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Light-cone sum rules for the $D_{(s)} \to \eta^{(\prime)} l \nu_l$ form factor

N. Offen, F. A. Porkert, and A. Schäfer

Institut für Theoretische Physik, Universität Regensburg, D-93040 Regensburg, Germany

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We present an improved light-cone sum rule analysis of the decay form factors of D and D_s into η and η' and argue that these decays offer a very promising possibility to determine the leading Fock-state gluonic contribution of the η' at future experimental facilities such as FAIR or Super-KEKB. We also give the corresponding branching ratios for B decays.

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I. INTRODUCTION

With the advent of high luminosity accelerators, weak decays of hadrons containing valence charm or bottom quarks can be measured with very high precision. In fact, such decays might even offer one of the best chances for the discovery of beyond the standard model physics (see the recent reviews [1–3] and the citations given there). So, there is strong motivation to improve on the theoretical description of the QCD input needed for such searches. One of the most important quantities for such exclusive channels are the hadron distribution amplitudes (DAs), often also called wave functions, and form factors. For each hadron, DAs are characteristic nonperturbative quantities, just like parton distribution functions. As for the latter, moments of DAs can be calculated on the lattice, see e.g. [4,5], and rapid progress can be expected along these lines. Nevertheless, in the long run input from many sides will be needed to understand the basic systematics of hadron DAs, even for the most important standard hadrons. The controversial theoretical discussion spawned by the surprising *BABAR* data for the photon-pion transition form factor [6–9] has illustrated that this field is still in a pioneering phase. Another nonperturbative approach, besides lattice QCD, to DAs and form factors are light-cone sum rules (LCSR) [10]. As both approaches are conceptually completely different, the ideal situation is reached if both give the same results. We will show that this is what happens, e.g., for the decays $D_s \to \eta/\eta' + \ell + \nu_\ell$ we are analyzing in this contribution. This case is especially interesting because the singlet-octet mixing of the η and η' should be reflected by the respective form factors, e.g., by a substantially different size of the gluonic contribution (see e.g. [11,12] for a recent review). Since this debate has been ongoing for many years, it would be great news if the gluonic leading Fock-state contribution for the η' could be experimentally determined. (There always exist gluonic higher Fock-state components.) We will specify observables that are sensitive to this component and thus offer this opportunity.

From a theoretical point of view, B mesons would be better suited for our purpose. There, the light-cone expansion exhibits a stronger hierarchy due to the larger mass of the b quark, which in turn reduces the uncertainty coming from the truncation of this expansion. However, in practice, this uncertainty is not the dominant one.

As for all three cases (D, D_s and B decays), the required increase in experimental accuracy looks very feasible for next-generation experiments, and we hope that in a few years, data for this complete set of meson decays will provide undisputable experimental evidence for the gluonic component of the η'.

The decays $D_s \to \eta/\eta' + \ell + \nu_\ell$ have been analyzed before, both phenomenologically, e.g., [13,14], and using leading-order LCSRs with chiral currents including meson mass corrections [15]. We improved the LO twist-2 analysis by taking into account all two-particle twist-2 and twist-3 next to leading order (NLO) quark contributions and, in addition, the NLO twist-2 gluon contribution. The latter allows us to extract information on the leading gluon DA of the η'. To achieve this goal, we made heavy use of NLO results existing in the literature [16,17]. Our results for the decay form factors agree, within uncertainties, with those of [15]. While this is encouraging, we also feel that it is somewhat fortuitous, because we have some doubts concerning the benefits of the chiral currents used in that work, since they eliminate important nonperturbative information and do not couple only to the pseudoscalar mesons in the hadronic sum.

The decays $B \to \eta/\eta' + \ell + \nu_\ell$ were analyzed in [18] at leading order and in [16] at the same level of accuracy as in this note. We improve on the latter calculation by making an analysis of both the branching fractions and their ratios.

The paper is organized as follows: In Sec. II we discuss the $\eta - \eta'$ mixing scheme. In Sec. III we outline the derivation of the LCSRs for the different form factors. In Sec. IV we present our numerical results, and in Sec. V we summarize and conclude.

II. MIXING SCHEMES

Two different schemes for describing the $\eta - \eta'$ mixing are commonly used: the singlet-octet (SO) [19] and the quark-flavor (QF) schemes [20–24]. See also [25] for a mixing scheme independent sum rule determination of the

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couplings of the $\eta^{(\prime)}$ to the axial currents. The SO scheme defines two hypothetical pure singlet and octet states $|\eta_{1,8}\rangle$ and two mixing angles $\Theta_{1,8}$ to describe the four decay constants,

$$\begin{pmatrix} f_\eta^8 & f_\eta^1 \\ f_{\eta'}^8 & f_{\eta'}^1 \end{pmatrix} = \begin{pmatrix} \cos\theta_8 & -\sin\theta_1 \\ \sin\theta_8 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} f_8 & 0 \\ 0 & f_1 \end{pmatrix}, \quad (1)$$

defined as

$$\langle 0|J^i_{\mu 5}|P(p)\rangle = i f^i_P p_\mu, \qquad (i = 1, 8, P = \eta, \eta'). \quad (2)$$

In this scheme, f_1 describes the contribution of the $U(1)_A$ anomaly via the divergence of the singlet current $J^1_{\mu 5}$, and the difference $\theta_i \neq 0$ and $f_8 \neq f_\pi$ is given by $SU(3)_F$-violating effects. f_8 and θ_i are scale independent and f_1 renormalizes multiplicatively.

