

Kinetic theory and evolution of cosmological fluctuations with neutrino number asymmetry

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We derive a kinetic equation for chiral matter at nonzero chemical potential that governs the response of the parity-odd part of the distribution function to perturbations of the Robertson-Walker metric. The derivation is based on a recent evaluation of the gravitational polarization tensor at nonzero chemical potential. We also provide the equations for gravity waves that follow from the anisotropic stress tensor describing the lepton asymmetry. These equations can be used to assess the effects that a nonzero neutrino chemical potential would have on the evolution of cosmological perturbations.

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The implications of the existence of gauge and gravitational anomalies for relativistic hydrodynamics are currently being systematically explored (see e.g. Refs. [1,2] and references therein). Most of the work has focused on the new odd susceptibilities and transport coefficients related to the parity anomaly, and it has been realized that some of these quantities may be obtained from the variation of the equilibrium partition function in the presence of a time-independent background of the metric and gauge fields [3,4]. On the other hand, the study of time-dependent processes requires the evaluation of the appropriate Green's functions at nonzero frequency or, alternatively, the use of Boltzmann equations describing the evolution of distribution functions in momentum space. By considering the response to a background electromagnetic field, the authors of Ref. [5] have obtained a kinetic equation including the effects of triangle anomalies and have also discussed the interplay between the kinetic and field-theoretical approaches (see also Ref. [6]). As their analysis does not include the gravitational response, in order to complete the description it would be necessary to consider the gravitational correlation function between energy-momentum tensors.

The study of the thermal gravitational correlation function at nonzero frequency was performed some time ago by Rebhan [7]. At very small momenta $Q^\nu = (q^0, \mathbf{q})$ compared to the temperature, he showed that this quantity is proportional to the thermal energy density and has a universal tensorial structure that obeys the gravitational Ward identities for diffeomorphism and conformal transformations [7]. Later, in Refs. [8,9] these field-theoretical results were used to work out the evolution of cosmological perturbations, showing that they provide an equivalent description to that obtained from the kinetic approach based on the Vlasov equation. More recently, the authors of Ref. [10] have studied the parity-violating part of the gravitational response of an ideal gas of Weyl fermions at nonzero chemical potential μ . This next-to-leading contribution, which also satisfies the Ward identities, is

simply proportional to the net number density of chiral fermions. At zero frequency its form may be used to determine the modifications of the constitutive relations of hydrodynamics that give rise to macroscopic parity-violating effects, such as the chiral vortical effect. But, as far as we know, the implications of the parity-odd contributions for time-dependent gravitational perturbations have not been much explored. Previous studies have only focused on the effects of lepton asymmetry related to the dependence of the anisotropic inertia on even powers of the chemical potential [11], or at most have introduced effective interactions generating cosmological birefringence [12].

In this work we will use the knowledge of the parity-odd correlation function, denoted by $\Pi_\epsilon^{\mu\nu\rho\sigma}(q^0, \mathbf{q})$, to derive the Boltzmann equation that governs the evolution of the μ -dependent part of the chiral fermion distribution. The kinetic equation thus obtained turns out to be surprisingly simple. It includes a source term proportional to the chemical potential. In this way, we complete the treatment in Ref. [5]. Once adapted to the Robertson-Walker metric, these results could be used to assess the effects of neutrino asymmetry in the evolution of cosmological perturbations.

We first present the results of the thermal field-theoretical calculation of $\Pi_\epsilon^{\mu\nu\rho\sigma}(q^0, \mathbf{q})$ in flat space-time. The amount of the neutrino asymmetry for a given species is described by the degeneracy parameter, defined by the ratio of the chemical potential to the temperature $\xi_\nu = \mu_\nu^{(0)}/T_0$, which is assumed to be small $|\xi_\nu| \ll 1$. In terms of the Fermi-Dirac distribution functions

$$\bar{n}_\pm(p) \equiv \frac{1}{(2\pi)^3} \left[\exp\left(\frac{p}{T_0} \mp \xi_\nu\right) + 1 \right]^{-1}, \quad (1)$$

the net unperturbed neutrino number density reads

$$\begin{aligned} \bar{n}_{\nu-\bar{\nu}} &\equiv \int_0^\infty 4\pi p^2 (\bar{n}_+(p) - \bar{n}_-(p)) dp = \frac{T_0^3}{6} \left(\xi_\nu + \frac{\xi_\nu^3}{\pi^2} \right) \\ &\approx \frac{T_0^3 \xi_\nu}{6}. \end{aligned} \quad (2)$$

