

Expansions of τ hadronic spectral function moments in a nonpower QCD perturbation theory with tamed large order behavior

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(Received 25 July 2013; published 22 August 2013)

The moments of the hadronic spectral functions are of interest for the extraction of the strong coupling α_s and other QCD parameters from the hadronic decays of the τ lepton. Motivated by the recent analyses of a large class of moments in the standard fixed-order and contour-improved perturbation theories, we consider the perturbative behavior of these moments in the framework of a QCD nonpower perturbation theory, defined by the technique of series acceleration by conformal mappings, which simultaneously implements renormalization-group summation and has a tame large-order behavior. Two recently proposed models of the Adler function are employed to generate the higher-order coefficients of the perturbation series and to predict the exact values of the moments, required for testing the properties of the perturbative expansions. We show that the contour-improved nonpower perturbation theories and the renormalization-group-summed nonpower perturbation theories have very good convergence properties for a large class of moments of the so-called “reference model,” including moments that are poorly described by the standard expansions. The results provide additional support for the plausibility of the description of the Adler function in terms of a small number of dominant renormalons.

DOI: [10.1103/PhysRevD.88.034026](https://doi.org/10.1103/PhysRevD.88.034026)

PACS numbers: 12.38.Cy, 13.35.Dx, 11.10.Hi

I. INTRODUCTION

The strong coupling α_s is a fundamental parameter whose determination is crucial for the low- and high-energy precision predictions of the standard model. A variety of sources exist for an accurate determination of this quantity at different scales [1–3]. The hadronic decays of the τ lepton allow for one of the most precise determinations of the strong coupling and also provide a beautiful experimental test of the predicted QCD running [1,3]. Indeed, the recent calculation of the QCD Adler function to five loops in massless QCD [4] motivated a large number of new determinations of α_s from these processes [5–20]. It may however be noted that τ decays involve the strong coupling at a rather low scale, where the theoretical ambiguities inherent to perturbative QCD are expected to be large. An important ambiguity is related to the prescription chosen for implementing renormalization-group invariance [7,9,21–23]. Another serious problem is related to the fact that the coefficients of the perturbative series of the Adler function in QCD display a factorial growth, i.e. the series has a vanishing radius of convergence [24–28]. These two problems are in fact related; in particular, the inclusion of additional terms in the expansion does not reduce, but on the contrary increases the dependence of the results on the renormalization-group prescription. The nonperturbative power corrections and the effects of what is known as quark-hadron duality violation (DV), i.e. the breakdown of the operator product expansion near the timelike axis in the complex energy plane, generate

additional uncertainties. The effects of these ambiguities are important especially at the low scale M_τ , where the coupling is relatively large. The differences between the specific ways of treating them represent the main source of theoretical error in the extraction of $\alpha_s(M_\tau^2)$.

The τ hadronic width is a good observable for the determination of the strong coupling, since it receives small contributions from the power corrections and DV. Various other moments have been also used in the past for the extraction of the strong coupling. Depending on the structure of the relevant weight, some moments may receive larger contributions from the nonperturbative condensates and terms involving DV, allowing the simultaneous extraction from data of these quantities and the strong coupling. The most comprehensive analysis to date, reported in [18], attempted to include DV in a combined fit of several moments, which in particular lead to a substantial increase in the error of the nonperturbative contributions. To improve such analyses, however, also the properties of the perturbative expansions of the moments must be carefully examined.

Recently, the perturbative expansions of a large class of spectral function moments have been discussed, under different assumptions for the large-order behavior of the Adler function [29]. This work extends the investigation of the hadronic width within two standard QCD perturbative expansions, the fixed-order and the contour-improved perturbation theories (FOPT and CIPT), to moments defined by more general weights. One of the important conclusions of [29] (and of the further study reported in [30]) is that some moments that are commonly employed in α_s determinations

from τ decays should be avoided because of their perturbative instability. We emphasize however that this refers to the standard expansions, FOPT and CIPT. As we shall show in this paper, improved expansions with no perturbative instability can be defined.

It has been recently pointed out [15,16,19,20] that an alternative to FOPT and CIPT, which is placed somewhat between the two, but in practice is closer to CIPT, is one that sums the leading logarithms thereby accounting for the renormalization group invariance. In Refs. [16,19,20] we called this approach “renormalization-group-summed perturbation theory” (RGSPT). In the present work, we investigate the moments considered in [29] also in the frame of RGSPT. More significantly, in the present paper we investigate the moments also in the frame of a novel formulation of QCD perturbation theory, defined some time ago in [31–33] starting from the divergent character of the standard series. The method uses the idea of series acceleration by means of a conformal mapping [34], applied to the Borel plane of QCD correlators. In the new formulation, the standard powers of the coupling are replaced by new expansion functions which are singular at the origin of the coupling plane and have divergent perturbative expansions, resembling thereby the expanded function itself. To emphasize this essential feature, we named the new perturbation framework as “nonpower perturbation theory” (NPPT) [14,19,20].¹ Detailed studies of the Adler function in the frame of NPPT [11–16] show that the best version is obtained by simultaneously implementing renormalization-group invariance and the available knowledge about the divergent pattern of the series at large orders. These optimized expansions were denoted as “contour-improved nonpower perturbation theory” (CINPPT) and “renormalization-group-summed nonpower perturbation theory” (RGSNPPT), respectively [19,20].

Previous studies of the new expansions were focused on the extraction of α_s from the total hadronic width, which involves a particular moment of the spectral function. We now generalize the investigation to the class of moments considered in [29]. The main aim of the work is to check whether the good convergence properties of CINPPT and RGSNPPT, demonstrated in the case of the hadronic width, remain valid also for the more general class of weights discussed in [29].

The scheme of this article is as follows: In Sec. II we recall the definition of the spectral function moments, and specify the class of moments investigated in [29], that we consider also here. We then briefly review in Sec. III the standard perturbative expansions of the Adler function in massless QCD. In Sec. IV we discuss the large-order

behavior encoded in the singularities of the Borel transform. Here we point out the essential features of the Borel transform in the three schemes, namely FOPT, CIPT, and RGSPT. In Sec. V, using the technique of “optimal conformal mapping” (OCM) and “singularity softening” for convergence acceleration, we define, for each RG prescription, a class of new, nonpower expansions, where the powers of the coupling are replaced by more general functions. The models proposed in [7,29] for the physical Adler function, denoted as the reference model (RM) and the alternative model (AM), are briefly reviewed in Sec. VI. These models are used to compute the higher-order perturbative coefficients, as well as the exact value and the ambiguity of the moments. Our results on the perturbative expansions of the moments are presented in Sec. VII, which we split into several subsections to facilitate the discussion: we first consider moments defined by integrals up to $s_0 = M_\tau^2$, expanded in the frame of CINPPT and RGSNPPT based on the OCM. In the next subsection we explore a larger class of expansions based on different softening factors and different conformal mappings, and in the last subsection we consider moments defined by integrals up to an s_0 lower than M_τ^2 . Section VIII contains discussions and conclusions.

II. MOMENTS OF THE SPECTRAL FUNCTIONS

We consider the moments of the spectral function $\text{Im}\Pi^{(0+1)}(s)$ defined as [29]

$$M_{w_i}(s_0) = \frac{2}{\pi} \int_0^{s_0} w_i(s/s_0) \text{Im}\Pi^{(0+1)}(s) ds, \quad (1)$$

where $s_0 \leq M_\tau^2$ and $w_i(x)$ are arbitrary nonnegative weights. We are interested in the perturbative contribution to M_{w_i} dependent on α_s , denoted as $\delta_{w_i}^{(0)}$, obtained by subtracting from (1) the perturbative tree values $\delta_{w_i}^{\text{tree}}(s_0)$. We adopt the set of weights $w_i(x)$, $i = 1, 17$ investigated in [29]. For the purposes to follow, we need to define in terms of the w_i , the corresponding

$$W_i(x) = 2 \int_x^1 dz w_i(z). \quad (2)$$

For completeness, we list in Table I the functions $W_i(x)$ for the weights $w_i(x)$ adopted in [29]. We recall that $i = 12$ gives the kinematical weight w_τ relevant for the hadronic decay width. According to the terminology of [29], the first class in Table I contains functions $W_i(x)$ generated from weights $w_i(x)$ equal to the monomials x^{i-1} , the second class contains moments generated by “pinched” weights (i.e. weights that vanish at $x = 1$) and include a “1” term, and the third class contains “pinched” weights without a “1” term, respectively. Some of the moments listed in Table I of the first and second classes were investigated in Ref. [14].