In the QF scheme, the basic currents and couplings are given by

$$\begin{aligned} &\langle 0|J^a_{\mu 5}|\eta(p)\rangle =: i f^a_\eta p_\mu, \\ &\langle 0|J^a_{\mu 5}|\eta'(p)\rangle =: i f^a_{\eta'} p_\mu, \\ &J^a_{\mu 5} = \begin{cases} \frac{1}{\sqrt{2}}(\bar u \gamma_\mu \gamma_5 u + \bar d \gamma_\mu \gamma_5 d), & a = q \\ \bar s \gamma_\mu \gamma_5 s, & a = s. \end{cases} \end{aligned} \quad (3)$$

Here the angles are scale dependent and their difference is given by Okubo-Zweig-Iizuka (OZI) rule–violating contributions. Phenomenologically, this difference is very small. Thus, the authors of [20] proposed to use, within the QF scheme, the approximation

$$\phi \equiv \phi_{q,s}, \qquad \phi_q - \phi_s = 0, \quad (4)$$

which has only three parameters with the phenomenological values

$$\begin{aligned} f_q &= (1.07 \pm 0.02) f_\pi, \\ f_s &= (1.34 \pm 0.06) f_\pi, \\ \phi &= 39.3° \pm 1.0°, \end{aligned} \quad (5)$$

and where the mixing of the states follows the same pattern as for the decay constants,

$$\begin{pmatrix} |\eta(p)\rangle \\ |\eta'(p)\rangle \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} |\eta_q(p)\rangle \\ |\eta_s(p)\rangle \end{pmatrix}. \quad (6)$$

The masses of the states, to the order in which we perform our calculations, are given by [20]

$$m^2_{qq} = m^2_\pi, \qquad m^2_{ss} = 2m^2_K - m^2_\pi. \quad (7)$$

One important point to note is that in this version of the QF scheme, there is no scale dependence left in the parameters. Since the mixing of the two different flavor states is given by OZI rule–violating contributions,

$$|\eta_q(p)\rangle \propto \phi_2^q(u)|q\bar q\rangle + \phi_2^{\rm OZI}(u)|s\bar s\rangle + \cdots, \quad (8)$$

$$|\eta_s(p)\rangle \propto \phi_2^{\rm OZI}(u)|q\bar q\rangle + \phi_2^s(u)|s\bar s\rangle + \cdots, \quad (9)$$

where

$$\begin{aligned} \phi_2^q &= \frac{1}{3}(\phi_2^8 + 2\phi_2^1), \\ \phi_2^s &= \frac{1}{3}(2\phi_2^8 + \phi_2^1), \\ \phi_2^{\rm OZI} &= \frac{\sqrt{2}}{3}(\phi_2^1 - \phi_2^8) \end{aligned} \quad (10)$$

are leading twist distribution amplitudes, a consistent implementation requires us to set

$$\phi_2^{\rm OZI} = \frac{\sqrt{2}}{3}(\phi_2^1 - \phi_2^8) = 0.$$

This implies that one has to ignore the different scale dependence of the singlet and octet distribution amplitudes, because otherwise their evolution would generate a nonzero $\phi_2^{\rm OZI}$. We followed [16] and set $\phi_2^1 = \phi_2^8$ and evolved their lowest moment a_2, according to the octet scaling law. We confirm that the induced difference due to different renormalization behavior is very small. We also confirm their finding that the mixing of the leading Gegenbauer moment in the conformal expansion of the twist-2 quark and gluon distribution amplitudes

$$\begin{aligned} \phi^1_{2;\eta}(u,\mu) &= 6u\bar u\Big(1 + \sum_{n=1}^\infty a_{2n}^{\eta,1}(\mu) C_{2n}^{3/2}(2u-1)\Big), \\ \psi^g_{2;\eta}(u,\mu) &= u^2\bar u^2 \sum_{n=1}^\infty B_{2n}^{\eta,g}(\mu) C_{2n-1}^{5/2}(2u-1), \end{aligned} \quad (11)$$

given by [26],

$$\mu\frac{d}{d\mu}\begin{pmatrix} a_2^{\eta,1} \\ B_2^{\eta,g} \end{pmatrix} = \begin{pmatrix} \frac{100}{9} & -\frac{10}{81} \\ -36 & 22 \end{pmatrix} \begin{pmatrix} a_2^{\eta,1} \\ B_2^{\eta,g} \end{pmatrix},$$

has only small numerical influence. This led us to neglect this effect in accordance with the remarks made above. Higher Gegenbauer moments turned out to give only negligible contributions as well, and therefore we restrict our analysis to the lowest moments. On the whole, these effects are smaller than 3%. The main difference with respect to [16], besides using the $\overline{\rm MS}$ mass for m_c, is that for the $D_s \to \eta^{(\prime)}$ decays we probe the $\bar s s$ content of the $\eta^{(\prime)}$, which leads to a different dependence on the mixing angle [see Eq. (20)], while for the $D \to \eta^{(\prime)}$ the only difference is the change of Borel parameter, continuum threshold and masses $m_c \leftrightarrow m_b$, $m_D \leftrightarrow m_B$.

III. OUTLINE OF THE LCSR METHOD

The idea behind LCSR calculations for decay matrix elements from heavy into light quark hadrons is illustrated in Fig. 1. For a detailed discussion of the original two-point sum rules and their extension, consult, e.g., [27–32].

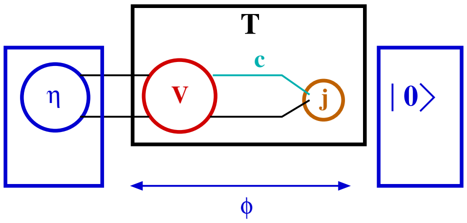

FIG. 1 (color online). Structure of the light-cone sum rule calculation: j is the interpolating current for the heavy meson. The weak matrix element is contained in V, and the charm quark propagator is treated perturbatively. Thus, the factor T can be calculated purely perturbatively, which is done at NLO accuracy. At that level, the parton lines coupling into the η can be either quark-antiquark or two gluons. The occurring matrix elements are parametrized in terms of the distribution amplitudes. A Borel transform serves to filter out the D and D_s contributions from T.

In short, one uses the twofold nature of the correlation function to equate two different representations: First, one inserts a complete set of hadronic states, separates the ground state and expresses the rest via a dispersion integral over the hadronic spectral density. Second, one uses that for large negative virtualities the correlation function is dominated by lightlike distances and makes an expansion around the light cone, leading to a convolution of perturbatively calculable hard-scattering amplitudes and universal soft-distribution amplitudes. After an analytic continuation of the light-cone expansion to physical momenta using a dispersion relation, one equates these two representations by the assumption of quark-hadron duality. Finally, it is customary to use a Borel transformation to suppress higher states in the hadronic sum and to get rid of subtraction terms that are necessary if the dispersion relation is divergent. We will illustrate these steps below. The starting point for the $D^+_{(s)} \to \eta^{(\prime)} l^+ \nu_l$ form factor,

