermediate value between r_0 and $r_\star$. In the case when $[r^2\dot{r}]^{\cdot}_{r_0} = 0$ at the center, we recover the result of Di Prisco *et al.* [39], establishing the vanishing of the radial acceleration, i.e. $\dot{\Theta} = 0$, at some $0 < r < r_\star$. However this result is derived here in a nonperturbative and a more general way than in Ref. [39].

C. Dynamics around the shell

In this section, we will address the dynamics of the system under consideration, adding various restrictions of interest for the rest of the paper and, in each case, examining the dynamics of the matter-trapped shell.

1. Dynamical system of the imperfect fluid

Governing equations.—The dynamical system of partial differential equations (PDEs) that results from the 3 + 1 splitting and the use of the local kinematical and geometric quantities is given by Eqs. (3.11), (2.14), (2.15), (3.8), (2.21), (2.16), and (2.17) and a constraint, Eq. (3.24), on the Weyl tensor. In turn, this constraint is induced by the differences in the shear equation obtained by projections both from the Einstein field equations [Eq. (2.14) from $\frac{P^{dc}}{6}G_{cd}$] and from the Ricci identities [Eq. (3.22) from the projection $-\frac{1}{6}P^{ac}u^b$]. We restate the whole system as

$$\left(\frac{\Theta}{3}+\sigma\right)^{\cdot} = \frac{1}{3\alpha}D^kD_k\alpha + \epsilon - \left(\frac{\Theta}{3}+\sigma\right)^2 - \frac{\kappa^2}{6}(\rho + 3(P-2\Pi)) + \frac{\Lambda}{3} - \left(\Xi + \frac{\kappa^2}{2}\Pi\right), \quad (3.21)$$

$$\dot{\sigma} = -\sigma\Theta + \sigma\left(\frac{\Theta}{3}+\sigma\right) + \left[\epsilon - \left(\Xi - \frac{\kappa^2}{2}\Pi\right)\right], \quad (3.22)$$

$$\left(\Xi + \frac{\kappa^2}{2}\Pi\right)^{\cdot} = -\frac{\kappa^2}{2}\sigma(\rho + P - 2\Pi) - \left[2\left(\Xi + \frac{\kappa^2}{2}\Pi\right) + \left(\Xi - \frac{\kappa^2}{2}\Pi\right)\right] \times \left(\frac{\Theta}{3}+\sigma\right), \quad (3.23)$$

$$\Xi + \frac{\kappa^2}{2}\Pi = \eta + \sigma\left(\frac{\Theta}{3}+\sigma\right), \quad (3.24)$$

$$\left(\frac{\Theta}{3}+\sigma\right)^2 = \frac{\kappa^2}{3}\rho - \frac{{}^{(3)}R}{6} + \frac{\Lambda}{3} + 2\sigma\left(\frac{\Theta}{3}+\sigma\right), \quad (3.25)$$

$$(P-2\Pi)' = 6\Pi\frac{r'}{r} - (\rho + P - 2\Pi)\frac{\alpha'}{\alpha}, \quad (3.26)$$

$$\left(\frac{\Theta}{3}+\sigma\right)' = -3\sigma\frac{r'}{r}, \quad (3.27)$$

$$\frac{\kappa^2}{6}\rho' = -\frac{\left(\left(\Xi + \frac{\kappa^2}{2}\Pi\right)r^3\right)'}{r^3}. \quad (3.28)$$

In this formulation the equations reveal[12] the fundamental role played by some combinations of gauge invariant quantities like expansion and shear, electric Weyl and anisotropic stresses. In the latter case, they emerge in two different combinations that play different and important roles in the governing equations, as we will discuss in what follows. $\Xi + \frac{\kappa^2}{2}\Pi$ acts as a source for density inhomogeneities as seen in Eq. (3.28). From Eq. (3.24) we see that this is related to the 3-curvature distortion of the hypersurfaces as well as to the distortion of the extrinsic curvature, as expected. The role of the other combination is clearly revealed in subsection III C 2, which follows.

Alternatively, one can present Eq. (2.16) in a form parallel to that of Eq. (3.28)

[12]Notice that Eq. (3.21) can also be noted

$$\left(\frac{\Theta}{3}+\sigma\right)^{\cdot} = \frac{1}{3\alpha}D^kD_k\alpha + \epsilon - \left(\frac{\Theta}{3}+\sigma\right)^2 - \frac{\kappa^2}{6}(\rho+3P) + \frac{\Lambda}{3} - \left(\Xi - \frac{\kappa^2}{2}\Pi\right). \quad (3.29)$$

$$\frac{\Theta'}{3} + \frac{(\sigma r^3)'}{r^3} = 0. \tag{3.30}$$

We can note here that the above set regulates the fully nonlinear evolution of a spherically symmetric fluids with anisotropic stress expressed in terms of scalar equations, and thus represents a convenient framework to be used in numerical studies.

Dynamics of the separating shell.—On the separating shell, the dynamics can be expressed from the EFE and Bianchi identities. It takes the form of the residual constraint from the Raychaudhuri equation, Eqs. [(2.13) + (2.18)/2]/6,

$$-\frac{\dot{\Theta}_\star}{3} - \left(\frac{\Theta_\star}{3}\right)^2 + \frac{1}{3\alpha_\star} D^\mu D_\mu \alpha_\star = \frac{\kappa^2}{6}(\rho_\star + 3P_\star) - \frac{\Lambda}{3}, \tag{3.31}$$

and the "generalized" Raychaudhuri equation (3.11),

$$\epsilon_\star + \frac{1}{3\alpha_\star} D^\mu D_\mu \alpha_\star = \frac{\kappa^2}{6}(\rho_\star + 3P_\star) + \left(\Xi_\star - \frac{\kappa^2}{2}\Pi_\star\right) - \frac{\Lambda}{3}. \tag{3.32}$$