Here T_0 is any arbitrary reference temperature, which in the cosmological setting will be related to the equilibrium

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temperature at the present time $\bar{T}(t_0)$ through $T_0 = \bar{T}(t_0)a(t_0)$, with $a(t)$ the Robertson-Walker scale factor. The Fourier components of the perturbations to the energy-momentum tensor are connected with metric perturbations $h_{\mu\nu}(t, \mathbf{x}) = g_{\mu\nu}(t, \mathbf{x}) - \eta_{\mu\nu}$ ¹ by

$$\delta\langle T^{\mu\nu}(Q) \rangle = -\frac{1}{2}\Pi^{\mu\nu\rho\sigma}(q^0, \mathbf{q})h_{\rho\sigma}(Q), \quad (3)$$

where the retarded graviton self-energy has been defined by

$$\begin{aligned} \Pi^{\mu\nu\rho\sigma}(x-y) &\equiv -i\Theta(x^0-y^0)\langle [T^{\mu\nu}(x), T^{\rho\sigma}(y)] \rangle \\ &\quad - 2\left\langle \frac{\delta(\sqrt{-g(x)}T^{\mu\nu}(x))}{\delta g_{\rho\sigma}(y)} \Big|_{g=\eta} \right\rangle. \end{aligned} \quad (4)$$

The calculation from thermal field theory shows that the thermal part of this response function receives a parity-violating contribution proportional to the totally antisymmetric symbol ϵ . This contribution is tied to the helicity of the equilibrium thermal state [13], and for $Q \ll |\mu_\nu|$, T , it is suppressed by a factor $\xi_\nu Q/T$ with respect to the leading-order temperature contribution proportional to the energy density $\bar{\rho}_{\nu+\bar{\nu}} \approx 7\pi^2 T^4/120$. Because the one-point function $\langle T^{\rho\sigma} \rangle$ does not have any odd-parity contribution, the graviton self-energy tensor verifies the Ward identity $Q_\mu \Pi^{\mu\nu\rho\sigma}(Q) = 0$. Its explicit form is given by

$$\begin{aligned} \Pi^{\mu\nu\rho\sigma}(q^0, \mathbf{q}) &= ic_V(q^0, q)\frac{Q^2}{(u \cdot Q)^2 + Q^2}u_\alpha Q_\beta [\epsilon^{\alpha\beta\mu\rho}P_V^{\nu\sigma} \\ &\quad + \epsilon^{\alpha\beta\nu\rho}P_V^{\mu\sigma} + (\rho \leftrightarrow \sigma)] \\ &\quad + ic_T(q^0, q)u_\alpha Q_\beta [\epsilon^{\alpha\beta\mu\rho}P_T^{\nu\sigma} + \epsilon^{\alpha\beta\nu\rho}P_T^{\mu\sigma} \\ &\quad + (\rho \leftrightarrow \sigma)], \end{aligned} \quad (5)$$

where $u_\nu = \delta_0^\nu$ is the velocity of the plasma, and $P_{L,T}$ are two projectors given by

$$\begin{aligned} P_T^{\mu\nu} &= \eta^{\mu\nu} - \frac{1}{(u \cdot Q)^2 + Q^2}[u \cdot Q(u^\mu Q^\nu + u^\nu Q^\mu) \\ &\quad + Q^\mu Q^\nu - Q^2 u^\mu u^\nu], \\ P_V^{\mu\nu} &= \eta^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2} - P_T^{\mu\nu}. \end{aligned} \quad (6)$$

The two scalar functions $c_V(q^0, q)$ and $c_T(q^0, q)$ are

$$c_V(q^0, q) = \bar{n}_{\nu-\bar{\nu}}\left(\frac{3}{10}Q_1(q^0/q) - \frac{3}{10}Q_3(q^0/q)\right), \quad (7)$$

$$\begin{aligned} c_T(q^0, q) &= \bar{n}_{\nu-\bar{\nu}}\frac{q^0}{q}\left(-\frac{1}{10}Q_0(q^0/q) + \frac{1}{7}Q_2(q^0/q)\right. \\ &\quad \left.- \frac{3}{70}Q_4(q^0/q)\right), \end{aligned} \quad (8)$$

¹ $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, $\epsilon^{0123} = 1$.

