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Froissart-Martin bound for $\pi\pi$ scattering in QCD

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The Froissart-Martin bound for total $\pi\pi$ scattering cross sections is reconsidered in the light of QCD properties such as spontaneous chiral symmetry breaking and the counting rules for a large number of colors $\mathrm{N_c}$.

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I. INTRODUCTION

The early works by Froissart [1], Martin [2] and colleagues [3] have shown in a series of seminal papers[1] that under very general assumptions, total cross sections for $\pi\pi$, $K\bar{K}$, πK, πN and $\pi\Lambda$ scattering cannot grow faster than

$$\sigma^{\mathrm{tot}}(s) \underset{s\to\infty}{\sim} \frac{4\pi}{t_0} \log^2 \frac{s}{s_0}, \tag{1.1}$$

where s is the total center-of-mass–energy squared, t_0 denotes the lowest mass squared singularity in the t channel, which for the processes mentioned above occurs at $4m_\pi^2$, and the normalization s_0 in the $\log^2 s$ is arbitrary indicating where the asymptotic behavior sets in. Although several hadronic models have been shown to saturate the Froissart-Martin (FM) bound,[2] it is quite frustrating that the advent of QCD as the theory of the strong interactions has not added anything new, at least so far, on the FM bound. Some obvious questions which one would like to answer are the following:

(1) What happens in QCD in the chiral limit where pions, the Nambu-Goldstone states of the chiral $SU(2)$ flavor symmetry of QCD, become massless? Does the bound become irrelevant, as the presence of the pion mass in the denominator in Eq. (1.1) seems to indicate?
(2) What becomes of the FM bound in the large-N_c limit of QCD? The large-N_c counting rules fix $\sigma^{\mathrm{tot}}(s)$ in Eq. (1.1) to be of $\mathcal{O}(1/\mathrm{N_c})$, while the FM bound appears to be of $\mathcal{O}(1)$.
(3) Independently of the previous questions concerning the chiral limit and the large-N_c limit, one would also like to know if the $\log^2 s$ behavior of the FM bound is saturated in QCD.

The purpose of this paper is to set the path to an investigation of these questions. Here we shall limit ourselves to the case of total cross sections for $\pi\pi$ scattering. In the next section we summarize the well-known properties of the elastic $\pi\pi$ scattering amplitudes which we shall need for our discussion. The framework of our analyses uses a Mellin-Barnes representation for the $\pi\pi$ amplitudes which we present in Sec. III. This will allow us to fix the discussion concerning the first question above. Section IV is dedicated to a discussion of the FM bound within the framework of the QCD large-N_c limit. Our conclusions are given in Sec. V.

[1] See e.g. Ref. [4] where earlier references can be found.
[2] See e.g. Refs. [5,6] and references therein.

II. ELASTIC PION-PION SCATTERING

Elastic $\pi\pi$ scattering in the isospin symmetry limit is described by a single invariant Lorentz amplitude $A(s,t,u)$,[3]

$$\begin{aligned}\langle \pi^d(p_4)\pi^c(p_3)\,\mathrm{out}|\pi^a(p_1)\pi^b(p_2)\,\mathrm{in}\rangle \\ = \mathbf{1} + i(2\pi)^4\delta^4(p_3+p_4-p_1-p_2)\{\delta^{ab}\delta^{cd}A(s,t,u) \\ + \delta^{ac}\delta^{bd}A(t,u,s) + \delta^{ad}\delta^{bc}A(u,s,t)\},\end{aligned} \tag{2.1}$$

where a, b, c, d denote the 1, 2, 3 components of the adjoint representation of the pion fields in $SU(2)$ and s, t and u the usual Mandelstam variables constrained by

$$s + t + u = 4m_\pi^2. \tag{2.2}$$

Because of the optical theorem which relates the absorptive part of an elastic amplitude to a total cross section we shall only consider elastic scattering amplitudes with the same in and out quantum numbers, i.e.,

$$\begin{aligned}\mathcal{A}_{\pi^\pm\pi^0\to\pi^\pm\pi^0}(s,t) &= A(t,u,s), \\ \mathcal{A}_{\pi^0\pi^0\to\pi^0\pi^0}(s,t) &= A(s,t,u)+A(t,u,s)+A(u,s,t), \\ \mathcal{A}_{\pi^+\pi^-\to\pi^+\pi^-}(s,t) &= A(s,t,u)+A(t,u,s), \\ \mathcal{A}_{\pi^\pm\pi^\pm\to\pi^\pm\pi^\pm}(s,t) &= A(t,u,s)+A(u,s,t).\end{aligned} \tag{2.3}$$