The analytic properties of the polarization function and the Cauchy theorem allow one to write equivalently (1) as an integral along a contour in the complex s plane, chosen for

¹We mention here that a different type of nonpower expansion, called “Analytic perturbation theory,” which is not based on the idea of optimal conformal mapping but exploits the dispersion relations satisfied by the QCD correlators, has been proposed in [35].

TABLE I. Functions $W_i(x)$ defined in (2) for the weights $w_i(x)$ listed in Table 2 of [29].

i	$W_i(x)$
1	$2(1-x)$
2	$1-x^2$
3	$\frac{2}{3}(1-x^3)$
4	$\frac{1}{2}(1-x^4)$
5	$\frac{2}{5}(1-x^5)$
6	$(1-x)^2$
7	$\frac{2}{3}(1-x)^2(2+x)$
8	$\frac{1}{2}(3-4x+x^4)$
9	$\frac{1}{4}(1-x)^3(3+x)$
10	$\frac{2}{3}(1-x)^3$
11	$\frac{1}{2}(1-x)^4$
12	$(1-x)^3(1+x)$
13	$\frac{1}{10}(1-x)^4(7+8x)$
14	$\frac{1}{6}(1-x)^3(1+3x)$
15	$\frac{1}{6}(1-x)^4(1+2x)^2$
16	$\frac{1}{210}(1-x)^4(13+52x+130x^2+120x^3)$
17	$\frac{1}{70}(1-x)^4(2+8x+20x^2+40x^3+35x^4)$

convenience to be the circle $|s| = s_0$. After an integration by parts, the perturbative contribution $\delta_{w_i}^{(0)}$ can be written as

$$\delta_{w_i}^{(0)}(s_0) = \frac{1}{2\pi i} \oint_{|s|=s_0} \frac{ds}{s} W_i(s/s_0) \hat{D}_{\text{pert}}(s), \quad (3)$$

where the weights $W_i(x)$ are defined in (2) and \hat{D}_{pert} is the perturbative part of the reduced Adler function

$$\hat{D}(s) \equiv -s d\Pi^{(1+0)}(s)/ds - 1. \quad (4)$$

This sets the stage for the computation of the moments of interest in this work.

III. RENORMALIZATION-GROUP SUMMATION: FOPT, CIPT, AND RGSPT

In our notation the standard perturbative expansion of the Adler function in a definite renormalization scheme, denoted usually as FOPT [7], is written as

$$\hat{D}_{\text{FOPT}}(s) = \sum_{n \geq 1} (a_s(\mu^2))^n \left[c_{n,1} + \sum_{k=2}^n k c_{n,k} \left(\ln \frac{-s}{\mu^2} \right)^{k-1} \right], \quad (5)$$

where $a_s(\mu^2) = \alpha_s(\mu^2)/\pi$. In (5) the renormalization scale μ^2 is chosen close to s_0 , the leading coefficients $c_{n,1}$ are computed from Feynman diagrams, and $c_{n,k}$ for $2 \leq k \leq n$ depend on $c_{n,1}$ and the perturbative coefficients β_k of the renormalization-group (RG) β function, which are known at present to four loops [36,37]. In the $\overline{\text{MS}}$ scheme for $n_f = 3$ flavors the coefficients $c_{n,1}$ calculated up to fourth order (cf. [4] and references therein) are

$$c_{1,1} = 1, \quad c_{2,1} = 1.64, \quad c_{3,1} = 6.371, \quad c_{4,1} = 49.079. \quad (6)$$

By setting $\mu^2 = -s$ in the expansion (5), one obtains the CIPT expansion of the Adler function [9,21–23]:

$$\hat{D}_{\text{CIPT}}(s) = \sum_{n \geq 1} c_{n,1} (a_s(-s))^n, \quad (7)$$

where the running coupling $a_s(-s)$ is determined by solving the RG equation

$$s \frac{da_s(-s)}{ds} = \beta(a_s). \quad (8)$$

For the evaluation of the integral (3), this equation is solved numerically in an iterative way along the contour $|s| = s_0$, starting with the input value $a_s(M_\tau^2)$ at $s = -M_\tau^2$.

The properties of the above expansions, in particular their convergence and the behavior in the complex energy plane, have been examined critically in several recent papers [5,7,9,10,29], where arguments in favor of one or another expansion have been given.

We mention also another prescription, proposed in [38,39] for timelike observables and applied in [15,16] to the Adler function in the complex energy plane. It generalizes the summation of leading logarithms to nonleading logs, by summing all the terms available from RG invariance. We refer to it as RGSPT. It can be shown [16] that the perturbative expansion (5) can be written as

$$\hat{D}_{\text{RGSPT}}(s) = \sum_{n \geq 1} (\tilde{a}_s(-s))^n \left[c_{n,1} + \sum_{j=1}^{n-1} c_{j,1} d_{n,j}(y) \right], \quad (9)$$

where

$$\tilde{a}_s(-s) = \frac{a_s(\mu^2)}{1 + \beta_0 a_s(\mu^2) \ln(-s/\mu^2)} \quad (10)$$

is the solution of the RG equation (8) to one loop, and $d_{n,j}(y)$ are calculable functions depending on the variable $y \equiv 1 + \beta_0 a_s(\mu^2) \ln(-s/\mu^2)$ and the coefficients β_j . These functions are shown to vanish for $y = 1$ or in the limit $\beta_j = 0$, $j \geq 1$. They have analytically closed, but quite lengthy expressions, given in [16]. As an effective series in powers of the one-loop running coupling, with coefficients that depend still on the coupling at a fixed scale, and also on the nonleading β_j , the expansion (9) appears to be placed “in-between” FOPT and CIPT: it resembles FOPT as it contains only analytical closed expressions, but makes a summation of higher terms known from renormalization group invariance, like CIPT. In practice, since the running of α_s in QCD is largely dominated by β_0 , the RGS expansion and CIPT are very similar. This feature is confirmed numerically, as discussed in detail in [15].

IV. LARGE-ORDER BEHAVIOR AND THE BOREL TRANSFORM

From special classes of Feynman diagrams it is known that the perturbative coefficients $c_{n,1}$ display a factorial increase, $c_{n,1} \sim n!$, so the perturbative expansions written above are divergent series [25–28]. This property follows also indirectly from the arguments given in [24], which infer that the expanded amplitude, viewed as a function of the coupling α_s , is singular at $\alpha_s = 0$. The divergent series in field theory are often interpreted as asymptotic series [25,28,40].

The large-order behavior of the CIPT series (7) is encoded in the properties of the Borel transform $B(u)$, defined by the expansion²

$$B(u) = \sum_{n=0}^{\infty} c_{n+1,1} \frac{u^n}{\beta_0^n n!}. \quad (11)$$

The original function $\hat{D}_{\text{CIPT}}(s)$ is recovered from $B(u)$ by a Laplace-Borel integral. Actually, in the present case $B(u)$ is known to have singularities on the real axis of the u plane, more precisely along the lines $u \geq 2$ (infrared renormalons) and $u \leq -1$ (ultraviolet renormalons) [28], so the integral requires a prescription. We adopt the principal value (PV) prescription [7,25,28]

$$\hat{D}_{\text{CIPT}}(s) = \frac{1}{\beta_0} \text{PV} \int_0^{\infty} \exp\left(\frac{-u}{\beta_0 a_s(-s)}\right) B(u) du, \quad (12)$$

which is preferred from the point of view of momentum-plane analyticity [41].

Similarly, one defines the Borel transforms $B_{\text{FO}}(u, s)$ and $B_{\text{RGS}}(u, y)$ of the FOPT and RGSPT expansions, (5) and (9) respectively, which can be written as [16]

$$B_{\text{FO}}(u, s) = B(u) + \sum_{n=0}^{\infty} \frac{u^n}{\beta_0^n n!} \sum_{k=2}^{n+1} k c_{n+1,k} \left(\ln \frac{-s}{M_\tau^2}\right)^{k-1}, \quad (13)$$

$$B_{\text{RGS}}(u, y) = B(u) + \sum_{n=0}^{\infty} \frac{u^n}{\beta_0^n n!} \sum_{j=1}^n c_{j,1} d_{n+1,j}(y). \quad (14)$$

The functions $\hat{D}_{\text{FOPT}}(s)$ and $\hat{D}_{\text{RGS}}(s)$ are recovered from their Borel transforms by Laplace-Borel integrals similar to (12):

$$\hat{D}_{\text{FOPT}}(s) = \frac{1}{\beta_0} \text{PV} \int_0^{\infty} \exp\left(\frac{-u}{\beta_0 a_s(s_0)}\right) B_{\text{FO}}(u, s) du, \quad (15)$$

$$\hat{D}_{\text{RGSPT}}(s) = \frac{1}{\beta_0} \text{PV} \int_0^{\infty} \exp\left(\frac{-u}{\beta_0 \tilde{a}_s(-s)}\right) B_{\text{RGS}}(u, y) du. \quad (16)$$

²For consistency with our subsequent notations, this Borel transform should have the index CI. However, we prefer the standard notation $B(u)$, which is used by most authors.