where $Q_j(x)$ are Legendre functions of the second kind.² The functions above turn out to be the coefficients of the gauge-invariant combinations of vector and tensor metric perturbations in Eq. (3). In particular, for vector perturbations, the asymmetry gives a nonzero contribution to the energy-momentum tensor,

$$\begin{aligned} \delta\langle T^{0i} \rangle &= c_V(q^0, q)i\epsilon^{ijk}q^j(G_k + iq^0 C_k), \\ \delta\langle T^{ij} \rangle &= c_V(q^0, q)iq^0(\epsilon^{imn}\hat{q}^m\hat{q}^j + \epsilon^{jmn}\hat{q}^m\hat{q}^i)(G_n + iq^0 C_n), \end{aligned} \quad (9)$$

where $\hat{q}^j = q^j/q$, while for tensor perturbations the induced contribution takes the form

$$\delta\langle T^{ij} \rangle = -c_T(q^0, q)\epsilon^{ilm}\delta^{jn}iq^l D_{mn} + (i \leftrightarrow j). \quad (10)$$

In these expressions we have followed the notation of Ref. [14] for metric perturbations

$$h_{0i} = G_i, \quad h_{ij} = \frac{\partial C_i}{\partial x^j} + \frac{\partial C_j}{\partial x^i} + D_{ij}, \quad (11)$$

where $G_j(t, \mathbf{x})$ and $C_j(t, \mathbf{x})$ are solenoidal vector fields describing the vector perturbation, and the traceless field $D_{ij}(t, \mathbf{x})$ satisfying $\partial_i D_{ij} = 0$ describes the tensor perturbation.

In order to obtain a kinetic formulation of these results, it is necessary to introduce the distribution function in momentum space and then derive the Boltzmann equation that governs it. The possible contributions to the perturbed energy-momentum tensor from the neutrino asymmetry may be written as

$$\delta T^{\mu\nu}(t, \mathbf{x}) = \int \frac{d^3 p}{p} \delta n_{\nu-\bar{\nu}}(\mathbf{x}, \mathbf{p}, t) p^\mu p^\nu, \quad (12)$$

where $\delta n_{\nu-\bar{\nu}}(t, \mathbf{x}, \mathbf{p})$ is the perturbation to the equilibrium neutrino distribution given by

$$n_{\nu-\bar{\nu}}(t, \mathbf{x}, \mathbf{p}) = \bar{n}_+(p) - \bar{n}_-(p) + \delta n_{\nu-\bar{\nu}}(t, \mathbf{x}, \mathbf{p}), \quad (13)$$

and $p^\mu = p(1, \hat{\mathbf{p}})$. A comparison with Eq. (3) suggests that the integral over \mathbf{p} in the Fourier transform of Eq. (12) may be viewed as the one-loop integral defining $-\frac{1}{2}\Pi^{\mu\nu\rho\sigma}(Q)h_{\rho\sigma}(Q)$. The thermal field theory computation shows that, for $Q \ll |\mu|$, T , the corresponding integrand is proportional to $(q^0 - \hat{\mathbf{p}} \cdot \mathbf{q})^{-1}$. Thus, in view of Eq. (12) we identify it with $\delta n_{\nu-\bar{\nu}}(Q, \mathbf{p})$; then we are left with

$$(-iq^0 + i\hat{\mathbf{p}} \cdot \mathbf{q})\delta n_{\nu-\bar{\nu}}(Q, \mathbf{p}) = S^{\mu\nu}(Q, \mathbf{p})h_{\rho\sigma}(Q), \quad (14)$$

where the function $S^{\mu\nu}(Q, \mathbf{p})$ is determined by the numerator of the one-loop integrand defining c_V and c_T . This relation is indeed the Fourier transform of a kinetic equation of the Vlasov type in flat space-time. It may be worth pointing

²In Ref. [10] the expressions of c_V and c_T were written in terms of $Q_1(q^0/q)$ solely, but for our purposes it is advantageous to use this equivalent form.

out that this connection between the kinetic and thermal-field treatments is quite common within the hard-thermal-loop approximation.