It is convenient to work with the three s-channel isospin components $\mathbf{T} = (T^0, T^1, T^2)$ of the amplitudes in Eq. (2.1) given by

[3] For a modern review see Ref. [7].

$$\begin{aligned} T^0(s,t) &= 3A(s,t,u) + A(t,u,s) + A(u,s,t),\\ T^1(s,t) &= A(t,u,s) - A(u,s,t),\\ T^2(s,t) &= A(t,u,s) + A(u,s,t). \end{aligned} \tag{2.4}$$

These amplitudes obey fixed-t dispersion relations, valid in the interval $-28m_\pi^2 < t < 4m_\pi^2$. They are the so-called Roy equations [8] which we shall consider at $t = 0$ and, because of our first question in the Introduction concerning the chiral limit of the FM bound, at $m_\pi \to 0$. The Roy equations simplify then as follows:

$$\begin{aligned} \mathrm{Re}\begin{pmatrix} T^0(s,0)\\ T^1(s,0)\\ T^2(s,0)\end{pmatrix} &= \frac{s}{f_\pi^2}\begin{pmatrix} 2\\ 1\\ -1\end{pmatrix} + s^2\int_0^\infty ds'\frac{1}{s'^2}\left[\frac{1}{s'-s}\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right.\\ &\quad\left. + \frac{1}{s'+s}\begin{pmatrix} 1/3 & -1 & 5/3\\ -1/3 & 1/2 & 5/6\\ 1/3 & 1/2 & 1/6\end{pmatrix}\right]\\ &\quad\times\frac{1}{\pi}\mathrm{Im}\begin{pmatrix} T^0(s',0)\\ T^1(s',0)\\ T^2(s',0)\end{pmatrix}. \end{aligned} \tag{2.5}$$

The term in the rhs in the first line of this equation reflects the two subtractions which have been made, as required by the Froissart bound.[4] In QCD, the explicit values of these subtractions are fixed by lowest order χPT [9]. We recall that in chiral $SU(2)$ the amplitude $A(s,t,u)$ in the limit we are considering is given by the χPT expansion (see Ref. [10] and earlier references therein):

$$\begin{aligned} A(s,t,u) \underset{s,t,u\to 0}{\sim} &\frac{s}{f_\pi^2} + \frac{1}{f_\pi^4}\left[2s^2 l_1^{\mathrm{r}} + [s^2 + (t-u)^2]\frac{1}{2}l_2^{\mathrm{r}}\right]\\ &+ \frac{1}{96\pi^2 f_\pi^4}\left[3s^2\left(\log\frac{\mu^2}{-s} + \frac{5}{6}\right)\right.\\ &+ t(t-u)\left(\log\frac{\mu^2}{-t} + \frac{7}{6}\right)\\ &\left.+ u(u-t)\left(\log\frac{\mu^2}{-u} + \frac{7}{6}\right)\right] + \mathcal{O}(p^6), \end{aligned} \tag{2.6}$$

where $l_{1,2}^{\mathrm{r}}$ are renormalized coupling constants of the $\mathcal{O}(p^4)$ effective chiral Lagrangian at the scale μ. The terms in the first line of Eq. (2.6) are leading in the QCD large-N_c limit but so far, in this section, we are not restricting ourselves to this limit. The terms in the second line are induced by the chiral loops generated by the lowest order Lagrangian renormalized at the scale μ. The overall contribution of $\mathcal{O}(p^4)$ is μ-scale independent and well defined in the chiral limit. The relation between the l_i^{r} constants and the more conventional L_i^{r} constants of the chiral $SU(3)$ Lagrangian [11] is as follows:

$$l_1^{\mathrm{r}}(\mu) = 4L_1^{\mathrm{r}}(\mu) + 2L_3 - \frac{1}{96\pi^2}\frac{1}{8}\left(\log\frac{M_K^2}{\mu^2} + 1\right), \tag{2.7}$$

$$l_2^{\mathrm{r}}(\mu) = 4L_2^{\mathrm{r}}(\mu) - \frac{1}{96\pi^2}\frac{1}{4}\left(\log\frac{M_K^2}{\mu^2} + 1\right), \tag{2.8}$$

where here, kaon particles have been treated as massive and integrated out, hence the dependence on their mass M_K.