$$\langle P(p)|\bar{q}\gamma_\mu c|D_{(s)}(p+q)\rangle = 2f^+_{D_{(s)}P}(q^2)p_\mu + (f^+_{D_{(s)}P}(q^2) + f^-_{D_{(s)}P}(q^2))q_\mu,$$

is the correlation function,

$$F^{HP}_\mu(p,q) = i\int d^4x e^{iqx}\langle P(p)|\mathrm{T}\{V^P_\mu(x), j^\dagger_H(0)\}|0\rangle = F^{HP}(q^2,(p+q)^2)p_\mu + \tilde{F}^{HP}(q^2,(p+q)^2)q_\mu, \tag{12}$$

where P is the on-shell pseudoscalar meson (in our case $P = \eta, \eta', H = B, D_{(s)}$), V^P is the local weak interaction vertex and j_H is a local interpolating current for the heavy quark system. In the present case we deal with the expressions collected in Table I. The scalar form factor,

$$f^0_{D_{(s)}P}(q^2) = f^+_{D_{(s)}P}(q^2) + \frac{q^2}{m^2_{D_{(s)}} - m^2_{\eta^{(\prime)}}} f^-_{D_{(s)}P}(q^2),$$

TABLE I. Currents entering the correlation function Eq. (12).

Decay	Interpolation current	Weak current
$D^+_s \to \eta^{(\prime)} l \nu_l$	$j_{D^+_s} = m_c \bar{s} i\gamma_5 c$	$V^{(\eta,\eta')}_\mu = \bar{s}\gamma_\mu c$
$D^+ \to \eta^{(\prime)} l^+ \nu_l$	$j_{D^+} = m_c \bar{d} i\gamma_5 c$	$V^{(\eta,\eta')}_\mu = \bar{d}\gamma_\mu c$

enters the leptonic spectrum only with factors proportional to m^2_l. Therefore, we do not consider $\tilde{F}^P$, which is needed to calculate $f^0_{D_{(s)}\eta^{(\prime)}}$.

Inserting a complete set of hadronic states between the two currents Eq. (12) and separating the ground state leads to

$$F^{D_{(s)}P}(q^2,(p+q)^2) = \frac{2m^2_{D_{(s)}} f_{D_{(s)}} f^+_{D_{(s)}P}(q^2)}{(m^2_{D_{(s)}} - (p+q)^2)} + \int_{s_0^{h_{(s)}}}^\infty ds \frac{\rho^{h_{(s)}}(q^2,s)}{s-(p+q)^2}, \tag{13}$$

where $s_0^{h_{(s)}}$ is a hadronic threshold, $\rho^{h_{(s)}}(s)$ is the hadronic spectral density and $f_{D_{(s)}}$ is the decay constant of the $D(D_s)$ meson. Since the Borel transform will take care of subtraction terms in the end, we will not write them anywhere.

The light-cone expansion for $q^2, (p+q)^2 \ll m^2_c$ can be written in the general form

$$[F^{D_{(s)}P}(q^2,(p+q)^2))]_{\mathrm{OPE}} = \sum_{t=2,3,4} F^{P,t}_0(q^2,(p+q)^2) + \frac{\alpha_s C_F}{4\pi}\sum_{t=2,3} F^{P,t}_1(q^2,(p+q)^2) + \cdots. \tag{14}$$

Here t denotes the twist which is taken into account at the current accuracy. The leading- and next-to-leading order expressions $F_{0,1}$ are given as convolutions of hard-scattering amplitudes and distribution amplitudes (see Fig. 1),

$$F^{D_{(s)}P,t}_{0,1}(q^2,(p+q)^2) = \int du T^{(t)}_{0,1}(q^2,(p+q)^2,m^2_c,u,\mu)\phi^{(t)}_{\eta^{(\prime)}}(u,\mu). \tag{15}$$

u denotes a generic expression for the momentum fractions of the partons in the meson and μ, the factorization scale. The leading-order term is given by contracting the c quarks to generate the free propagator and taking into account only the twist-2 distribution amplitude [see Eq. (A1)],

$$F^{D_{(s)}\eta^{(\prime)},2}_0(q^2,(p+q)^2) = f_\eta m^2_c \int_0^1 \frac{du\phi_{\eta^{(\prime)}}(u)}{m^2_c - q^2\bar{u} - (p+q)^2 u}. \tag{16}$$

Analytic continuation of the momentum $(p+q)^2$ flowing through the interpolating current leads to

$$[F^{D_{(s)}P}(q^2,(p+q)^2)]_{\rm OPE} = \frac{1}{\pi}\int_{m_c^2}^{\infty}\frac{ds}{s-(p+q)^2}\,{\rm Im}[F^{D_{(s)}P}(q^2,s)]_{\rm OPE}. \quad (17)$$

Now the two representations can be equated by using the semilocal quark-hadron duality assumption that from a certain continuum threshold $s_0^{D_{(s)}}$ on, the integral over the hadronic spectral density and over the partonic result should be the same,

$$\int_{s_0^{D_{(s)}}}^{\infty} ds\frac{{\rm Im}[F^{D_{(s)}P}(q^2,s)]_{\rm OPE}}{s-(p+q)^2} = \int_{s_0^{h_{(s)}}}^{\infty} ds\frac{\rho^{h_{(s)}}(q^2,s)}{s-(p+q)^2}.$$

This assumption and the final Borel transformation

$$B_{M^2}\frac{1}{s-(p+q)^2}\longrightarrow e^{-\frac{s}{M^2}}$$

lead to the sum rule,

$$f^+_{D_{(s)}P}(q^2) = \frac{1}{2m^2_{D_{(s)}}f_{D_{(s)}}}e^{\frac{m^2_{D_{(s)}}}{M^2}}\frac{1}{\pi}\int_{m_c^2}^{s_0^{D_{(s)}}} ds\,{\rm Im}[F^{D_{(s)}P}(q^2,s)]_{\rm OPE}\,e^{-\frac{s}{M^2}}, \quad (18)$$

where M^2 is the Borel parameter. It is important to note that every additional two units of twist are accompanied by another power of the denominator,

$$D = m_c^2 - q^2\bar{u} - (p+q)^2 u, \quad (19)$$

which shows that for the processes in question, the momentum transfer q^2 is severely constrained in order to have a converging light-cone expansion. Another point worth mentioning is that odd twists $(3, 5, \ldots)$ come from the mass term of the c-quark propagator and are formally subleading in $\frac{1}{m_c}$ compared to their even counterparts. However, due to chiral enhancement coming from the prefactor μ_η of the twist-3 distribution amplitudes, they numerically exceed these. This would imply that the unknown twist-5 contributions might be larger than the twist-4 ones, which we analyze, and convergence cannot be taken for granted. To really assess the situation, a dedicated study of these higher twist contributions would be needed, which is a formidable task, far exceeding the scope of this paper. To have at least a rough guess of the resulting uncertainty, we follow [33] and assume that the ratio of the unknown twist-5 term to the twist-3 term is the same as the ratio of the twist-4 term to the twist-2 term. This gives an additional uncertainty, varying from 4% for $q^2 = -2$ GeV to 2.5% for $q^2 = 0$.