Let us now derive the specific form of the kinetic equation corresponding to Eq. (14). As all the dependence of the perturbation $\delta n_{\nu-\bar{\nu}}(Q, \mathbf{p})$ on p is contained in the factor $(\bar{n}'_+(p) - \bar{n}'_-(p))$, the radial integration over p , including the factor $p^\mu p^\nu / p \propto p$, produces the unperturbed distribution $\bar{n}_{\nu-\bar{\nu}}$ of Eqs. (7) and (8). Thus, using a notation similar to that of Ref. [14], it is convenient to define a direction-dependent intensity³ $K(Q, \hat{\mathbf{p}})$ through

$$\bar{n}_{\nu-\bar{\nu}} K(Q, \hat{\mathbf{p}}) \equiv \int_0^\infty \delta n_{\nu-\bar{\nu}}(Q, \mathbf{p}) 4\pi p^3 dp. \quad (15)$$

In view of Eq. (12), it follows in particular that $K(Q, \hat{\mathbf{p}})$ must satisfy

$$\bar{n}_{\nu-\bar{\nu}} \int \frac{d^2 \hat{\mathbf{p}}}{4\pi} K(Q, \hat{\mathbf{p}}) \hat{\mathbf{p}}_j = \delta T^0_j(Q), \quad (16)$$

$$\bar{n}_{\nu-\bar{\nu}} \int \frac{d^2 \hat{\mathbf{p}}}{4\pi} K(Q, \hat{\mathbf{p}}) \hat{\mathbf{p}}_i \hat{\mathbf{p}}_j = \delta T^i_j(Q), \quad (17)$$

where $\delta T^\mu_j = \delta T^{\mu j}$ are given by Eqs. (9) and (10).⁴ To find the kinetic equation for $K(Q, \hat{\mathbf{p}})$, we note that the integral

$$\int \frac{d^2 \hat{\mathbf{p}}}{4\pi} \hat{\mathbf{p}}_i \hat{\mathbf{p}}_n \frac{\hat{\mathbf{p}} \cdot \mathbf{q}}{q^0 - \hat{\mathbf{p}} \cdot \mathbf{q} + i0^+} = A_2 \delta_{in} + B_2 \hat{q}_i \hat{q}_n \quad (18)$$

has the property that the coefficient A_2 is exactly proportional to the one-loop angular integral that yields the coefficient $c_V(q^0, q)$,

$$\begin{aligned} A_2 &= \frac{q}{2} \int \frac{d^2 \hat{\mathbf{p}}}{4\pi} \frac{(1 - \hat{\mathbf{p}} \cdot \hat{\mathbf{q}}^2) \hat{\mathbf{p}} \cdot \hat{\mathbf{q}}}{q^0 - \hat{\mathbf{p}} \cdot \mathbf{q} + i0^+} \\ &= \frac{1}{5} Q_1(q^0/q) - \frac{1}{5} Q_3(q^0/q). \end{aligned} \quad (19)$$

Hence, the multiplication of Eq. (18) by $e^{\mu j k} q^j (-a_k + i q^0 F_k)$ yields the same structure proportional to T^{0i} in Eq. (9). Therefore, we may identify the contribution to $K(Q, \hat{\mathbf{p}})$ that reproduces the effect from the vector perturbation,

$$(-i q^0 + i \hat{\mathbf{p}} \cdot \mathbf{q}) K(Q, \hat{\mathbf{p}}) = \frac{3}{2} \hat{\mathbf{p}} \cdot \mathbf{q} \hat{\mathbf{p}}_n \epsilon_{nj k} q_j (G_k(Q) + i q^0 C_k(Q)). \quad (20)$$

It can be checked that, upon integration with $\hat{\mathbf{p}}_i \hat{\mathbf{p}}_j$, this form of $K(Q, \mathbf{p})$ reproduces the correct $\delta \langle T^{ij} \rangle$ for the vector perturbation in Eq. (9).

³The leading perturbation $\delta n_{\nu+\bar{\nu}}(Q, \mathbf{p})$ depends on p through the combination $p(\bar{n}'_+(p) + \bar{n}'_-(p))$, and the radial integral of $\delta n_{\nu+\bar{\nu}}(Q, \mathbf{p}) p^3$ yields a factor proportional to the unperturbed energy density $\bar{\rho}_{\nu+\bar{\nu}}$.

⁴The spatial indices may be lowered with δ_{jk} , so that $\hat{p}^i = \hat{p}_i$.