It is important to recall that not only the location, but also the nature of the leading singularities of $B(u)$ is known. Namely, near the points $u = -1$ and $u = 2$ $B(u)$ behaves as

$$B(u) \sim (1 + u)^{-\gamma_1}, \quad B(u) \sim (1 - u/2)^{-\gamma_2}, \quad (17)$$

where $\gamma_1 = 1.21$ and $\gamma_2 = 2.58$ [7,25,28,29,42]. As argued in [16,25], the leading singularities in the u plane of the Borel transforms $B_{\text{FO}}(u, s)$ and $B_{\text{RGS}}(u, y)$ have the same positions and nature as those of $B(u)$.

Starting from the divergent character of the standard perturbative series in QCD, the need of a new perturbation theory was advocated in [31]. Since the powers of a_s are holomorphic, while the function \hat{D}_{pert} is expected to be singular at the expansion point $a_s = 0$, no finite-order standard perturbative approximant can share this singularity with the expanded function: singularities can emerge only from the infinite series as a whole, which is not defined unambiguously since the perturbation series is divergent.

As discussed in [14,19], a perturbation series would be more instructive if the finite-order approximants could retain some information about the known singularities of the expanded function. Such approximants would tell us more about the function also from the numerical point of view. In the next section we shall review, following [14,16,19,31], the properties of these improved expansions based on the technique of series acceleration by the conformal mappings of the complex plane.

V. NONPOWER PERTURBATIVE EXPANSIONS

As discussed in Ref. [14], the method of conformal mappings is not applicable to the (formal) perturbative series in powers of a_s , because the expanded correlators are singular at the point of expansion, $a_s = 0$. However, the method can be applied in the Borel plane, where a holomorphy domain around the origin $u = 0$ is known to exist.

The starting point in the derivation is the remark that the expansion (11) converges only in the disk $|u| < 1$, whose boundary passes through the singularity of $B(u)$ closest to $u = 0$. However, the function $B(u)$ is holomorphic in a larger domain, assumed in general to be the whole complex u plane cut along the lines $u \geq 2$ and $u \leq -1$. It would be useful to insert in (12) an expansion of $B(u)$ that is convergent also outside the disk $|u| < 1$. Such an expansion is easily obtained: since the disk is the natural domain of convergence for power series, it suffices to expand the function in powers of variables that perform the conformal mapping of a larger part of its holomorphy domain onto a disk. Intuitively, one expects that a larger domain of convergence is related also to a better convergence rate. This expectation turns out to be correct: as shown a long time ago in [34], the variable that maps the entire holomorphy domain of the expanded function onto a disk has the remarkable property that the expansion in powers of this variable has the fastest large-order convergence rate at all

points inside the holomorphy domain (we assume here that the holomorphy domain is simply connected). This mapping was called ‘‘optimal conformal mapping’’ for series expansions [34]. More detailed arguments are given in two lemmas formulated and proved in [13,14]. For the Adler function in QCD, the optimal mapping $\tilde{w}(u)$ and the corresponding perturbative expansion were defined and investigated in [31–33].

An additional improvement is obtained by exploiting the known behavior (17) of $B(u)$ near the first singularities. If one multiplies $B(u)$ with a suitable factor $S(u)$, which fully compensates, or at least ‘‘softens’’ the dominant singularities, the expansions will have a more rapid convergence even at low orders [43]. In fact, a mild branch point, where the function vanishes instead of becoming infinite, is expected to influence the power expansions of the function only at larger orders. Hence, one can expand the product $S(u)B(u)$ in powers of conformal mappings that account only for the nonleading, i.e. the more distant, singularities, and contain a residual ‘‘mild’’ cut inside the convergence disks. In view of these remarks, it was useful to consider the general class of conformal mappings [12–14,31]:

$$w_{jk} \equiv \tilde{w}_{jk}(u) = \frac{\sqrt{1+u/j} - \sqrt{1-u/k}}{\sqrt{1+u/j} + \sqrt{1-u/k}}, \quad (18)$$

where j, k are positive integers satisfying $j \geq 1$ and $k \geq 2$. The function $\tilde{w}_{jk}(u)$ maps the u plane cut along $u \leq -j$ and $u \geq k$ onto the disk $|w_{jk}| < 1$ in the plane $w_{jk} \equiv \tilde{w}_{jk}(u)$, such that $\tilde{w}_{jk}(0) = 0$, $\tilde{w}_{jk}(-j) = -1$, and $\tilde{w}_{jk}(k) = 1$. The OCM defined above is $\tilde{w}(u) \equiv \tilde{w}_{12}(u)$, for which the entire holomorphy domain of the Borel transform, i.e. the u plane cut along $u \geq 2$ and $u \leq -1$, is mapped onto the interior of the unit circle in the plane $w_{12} \equiv \tilde{w}_{12}(u)$.

Using the above ideas, we consider the following expansion [12–14,31]:

$$S(u)B(u) = \sum_{n \geq 0} c_{n,CI}^{(jk)} (\tilde{w}_{jk}(u))^n. \quad (19)$$

In practice, this series is obtained by inserting in the product $S(u)B(u)$ the series (11) truncated at the order N , with u replaced by the inverse \tilde{u}_{jk} of (18). Then we expand the product in powers of $\tilde{w}_{jk}(u)$ and keep N terms in the series.

As discussed in [11,14], unlike the OCM which is unique, the choice of the softening factor $S(u)$, i.e. the implementation of the known nature of the first branch points, is to a large extent arbitrary. For a large number of terms in the expansion (19), the form of this factor should be irrelevant, but at low orders one prescription may be better than another.

In Refs. [14,16] the factor $S(u)$ was chosen as a simple expression of the expansion variable $\tilde{w}_{jk}(u)$ itself

$$S(u) \equiv S_{jk}(u) = \left(1 - \frac{\tilde{w}_{jk}(u)}{\tilde{w}_{jk}(-1)}\right)^{\gamma'_1} \left(1 - \frac{\tilde{w}_{jk}(u)}{\tilde{w}_{jk}(2)}\right)^{\gamma'_2}, \quad (20)$$

where γ'_j , $j = 1, 2$, are suitable exponents, given in [14], defined such as to preserve the behavior (17) of $B(u)$. This choice ensures a good convergence of the expansion (19), as noted by extensive numerical calculations [14]. Of course, other choices are possible, for instance the simple expression

$$S(u) = (1+u)^{\gamma_1} (1-u/2)^{\gamma_2}. \quad (21)$$

The expansions (19) converge in a domain larger than the convergence disk $|u| < 1$ of the original series (11), and according to the lemmas proven in [14], have a better convergence rate, in particular at points u close to the origin, which are dominant in the Laplace-Borel integral (12). The use of several conformal mappings and different softening factors reduces the bias related to the implementation of the threshold behavior (17), which is not unique, as we mentioned above. As discussed in [14], useful choices of the expansion variables are, besides the OCM $\tilde{w}_{12}(u)$, the conformal mappings $\tilde{w}_{13}(u)$, $\tilde{w}_{1\infty}(u)$ and $\tilde{w}_{23}(u)$.

From (12) and (19) one obtains the CINPPT [14]

$$\hat{D}_{\text{CINPPT}}(s) = \sum_{n \geq 0} c_{n,CI}^{(jk)} \mathcal{W}_{n,\text{CINPPT}}^{jk}(s), \quad (22)$$

where the expansion functions are defined as

$$\mathcal{W}_{n,\text{CINPPT}}^{jk}(s) = \frac{1}{\beta_0} \text{PV} \int_0^\infty e^{-u/(\beta_0 a_s(-s))} \frac{(\tilde{w}_{jk}(u))^n}{S(u)} du. \quad (23)$$

Similarly, the ‘‘fixed-order nonpower perturbation theory’’ (FONPPT) and the renormalization-group-summed non-power perturbation theory are defined as [14,16]

$$\hat{D}_{\text{FONPPT}}(s) = \sum_{n \geq 0} c_{n,FO}^{(jk)}(s) \mathcal{W}_{n,\text{FONPPT}}^{jk}(s_0), \quad (24)$$

$$\hat{D}_{\text{RGSNPPT}}(s) = \sum_{n \geq 0} c_{n,\text{RGS}}^{(jk)}(y) \mathcal{W}_{n,\text{RGSNPPT}}^{jk}(s), \quad (25)$$

where the coefficients are obtained from expansions similar to (19) of the Borel transforms $B_{\text{FO}}(u, s)$ and $B_{\text{RGS}}(u, y)$, and the expansion functions $\mathcal{W}_{n,\text{FONPPT}}^{jk}(s_0)$ and $\mathcal{W}_{n,\text{RGSNPPT}}^{jk}(s)$ are obtained from (23) by replacing the running coupling $a_s(-s)$ in the exponent through the fixed-scale coupling $a_s(s_0)$ and the one-loop running coupling $\tilde{a}_s(-s)$ defined in (10), respectively.