The linear combinations of the isospin amplitudes $T^I(s,0)$ which diagonalize the crossing matrix in the second line of Eq. (2.5) are

$$\begin{aligned} F_1(s,0) &= -\frac{1}{6}T^0(s,0) - \frac{1}{4}T^1(s,0) + \frac{5}{12}T^2(s,0),\\ F_2(s,0) &= +\frac{1}{6}T^0(s,0) + \frac{1}{4}T^1(s,0) + \frac{7}{12}T^2(s,0),\\ F_3(s,0) &= -\frac{1}{6}T^0(s,0) + \frac{3}{4}T^1(s,0) + \frac{5}{12}T^2(s,0), \end{aligned} \tag{2.9}$$

and the physical elastic forward scattering amplitudes we are concerned with are then given by

$$\begin{aligned} \mathcal{A}_{\pi^\pm\pi^0\to\pi^\pm\pi^0}(s,0) &= \frac{1}{2}[F_2(s,0) + F_3(s,0)]\\ &= \frac{1}{2}[T^1(s,0) + T^2(s,0)],\\ \mathcal{A}_{\pi^0\pi^0\to\pi^0\pi^0}(s,0) &= \frac{1}{2}[3F_2(s,0) - F_3(s,0)]\\ &= \frac{1}{3}[T^0(s,0) + 2T^2(s,0)],\\ \mathcal{A}_{\pi^+\pi^-\to\pi^+\pi^-}(s,0) &= -F_1(s,0) + F_2(s,0)\\ &= \frac{1}{3}T^0(s,0) + \frac{1}{2}T^1(s,0) + \frac{1}{6}T^2(s,0),\\ \mathcal{A}_{\pi^\pm\pi^\pm\to\pi^\pm\pi^\pm}(s,0) &= F_1(s,0) + F_2(s,0) = T^2(s,0). \end{aligned} \tag{2.10}$$

The Roy equations for the $F_i(s,0)$ amplitudes are then

$$\begin{aligned} \mathrm{Re}\begin{pmatrix} F_1(s,0)\\ F_2(s,0)\\ F_3(s,0)\end{pmatrix} &= \frac{s}{f_\pi^2}\begin{pmatrix} -1\\ 0\\ 0\end{pmatrix} + s^2\int_0^\infty ds'\frac{1}{s'^2}\left[\frac{1}{s'-s}\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right.\\ &\quad\left.+\frac{1}{s'+s}\begin{pmatrix} -1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right]\frac{1}{\pi}\mathrm{Im}\begin{pmatrix} F_1(s',0)\\ F_2(s',0)\\ F_3(s',0)\end{pmatrix}. \end{aligned} \tag{2.11}$$

From these equations there follows that the amplitudes F_2 and F_3 obey the same dispersion relation

[4] Notice, however, that the presence of the two powers of $\log s$ in the asymptotic behavior of the absorptive amplitudes does not restrict any further the two subtractions which are already required for a cross section going as a constant at $s \to \infty$.

$$\mathrm{Re}F_{2,3}(s,0)=s^2\int_0^\infty\frac{ds'^2}{s'^2}\frac{1}{s'^2-s^2}\frac{1}{\pi}\mathrm{Im}F_{2,3}(s',0),\quad(2.12)$$

and are even under $s\leftrightarrow -s$, while the amplitude $F_1(s,t)$ obeys the dispersion relation

$$\mathrm{Re}\,F_1(s,0)=-\frac{s}{f_\pi^2}+2s^3\int_0^\infty\frac{ds'}{s'^2}\frac{1}{s'^2-s^2}\frac{1}{\pi}\mathrm{Im}F_1(s',0),\quad(2.13)$$

and is odd under $s\leftrightarrow -s$. Indeed, one can check that there is no contribution of $\mathcal{O}(s^2)$ to the $F_1(s,0)$ amplitude in χPT, while the contributions of that order from χPT to the $F_2(s,0)$ and $F_3(s,0)$ amplitudes are

$$\mathrm{Re}F_2(s,0)\underset{s\to0}{=}\frac{s^2}{f_\pi^4}\left[2l_1^r+3l_2^r+\frac{1}{12\pi^2}\left(\log\frac{\mu^2}{s}+\frac{25}{24}\right)\right]+\mathcal{O}(s^4),\quad(2.14)$$

$$\mathrm{Re}\,F_3(s,0)\underset{s\to0}{=}\frac{s^2}{f_\pi^4}\left[-2l_1^r+l_2^r+\frac{1}{96\pi^2}\right]+\mathcal{O}(s^4).\quad(2.15)$$