We can use a similar argument to find the contribution to K from tensor perturbations. Now the integral

$$\begin{aligned} &\int \frac{d^2 \hat{\mathbf{p}}}{4\pi} \hat{\mathbf{p}}_i \hat{\mathbf{p}}_j \hat{\mathbf{p}}_l \hat{\mathbf{p}}_m \frac{q}{q^0 - \hat{\mathbf{p}} \cdot \mathbf{q} + i0^+} \\ &= A_4 (\delta_{ij} \delta_{lm} + \text{two similar}) + B_4 (\delta_{ij} \hat{q}_l \hat{q}_m + \text{five similar}) \\ &\quad + C_4 \hat{q}_i \hat{q}_j \hat{q}_l \hat{q}_m \end{aligned} \quad (21)$$

has the coefficient A_4 proportional to $c_T(q^0, q)$,

$$\begin{aligned} A_4 &= \frac{q}{8} \int \frac{d^2 \hat{\mathbf{p}}}{4\pi} \frac{(1 - \hat{\mathbf{p}} \cdot \hat{\mathbf{q}}^2)^2}{q^0 - \hat{\mathbf{p}} \cdot \mathbf{q} + i0^+} \\ &= \frac{1}{15} Q_0(q^0/q) - \frac{2}{21} Q_2(q^0/q) + \frac{1}{35} Q_4(q^0/q), \end{aligned} \quad (22)$$

while the others will not contribute to the contraction with ϵq , because of the (anti)symmetry in the indices and the transversality property $q_k D_{kn} = 0$. Including the prefactor q^0/q of Eq. (8) and using Eq. (17), one obtains the kinetic equation for the intensity perturbation that reproduces the parity-violating effects from tensor fluctuations,

$$(-i q^0 + i \hat{\mathbf{p}} \cdot \mathbf{q}) K(Q, \hat{\mathbf{p}}) = \frac{3}{2} q^0 \hat{\mathbf{p}}_i \epsilon_{im n} q_m \hat{\mathbf{p}}_j D_{nj}(Q). \quad (23)$$

The extension of these results to perturbations of the Robertson-Walker metric can be made by exploiting the invariance under conformal transformations. The graviton self-energy is defined by

$$\begin{aligned} \Pi^{\mu\nu\rho\sigma}(x, y) &= -4 \frac{\delta \Gamma}{\delta g_{\mu\nu}(x) \delta g_{\rho\sigma}(y)} \Big|_{g=\bar{g}} \\ &= -2 \frac{\delta}{\delta g_{\mu\nu}(x)} \left(\sqrt{-g(y)} \langle T^{\rho\sigma}(y) \rangle \right) \Big|_{g=\bar{g}}, \end{aligned} \quad (24)$$

where \bar{g} is a background metric. Since the thermal contribution to the underlying effective action $\Gamma[g_{\mu\nu}]$ is conformally invariant, the graviton self-energy for a conformally flat background $g_{\mu\nu}(x) = \Omega^2(x) \eta_{\mu\nu}$ reads

$$\Pi^{\mu\nu\rho\sigma}(x, x') = \Omega^{-2}(x) \Pi^{\mu\nu\rho\sigma}(x - x') \Big|_{g=\eta} \Omega^{-2}(x'). \quad (25)$$

As a consequence, the combination $\sqrt{-g(x)} \delta T^\mu_\nu(x)$ is conformally invariant, and may be evaluated from the already computed $\delta T^\mu_\nu(x) \Big|_{g=\eta}$. Therefore, it is convenient to write the perturbed metric of the expanding universe as

$$ds^2 = \Omega^2(\tau) (\eta_{\mu\nu} + h_{\mu\nu}(\tau, \mathbf{x})) dx^\mu dx^\nu, \quad (26)$$

where $\tau = \int dt a^{-1}(t)$ is the conformal time, and $\Omega(\tau) = a(t)$. By making the replacements $-i q^0 = \partial_\tau \rightarrow a(t) \partial_t$ and $i q_j \rightarrow \partial_j$, we are left with the kinetic equation for the intensity perturbation $K(t, \mathbf{x}, \hat{\mathbf{p}})$,

$$\begin{aligned} & \frac{\partial K(t, \mathbf{x}, \hat{\mathbf{p}})}{\partial t} + \frac{\hat{p}_i}{a(t)} \frac{\partial K(t, \mathbf{x}, \hat{\mathbf{p}})}{\partial x^i} \\ &= \frac{3}{2} \hat{p}_i \hat{p}_j \epsilon_{imn} \frac{\partial}{\partial x^m} \left(\frac{\partial D_{jn}}{\partial t} + \frac{\partial^2 C_n}{\partial x^j \partial t} - \frac{1}{a(t)} \frac{\partial G_n}{\partial x^j} \right). \end{aligned} \quad (27)$$