The inclusion of the gluonic part of the $\eta^{(\prime)}$ in the sum rules was already discussed in [16], and we do not repeat it here. It boils down to using relation (6) to calculate the correlation functions,

$$\begin{aligned} F^{D_s\eta} &= -F^{D_s\eta_s}\sin\phi + F^{D_s\eta_q}\cos\phi,\\ F^{D_s\eta'} &= F^{D_s\eta_s}\cos\phi + F^{D_s\eta_q}\sin\phi,\\ F^{D\eta} &= F^{D\eta_q}\cos\phi - F^{D\eta_s}\sin\phi,\\ F^{D\eta'} &= F^{D\eta_q}\sin\phi + F^{D\eta_s}\cos\phi, \end{aligned} \quad (20)$$

and inserting these into Eq. (18).

The second summand in each equation of (20) gets only contributions at NLO from the gluonic part, while the first summand is a combination of quark and gluonic contributions. The quark contribution we take from [17,33] with the replacements $f_\pi \rightarrow f_{q(s)}$, $f_\pi\frac{m_\pi^2}{2m_q}\rightarrow f_q\frac{m_\pi^2}{2m_q}$, $f_\pi\frac{m_\pi^2}{2m_q}\rightarrow f_s\frac{2m_K^2-m_\pi^2}{2m_s}$, which means that we take $SU(3)$-flavor violation into account only via the decay constants. In [33,34] it was shown that for decays into kaons and pions, this is indeed a good approximation. We checked that our results do not change significantly if we include meson and quark mass corrections. But keeping all $SU(3)$-violating effects would force us not only to keep all quark and meson mass dependences in the correlation function but also to use

$$\begin{aligned} h_q &= f_q(m_\eta^2\cos^2\phi + m_{\eta'}^2\sin^2\phi)\\ &\quad - \sqrt{2}f_s(m_{\eta'}^2 - m_\eta^2)\sin\phi\cos\phi,\\ h_s &= f_s(m_{\eta'}^2\cos^2\phi + m_\eta^2\sin^2\phi)\\ &\quad - \frac{f_q}{\sqrt{2}}(m_{\eta'}^2 - m_\eta^2)\sin\phi\cos\phi, \end{aligned} \quad (21)$$

[35] instead of $f_q m_\pi^2$ and $f_s(2m_K^2 - m_\pi^2)$, respectively. These quantities are, due to cancellations, very weakly constrained, which would lead to uncertainties at the level of 200% if one assumes uncorrelated errors in the twist-3 part (see e.g. [16]). In the ratios, these uncertainties cancel for the largest part, but for the form factors and decay rates this seems to be a huge overestimation.

IV. NUMERICS

A. Choice of input

We follow [17,33] in using the $\overline{\rm MS}$ scheme and one universal scale throughout our calculation. The scale is set to be $\mu \approx \sqrt{m^2_{D_{(s)}} - m_c^2} = 1.4(1.5)$ GeV, and all quantities are evolved to this scale using one-loop running for the quark masses and distribution amplitude parameters and two-loop running for α_s.

The values for the Gegenbauer moments need some discussion. In a recent perturbative analysis [36,37] of the $\eta^{(\prime)}$ transition form factors, P. Kroll and K. Passek-Kumerički got the values (for $\mu = 1$ GeV)

$$a_2^8 = -0.05 \pm 0.02,\quad a_2^1 = -0.12 \pm 0.01,\quad a_2^g = 19 \pm 5, \tag{22}$$

similar to their older results in [26] (see also [38,39]). Unfortunately, these numbers are at first sight in contradiction with the sum rule value $a_2^8 \approx 0.2$. The authors of [36,37] state that their values are effective ones, contaminated by higher Gegenbauer moments, while the effect of power corrections is neglected. Both effects were shown to be large in the accessible Q^2 region for the pion transition form factor in [8,9], where the value $a_2^\pi = 0.13$–0.16 was obtained, in stark contrast to the value $a_2^\pi = -0.02 \pm 0.02$, obtained in [36,37]. Including generic power corrections leads to $a_2^8 = 0.06 \pm 0.05$, which also suggests that the values given in (22) should be taken with a grain of salt. As we do not see how to correct for these effects, we decided to ignore Eq. (22) and to use the average over sum rule fits to experimental data and direct lattice and sum rule calculations instead, leading to

$$a_2^8(1\ \text{GeV}) = 0.25 \pm 0.15. \tag{23}$$

We implement the quark-flavor scheme by setting $a_2^1(1\ \text{GeV}) = a_2^8(1\ \text{GeV})$ and evolving both via the renormalization of the octet moment. This, in turn, implies $a_2^q = a_2^s$, see (10). There is no hint of large $SU(3)$-flavor violation in the even Gegenbauer moments, where one finds, e.g., $a_2^\pi \approx a_2^K$, which should be an acceptable approximation. Since the impact of the mixing between a_2^1 and B_2^g is rather small, we treat the latter as a free parameter and vary it over the same very conservative range $B_2^g = 0 \pm 20$ as in [16]. We take the quark and meson masses from the Particle Data Group [40]. Their current values are

$$\bar{m}_c(\bar{m}_c) = (1.275 \pm 0.025)\ \text{GeV}, \tag{24}$$

$$m_u(\mu = 2\ \text{GeV}) = \left(2.3^{+0.7}_{-0.5}\right)\ \text{MeV}, \tag{25}$$

$$m_d(\mu = 2\ \text{GeV}) = \left(4.8^{+0.7}_{-0.3}\right)\ \text{MeV}, \tag{26}$$