The properties of the expansions (22)–(25) have been discussed in [14,31,33]. When reexpanded in powers of a_s , they reproduce order by order the known perturbative coefficients calculated from Feynman diagrams. On the other hand, the expansion functions resemble the expanded

function, being singular at $a_s = 0$ and having divergent series in powers of a_s . Therefore, the divergent pattern of the expansion of the QCD correlators in terms of these new functions is expected to be tamer. This expectation is fully confirmed for the expansions that implement also RG summation, i.e. CINPPT and RGSNPPT, which give a very good description of suitable models of the Adler function in the complex s plane [14,16]. By contrast, FONPPT gives a very good description near the spacelike axis, which gradually deteriorates for points closer to the timelike axis. As discussed in [11,14], this behavior is due to the large imaginary parts of the logarithms in the coefficients $c_{n,FO}^{(jk)}(s)$ near the timelike axis, which follow from (5) and (13). Therefore, FONPPT describes well “pinched” moments for which the weight function suppresses this region, but the description is not so good for other moments. For this reason we shall concentrate in this paper mainly on the CINPPT and RGSNPPT frameworks, which simultaneously sum the large logarithms by RG invariance and tame the large-order behavior, by accelerating the convergence through conformal mappings. We shall present some results obtained with FONPPT only to illustrate the statement made above.

VI. MODELS FOR THE ADLER FUNCTIONS

In order to test numerically the convergence properties of the perturbative expansions, a model for the higher-order coefficients of the Adler function, $c_{n,1}$ for $n > 4$, is necessary. We follow the approach adopted recently in the literature [7,11,14,29,44], in which the physical function is expressed in terms of a few dominant singularities in the Borel plane. Unfortunately, even in this rather limited class of models, considerable freedom still exists: while the nature of the leading singularities is known, the residues cannot be determined from theory and an ansatz must be adopted. As discussed in [9,29,44], depending on the assumed strength pattern of the dominant singularities, either FOPT or CIPT turns out to be the preferred scheme.

In the RM proposed in [7,29], the Adler function $\hat{D}(s)$ is defined as the PV-regulated Laplace-Borel integral

$$\hat{D}(s) = \frac{1}{\beta_0} \text{PV} \int_0^\infty \exp\left(\frac{-u}{\beta_0 a_s(-s)}\right) B(u) du, \quad (26)$$

where the Borel transform $B(u) \equiv B_{\text{RM}}(u)$ is parametrized in terms of a few ultraviolet (UV) and infrared (IR) renormalons, and a regular, polynomial part. In our notations, it reads

$$\frac{B_{\text{RM}}(u)}{\pi} = B_1^{\text{UV}}(u) + B_2^{\text{IR}}(u) + B_3^{\text{IR}}(u) + d_0^{\text{PO}} + d_1^{\text{PO}} u, \quad (27)$$

with the renormalons parametrized as [7,29]

$$\begin{aligned} B_p^{\text{IR}}(u) &= \frac{d_p^{\text{IR}}}{(p-u)^{\gamma_p}} [1 + \tilde{b}_1(p-u) + \dots], \\ B_p^{\text{UV}}(u) &= \frac{d_p^{\text{UV}}}{(p+u)^{\tilde{\gamma}_p}} [1 + \bar{b}_1(p+u) + \dots]. \end{aligned} \quad (28)$$

The free parameters of the model, namely the residues d_1^{UV} , d_2^{IR} , and d_3^{IR} of the first renormalons and the coefficients d_0^{PO} , d_1^{PO} of the polynomial in (27), were determined by the requirement of reproducing the perturbative coefficients $c_{n,1}$ for $n \leq 4$ from (6) and the estimate $c_{5,1} = 283$. Their numerical values are [7,29]

$$\begin{aligned} d_0^{\text{PO}} &= 0.781, & d_1^{\text{PO}} &= 7.66 \times 10^{-3}, & d_2^{\text{IR}} &= 3.16, \\ d_3^{\text{IR}} &= -13.5, & d_1^{\text{UV}} &= -1.56 \times 10^{-2}. \end{aligned} \quad (29)$$

Then all the higher-order coefficients $c_{n,1}$ are fixed and exhibit a factorial increase. Their numerical values up to $n = 18$ are listed in [7,11].

This model is considered in [7,29] as most natural from the point of view of the strengths of the leading singularities, as no residue is fixed *a priori* by hand. If this model is adopted, FOPT provides the preferred framework for implementing RG invariance of the spectral function moments [29].

On the other hand, models with a smaller residue d_2^{IR} of the first IR renormalon are described better by CIPT [14,29,44]. As an extreme case, in the AM considered in [29] the first IR renormalon at $u = 2$ was removed by hand. Thus, the “extreme” AM proposed in [29] is defined by a Borel transform $B(u) \equiv B_{\text{AM}}(u)$ containing no singularity at $u = 2$ and an additional singularity at $u = 4$:

$$\frac{B_{\text{AM}}(u)}{\pi} = B_1^{\text{UV}}(u) + B_3^{\text{IR}}(u) + B_4^{\text{IR}}(u) + d_0^{\text{PO}} + d_1^{\text{PO}} u. \quad (30)$$

The five parameters found by matching the coefficients $c_{n,1}$ for $n \leq 5$ are

$$\begin{aligned} d_0^{\text{PO}} &= 2.15, & d_1^{\text{PO}} &= 4.01 \times 10^{-1}, & d_3^{\text{IR}} &= 66.18, \\ d_4^{\text{IR}} &= -289.71, & d_1^{\text{UV}} &= -5.21 \times 10^{-3}. \end{aligned} \quad (31)$$

Intermediate models, where the IR renormalon at $u = 2$ is present, but has a prescribed residue smaller (or larger) than the value in (29), were discussed in [8,14,44].

The properties of the perturbative expansions of the quantities $\delta_{w_i}^{(0)}$ in the standard FOPT and CIPT, using the models described above, were discussed in detail in [29,30]. The parameter s_0 was set equal to M_τ^2 in [29], while lower values of s_0 were investigated in [30]. For each class of weights defined in Table I, specific features of the perturbative expansions were identified. A bad perturbative behavior was found for some moments, in particular from the last class in Table I and for weights $w_i(x)$ having a linear term in x .

VII. RESULTS

In the present analysis we consider, in addition to the standard FOPT and CIPT series, the standard RGSPT and the nonpower expansions, CINPPT and RGSNPPT, defined in Sec. V. For illustration we selected several moments representative for each family of weights listed in Table I. The expressions given in the previous section for RM and AM were used to compute the exact values of the moments and their theoretical uncertainties, obtained from the imaginary part of the Laplace-Borel integral (26) using Eq. (A.8) of [7]. The results are presented in Figs. 1–4, where we compare the perturbative expansions with the exact values and their theoretical uncertainties, represented as bands. To facilitate the comparison with Refs. [29,30], we have used in the calculations $\alpha_s(M_\tau^2) = 0.3186$.

A. Results obtained with the OCM for $s_0 = M_\tau^2$

We investigate first the nonpower expansions obtained from (22) and (25) with the choice $j = 1$ and $k = 2$, which corresponds to the OCM defined in Sec. V. For the softening factor $S(u)$ we adopt the expression given in (20) for the same values of j and k . In Fig. 1 we show the perturbative behavior of the chosen sample of moments in the case of RM. For each moment we show the behavior of the standard expansions FOPT, CIPT, and RGSPT, together with the optimal nonpower version of each expansion.

As noted already from studies of the hadronic width [15,20], RGSPT gives results very close to CIPT, and this is confirmed for all the moments shown in Fig. 1. As concerns the FO nonpower expansions, the previous studies [11,14,19] showed that they achieve a good approximation of the Adler function near the spacelike axis, taming the large-order increase of the leading coefficients $c_{n,1}$, but have a bad behavior near the timelike axis, where the unsummed s -dependent logarithms present in the coefficients are large. Therefore, we expect good perturbative results only for “pinched” moments, like the 6th or the 12th, where the weight suppresses the region near the timelike axis. The results presented in Fig. 1 confirm this expectation. Incidentally, the improvement of the large-order behavior may destroy some suitable compensations of terms that take place in the standard FOPT and explain the good perturbative behavior of this scheme for some moments. Therefore, as already concluded in [11,14], the FO nonpower perturbative scheme is not suitable, because it cures only one facet of the problem, i.e. the large-order behavior, leaving the renormalization-group coefficients unsummed. To optimize the perturbative expansion, we must improve both aspects, as done within the CI and RGS nonpower frameworks.