By assuming that the degeneracy parameter ξ_ν is preserved in the cosmic expansion, the relation (15) between the intensity $K(t, \mathbf{x}, \hat{\mathbf{p}})$ with dimensions of energy and $\delta n_{\nu-\bar{\nu}}$ may be written as

$$a^3(t) \bar{n}_{\nu-\bar{\nu}}(t) K(t, \mathbf{x}, \hat{\mathbf{p}}) = \int_0^\infty \delta n_{\nu-\bar{\nu}}(t, \mathbf{x}, \mathbf{p}) 4\pi p^3 dp, \quad (28)$$

where the fermion asymmetry $\bar{n}_{\nu-\bar{\nu}}(t) \equiv \bar{T}^3(t) \xi_\nu / 6$ has now been expressed in terms of the equilibrium temperature $\bar{T}(t) = T_0/a(t)$ in the comoving system. With this relation and Eq. (27), one obtains the Boltzmann equation for the perturbation $\delta n_{\nu-\bar{\nu}}(t, \mathbf{x}, \mathbf{p})$,

$$\begin{aligned} & \frac{\partial \delta n_{\nu-\bar{\nu}}(t, \mathbf{x}, \mathbf{p})}{\partial t} + \frac{\hat{p}_i}{a(t)} \frac{\partial \delta n_{\nu-\bar{\nu}}(t, \mathbf{x}, \mathbf{p})}{\partial x^i} \\ &= -\frac{1}{2} (\bar{n}'_+(p) - \bar{n}'_-(p)) \hat{p}_i \hat{p}_j \epsilon_{imn} \frac{\partial}{\partial x^m} \\ & \quad \times \left(\frac{\partial D_{jn}}{\partial t} + \frac{\partial^2 C_n}{\partial x^j \partial t} - \frac{1}{a(t)} \frac{\partial G_n}{\partial x^j} \right). \end{aligned} \quad (29)$$

To determine the components $\delta T^0_j(t, \mathbf{x})$ and $\delta T^i_j(t, \mathbf{x})$ in the usual comoving coordinates, we can use the relations

$$\Omega^4(\tau) \delta T^\tau_j(\tau, \mathbf{x}) = a^3(t) \delta T^0_j(t, \mathbf{x}) = \delta T^\tau_j(\tau, \mathbf{x})|_{g=\eta}, \quad (30)$$

$$\Omega^4(\tau) \delta T^i_j(\tau, \mathbf{x}) = a^4(t) \delta T^i_j(t, \mathbf{x}) = \delta T^i_j(\tau, \mathbf{x})|_{g=\eta}, \quad (31)$$

which lead to

$$\delta T^0_j(t, \mathbf{x}) = \bar{n}_{\nu-\bar{\nu}}(t) \int \frac{d^2 \hat{\mathbf{p}}}{4\pi} K(t, \mathbf{x}, \hat{\mathbf{p}}) \hat{p}_j, \quad (32)$$

$$\delta T^i_j(t, \mathbf{x}) = \frac{\bar{n}_{\nu-\bar{\nu}}(t)}{a(t)} \int \frac{d^2 \hat{\mathbf{p}}}{4\pi} K(t, \mathbf{x}, \hat{\mathbf{p}}) \hat{p}_i \hat{p}_j. \quad (33)$$

The simplicity of the source terms in Eq. (27) or Eq. (29) is remarkable. A nice feature of this result is that, in the absence of G_n , the effect of the coefficients c_V and c_T in the kinetic equation enter through the single combination of vector and tensor quantities corresponding to the spatial perturbation of the metric, $a^{-2} \delta g_{ij} = D_{ij} + \partial_j C_i + \partial_i C_j$. This is similar to what happens in the Boltzmann equation [9,15] for the leading even-parity density perturbation $\delta n_{\nu+\bar{\nu}}(t, \mathbf{x}, \mathbf{p})$,

$$\begin{aligned} & \frac{\partial \delta n_{\nu+\bar{\nu}}(t, \mathbf{x}, \mathbf{p})}{\partial t} + \frac{\hat{p}_i}{a(t)} \frac{\partial \delta n_{\nu+\bar{\nu}}(t, \mathbf{x}, \mathbf{p})}{\partial x^i} \\ &= \frac{1}{2} p (\bar{n}'_+(p) + \bar{n}'_-(p)) \hat{p}_j \hat{p}_n \frac{\partial}{\partial t} \left(D_{jn} + \frac{\partial C_n}{\partial x^j} + \frac{\partial C_j}{\partial x^n} \right). \end{aligned} \quad (34)$$

The relation between the field theory approach and the one based on kinetic theory has been recently established in Ref. [5], where the authors have considered the effects of triangle anomalies without any metric perturbation. The previous treatment completes the derivation of the kinetic equation by including parity-violation effects from chiral matter in the presence of a weak time-dependent gravitational field.