The Hamiltonian constraint yield the local curvature of the shell, ${}^3R_\star = 2\kappa^2\rho_\star + 2\Lambda$, the momentum constraint governs the expansion and shear transfer across the shell, $(\frac{\Theta}{3} + \sigma)'_\star = -3\sigma_\star \frac{r'_\star}{r_\star}$, the Weyl constraint from the shear equations links it directly to the 3-curvature residual

$$\Xi_\star + \frac{\kappa^2}{2}\Pi_\star = \eta_\star, \tag{3.33}$$

the density remains conserved by Eq. (2.19) which, with Eq. (3.6), reads now

$$\dot{\rho}_\star = -\Theta_\star(\rho + P - 2\Pi)_\star. \tag{3.34}$$

Equation (2.21) gives a part of the gTOV staticity condition. The Weyl constraint Eq. (2.17) governs the balance of anisotropic stresses and energy density across the shell. But most interestingly, the evolution of the electric part of the Weyl tensor is bound to that of the anisotropic stresses by Eq. (3.23) which reduces here to

$$\left(\Xi + \frac{\kappa^2}{2}\Pi\right)^{\cdot}_\star = \frac{\kappa^2}{6}(\rho + P - 2\Pi)_\star\Theta_\star. \tag{3.35}$$

This is to be related with the studies on cracking by Herrera *et al.* [21,32,33,39]. We now restrict to shear-free flows.

2. Dynamical system restricted to shear-free flows

We set out to restrict to shear-free flows as they constitute an important subcase in many studies, i.e. as in [3,26,40–42] or even in cosmological FLRW models.

Governing equations.—Equation (2.14) reveals the necessary and sufficient condition for a shear-free flow to be

$$\epsilon - \left(\Xi - \frac{\kappa^2}{2}\Pi\right) = 0. \tag{3.36}$$

Here the combination of electric Weyl and anisotropic stresses that govern the shear evolution appears clearly in its role. This generalizes the result of [29] to the case of nonvanishing acceleration.

The remaining equations of the system (assuming (3.36)) that differ from the general case reduce to

$$\dot{\Theta} = \frac{1}{\alpha} D^k D_k \alpha - \frac{\Theta^2}{3} - \frac{\kappa^2}{2}(\rho + 3P) + \Lambda, \tag{3.37}$$

$$\left(\Xi + \frac{\kappa^2}{2}\Pi\right)^{\cdot} = -\left[2\left(\Xi + \frac{\kappa^2}{2}\Pi\right) + \left(\Xi - \frac{\kappa^2}{2}\Pi\right)\right]\frac{\Theta}{3}, \tag{3.38}$$

$$\Xi + \frac{\kappa^2}{2}\Pi = \eta, \tag{3.39}$$

$$\frac{\Theta^2}{3} = \kappa^2\rho - \frac{{}^3R}{2} + \Lambda, \tag{3.40}$$

$$\Theta' = 0. \tag{3.41}$$

Notice that Eq. (3.39) shows that in the shear-free case, the $\Xi + \frac{\kappa^2}{2}\Pi$ combination of electric Weyl and anisotropic stresses relates only to the 3-curvature distortion of the hypersurfaces. It also is a generalization of the constraint $\kappa^2\Pi = \eta = 2\Xi$ found in [29–31]. Moreover, Eq. (3.38) that governs its evolution can be reexpressed, using Eqs. (3.39) and (3.36) as

$$\dot{\eta} = -[2\eta + \epsilon]\frac{\Theta}{3},$$

so η is damped by $\frac{2\Theta}{3}$. Therefore the sign of Θ determines the increase or decrease of the 3-curvature distortion. Expansion dampens the distortion while collapse enhances it. More importantly Eq. (3.41) implies that Θ does not depend on R, and therefore

$$\alpha\dot{\Theta} = \frac{\partial\Theta}{\partial t} = D^k D_k \alpha - \alpha\left[\frac{\Theta^2}{3} + \frac{\kappa^2}{2}(\rho + 3P) - \Lambda\right] = \chi(t), \tag{3.42}$$

where χ is a function of just the time coordinate t. The Hessian trace is thus determined by the Friedmann acceleration sources and a time dependent term: $\frac{1}{\alpha}D^k D_k\alpha = \frac{\kappa^2}{2}(\rho + 3P) - \Lambda + \frac{\Theta^2(t)}{3} + \frac{\chi(t)}{\alpha}$.

Dynamics of shear-free separating shell.—Since at $r_\star$ we further have $\Theta_\star = 0$, the remaining changed equations of the system reduce at that locus to

$$\alpha_\star \dot{\Theta}_\star = \frac{\partial \Theta_\star}{\partial t} = D^k D_k \alpha_\star - \alpha_\star \frac{\kappa^2}{2}(\rho + 3P)_\star + \alpha_\star \Lambda = -3\alpha_\star \dot{\sigma}_\star = 0, \tag{3.43}$$

$$\left(\Xi + \frac{\kappa^2}{2}\Pi\right)^{\cdot}_\star = 0, \tag{3.44}$$

$$\Theta'_\star = 0. \tag{3.45}$$

From Eq. (3.44) we realize then that $(\Xi + \frac{\kappa^2}{2}\Pi)_\star = \eta_\star$ is a constant of the motion along the flow u^a (timelike vector fields).

In the shear-free case the expansion scalar throughout is only a function of time, and the relation between the local values of the electric part of the Weyl tensor and of the anisotropic stresses does not change along the orbits of the shells. It is also worth noticing that the 3-curvature of the separating shell is completely determined by the local energy density.

A "limit" case is the case of a static initial configuration $\Theta = 0$ in addition to the vanishing of the shear. We will consider this case in the subsection on the cracking phenomena.