The remaining two curves in each figure, denoted as CINPPT and RGSNPPT, prove in an impressive way the excellent approximation achieved with the CI and RGS nonpower expansions based on the OCM $\tilde{w}_{12}(u)$, even for

moments for which the standard CI, FO, and RGS expansions fail badly. The only moment for which the perturbative description is less impressive at low and moderate orders is the one obtained with the weight W_{16} . However, this moment is very small and has a large uncertainty, so the description may be considered good in this case as well. In all the cases, one may note a slightly better description achieved by CINPPT compared to RGSNPPT. The other moments defined in Table I, for which we do not explicitly exhibit the results, have a behavior similar to that of the representative moment of their class.

The good convergence of CINPPT and RGSNPPT for the moments shown in Fig. 1 can be understood from previous studies [14,16,19], which demonstrated that these expansions provide a very good approximation of the exact Adler function itself in the complex plane along the whole circle $|s| = M_\tau^2$. The good pointwise convergence of these expansions implies a good convergence to the true values also for contour integrals defined in (3), for all types of weights $W_i(x)$.

We consider now the AM discussed in [29], specified above in Eqs. (30) and (31). We recall that in this extreme model the first singularity of the Borel transform at $u = 2$ is completely removed. On the other hand, the nonpower expansions defined in Sec. V explicitly implement both the position and the nature of this singularity, known theoretically. In particular, the expansion functions (23) explicitly contain the singularity at $u = 2$ in the Borel plane (known actually to be present in the true, physical Adler function), while the function that we want to approximate does not have such a singularity. This means that the expansion functions defined in Sec. V are not mathematically optimal for this extreme model. We expect therefore a slower convergence and a poorer description of the true values at low orders.

On the other hand, after the conformal mapping of the cut u plane onto the unit disk, the expansion (19) of the Borel function converges in a larger domain. This leads also to a better convergence at large orders for points u on the real axis near the origin, which dominate the Laplace-Borel integral. Therefore, we expect the nonpower expansions defined in Sec. V to exhibit a tame behavior at large orders also in the case of the AM.

These expectations are confirmed by the results shown in Fig. 2, where we present the moments considered in Fig. 1 for the AM: at high orders the nonpower expansions tend to the exact value, illustrating the series acceleration by the OCM [14,34]. The description is relatively good even at low orders for weights like W_2 and W_6 , for which the exact values of the moments in RM and AM are rather close (however the uncertainty of these moments in AM is much smaller, requiring a better precision). For other moments, for which the true values in AM are quite different from those in RM, the approximation at low orders is worse in AM compared to RM.

In order to gain further insight, we have also carried out a study of the CINPPT series for the Adler function in the complex plane, along the contour $|s| = s_0$. Note that the true Adler function defined by the AM has a more oscillating behavior along the circle compared to the RM (this was noted also for other models in Ref. [29]). For

large perturbative orders N , the series approaches the true values for both the real and imaginary parts of $\hat{D}(s)$ quite uniformly along the circle. On the other hand, at low orders, the expansions (which are the same for all models up to $N = 5$) stay quite close to the true function defined by the RM, departing therefore from the AM. In particular,

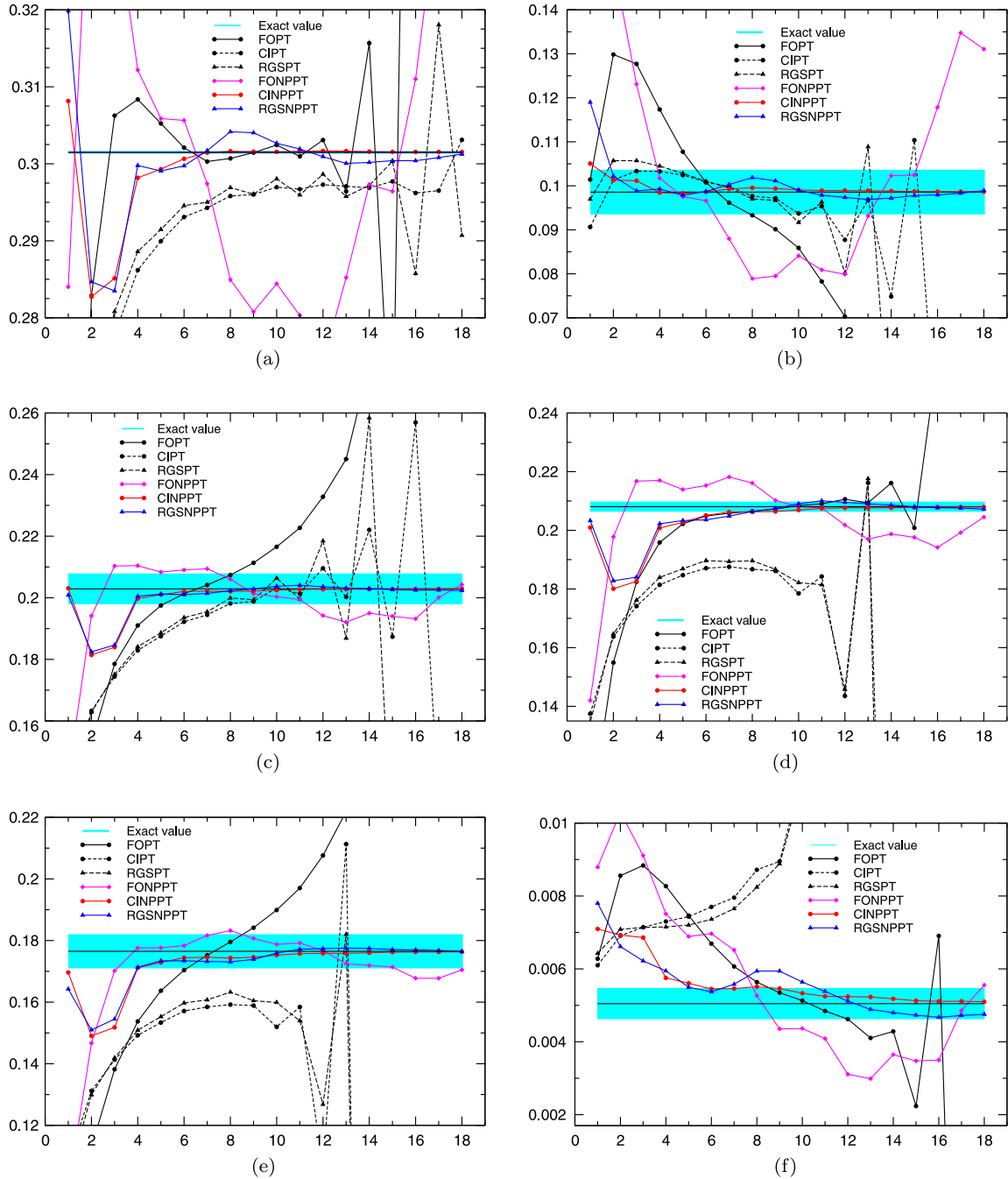


FIG. 1 (color online). $\delta_{W_i}^{(0)}$ defined in (3) for the weights $W_1, W_2, W_6, W_{12}, W_{13}$, and W_{16} , calculated for the RM defined in [7,29] with the standard and nonpower versions of FO, CI, and RGS expansions, as functions of the perturbative order up to which the series was summed. The horizontal bands give the uncertainties of the exact values. As in [29], we use $\alpha_s(M_Z^2) = 0.3186$. (a) W_1 , reference model. (b) W_2 , reference model. (c) W_6 , reference model. (d) W_{12} , reference model. (e) W_{13} , reference model. (f) W_{16} , reference model.