It is instructive to write the explicit form of the odd-parity corrections to the anisotropic inertia and to compare them with the leading contributions proportional to the energy density $\bar{\rho}_{\nu+\bar{\nu}}(t)$. Here we reproduce for convenience the governing equations for these quantities [14],

$$\begin{aligned} & \frac{\partial J(t, \mathbf{x}, \hat{\mathbf{p}})}{\partial t} + \frac{\hat{p}_i}{a(t)} \frac{\partial J(t, \mathbf{x}, \hat{\mathbf{p}})}{\partial x^i} \\ &= -2 \hat{p}_j \hat{p}_n \left(\frac{\partial D_{jn}}{\partial t} - \frac{2}{a(t)} \frac{\partial \tilde{G}_j}{\partial x^n} \right), \end{aligned} \quad (35)$$

$$\delta T^0_j(t, \mathbf{x}) = a(t) \bar{\rho}_{\nu+\bar{\nu}}(t) \int \frac{d^2 \hat{\mathbf{p}}}{4\pi} J(t, \mathbf{x}, \hat{\mathbf{p}}) \hat{p}_j, \quad (36)$$

$$\delta T^i_j(t, \mathbf{x}) = \bar{\rho}_{\nu+\bar{\nu}}(t) \int \frac{d^2 \hat{\mathbf{p}}}{4\pi} J(t, \mathbf{x}, \hat{\mathbf{p}}) \hat{p}_i \hat{p}_j, \quad (37)$$

where $\tilde{G}_j \equiv G_j - a \partial_t C_j$. In order to find the time dependence of δT^μ_ν , we could use Eqs. (9) and (10), and evaluate the inverse Fourier transforms. But it is better to integrate the Vlasov equations, and then compute Eqs. (32) and (36), because the initial conditions are more clearly introduced in this way. With the standard expansion in plane waves $e^{i\mathbf{q}\cdot\mathbf{x}}$, this procedure yields the time dependence of $K(t, \mathbf{q}, \hat{\mathbf{p}})$ and $J(t, \mathbf{q}, \hat{\mathbf{p}})$, which upon evaluation of the integrals in Eqs. (32) and (36) for a vector perturbation yields

$$\begin{aligned} \delta T^0_j(t, \mathbf{q}) &= \tilde{g}_j(t, \mathbf{q}) + 4a(t) \bar{\rho}_{\nu+\bar{\nu}}(t) \int_0^u \frac{j_2(u-u')}{u-u'} \tilde{G}_j(t', \mathbf{q}) du' \\ & \quad - \frac{3}{2} \bar{n}_{\nu-\bar{\nu}}(t) q \int_0^u \frac{j_2(u-u')}{u-u'} \epsilon_{jmn} i \hat{q}_m \tilde{G}_n(t', \mathbf{q}) du', \end{aligned} \quad (38)$$

where u is proportional to the conformal time,

$$u = q \int_{t_1}^t \frac{dt'}{a(t')}, \quad (39)$$

and $\tilde{g}_i(t, \mathbf{q})$ is any arbitrary invariant contribution satisfying $q_j \tilde{g}_j = 0$. This may be traced to the solution of the Vlasov equation in the absence of sources for a specific initial condition $J(t_1, \mathbf{q}, \hat{\mathbf{p}})$,

$$\tilde{g}_i(t, \mathbf{q}) = a(t) \int \frac{d^2 \hat{\mathbf{p}}}{4\pi} \exp\left(-i \hat{\mathbf{p}} \cdot \mathbf{q} \int_{t_1}^t \frac{dt'}{a(t')}\right) J(t_1, \mathbf{q}, \hat{\mathbf{p}}) \hat{p}_i. \quad (40)$$