3. Dynamical system restricted to geodesic flow

The following case of geodesic flow is defined by no acceleration. This implies $\alpha' = 0$, and therefore $\epsilon = 0$, $D^k D_k \alpha = 0$, and it is advisable to set $\alpha(t) = 1$, and use gLTB coordinates (generalization from the LTB coordinates). As a remark, the more restrictive geodesic, shear-free flows are subject to

$$\frac{\kappa^2}{2}\Pi = \Xi \tag{3.46}$$

recovering the Mimoso and Crawford result [29], and the subsequent discussion of Coley and McManus [30,31].

Governing equations.—The equations for geodesic flows that differ from the general case now reduce to

$$\left(\frac{\Theta}{3} + \sigma\right)^{\cdot} = -\left(\frac{\Theta}{3} + \sigma\right)^2 - \left\{\frac{\kappa^2}{6}(\rho + 3P) + \left(\Xi - \frac{\kappa^2}{2}\Pi\right)\right\} + \frac{\Lambda}{3}, \tag{3.47}$$

$$\dot{\sigma} = -\sigma\Theta + \sigma\left(\frac{\Theta}{3} + \sigma\right) - \left(\Xi - \frac{\kappa^2}{2}\Pi\right), \tag{3.48}$$

$$(P - 2\Pi)' = 6\Pi \frac{r'}{r}. \tag{3.49}$$

This case is interesting as it corresponds to the generalization of the classic LTB model [36,43] as well as to the Sussman and Pavón [20,44] example we will use later. The major difference from the general case is the absence of Hessian trace, $\frac{1}{\alpha} D^a D_a \alpha$, and traceless Hessian, ϵ, in the equations (3.47)–(3.49). In particular Eq. (3.49) displays a completely different radial constraint: not only the inertial mass is no longer involved, as the acceleration vanishes, but also in this way the anisotropic stress is the only source for the inhomogeneity of the pressure. For a perfect fluid the pressure should be spatially homogeneous, as found in [45].[13]

Dynamics of geodesic separating shells.—Further restricting to the shell $r_\star$ we have the remaining changed equations

$$\left(\frac{\Theta}{3} + \sigma\right)^{\cdot}_\star = 0 = -\left\{\frac{\kappa^2}{6}(\rho + 3(P - 2\Pi)) + \left(\Xi + \frac{\kappa^2}{2}\Pi\right)\right\}_\star + \frac{\Lambda}{3}, \tag{3.50}$$

$$\dot{\sigma}_\star = \frac{\Theta^2_\star}{3} - \left(\Xi - \frac{\kappa^2}{2}\Pi\right)_\star, \tag{3.51}$$

$$(P - 2\Pi)'_\star = 6\Pi_\star \frac{r'_\star}{r_\star}, \tag{3.52}$$

$$\left(\frac{\Theta}{3} + \sigma\right)'_\star = \Theta_\star \frac{r'_\star}{r_\star}. \tag{3.53}$$

The definition of the matter-trapped shell then implies Eq. (3.50) which is the local version of the gTOV, i.e., the local gRAY $= 0$ equation. If in addition we have the shear-free condition (3.46), we see that the matter-trapped shell imposes that $(\rho + 3P)$ is locally constant or vanishing (if $\Lambda = 0$). On the other hand Eq. (3.53) shows that the value of $(\frac{\Theta}{3} + \sigma)$ in the neighborhood of the separating shell is nonvanishing and that $(\frac{\Theta}{3} + \sigma)'_\star > 0$ provided $\Theta_\star r'_\star > 0$.

Geodesic Misner-Sharp mass and electric Weyl.—For the geodesic flow, we have, from Eqs. (3.2) and (3.11), the latter in the form of Eq. (3.47), a relation between the Misner-Sharp mass and the electric Weyl,

$$\frac{M}{r^3} = \left\{\frac{\kappa^2}{6}[\rho + 3\Pi] - \Xi\right\}. \tag{3.54}$$

D. Separation and expansion

In cosmology, the expansion of the background universe is understood as the condition on the universal fluid flow of $\Theta > 0$. In Sec. III, we have extended the definition of [10] for matter-trapped surfaces separating expansion from collapse. However it should be noted that, because of its

[13]There, spherically symmetric, inhomogeneous models are studied, and it is shown that there are no exact, geodesic, nonstatic perfect fluid solutions with nonzero shear.

definition (3.6), the expansion of the outside region does not precisely cover the usual expansion region: the separating shell itself can have nonzero expansion and thus one of the said collapsing or expanding region may contain the $\Theta = 0$ shell. However, the focus of our study was not laid on such shell because the present definition yields the staticity condition on that surface [Eq. (3.3)] which is not, in general the case for turnaround shells ($\Theta = 0$).

The meaning of expansion in the terms of Sec. III is linked with the areal radius: the luminosity distance of a shell to the centre. Thus the static shell keeps its luminosity distance to the centre while expanding regions appear so in the luminosity distance space.

Isolating the Ricci curvature of spatial hypersurfaces ${}^{(3)}R$ in Eqs. (2.18) and (3.8), however, reveals that both $\Theta = 0$ and Eq. (3.6) require ${}^{(3)}R > 0$, placing the respective surfaces both in the positively curved region of spacetime, where the region between the two must lie. In models with negatively curved regions, those expanding shells will therefore be contained in the expansion regions defined for both expansion scalar and areal radius. The flat and closed background can still present expansion infinities in both senses, as seen in [14], although the general treatment can be more complex.

IV. APPLICATIONS OF OUR RESULTS

A. Sussman-Pavón exact solution: radiation and matter

To illustrate our results we turn our attention to an exact solution derived by Sussman and Pavón for a spherically symmetric model with a matter content consisting of a combination of dust and radiation that exhibits anisotropic stresses, but no heat fluxes and $\Lambda = 0$ [20].