III. MELLIN-BARNES REPRESENTATION FOR THE $F_i(s,0)$ AMPLITUDES

The optical theorem relates the amplitudes $\mathrm{Im}F_i(s,0)$ to the total $\pi\pi$ cross sections as follows (massless pions):

$$\begin{aligned}\mathrm{Im}F_1(s,0)&=\frac{1}{2}[s\sigma^{\mathrm{tot}}_{\pi^+\pi^+}-s\sigma^{\mathrm{tot}}_{\pi^+\pi^-}],\\ \mathrm{Im}F_2(s,0)&=\frac{1}{2}[s\sigma^{\mathrm{tot}}_{\pi^+\pi^+}+s\sigma^{\mathrm{tot}}_{\pi^+\pi^-}]\\ &=\frac{1}{2}[s\sigma^{\mathrm{tot}}_{\pi^+\pi^0}+s\sigma^{\mathrm{tot}}_{\pi^0\pi^0}],\\ \mathrm{Im}F_3(s,0)&=\frac{1}{2}[3s\sigma^{\mathrm{tot}}_{\pi^+\pi^0}-s\sigma^{\mathrm{tot}}_{\pi^0\pi^0}].\end{aligned}\quad(3.1)$$

Let us then consider the Mellin transforms of the $\frac{1}{\pi}\,\mathrm{Im}F_i(s,0)$ amplitudes,

$$\Sigma_i(\xi)=\int_0^\infty d\left(\frac{s}{M^2}\right)\left(\frac{s}{M^2}\right)^{\xi-1}\frac{1}{\pi}\,\mathrm{Im}F_i(s,0),\quad(3.2)$$

and the corresponding inverse Mellin transforms,

$$\frac{1}{\pi}\,\mathrm{Im}F_i(s,0)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}d\xi\left(\frac{s}{M^2}\right)^{-\xi}\Sigma_i(\xi),\quad(3.3)$$

where, for convenience, we have introduced an arbitrary mass scale M (e.g. the ρ mass) so as to normalize the dimensions of the s variable. We then observe the following facts:

(i) According to Eq. (1.1), a FM-like asymptotic behavior for the physical $\sigma^{\mathrm{tot}}_{\pi\pi}(s)$ cross sections implies

$$\begin{aligned}\sigma^{\mathrm{tot}}_{\pi^+\pi^+}(s)&\underset{s\to\infty}{\sim}A_{\pi^+\pi^+}\frac{\pi}{M^2}\log^2\frac{s}{M^2},\\ \sigma^{\mathrm{tot}}_{\pi^+\pi^-}(s)&\underset{s\to\infty}{\sim}A_{\pi^+\pi^-}\frac{\pi}{M^2}\log^2\frac{s}{M^2},\\ \sigma^{\mathrm{tot}}_{\pi^+\pi^0}(s)&\underset{s\to\infty}{\sim}A_{\pi^+\pi^0}\frac{\pi}{M^2}\log^2\frac{s}{M^2},\\ \sigma^{\mathrm{tot}}_{\pi^0\pi^0}(s)&\underset{s\to\infty}{\sim}A_{\pi^0\pi^0}\frac{\pi}{M^2}\log^2\frac{s}{M^2},\end{aligned}\quad(3.4)$$

where the $A_{\pi\pi}$ are some appropriate constants. According to the normalization implied by Eq. (1.1) they should all be fixed to

$$A_{\pi\pi}|_{\mathrm{FM}}=\frac{M^2}{m_\pi^2},\quad(3.5)$$

but here we consider the $A_{\pi\pi}$ constants as *a priori* unknown.

The *inverse mapping theorem* [12] requires then that if the asymptotic behaviors in Eq. (3.4) are satisfied, the Mellin transforms of the $\frac{1}{\pi}\,\mathrm{Im}F_i(s,0)$ amplitudes must have a triple pole at $\xi\to -1$:

$$\Sigma_i(\xi)\underset{\xi\to-1}{\sim}\frac{-2a_i}{(\xi+1)^3},\quad(3.6)$$

where

$$\begin{aligned}a_1&=\frac{1}{2}[A_{\pi^+\pi^+}-A_{\pi^+\pi^-}],\\ a_2&=\frac{1}{2}[A_{\pi^+\pi^+}+A_{\pi^+\pi^-}]=\frac{1}{2}[A_{\pi^+\pi^0}+A_{\pi^0\pi^0}],\\ a_3&=\frac{1}{2}[3A_{\pi^+\pi^0}-A_{\pi^0\pi^0}].\end{aligned}\quad(3.7)$$

The leading singularity of $\Sigma_i(\xi)$ in the Mellin plane at the *right* of the *fundamental strip* [which fixes the integration boundary c in the inverse Mellin transform in Eq. (3.3)] must then be at $\xi=-1$ and it must be a triple pole. If all the $A_{\pi\pi}$ constants are equal, then $a_1=0$ and the corresponding pole at $\xi=-1$ becomes, at most, a double pole. We assume however, for the sake of generality, that all the $a_i\neq 0$.