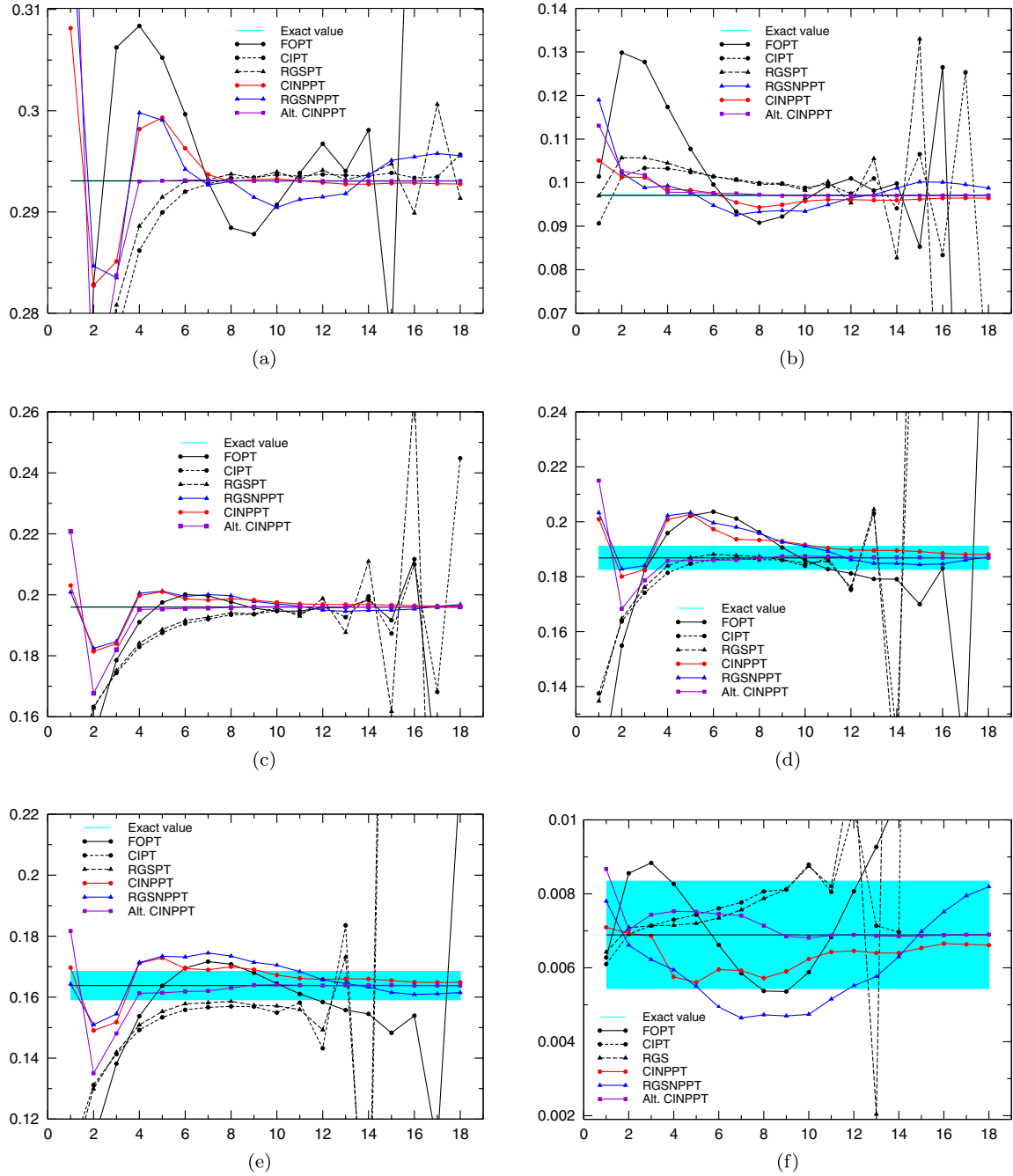


FIG. 2 (color online). Perturbative expansions of the moments of the AM adopted in [29]. “Alt. CINPPT” denotes the specific optimal expansion devised for the AM, as explained in the text. (a) W_1 , alternative model. (b) W_2 , alternative model. (c) W_6 , alternative model. (d) W_{12} , alternative model. (e) W_{13} , alternative model. (f) W_{16} , alternative model.

they are not able to reproduce the oscillations of the model along the circle. This shows that the CINPPT expansions approximate better the exact Adler function defined by the RM than the function defined by the AM.

As we mentioned above, the expansion that we used is not optimal for the AM. One can actually define an optimal expansion for this model, using the fact that its first singularities are situated at $u = -1$ and $u = 3$, and have a known nature [7,29]. The optimal mapping is obtained

by setting $j = 1$ and $k = 3$ in (18). Moreover, the softening factor $S(u)$ must vanish at $u = -1$ and $u = 3$. Adopting for $S(u)$ the expression (20), we obtain the proper factor by replacing $\tilde{w}_{12}(2)$ and γ'_2 by $\tilde{w}_{13}(3)$ and the value of γ'_3 derived from the parameter γ_3 given in [7,29]. It is instructive to show also the results obtained with this optimal perturbative expansion suitable for the AM. It is denoted as “Alt. CINPPT” in Fig. 2, and exhibits a very rapid convergence to the true values of all the moments.

This exercise demonstrates the mathematical power of the technique of the singularity softening and conformal mappings for series acceleration when the position and the nature of the leading singularities is known. Of course, for the physical Adler function, where the first IR renormalon is known to be present, the softening factor $S(u)$ must vanish at the dominant branch points $u = -1$ and $u = 2$, and the optimal expansion variable must map onto a disk the u plane cut along $u \leq -1$ and $u \geq 2$.

The results presented in Fig. 2 and the numerical studies performed in the previous works show nevertheless that the optimal CINPPT and RGSNPPT may have a slower convergence for models of the Adler functions with a residue of the first IR singularity significantly smaller than the value it has in the RM. It turns out that the description is less precise at low orders also for models where this residue is larger than the RM value (for such models the standard FOPT and CIPT are both quite poor). Indeed, the perturbative curves are the same for $N \leq 5$ for all models, while by adjusting the residues of the leading singularities one can shift up or down, by a certain amount, the exact values of the moments.

The good convergence of the expansions based on the OCM for the RM starting from relatively low orders may suggest that this model has a preferred place among models. Indeed, CINPPT and RGSNPPT have a solid theoretical basis, exploiting simultaneously RG invariance and the known large-order behavior of the expanded function. However, as mentioned above, there is still a certain arbitrariness in defining these expansions, since the implementation of the singular behavior at the leading branch points is not unique. The preference for the RM might well be a consequence of the specific choice of the softening factor $S(u)$ given by (20), for the OCM defined by $j = -1$ and $k = 2$. In order to reduce the possible bias, we must investigate also other expansions, with a different implementation of the threshold behavior. Moreover, as discussed in Sec. V, for mild singularities the expansions based on different conformal mappings are expected to have properties similar to those based on the optimal mapping. The investigation of a more general class of expansions is the subject of the next subsection.

B. Results obtained with various softening factors and other conformal mappings

An investigation of CINPPT with different choices of the softening factor $S(u)$ was performed already in [12] for the RM and the particular moment relevant for the τ hadronic width. For instance, the dominant behavior (17) was implemented by singular factors expressed in terms of the u variable, like in (21), and the leading factors were multiplied by other functions analytic in the u -complex plane cut along the real axis for $u \geq 2$ and $u \leq 1$. In particular, singularities on an unphysical Riemann sheet, or placed at $u = 3$ and $u = 2$, were included, the additional factors

being expressed either in the variable u or in the variable $\tilde{w}(u)$. As reported in [12], the results for the τ hadronic width are very stable and reproduce well the exact value of the RM for relatively low perturbative orders, of interest for the extraction of $\alpha_s(M_\tau^2)$ from the perturbative calculations available so far.

In the present work we consider the class of expansions defined in Sec. V. As in [13,14,16,19], where we investigated the τ hadronic width, we adopt besides the OCM $\tilde{w}_{12}(u)$ also the variables $\tilde{w}_{13}(u)$, $\tilde{w}_{1\infty}(u)$, and $\tilde{w}_{23}(u)$ (some of these conformal mappings have been used also by other authors; see [14] for earlier references). For each expansion variable $\tilde{w}_{jk}(u)$ we chose also a different form of the singularity softening factors $S(u)$, as the simple expression of $\tilde{w}_{jk}(u)$ given in (20). The only requirement is to reproduce the branch-point behavior (17). To further enlarge the class, we consider also the softening factor $S(u)$ given by (21).

In Fig. 3 we show the results obtained with this general class of perturbative expansions. We consider the same moments of the RM as in the previous subsection. For simplicity, we give only the results obtained in the frame of CINPPT. The RGSNPPT expansions exhibit a similar behavior. For comparison we show also the standard CIPT and FOPT.

The results show that at very low perturbative orders the various nonpower expansions are different, but starting from an order N around 5 they give very similar predictions, which agree also quite well with the exact values of the RM moments. One can see that the expansion based on the softening factor (21) gives slightly poorer results for some moments (for instance the 1st and the 16th), compared to the expansions based on the softening factors (20). We mention that the softening factor (21) leads to a worse approximation compared to the choice (20) also in the case of the AM. From these results and other numerical tests [11,14] it follows that the choice of the softening factor as a simple expression (20) of the variable used in the expansion (19) ensures a good convergence.