$$m_s(\mu = 2\ \text{GeV}) = (95 \pm 5)\ \text{MeV}, \tag{27}$$

and

$$m_{D^+} = 1869.6\ \text{MeV},\qquad m_{D_s^+} = 1968.5\ \text{MeV}, \tag{28}$$

$$m_{\pi^0} = 134.98\ \text{MeV},\qquad m_{K^0} = 497.61\ \text{MeV}. \tag{29}$$

The latter ones are related via flavor symmetry to the masses of the $|\eta_{q(s)}\rangle$ states as given in Eq. (7). For the pion decay constant, we use $f_\pi = 130.4$ MeV and for the $D_{(s)}$ decay constant we take the experimental values from [40],

$$f_D = (206.7 \pm 8.5 \pm 2.5)\ \text{MeV},\quad f_{D_s} = (260 \pm 5.4)\ \text{MeV}, \tag{30}$$

while for the B meson, in view of the existing large discrepancies in determinations of $|V_{ub}|$, which is in turn needed for the extraction of f_B, we use a two-point sum rule at order α_s [41]. For the continuum threshold and the Borel parameter, we choose

$$s_0^D = (7 \pm 0.6)\ \text{GeV}^2,\qquad s_0^B = (35.75 \pm 0.25)\ \text{GeV}^2,$$
$$M^2_{D_{(s)}} = (4.4 \pm 1.1)\ \text{GeV}^2,\qquad M_B^2 = (18 \pm 3)\ \text{GeV}^2, \tag{31}$$

and for the two-point sum rule,

$$\bar{s}_0^B = (35.75 \pm 0.25)\ \text{GeV}^2,\qquad \bar{M}_B^2 = (5 \pm 1)\ \text{GeV}^2, \tag{32}$$

which fulfill the usual criteria for these parameters and are close to the ones used in [17,33]. The quark, gluon and mixed condensates are given by [42,43]

$$\langle \bar{q}q\rangle(2\ \text{GeV}) = \left(-0.246^{-0.019}_{+0.028}\right)^3\ \text{GeV}^3,$$
$$\left\langle \frac{\alpha_S}{\pi} GG\right\rangle(2\ \text{GeV}) = \left(0.012^{-0.012}_{+0.006}\right)\ \text{GeV}^4,$$
$$m_0^2 = \frac{g\langle \bar{q}\sigma_{\mu\nu}G^{\mu\nu}q\rangle}{\langle \bar{q}q\rangle}, = (0.8 \pm 0.2)\ \text{GeV}^2. \tag{33}$$

Finally we take for the twist-3 and -4 parameters at $\mu = 1$ GeV,

$$f_3^\pi = (0.0045 \pm 0.0015)\ \text{GeV}^2,\quad \omega_3^\pi = (-1.5 \pm 0.7)\ \text{GeV}^2,\quad \epsilon_\pi = \left(\frac{21}{8}\right)(0.2 \pm 0.1)\ \text{GeV}^2,\quad \delta_\pi^2 = (0.18 \pm 0.06)\ \text{GeV}^2. \tag{34}$$

B. Forms factors and their shape

As can be seen from Eq. (19), our sum rules for D and D_s decays are only applicable for $q^2 \ll m_c^2$. To be able to make a prediction for the shape of the form factor and for

TABLE II. Shape parameters for $f_+^{D^+_{(s)}\eta^{(\prime)}}(q^2)$ as input for the BZ model, Eq. (35).

Decay	r	α	$\lvert f_+(0)\rvert$
$D_s^+ \to \eta l^+ \nu_l$	$0.284^{+0.003}_{-0.002}$	$0.252^{+0.107}_{-0.082}$	$0.432^{+0.033}_{-0.033}$
$D_s^+ \to \eta' l^+ \nu_l$	$0.284^{+0.137}_{-0.095}$	$0.252^{+0.382}_{-0.395}$	$0.520^{+0.080}_{-0.080}$
$D^+ \to \eta l^+ \nu_l$	$0.174^{+0.001}_{-0.001}$	$-0.043^{+0.068}_{-0.052}$	$0.552^{+0.051}_{-0.051}$
$D^+ \to \eta' l^+ \nu_l$	$0.174^{+0.243}_{-0.142}$	$-0.043^{+0.526}_{-0.596}$	$0.458^{+0.105}_{-0.105}$

TABLE III. Form factors $f_+^{D_s^+\eta^{(\prime)}}(0)$, $f_+^{D^+\eta^{(\prime)}}(0)$ and $f_+^{B^+\eta^{(\prime)}}(0)$ calculated from LCSRs, Eq. (20).

Formfactor central value	M^2	μ	(s_0^D/s_0^B)	a_2	B_2^g	(fq, fs, ϕ)	twist-3	twist-4	(condensates, m_c/m_b)
$\lvert f_+^{D_s^+\eta}(0)\rvert = 0.432$	±0.003	±0.026	±0.010	±0.013	±0.001	±0.025	±0.014	±0.002	±0.005
$\lvert f_+^{D_s^+\eta'}(0)\rvert = 0.520$	±0.003	±0.032	±0.012	±0.015	±0.070	±0.028	±0.016	±0.002	±0.006
$\lvert f_+^{D^+\eta}(0)\rvert = 0.552$	±0.008	±0.034	±0.013	±0.016	±0.002	±0.015	±0.036	±0.002	±0.007
$\lvert f_+^{D^+\eta'}(0)\rvert = 0.458$	±0.007	±0.028	±0.011	±0.013	±0.096	±0.025	±0.030	±0.002	±0.006
$\lvert f_+^{B^+\eta}(0)\rvert = 0.238$	±0.002	±0.013	±0.002	±0.004	±0.001	±0.006	±0.011	±0.0002	±0.007
$\lvert f_+^{B^+\eta'}(0)\rvert = 0.198$	±0.001	±0.011	±0.002	±0.003	±0.061	±0.007	±0.009	±0.0001	±0.006

the value of the branching fractions, we follow [33]. We calculate the form factors at $q^2 < 0$, where the twist expansion of the sum rules works perfectly well, and then basically use a fit to extrapolate our results to $q^2 > 0$. We use the simple Ball-Zwicky parametrization [44], keeping in mind that all fit formulas work nearly equally well [33,45,46] and that unitarity constraints for more elaborate formulas are up to now not restrictive,

$$f_+^{\mathrm{BZ}}(q^2) = f_+(0)\left(\frac{1}{1-q^2/m^2_{D^*_{(s)}}} + \frac{rq^2/m^2_{D^*_{(s)}}}{(1-q^2/m^2_{D^*_{(s)}})(1-\alpha q^2/m^2_{D_{(s)}})}\right). \tag{35}$$

The idea of this fit formula is basically to take the dispersive representation of the form factor, take out the known lowest-lying resonance and approximate the dispersion integral over many particle states, starting from $(m_{D_{(s)}} + m_\pi)^2$ by an effective pole. r, α parametrize the residuum and position of this pole, while $f_+(0)$ gives the overall normalization. Despite the resonances $D^*_{(s)}$ being very close to the two-particle threshold, the fits are numerically perfectly stable.