In order to do that we need to translate the metric (2.1) in the LTB form used in [20], following their assumption of comoving, i.e. geodesic, flow (here the velocity of light c is reintroduced)

$$u^a_{SP} = c\delta^a_t.$$

That imposed condition translates into a flow without acceleration, which leads in terms of metric components to $\alpha = \alpha(t)$. Thus, the time function can always be rescaled to absorb the lapse $\alpha\dot{t} = c$. In this section we will use the notation $\partial_T X = \dot{X}$, as the LTB coordinates are comoving.

1. Coordinate transform from GPG to LTB

Canceling the metric crossed term gives a similar relation than the perfect fluid generalized LTB formulation of [16], while the radial term imposes

$$(\beta\dot{t} + \dot{R})R' = 0, \tag{4.1}$$

$$R'^2 = r'^2, \tag{4.2}$$

(with $R = R_{\mathrm{GPG}}$, the $'$ and $\dot{}$ denoting derivatives in the gLTB frame). Compared with [17], the absence of heat flux suggests that the extra degree of freedom provided by a spacetime dependent lapse in the GPG frame becomes superfluous. The new areal radius, from Eq. (4.2) is reset to the GPG radial coordinate $r = R_{\mathrm{GPG}} = \mathcal{R}$ (here for convenience we change notation for the GPG radial coordinate), yet is still spacetime dependent. Then, as in the perfect fluid case in [16], the coordinate transformation (4.1) is such that $\beta dt + d\mathcal{R} \propto dR$. Taking $t(T) = c\int \frac{dT}{\alpha}$ and $r(T, R)$, we have then the condition

$$\beta\partial_T t + \partial_T \mathcal{R} = 0, \tag{4.3}$$

which becomes (in GPG coordinates)

$$\frac{\beta}{\alpha} = -\frac{\dot{\mathcal{R}}}{c}. \tag{4.4}$$

Moreover, Eq. (2.23) implies that, in the new gLTB coordinates, $E = E(R)$. Consequently, the line element (2.1) can be rewritten as

$$ds^2 = -c^2dT^2 + \frac{(\partial_R r)^2}{1 + E(R)}dR^2 + r^2d\Omega^2, \tag{4.5}$$

as in [20].

2. Restricted dynamical equations

The crucial ansatz adopted by Sussman and Pavón was the assumption that the flow is geodesic, keeping as close as possible to the case where dust is the only component present, i.e., as in the original LTB case. The Bianchi contracted identity (2.20), together with the geodesic condition $\alpha' = 0 \Leftrightarrow \dot{u}^a = 0$, imply that

$$D^b(h_{ib}P + \Pi_{ib}) - u_i[\Theta P + 6\Pi\sigma] = 0, \tag{4.6}$$

that is, in gLTB coordinates,

$$(P - 2\Pi)' - 6\Pi\frac{r'}{r} = 0, \tag{4.7}$$

where the prime stands for differentiation with respect to the geodesic R. For practical purposes this amounts to have

$$M_{\mathrm{dust}} = M(R), \tag{4.8}$$

and

$$M_{\mathrm{rad}} = \frac{W(R)r_i(R)}{2r(T,R)}, \tag{4.9}$$

so that Eq. (2.25), including c, now reads

$$\dot{r}^2 = c^2\left(2\frac{M}{r} + \frac{Wr_i}{r^2} + E\right). \tag{4.10}$$

From Eq. (4.10), one is led to the following solution,[14] generalized from [20] to encompass the cases where $E \neq 0$,

[14] The elliptic integral, from Eq. (4.10), reads $\pm c\int \mathrm{d}t = \int\frac{rdr}{\sqrt{Er^2+2Mr+Wr_i}} = \int\frac{dY}{2\sqrt{E}\sqrt{Y+cst}} - \frac{M}{E^{3/2}}\int\frac{dX}{\sqrt{X^2+cst}}$.

$$\pm c \int dt = \left\{ \ln\left[\left(\frac{\sqrt{Er^2 + 2Mr + Wr_i}}{E^{1/2}} + r + \frac{M}{E} \right)^{\frac{-M}{E^{3/2}}} \right] + \frac{\sqrt{Er^2 + 2Mr + Wr_i}}{E} \right\}_{r_i}^{r}. \tag{4.11}$$

A more complex form of this solution was previously found in [46].

3. Existence of a separating shell in the generalized Sussman-Pavón solutions

We see that the vanishing of the right-hand side of Eq. (4.10) provides one of the conditions for the separating shell, while the other condition that corresponds to the gTOV equation will be derived from the radial acceleration

$$\frac{\ddot{r}}{c^2} = -\frac{M}{r^2} - \frac{Wr_i}{r^3}, \tag{4.12}$$

or directly from combining Eqs. $[(2.13)/6 + (2.14)]$ in the gLTB frame with Eq. (4.10). So we find that

$$\frac{Wr_i}{2r^3} = \frac{\kappa^2}{2}(P - 2\Pi)r. \tag{4.13}$$

The form of the gTOV condition in the gLTB frame [Eqs. (2.27) and (2.26) with conditions $\alpha' = 0$] can then be recognized in Eq. (4.12) using the Raychaudhuri constraint (4.13).