At larger orders the description is very precise, and this feature remains stable up to the large order, $N = 18$, shown in the figure, and even to larger orders investigated numerically. Only the expansion based on the choice $j = 2, k = 3$ starts to exhibit oscillations at large orders (especially for the 16th moment). As explained in detail in [14], this behavior is due to the effect of the mild (after singularity softening) singularity at $u = -1$, which is still present inside the unit disk $|\tilde{w}_{23}(u)| < 1$. This singularity affects the convergence of the corresponding power series at points u larger than unity, but still small enough such as to bring a non-negligible contribution to the Laplace-Borel integral.

We conclude that the moments of the RM have very stable perturbative expansions in the frame of CINPPT and RGSNPPT, for various prescriptions of singularity softening and various conformal mappings. These expansions

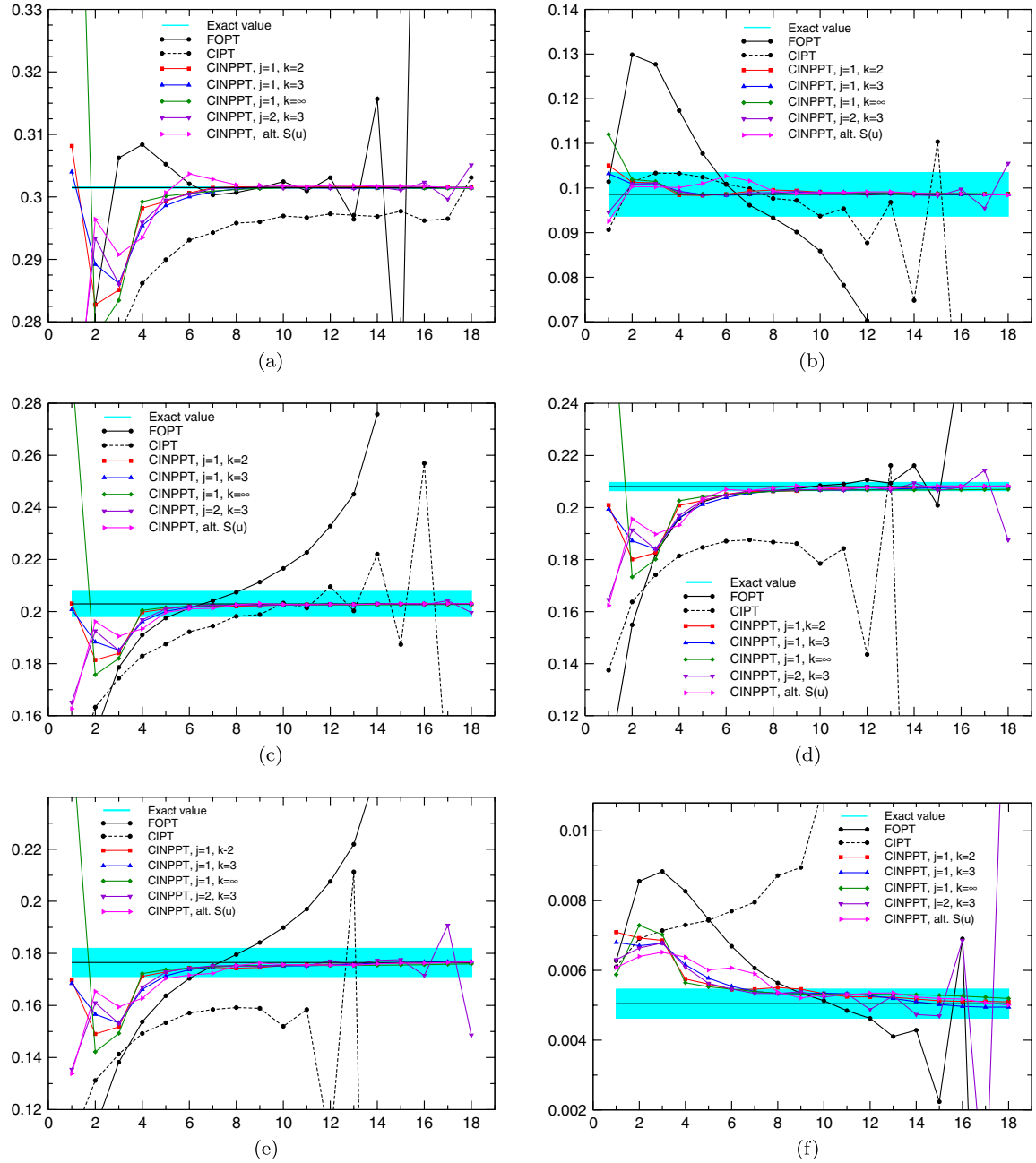


FIG. 3 (color online). Several CINPPT expansions of the moments shown in Fig. 1, compared with the standard FOPT and CIPT. The first four nonpower expansions are obtained with the choice (20) of the softening factors and several conformal mappings. The last expansion is obtained using in (19) the softening factor $S(u)$ from (21) and the OCM $\tilde{w}_{12}(u)$. (a) W_1 , reference model. (b) W_2 , reference model. (c) W_6 , reference model. (d) W_{12} , reference model. (e) W_{13} , reference model. (f) W_{16} , reference model.

reproduce the exact moments of RM starting from rather low perturbative orders. We emphasize that no assumption about the magnitude of the residues of the singularities is made in defining these expansions.

C. Results for $s_0 < M_\tau^2$

In several moment analyses for the extraction of the strong coupling and other fundamental parameters of QCD, values of s_0 less than M_τ^2 , but sufficiently large so as

to ensure the validity of the perturbation theory, have been also employed. For lower values of s_0 the convergence of the standard perturbative expansions along the circle $|s| = s_0$ is expected to be slower due to the fact that the coupling is larger. The study of the standard expansions FOPT and CIPT performed in [29] was extended to lower values of s_0 in [30], where it was shown that the conclusions of [29] about the bad perturbative behavior of some moments and the preference for FOPT are still valid for $s_0 < M_\tau^2$.