The results for $f_+^{D_s\eta}(q^2)$ and $f_+^{D_s\eta'}(q^2)$ are shown in Figs. 3 and 4. To get the error bands, we made a statistical analysis of all input parameters at each $q^2 \leq 0$, assuming Gaussian uncertainties, and then extrapolated them in the same way as the central values. As can be seen, the uncertainty coming from the unknown gluon distribution amplitude is nearly negligible for the $f_+^{D_s\eta}(q^2)$ form factor, which holds for $f_+^{D\eta}(q^2)$ and $f_+^{B\eta}(q^2)$ as well, supporting the notion of a nearly total octet nature of the η. On the other hand, there is a considerable impact on the $D_{(s)}(B) \to \eta'$-form factors from the gluonic part. The fit parameters can be found in Table II. Figures 3 and 4 also contain results from a first lattice simulation for this quantity [47], which were corrected in accordance with a private communication from the author. (The fact that one has to calculate disconnected contributions makes such lattice simulations very demanding [48].)

Our results for $q^2 = 0$ are shown in Table III. For illustration, we show the dependence of the $D_s \to \eta^{(\prime)}$ form factors on the Borel parameter in Fig. 2.

As can be seen, the sum rules are stable for a very large range of parameter values.

Especially interesting are the ratios of the η' to η form factors, since for such ratios most of the uncertainties cancel. For the gluonic part, we made the assumption

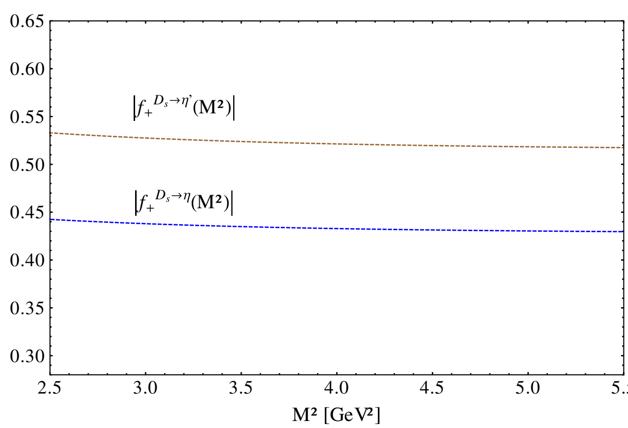

FIG. 2 (color online). $\lvert f_+^{D_s\eta}(q^2=0)\rvert$, $\lvert f_+^{D_s\eta'}(q^2=0)\rvert$ plotted as a function of the Borel parameter M^2. The blue dashed line corresponds to $\lvert f_+^{D_s\eta}(q^2=0)\rvert$ and the brown dashed line to $\lvert f_+^{D_s\eta'}(q^2=0)\rvert$.

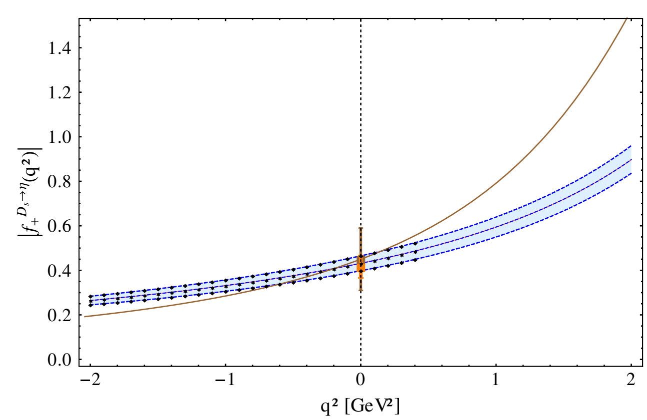

FIG. 3 (color online). $f_+^{D_s\eta'}(q^2)$ plotted as a function of q^2. The black dots are the calculated sum rule values. The blue straight line is the fit to the central values. Blue dashed band: Full uncertainties of our result. Red lines: Uncertainty coming from the gluonic contribution, which due to a very small impact nearly conceals the blue line. Brown line: Results of [15]. Orange point: corrected lattice result from [47] in accordance with a private communication from the author.

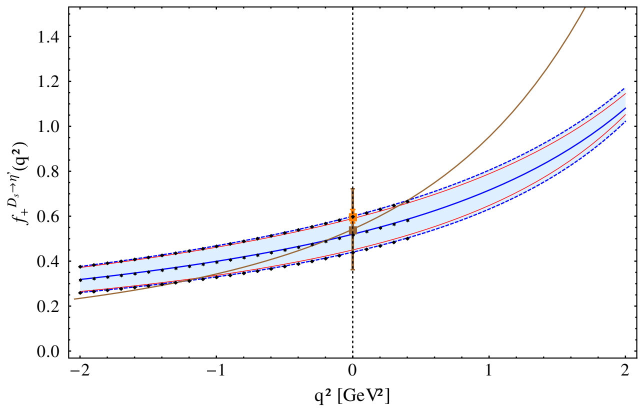

FIG. 4 (color online). $f_+^{D_s\eta'}(q^2)$ plotted as a function of q^2. Same convention as in Fig. 3.

$B_2^{g,\eta} = B_2^{g,\eta'}$, since no large $SU(3)$-breaking is expected in this Gegenbauer moment. Note, however, that the contribution to the form factors is vastly different, due to the different admixture of the singlet part, which is given by the decay constants

$$f_\eta^1 = \sqrt{\frac{2}{3}}\cos\phi f_q - \sqrt{\frac{1}{3}}\sin\phi f_s,$$
$$f_{\eta'}^1 = \sqrt{\frac{2}{3}}\sin\phi f_q + \sqrt{\frac{1}{3}}\cos\phi f_s, \qquad (36)$$

[see Eqs. (A3) and (A4)].