The existence of a matter-trapped shell in the solution inspired by [20] requires both Eqs. (4.10) and (4.12) to be zero on some $r = r_\star$. This implies the existence of $r_\star$ and, from Eq. (4.12), and then Eq. (4.13),

$$W_\star = -M_\star \frac{r_\star}{r_{i\star}} \Rightarrow W(R_\star) < 0 \Leftrightarrow P_\star < 2\Pi_\star. \tag{4.14}$$

This latter condition shows that, in order to allow the separating shell to exist locally, the model of Sussman and Pavón [20] must contain regions where the transverse pressures balance the radial pressure. This can be understood with Eq. $[(4.13)\times r^2]$: the radiation Misner-Sharp mass corresponds to the flux of the pressures across the shell. Setting Eq. (4.10) to zero implies, using again Eq. (4.13),

$$E_\star = -2\frac{M_\star}{r_\star} - \frac{W_\star r_{i\star}}{r_\star^2} = -2\frac{M_\star}{r_\star} - \kappa^2(P_\star - 2\Pi_\star)r_\star^2, \tag{4.15}$$

while Eq. (4.12) gives

$$r_\star = \sqrt[3]{\frac{M_\star}{\kappa^2(2\Pi_\star - P_\star)}} \tag{4.16}$$

so with (4.13), the energy/curvature parameter E reads

$$E_\star = -\frac{M_\star}{r_\star} = -\sqrt[3]{\kappa^2(2\Pi_\star - P_\star)}M_\star^{\frac{2}{3}} = \frac{M_\star^{\frac{2}{3}} W_\star^{\frac{1}{3}} r_{i\star}^{\frac{1}{3}}}{r_\star^{(4/3)}} < 0. \tag{4.17}$$

Again, as in the perfect fluid case [10], the separating shell only exists in elliptic regions ($E < 0$). Finally with Eq. (4.14) we have

$$r_\star = -\frac{W}{M} r_i, \tag{4.18}$$

and thus

$$E_\star = \frac{M^2}{Wr_i}. \tag{4.19}$$

For outward initial flows, this requires $r_i \leq r_\star$, thus the additional condition $W \leq -M < 0$.

Note that, similarly as for the LTB dust model, from Eqs. (4.10) and (4.12), conditions (3.6) and (3.7) are met for $E_\star = 0$, in the limits of both $t \to \infty$ and $r_\star \to \infty$. As this happens at radial infinity, we do not consider that shell to be a separating shell, since it does not separate distinct radial domains, given that the inner region will extend up to infinity.

4. Dynamical analysis and global shell

As for the examples of Ref. [10], a dynamical analysis of Eq. (4.10) can be performed in the regions where $W < 0$ (see Fig. 1), required by Eq. (4.14). Initial conditions with cosmological outwards initial areal radius velocity flow, FLRW outer behavior (M, $W_{r \simeq \infty} r^3$, $E_{r \simeq \infty} r^2$) and an intermediate $W \leq -M < 0$ region can be qualitatively

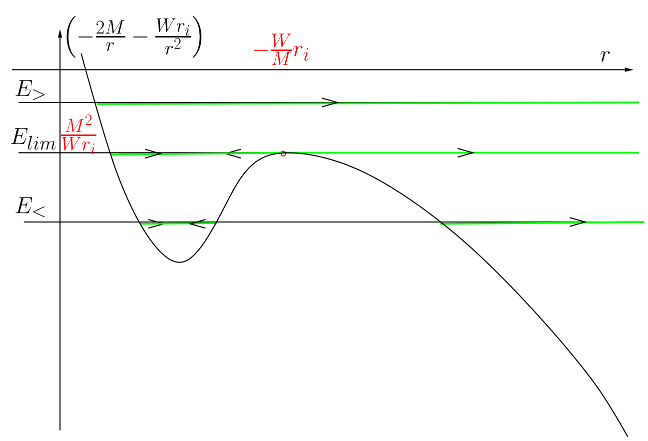

FIG. 1 (color online). Dynamical analysis of a local $W < 0$ shell. The dynamic for a given shell (fixed M, W and E without shell crossing) obeys Eq. (4.10). It then behaves as a one dimensional particle in an effective potential, following [14]. We draw a qualitative energy diagram to illustrate the definition of the critical curvature/energy $E_{\rm lim}$, when it lies in a region of $W < 0$. The various cases of $E_> > E_{\rm lim}$, $E_< < E_{\rm lim}$ and $E = E_{\rm lim}$ yield unbound, bound and marginally bound behaviors.

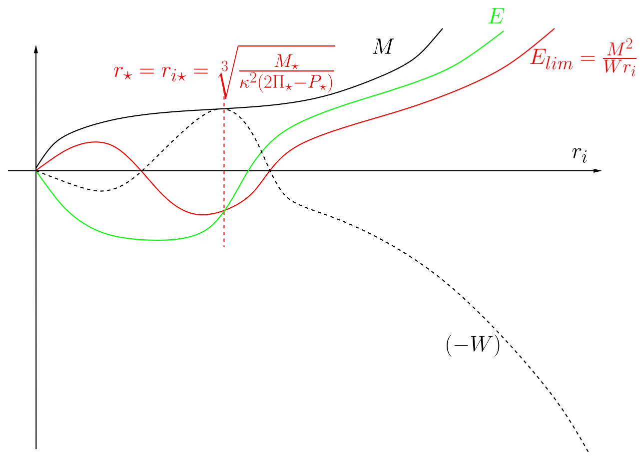

FIG. 2 (color online). Global, qualitative, analysis yielding the separating shell at the intersection of E with $E_{\rm lim}$. Following [14], we construct initial cosmological conditions with a $W < 0$ region such that Eqs. (4.10) and (4.12) are 0 simultaneously, that is $E = E_{\rm lim}$ and $W = -M$, so the intersection of the curves gives a global dynamical separation.

obtained (Fig. 2). Then using $E_{\rm lim} = \frac{M^2}{W r_i}$, and choosing initial velocities such as E crosses $E_{\rm lim}$ in the $W < 0$ region, one gets the global separation (see Fig. 2). Although we allow for a region where the Misner-Sharp mass of the radiation fluid is negative, we remind that only the total density of the fluid is actually meaningful and point out that we should keep

$$M' + W' = \frac{\kappa^2}{2} r_i^2 r_i' (\rho_m + \rho_r)_i \geq 0, \qquad \text{so } M + W \geq 0.$$

This implies that only a static global separation can fulfill both $W \leq -M$ and $M + W \geq 0$: $W = -M$, obtained by crossing E with $E_{\rm lim}$ at the radius where $M = -W$, as shown on Fig. 2. Note that the energy conditions does prevent initial conditions with inward going initial flow in the neighborhood of the global separation if the no shell crossing condition is to be maintained. In that case it is allowed to have initial radius outside $r_\star \leq r_i$ but then the shells just outside the separating one should be ingoing and unbound. This would result in shell crossing after some time in a symmetric way as found in [14] in the case of the analysis of a ΛLTB model.