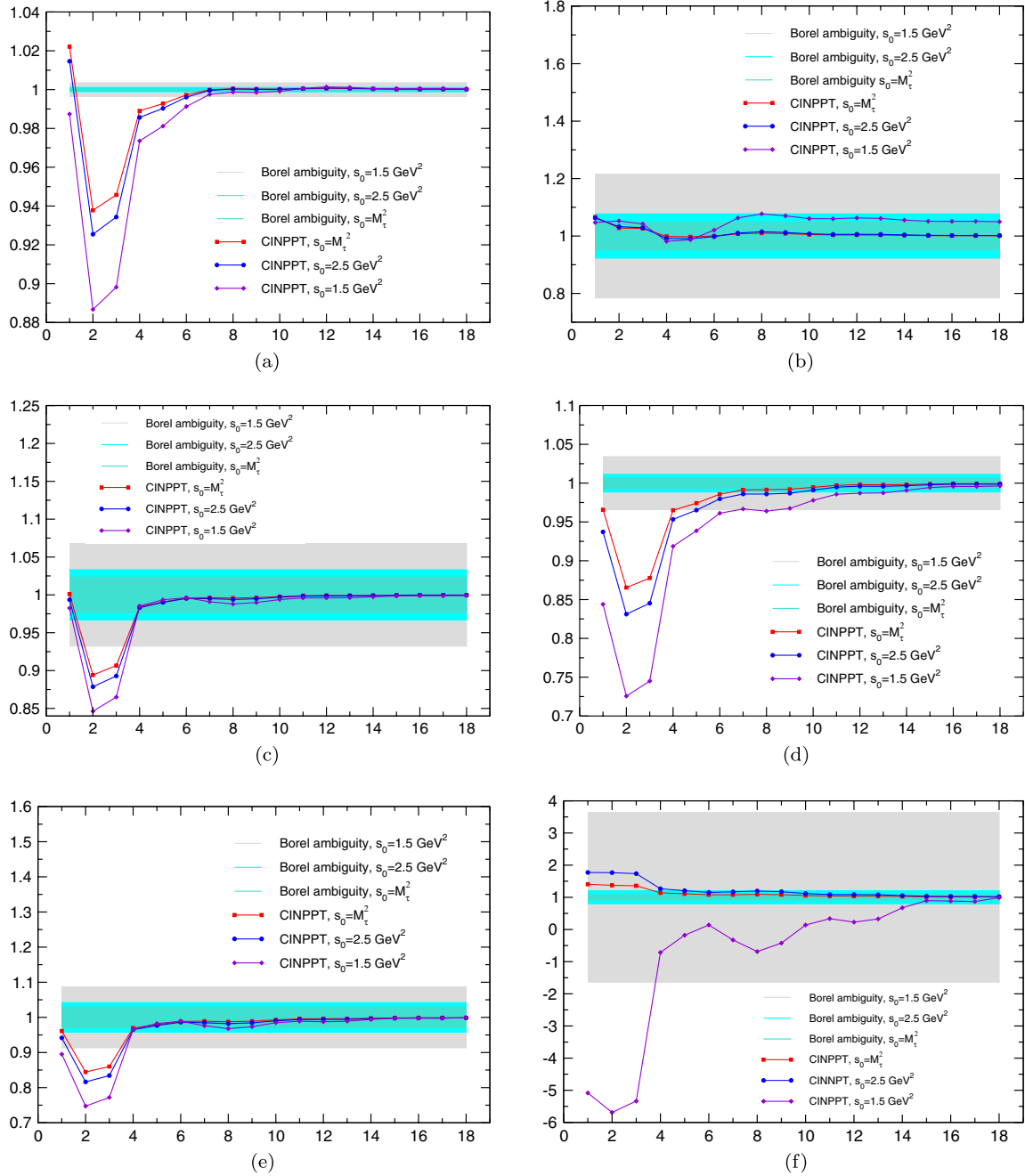


FIG. 4 (color online). $\delta_{w_i}^{(0)}$ defined in (3) for the RM, for $s_0 = 1.5 \text{ GeV}^2$, 2.5 GeV^2 , and M_τ^2 , expanded in the optimal CINPPT normalized to the exact value. The horizontal bands show the uncertainties of the exact values. (a) W_1 , reference model. (b) W_2 , reference model. (c) W_6 , reference model. (d) W_{12} , reference model. (e) W_{13} , reference model. (f) W_{16} , reference model.

Here we present the results of our analysis for the optimal CINPPT expansions at lower s_0 . As in [30], in order to compare the results for various s_0 we normalize the expansions to the exact value of the moment given by the model. In Fig. 4 we present the CINPPT expansions for the representative moments chosen in this work. To keep the figures simple, we do not show now the standard expansions (for some of them see [30]). As expected, the perturbative behavior becomes poorer at lower

s_0 , but the extent to which this happens depends very much on the moment. On the other hand, the ambiguity of the exact value also increases for smaller s_0 , due to the larger value of $\alpha_s(s_0)$ [we use as before, $\alpha_s(M_\tau^2) = 0.3186$, which corresponds to $\alpha_s(2.5 \text{ GeV}^2) = 0.3415$ and $\alpha_s(1.5 \text{ GeV}^2) = 0.4078$].

For the 2nd, 6th, and 13th moments the perturbative behavior is very stable with s_0 and within the chosen uncertainty starting from low perturbative orders, $N \geq 4$.

Therefore, these moments are good candidates for moment analyses with lower s_0 in the framework of CINPPT. The 1st and the 12th moment show stability for s_0 down to 2.5 GeV^2 , while at lower s_0 the agreement with the true value is reached only at higher orders. In fact, for these moments the ambiguity of the Borel integral is rather small for the RM. Therefore, if we take this uncertainty seriously, the perturbative expansions require slightly higher orders, $N \geq 6$, for all s_0 , to become acceptable. Finally, for the 16th moment the CINPPT expansion is quite poor at low orders for $s_0 = 1.5 \text{ GeV}^2$, but in this case the ambiguity of the exact value is also very large. At higher orders the convergence is good in all the cases.

VIII. DISCUSSION AND CONCLUSIONS

In this work we have investigated several spectral function moments of the massless Adler function in the frame of a new class of “nonpower” perturbative expansions in QCD, where the powers of the coupling are replaced by more adequate functions [11,14,16,31–33]. The new expansions simultaneously implement RG summation, either in the “contour-improved” or in the “renormalization-group-summed” form, and the known location and nature of the first singularities of the expanded function in the Borel plane. Mathematically, the definition is based on the acceleration of series convergence by the technique of conformal mappings [34] applied in the Borel plane [31–33]. When reexpanded in powers of a_s , the new series reproduce order by order the perturbative coefficients known from Feynman diagrams. On the other hand, they exhibit a much tamer behavior at larger orders, allowing a more reasonable estimate of the truncation error, which accounts for the unknown higher terms in the expansions.

In our earlier works [11–16], the new expansions were used mainly for the extraction of the strong coupling from the τ hadronic width. In this work we go further by employing them in a study of other spectral function moments that are relevant for the extraction of the strong coupling and other QCD parameters from τ decays. Our work is motivated by the recent papers [29,30], which performed a detailed analysis of the moments in the framework of standard CIPT and FOPT. The main aim of our research was to see whether the good behavior of CINPPT and RGSNPPT, already established in the case of τ hadronic width, remains valid also for other moments.

In order to assess the quality of various perturbative frameworks, the larger-order pattern of the perturbative coefficients of the Adler function must be known. Of course, this knowledge is not available and an ansatz must be adopted. The description of the function in terms of its dominant singularities in the Borel plane is a natural choice, consistent with the general principles of analyticity. However, a considerable ambiguity still remains because, while the position and nature of the leading singularities are known theoretically [7,25,28,42], nothing can be said from

theory about their strengths. The recent claims in favor of either CIPT or FOPT are based on different views about the magnitude of the residues of the leading singularities (the IR renormalon at $u = 2$ and the UV renormalon at $u = -1$). The situation was analyzed in detail in [29], where some arguments in favor of a “reference model,” defined in [7], were put forth. Moreover, as discussed in [29], the reference model favors FOPT compared to CIPT.

Our analysis confirms first the similarity of the contour-improved and renormalization-group-summed prescriptions, both in the standard form (CIPT and RGSPT) and the nonpower frameworks (CINPPT and RGSNPPT), for all the moments investigated. The essential feature of these prescriptions is that they sum the large logarithms present in the coefficients into the running coupling, calculated either numerically (in CIPT) or by explicit expressions (in RGSPT). In the CINPPT and RGSNPPT frameworks the series is further optimized in order to tame the large-order behavior.

The results reported in Sec. VII show that CINPPT and RGSNPPT describe very well the spectral function moments of the RM considered in [29], including those that are poorly described by the standard expansions, FOPT, CIPT, and RGSPT. We have demonstrated a good convergence of CINPPT for various conformal mappings used as expansion variables after softening the leading singularities. The description continues to remain good also at lower values of s_0 , within the uncertainties adopted for the true values.

For the extreme AM defined in [29], where the first IR renormalon is removed by hand, the approximation achieved with the nonpower expansions defined in Sec. V is less precise for some moments at low orders. This is due to the fact that the expansions are optimally devised for the physical Adler function, exploiting in a manifest way its first singularities. However, they are not optimal for the alternative model, where one of the dominant singularities is absent. On the other hand, at higher orders the nonpower expansions have a tame behavior tending to the true values for both models, nicely illustrating the theorem of series acceleration by conformal mappings [14,34].

Our analysis shows that the class of nonpower expansions (22) and (25), based on different softening factors and different conformal mappings, agree among them and with the exact moments of the RM of the Adler function defined in [7,29] starting from rather low perturbative orders, $N = 4$ or 5 . This may be a coincidence, but may also signal a special place of this model among other models of the Adler function. Of course, such a conclusion is not fully rigorous, because the nonpower expansions contain some arbitrariness in the implementation of the dominant singular behavior. However, we have investigated several reasonable expansions to reduce the bias, and the results are quite stable. Thus, the rapid convergence and the stability of CINPPT and RGSNPPT for all the

moments of the RM might be an argument in favor of the naturalness of this model.

In conclusion, the contour-improved nonpower perturbation theory (CINPPT) and the renormalization-group-summed nonpower perturbation theory (RGSNPPT) provide a good perturbative description of a large class of τ hadronic spectral function moments, including some for which all the standard expansions fail. In contrast to standard perturbation theory, we do not use series in powers of the strong coupling, which are mostly chosen for their “simplicity.” A fundamental merit of our approach is the fact that, to expand a singular (Adler, e.g.) function, we make use of a set of expansion functions possessing singularities that resemble those of the expanded function itself. These expansions also give confidence in a more realistic estimate of the truncation

error. As a consequence, our perturbation expansions CINPPT and RGSNPPT provide solid theoretical frameworks for the perturbative part in moment analyses. A program that employs these expansions for the simultaneous determination of the strong coupling and other parameters of QCD from hadronic τ decays is of interest for future investigations.

ACKNOWLEDGMENTS

I.C. acknowledges support from the Ministry of Education under Contracts No. PN 09370102/2009 and No. Idei-PCE 121/2011. The work was supported also by the Projects No. LA08015 and No. LG130131 of the Ministry of Education of the Czech Republic.

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