What can be seen from Table IV is that almost the whole uncertainty comes from B_2^g, which would give the possibility to constrain this quantity if more precise experimental data were available. The result for the D_s form factors in the considered q^2 region is shown in Fig. 5. As can be seen, the uncertainties are completely governed by the gluonic contribution. Table IV shows our results at $q^2 = 0$.

C. Branching fractions and experimental results

With an extrapolation of the form factors to the whole kinematic region, we are able to calculate the branching fractions and compare them to experimental results. For

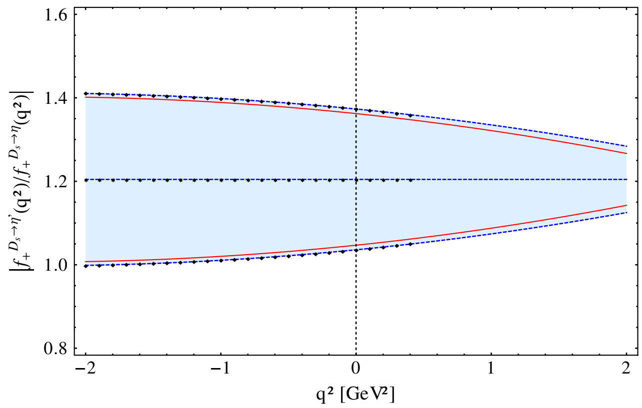

FIG. 5 (color online). $|f_+^{D_s\eta'}(q^2)/f_+^{D_s\eta}(q^2)|$ plotted as a function of q^2. Again the black dots are the calculated sum rule values. The blue dashed lines are fits to the sum rule results, where the upper and lower line correspond to the uncertainties of our calculation. They are completely dominated by the gluonic contribution (red lines).

massless leptons, the scalar form factor $f^0_{D_{(s)}\eta^{(\prime)}}$ does not contribute, so the decay rate is given by

$$\Gamma(D_s^+ \to \eta^{(\prime)} l^+ \nu_l) = \frac{G_F^2|V_{cs}|^2}{24\pi^3}\int_0^{q^2_{\max}} dq^2 \lambda^{\frac{3}{2}}(q^2)|f_+^{D_s^+\eta^{(\prime)}}(q^2)|^2, \qquad (37)$$

where the kinematical function $\lambda(q^2)$ is defined via

$$\lambda(x) = \frac{1}{4m_H^2}[(m_H^2 + m_M^2 - x)^2 - 4m_H^2 m_M^2], \qquad (38)$$

with ($P = \eta$, η'; $H = B^+$, D^+, D_s). After multiplication with the mean lifetime of the considered meson, we get the relevant branching fractions. To extract the uncertainties, we again assume Gaussian errors and extrapolate the error of $|f_+(q^2)|^2$ with different fit functions from $q^2 < 0$ to the physical region. The deviations found due to the change of the fit function are incorporated in the error budget. Our results and the experimental values are shown in Table V.

Again the ratios turn out to be especially interesting since most of the uncertainties in the theoretical calculation cancel, and they are dominated by the contribution of the

TABLE IV. Ratios $\left|\frac{f_+^{D^+_{(s)}\eta'}(0)}{f_+^{D^+_{(s)}\eta}(0)}\right|$ and $\left|\frac{f_+^{B^+\eta'}(0)}{f_+^{B^+\eta}(0)}\right|$ calculated from LCSRs, Eq. (20).

Formfactor central value	M^2	μ	(s_0^D/s_0^B)	a_2	B_2^g	(fq, fs, ϕ)	twist-3	twist-4	(condensates, m_c/m_b)
$\left\|\frac{f_+^{D_s^+\eta'}(0)}{f_+^{D_s^+\eta}(0)}\right\| = 1.20$	$\pm1\times10^{-13}$	$\pm1\times10^{-12}$	$\pm6\times10^{-13}$	$\pm7\times10^{-14}$	±0.16	±0.06	$\pm3\times10^{-12}$	$\pm3\times10^{-14}$	$\pm2\times10^{-14}$
$\left\|\frac{f_+^{D^+\eta'}(0)}{f_+^{D^+\eta}(0)}\right\| = 0.83$	$\pm5\times10^{-13}$	$\pm9\times10^{-13}$	$\pm2\times10^{-13}$	$\pm5\times10^{-15}$	±0.18	±0.04	$\pm8\times10^{-13}$	$\pm3\times10^{-14}$	$\pm5\times10^{-14}$
$\left\|\frac{f_+^{B^+\eta'}(0)}{f_+^{B^+\eta}(0)}\right\| = 0.83$	$\pm8\times10^{-13}$	$\pm6\times10^{-13}$	$\pm1\times10^{-13}$	$\pm1\times10^{-13}$	±0.26	±0.04	$\pm8\times10^{-13}$	$\pm2\times10^{-14}$	$\pm2\times10^{-13}$

TABLE V. Branching fractions for the different decays.

Decay	LCSRs (this work)	Experiment
$D_s \to \eta' e \nu_e$	$(0.75 \pm 0.23)\%$	$(0.91 \pm 0.33)\%$ [50]
$D_s \to \eta e \nu_e$	$(2.00 \pm 0.32)\%$	$(2.48 \pm 0.29)\%$ [50]
$D \to \eta' e \nu_e$	$(3.86 \pm 1.77) \times 10^{-4}$	$(2.16 \pm 0.53 \pm 0.07) \times 10^{-4}$ [51]
$D \to \eta e \nu_e$	$(24.5 \pm 5.26) \times 10^{-4}$	$(11.4 \pm 0.9 \pm 0.4) \times 10^{-4}$ [51]
$B \to \eta' e \nu_e$	$(0.36 \pm 0.22) \times 10^{-4}$	$(2.66 \pm 0.80 \pm 0.56) \times 10^{-4}$ [52]
		$(0.24 \pm 0.08 \pm 0.03) \times 10^{-4}$ [53]
$B \to \eta e \nu_e$	$(0.73 \pm 0.20) \times 10^{-4}$	$(0.44 \pm 0.23 \pm 0.11) \times 10^{-4}$ [52]
		$(0.36 \pm 0.05 \pm 0.04) \times 10^{-4}$ [53]