Finally, we would like to emphasize again our use of the Sussman-Pavón model: although considering the strict radiation fluid would constrain W to be positive [46], we are interested here in obtaining a model including a separating surface which lead us to relax the nature of W. The region of negative W can thus be seen as manifesting some dark energy properties that we leave for further analysis.

B. Cracking phenomenon of Herrera and co-workers

We find a second illustrative example of our results in the concept of cracking put forward by Herrera and collaborators [47] whereby a static spherical configuration is unstable to anisotropic perturbations and "cracks".

In order to discuss this concept within our framework we have to consider the set of EFEs as in Eqs. (3.21)–(3.28) in Sec. III C 1.

From these equations we see that the shear plays a central role. From the shear propagation equation (3.22) we realize that if the shear were to vanish initially, any deviations from constant curvature given by the term $\epsilon - (\Xi - \frac{\kappa^2}{2}\Pi)$ would indeed make the shear become nonvanishing at any later instant.

To recover the cracking phenomena envisaged by Herrera and collaborators we start assuming a static, isotropic, shear-free initial configuration. Thus we put $\Theta = 0$, $\Pi = 0$ and $\sigma = 0$ in some region of the initial hypersurface where we assume also that ρ and P only vary slowly. Then we can assess any future deviation from that configuration using the restriction to this hypersurface of Eqs. (3.21)–(3.28). The governing initial equations, for $\Lambda = 0$, $\Theta = 0$, $\Pi = 0$ and $\sigma = 0$, take the form

$$\dot{\Theta} = \frac{1}{\alpha} D^k D_k \alpha - \frac{\kappa^2}{2}(\rho + 3P), \tag{4.20}$$

$$\dot{\sigma} = \epsilon - \Xi, \tag{4.21}$$

$$\left(\Xi + \frac{\kappa^2}{2}\Pi\right)^{\cdot} = 0, \tag{4.22}$$

$$\Xi = q, \tag{4.23}$$

$$\frac{{}^3R}{2} = \kappa^2 \rho, \tag{4.24}$$

$$(P - 2\Pi)' = -(\rho + P)\frac{\alpha'}{\alpha}, \tag{4.25}$$

$$\left(\frac{\Theta}{3} + \sigma\right)' = 0, \tag{4.26}$$

$$\frac{\kappa^2}{6}\rho' = -\frac{\left(\left[\eta + \sigma\left(\frac{\Theta}{3} + \sigma\right)\right] r^3\right)'_{\Pi=\sigma=\Theta=0}}{r^3} = -\frac{(\eta r^3)'}{r^3}. \tag{4.27}$$

The form of the Eq. (4.27) uses the constraint [(4.23), actually (3.24)] while the Raychaudhuri Eq. (4.20) comes from the restriction of Eqs. (3.21) and (3.22) to the initial configuration. Further using Eq. (4.22) with the constraint (3.24) in the derivative, one can deduce the relations between the values and proper time evolutions of the electric Weyl scalar, the anisotropic stresses and traceless hypersurface curvature, as well as traceless Hessian scalar, shear and expansion on the initial hypersurface

$$-2\dot{\Xi} = \kappa^2 \dot{\Pi}, \tag{4.28}$$

$$\dot{\eta} = 0 \tag{4.29}$$

$$\Xi = \eta, \tag{4.30}$$

$$\sigma = \Theta = \Pi = 0, \tag{4.31}$$

where, in addition, the traceless Hessian scalar proper time evolution can be obtained with the derivative of the shear Eq. (3.22) on the hypersurface

$$\ddot{\sigma} = (\epsilon + \kappa^2\Pi)^{\cdot} = (\epsilon - 2\Xi)^{\cdot}. \tag{4.32}$$

From all this, the following can be deduced: (i) the perfect fluid source and Hessian combination in Eq. (4.20) drives, in general, the expansion away from 0; (ii) combining Eqs. (4.21) with (4.23), the shear, in general, is also driven away from 0 by the difference of traceless Hessian and curvature of the hypersurface, q, the latter mirroring the Weyl curvature, unless they are set equal, thus implying an imposed shear-free flow; (iii) anisotropy as well is driven away from 0, in parallel with the evolution both of the electric part of the Weyl, Ξ, and of the difference of traceless Hessian and hypersurface curvature, η, as we have

$$\dot{\Pi} = \frac{\ddot{\sigma} - \dot{\epsilon}}{\kappa^2}, \tag{4.33}$$

except if the flow is restricted to being shear-free and geodesic; (iv) the separation scalar defined in Eq. (3.6) is 0 on the initial hypersurface but will be driven away by

$$\left(\frac{\Theta}{3} + \sigma\right)^{\cdot} = -\eta + \frac{1}{3\alpha}D^k D_k \alpha + \epsilon - \frac{\kappa^2}{6}(\rho + 3P). \tag{4.34}$$

Therefore, if a shell where

$$\eta = -\frac{\kappa^2}{6}(\rho + 3P) + \frac{1}{3\alpha}D^k D_k \alpha + \epsilon \tag{4.35}$$

exists in the considered region, it satisfies locally the TOV equilibrium condition, where forces balance, and is thus surrounded by shells experiencing nonzero forces. That shell satisfies the conditions (3.6) and (3.7) of a separating shell. As a consequence, we realize that gTOV becomes nonvanishing in the neighborhood of the separating shell, inducing the appearance of the radial force responsible for the cracking phenomena under the following conditions: for $\Theta_\star r'_\star > 0$, the radial balance of the separation scalar will drive neighboring shells to the cracking condition of outer shells and inner shells experiencing positive, resp. negative areal expansion, as seen in Eq. (2.16), at some later time from those initial conditions. In conclusion, the cracking shell is a kind of separating shell.