In the case of tensor modes the total contribution to δT^k_j reads

$$\delta T^k_j(t, \mathbf{q}) = \tilde{d}_{kj}(t, \mathbf{q}) - 4\bar{\rho}_{\nu+\bar{\nu}}(t) \int_0^u \frac{j_2(u-u')}{(u-u')^2} \frac{\partial D_{kj}(t', \mathbf{q})}{\partial t'} dt' + \frac{3\bar{n}_{\nu-\bar{\nu}}(t)q}{2a(t)} \int_0^u \frac{j_2(u-u')}{(u-u')^2} \left(\epsilon_{kmn} i\hat{q}_m \frac{\partial D_{jn}(t', \mathbf{q})}{\partial t'} + (k \leftrightarrow j) \right) dt', \quad (41)$$

where the traceless divergenceless part $\tilde{d}_{kj}(t, \mathbf{q})$ plays the same role as before. The kernels with spherical Bessel functions arise from the integrals

$$\int \frac{d^2 \hat{p}}{4\pi} e^{-i\hat{p} \cdot \hat{q} u} \hat{p} \cdot \hat{q} \hat{p}_i \hat{p}_j = -i \frac{j_2(u)}{u} \delta_{ij} + \dots, \\ \int \frac{d^2 \hat{p}}{4\pi} e^{-i\hat{p} \cdot \hat{q} u} \hat{p}_i \hat{p}_j \hat{p}_l \hat{p}_m = \frac{j_2(u)}{u^2} (\delta_{ij} \delta_{lm} + \text{two similar}) + \dots, \quad (42)$$

and, as expected, they exactly agree with the inverse Fourier transform of c_V and c_T ,

$$\int_{-\infty+i0^+}^{\infty+i0^+} \frac{dq^0}{2\pi} e^{-iq^0 \tau} \left(\frac{3}{10} Q_1(q^0/q) - \frac{3}{10} Q_3(q^0/q) \right) \\ = -\frac{3}{2} \frac{j_2(q\tau)}{\tau} \Theta(\tau), \\ \int_{-\infty+i0^+}^{\infty+i0^+} \frac{dq^0}{2\pi} e^{-iq^0 \tau} \left(-\frac{1}{10} Q_0(q^0/q) + \frac{1}{7} Q_2(q^0/q) \right. \\ \left. - \frac{3}{70} Q_4(q^0/q) \right) = \frac{3i}{2q} \frac{j_2(q\tau)}{\tau^2} \Theta(\tau). \quad (43)$$

To conclude, let us consider in more detail the equation for the tensor modes. If we choose the polarization tensors $e_{jn}(\hat{\mathbf{q}}, \lambda)$ to be the ones produced by the rotation which takes $\hat{\mathbf{z}} \rightarrow \hat{\mathbf{q}}$, where

$$e_{jn}(\hat{\mathbf{z}}, \lambda = \pm 2) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i\lambda}{2\sqrt{2}} & 0 \\ \frac{i\lambda}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (44)$$

one can easily check the identity

$$\epsilon_{kmn} i\hat{q}_m e_{jn}(\mathbf{q}, \lambda) + \epsilon_{jmn} i\hat{q}_m e_{kn}(\mathbf{q}, \lambda) = \lambda e_{kj}(\mathbf{q}, \lambda). \quad (45)$$

Thus the decomposition of the tensor modes according to

$$D_{jn}(t, \mathbf{q}) = \sum_{\lambda} e_{jn}(\mathbf{q}, \lambda) \mathcal{D}(t, \mathbf{q}, \lambda) \quad (46)$$

leads to decoupled equations for the quantities $\mathcal{D}(t, \mathbf{q}, \lambda)$. In the absence of \tilde{d}_{jn} , the Einstein equations adopt the form

$$16\pi G \left(-4\bar{\rho}_{\nu+\bar{\nu}}(t) + \frac{3\lambda}{2} \frac{\bar{n}_{\nu-\bar{\nu}}(t)q}{a(t)} \right) \int_0^u \frac{j_2(u-u')}{(u-u')^2} \\ \times \frac{\partial \mathcal{D}(t', \mathbf{q}, \lambda)}{\partial t'} dt' = \frac{\partial^2 \mathcal{D}}{\partial t^2} + \frac{3\dot{a}(t)}{a(t)} \frac{\partial \mathcal{D}}{\partial t} + \frac{q^2}{a^2} \mathcal{D}. \quad (47)$$

Due to the nonzero net neutrino number, these equations are no longer independent of the helicity λ , but for each helicity the equation has the same previously known form [15], and the same techniques may be used to find solutions [16]. The main effect of $\bar{n}_{\nu-\bar{\nu}}$ is to produce birefringence or a splitting of the two helicities, which increases linearly with q . The relative size of this correction is therefore wave-number dependent, $\bar{n}_{\nu-\bar{\nu}} q / \bar{\rho} a \sim \xi_{\nu} q / \bar{T} a$. Whether this has a non-negligible impact on the spectrum of primordial gravity waves is an issue to be further considered.

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