gluonic Gegenbauer moment B_2^g. Here we made the same assumption, $B_2^{g,\eta} = B_2^{g,\eta'}$, as for the ratios of the form factors. Comparing them to the experimental values,

$$\frac{\Gamma(D_s^+ \to \eta' e^+ \nu_e)}{\Gamma(D_s^+ \to \eta e^+ \nu_e)} = 0.37 \pm 0.09(B_2^g) \pm 0.04(\text{rest}),$$
$$\text{Exp: } 0.36 \pm 0.14 \quad [49],$$
$$\frac{\Gamma(D^+ \to \eta' e^+ \nu_e)}{\Gamma(D^+ \to \eta e^+ \nu_e)} = 0.16 \pm 0.06(B_2^g) \pm 0.02(\text{rest}),$$
$$\text{Exp: } 0.19 \pm 0.09 \quad [50],$$
$$\frac{\Gamma(B \to \eta' e^+ \nu_e)}{\Gamma(B \to \eta e^+ \nu_e)} = 0.50 \pm 0.29(B_2^g) \pm 0.05(\text{rest}),$$
$$\text{Exp: } 0.67 \pm 0.24 \pm 0.1 \quad [52], \qquad (39)$$

one can see good overall agreement, but clearly the experimental precision is up to now not sufficient to draw any conclusion on B_2^g.

V. SUMMARY AND DISCUSSION

We have calculated the form factors and branching fractions of the decays $D_{(s)} \to \eta^{(\prime)} l \nu$ and $B \to \eta^{(\prime)} l \nu$ in the framework of light-cone sum rules for massless leptons. The form factors were shown to agree with available lattice results and the branching ratios, Eq. (39), with experiment. So the overall picture is nicely consistent. Our main result is, however, the error budget given in Eq. (39), clearly showing that B_2^g dominates the uncertainties in all cases. Therefore, even a moderate increase in experimental accuracy will allow us to determine the gluonic contribution to η and η' from all three ratios, providing a sensitive consistency check. FAIR and Super-KEKB should provide precision measurements of these ratios and thus allow us to settle this long-standing issue.

ACKNOWLEDGMENTS

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APPENDIX: DEFINITIONS OF DISTRIBUTION AMPLITUDES

Here we give the definitions of the distribution amplitudes used in the paper. We follow the notation of [49] (see also [33] for a minor correction) for the quark-antiquark,

$$\begin{aligned}\langle \eta(p)|\bar{q}^i_\omega(x_1) q^j_\xi(x_2)|0\rangle_{x^2\to 0} = \frac{i\delta^{ij}}{12} f_\eta \int_0^1 du e^{iup\cdot x_1 + i\bar{u}p\cdot x_2}\Big([\not{p}\gamma_5]_{\xi\omega}\varphi_\eta(u) - [\gamma_5]_{\xi\omega}\mu_\eta \phi^p_{3\eta}(u) \\ + \frac{1}{6}[\sigma_{\beta\tau}\gamma_5]_{\xi\omega} p_\beta (x_1 - x_2)_\tau \mu_\eta \phi^\sigma_{3\eta}(u) + \frac{1}{16}[\not{p}\gamma_5]_{\xi\omega}(x_1 - x_2)^2 \phi_{4\eta}(u) \\ - \frac{i}{2}[(\not{x}_1 - \not{x}_2)\gamma_5]_{\xi\omega} \int_0^u \psi_{4\eta}(v) dv\Big),\end{aligned} \qquad (A1)$$

and quark-antiquark-gluon distributions,

$$\begin{aligned}&\langle \eta(p)|\bar{q}^i_\omega(x_1) g_s G^a_{\mu\nu}(x_3) q^j_\xi(x_2)|0\rangle_{x^2\to 0} \\ &= \frac{\lambda^a_{ji}}{32}\int \mathcal{D}\alpha_i e^{ip(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)}\Big[i f_{3\eta}(\sigma_{\lambda\rho}\gamma_5)_{\xi\omega}(p_\mu p_\lambda g_{\nu\rho} - p_\nu p_\lambda g_{\mu\rho})\Phi_{3\eta}(\alpha_i) \\ &- f_\eta(\gamma_\lambda\gamma_5)_{\xi\omega}\Big\{(p_\nu g_{\mu\lambda} - p_\mu g_{\nu\lambda})\Psi_{4\eta}(\alpha_i) + \frac{p_\lambda(p_\mu x_\nu - p_\nu x_\mu)}{(p\cdot x)}(\Phi_{4\eta}(\alpha_i) + \Psi_{4\eta}(\alpha_i))\Big\} \\ &- \frac{i f_\eta}{2}\epsilon_{\mu\nu\delta\rho}(\gamma_\lambda)_{\xi\omega}\Big\{(p^\rho g^{\delta\lambda} - p^\delta g^{\rho\lambda})\tilde{\Psi}_{4\eta}(\alpha_i) + \frac{p_\lambda(p^\delta x^\rho - p^\rho x^\delta)}{(p\cdot x)}(\tilde{\Phi}_{4\eta}(\alpha_i) + \tilde{\Psi}_{4\eta}(\alpha_i))\Big\}\Big].\end{aligned} \qquad (A2)$$

For the gluon-gluon distribution amplitude, we take over the notation of [16],

$$\langle\eta^{(\prime)}(p)|G_{\mu x}(x)[x,-x]\tilde{G}^{\mu x}(-x)|0\rangle = f^1_{\eta^{(\prime)}}\frac{C_F}{2\sqrt{3}}(px)^2\int_0^1 due^{-i(2u-1)px}\psi^g_{2,\eta^{(\prime)}}(u), \tag{A3}$$

with

$$f^1_\eta = \sqrt{\frac{2}{3}}\cos\phi f_q - \sqrt{\frac{1}{3}}\sin\phi f_s, \qquad f^1_{\eta'} = \sqrt{\frac{2}{3}}\sin\phi f_q + \sqrt{\frac{1}{3}}\cos\phi f_s, \tag{A4}$$

which differs by a normalization factor of $\sigma = \sqrt{\frac{3}{C_F}}$ from the one used in [26]. The explicit conformal expansion of the different distribution amplitudes can be found in [16,26,33,49]. For conciseness, we do not write them here.

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