We can further extend our interpretation of cracking in this framework by considering shear-free flows. Then, (i) expansion is still driven away from 0 by $\frac{1}{\alpha}D^k D_k \alpha - \frac{\kappa^2}{2}(\rho + 3P)$; (ii) anisotropy is still driven away from 0 by the traceless Hessian; (iii) as seen in Sec. III C 2, the shear-free condition entails from Eq. (3.41) that the expansion should be of uniform sign in all spacetime. The departure from initial vanishing expansion by Eq. (4.20) will give its definite sign and thus the expansion is always either all collapsing or all expanding, as found in [40].

An important point to be emphasized at this stage is that our analysis draws on the full set on nonlinear equations and is therefore more general than that of the original works of Herrera an collaborators. Alternatively, a perturbative gauge invariant treatment of this issue can be done using the formalism developed by [48] and subsequently explored by others [44,49–52]. We will address this elsewhere.

We conclude this section emphasizing the importance of both the shear and the anisotropic stresses not only for the existence of a separating shell, but also for the cracking phenomena of Herrera and collaborators. In the latter case, this confirms their claims in an alternative way.

V. SUMMARY AND DISCUSSION

In the present work we have considered spherically symmetric, inhomogeneous universes with anisotropic stresses in order to investigate the existence and stability of a separating shell separating expanding and collapsing regions. With this endeavor we have gone one step further than in a previous work by considering a more realistic scenario where the matter is no longer a perfect fluid.

This shows up to be quite important in characterizing the contrasting dynamical behaviors of separate regions. This is relevant in relation with the present understanding of structure formation as the outcome of gravitational collapse of overdense patches within an overall expanding universe, since there is an underlying expectation that the two disparate behaviors decouple. This issue, is also related to the assessment of the influence of global physics on local physics.

In the present work we have addressed this issue by resorting to an ADM 3 + 1 splitting, utilizing the so-called Generalized-Painlevé-Gullstrand coordinates as developed in Refs. [15,16]. This enables us to follow a nonperturbative approach and to avoid having to consider the matching of the two regions with the contrasting behaviors. We have found local conditions characterizing the existence of a separating shell which generalizes our previous conditions for perfect fluids [10]. One is a condition establishing the precise balance between two energy quantities that are the analogues of the total and potential energies at the separating shell. (This amounts to the vanishing of the kinetic energy of the shell.) The second condition establishes that a generalized TOV equation is satisfied on that shell, and hence that this shell is in equilibrium, but one which now involves explicitly the anisotropic stresses. Moreover, the former condition also implies that there is no matter transfer across the separating shell, and hence we may call the region enclosed by the latter a trapped matter region.

The trapped matter is not in static equilibrium in contrast to the situations where the TOV equation is satisfied in the whole of the trapped region, and which are meant to describe stars.

We have also related these conditions to a gauge invariant definition of the properties of the separating shell. These require the vanishing of a combination of the expansion scalar and of the shear, on the shell. Finally, if we demand that the separating shell is static, in order to define an extreme case of reference, we obtain an additional equation of state that relates the gauge-invariant Hessian trace of the model with the quantity $\rho + 3P$ involved in the strong energy condition, on the separating shell. Naturally in this limit case, the expansion and shear will vanish on the shell.

The approach followed in this paper has allowed us to translate the Einstein field equations in terms of nonlocal quantities, such as the Misner-Sharp mass M and the energy/curvature parameter E, as well as into equations involving local quantities. The latter are convenient to describe the evolving behaviors separated by the separating shell. This procedure led us to relate the energy equation to the generalized Friedmann equation, and likewise we relate the generalized TOV equation with the generalized Raychaudhuri equation. Moreover, we present both the equations governing the flow behavior of the remaining quantities such as the shear, the electric part of the Weyl tensor, as well as the constraint equations that hold for them and for their radial gradients. We also give the constraints and evolution applied on the expansion and shear combination. This allowed us to discuss the dynamical behavior in the neighborhood of the separating shell. In particular we have obtained the condition for a shear-free flow that generalizes previous results [29,30,40]. Such dynamical description, presented as a fully nonlinear set of scalar equations, is particularly suitable for being used in a numerical studies of the evolution of the system.

We have considered two illustrations of our results, namely we have analyzed the existence of a separating shell in the class of matter and radiation solutions put forward by Sussman and Pavón. We showed that, in this case, the existence of a separating shell requires that the radiation exerts a repulsive role. And we have shown that our results allow a discussion of the emergence of the cracking phenomena put forward by Herrera and collaborators. We have described cracking initial conditions, their dynamics, and showed, within our gauge invariant formalism, how shear and anisotropic stresses trigger the phenomenon of cracking. Our approach also opens windows on the behaviors of the electric part of the Weyl, the quantities characterizing the 3-curvature. We also recover the properties discussed in [40] in shear-free flows.

In this paper we didn't do a thorough discussion of all dynamical possibilities offered by the system we discovered. This opens many possibilities for future work.

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APPENDIX A: METRIC ADM SCALAR FUNCTIONS

For clarity, we present the scalar gauge invariants involved in the ADM formulation in terms of the GPG metric functions, starting with the perfect fluid terms

$$\begin{aligned}\Theta &= 2\frac{\dot r}{r} - \frac{\beta'}{\alpha} - \frac{1}{2}\frac{\dot E}{1+E}\\ &= -\frac{1}{\alpha r^2}(r^2\beta)' - \frac{1}{2}\frac{\dot E}{1+E} + \frac{2\partial_t r}{\alpha r},\end{aligned} \tag{A1}$$

$$\begin{aligned}\sigma &= \frac{1}{3}\left[\frac{\dot r}{r} + \frac{\beta'}{\alpha} + \frac{1}{2}\frac{\dot E}{1+E}\right]\\ &= \frac{1}{3}\left[\frac{r}{\alpha}\left(\frac{\beta}{r}\right)' + \frac{1}{2}\frac{\dot E}{1+E} + \frac{\partial_t r}{\alpha r}\right],\end{aligned} \tag{A2}$$

which, combined with Eq. (2.23), yield

$$\Theta = \frac{\dot r'}{r'} + 2\frac{\dot r}{r}, \tag{A3}$$

$$\sigma = -\frac{1}{3}\left[\frac{\dot r'}{r'} - \frac{\dot r}{r}\right]. \tag{A4}$$

$$\begin{aligned}{}^{(3)}R &= -\frac{2}{r^2}[((1+E)(r^2)')' - r'((1+E)r)' - 1]\\ &= -2\left\{(1+E)\left(\frac{r'}{r}\right)^2 - \frac{1}{r^2} + 2\frac{\sqrt{1+E}}{r}(\sqrt{1+E}r')'\right\}\\ &= -\frac{2}{r^2}\{(Err')' + (1+E)rr'' + [(rr')' - 1]\},\end{aligned} \tag{A5}$$

$$\begin{aligned}\eta &= \frac{1}{6}\left\{r\left(\frac{Er'}{r^2}\right)' + E\frac{r''}{r} + \frac{2}{r^2}\left[1 + r^2\left(\frac{r'}{r}\right)'\right]\right\}\\ &= \frac{1}{6}\left[r\left(\frac{Er'}{r^2}\right)' + (2+E)\frac{r''}{r} + \frac{2}{r^2}(1 - r'^2)\right],\end{aligned} \tag{A6}$$

$$\frac{1}{\alpha}D^{\mu}D_{\mu}\alpha = \frac{\sqrt{1+E}}{\alpha r^2}(r^2\sqrt{1+E}\alpha')', \tag{A7}$$

$$\epsilon = -\frac{r\sqrt{1+E}}{3\alpha}\left(\frac{\sqrt{1+E}}{r}\alpha'\right)', \tag{A8}$$

and from the shear evolution comparing that from the EFE, Eq. (2.14), and that obtained from the Ricci identities, we get

$$\Xi = \frac{3\kappa^2}{2}\Pi - \eta - \sigma\left(\frac{\Theta}{3} + \sigma\right). \tag{A9}$$

APPENDIX B: CORRESPONDENCE BETWEEN ADM AND 1 + 3 COVARIANT VARIABLES

In this paper, the flow vector u^a corresponds to the normal to the ADM hypersurfaces. The correspondence between four-tensors $X^{fg}{}_{ij}$, defined on hypersurfaces S orthogonal to the flow, with 3-tensors $X^{\alpha\beta}{}_{\gamma\delta}$ defined on S, is

$$X^{\alpha\beta}{}_{\gamma\delta} := h^{\alpha}{}_f h^{\beta}{}_g h^{i}{}_{\gamma} h^{j}{}_{\delta} X^{fg}{}_{ij}. \tag{B1}$$

As the Hessian of the flow can be represented as $\dot{u}_{b;\bar{a}} = \frac{1}{\alpha}D_a D_b\alpha$, where the projected covariant derivative is $D_a = h^b{}_a\nabla_b$, the Hessian and traceless Hessian read

$$\frac{1}{\alpha}D^a D_a\alpha = \dot{u}^a{}_{;\bar{a}}, \tag{B2}$$

$$\frac{1}{\alpha}\left(D_a D_b - \frac{1}{3}h_{ab}D^c D_c\right)\alpha = \dot{u}_{b;\bar{a}} - \frac{1}{3}h_{ab}\dot{u}^a{}_{;\bar{a}}. \tag{B3}$$

In the present spherically symmetric case, we saw that any spatial two-tensor can be decomposed into its trace X and traceless tangential eigenvalue ξ. The correspondence between those two scalars and ADM tensors or the $1+3$ kinematics quantities can be seen through

$$X_{ab} = h^c{}_a h^d{}_b X_{cd} = X\frac{h_{ab}}{3} + \xi P_{ab}, \tag{B4}$$

$$X_{\alpha\beta} = X\frac{{}^{(3)}g_{\alpha\beta}}{3} + \xi P_{\alpha\beta}. \tag{B5}$$

We can thus, e.g., see from the Gauss-Codazzi Eq. (2.18) that the 3-Ricci trace is covariant. The traceless part of the 3-Ricci is given from Eq. (2.14) and the Hessian traceless tangential eigenvalue

$$\epsilon = \frac{1}{6}\left(\dot{u}_{b;\bar{a}} - \frac{1}{3}h_{ab}\dot{u}^a{}_{;\bar{a}}\right)P^{ab} \Leftrightarrow \epsilon P_{ab} = \dot{u}_{b;\bar{a}} - \frac{1}{3}h_{ab}\dot{u}^a{}_{;\bar{a}}, \tag{B6}$$

so that

$$\begin{aligned} {}^{(3)}R_{ab} - \frac{1}{3}h_{ab}{}^{(3)}R &= \eta P_{ab} \\ &= \dot{u}_{b;\bar{a}} - \frac{1}{3}h_{ab}\dot{u}^a{}_{;\bar{a}} + \kappa^2\Pi_{ab} \\ &\quad - \sigma_{ab}\Theta - P^{ab}\dot{\sigma}. \end{aligned} \tag{B7}$$

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