(ii) Let us next consider the Mellin-Barnes representation of the dispersion relations for the amplitudes $F_i(s,0)$ in Eq. (2.11). Using the relations

$$\frac{1}{1+A}=\frac{1}{2\pi i}\int_{c_\xi-i\infty}^{c_\xi+i\infty}d\xi A^{-\xi}\Gamma(\xi)\Gamma(1-\xi),\quad(3.8)$$

$$\frac{1}{1-A}=\frac{1}{2\pi i}\int_{c_\xi-i\infty}^{c_\xi+i\infty}d\xi A^{-\xi}\Gamma(\xi)\Gamma(1-\xi)\times\frac{\pi}{\Gamma(\frac{1}{2}+\xi)\Gamma(\frac{1}{2}-\xi)},\quad(3.9)$$

and respecting the $s\leftrightarrow -s$ symmetry properties of the $\mathrm{Re}F_i(s,0)$ amplitudes, one finds

$$\mathrm{Re}\begin{pmatrix} F_1(s,0) \\ F_2(s,0) \\ F_3(s,0) \end{pmatrix} = \frac{s}{f_\pi^2}\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2\pi i}\int_{c_\xi - i\infty}^{c_\xi + i\infty} d\xi \Gamma(\xi)\Gamma(1-\xi) \times \begin{pmatrix} \Sigma_1(\xi-2)\frac{s}{M^2}\left(\frac{|s|}{M^2}\right)^{1-\xi}\left[1 - \frac{\pi}{\Gamma(\frac{1}{2}+\xi)\Gamma(\frac{1}{2}-\xi)}\right] \\ \Sigma_2(\xi-2)\left(\frac{|s|}{M^2}\right)^{2-\xi}\left[1 + \frac{\pi}{\Gamma(\frac{1}{2}+\xi)\Gamma(\frac{1}{2}-\xi)}\right] \\ \Sigma_3(\xi-2)\left(\frac{|s|}{M^2}\right)^{2-\xi}\left[1 + \frac{\pi}{\Gamma(\frac{1}{2}+\xi)\Gamma(\frac{1}{2}-\xi)}\right] \end{pmatrix}, \qquad (3.10)$$

where we have used the fact that

$$\int_0^\infty d\left(\frac{s'}{M^2}\right)\left(\frac{s'}{M^2}\right)^{\xi-3}\frac{1}{\pi}\mathrm{Im}F_i(s',0) = \Sigma_i(\xi-2), \qquad (3.11)$$

with $\Sigma_i(\xi)$ the same Mellin transform as the one defined in Eq. (3.2). Notice that the *fundamental strip* in Eq. (3.10) is now defined by $c_\xi = \mathrm{Re}(\xi) \in]0,1[$. Again, the low energy behavior of the $F_i(s,0)$ amplitudes is governed by the singularities at the *left* of this *fundamental strip*, while their high energy behavior is governed by the singularities at the *right* of the same *fundamental strip*.

(iii) In particular, the leading low energy behaviors of the $F_i(s,0)$ amplitudes are governed by the values of the $\Sigma_i(\xi-2)$ at $\xi \to 0$ and leads to the results:

$$\mathrm{Re}F_1(s,0) \underset{s\to 0}{=} -\frac{s}{f_\pi^2} + \mathcal{O}(s^3), \qquad (3.12)$$

$$\mathrm{Re}F_2(s,0) \underset{s\to 0}{=} \frac{s^2}{M^4}\lim_{\xi\to 0}\left\{\frac{d}{d\xi}[2\xi\Sigma_2(\xi-2)]\log\frac{M^2}{s} + 2\xi\Sigma_2(\xi-2)\right\} + \mathcal{O}(s^4), \qquad (3.13)$$

$$\mathrm{Re}F_3(s,0) \underset{s\to 0}{=} \frac{s^2}{M^4}2\Sigma_3(-2) + \mathcal{O}(s^4). \qquad (3.14)$$

Comparison with the χPT expansion in Eq. (2.14) allows us then to fix the values of $\Sigma_{2,3}$ at $\xi = -2$ to

$$\Sigma_2(\xi) \underset{\xi\to -2}{\sim} \frac{M^4}{f_\pi^4}\left[l_1^{\mathrm{r}} + \frac{3}{2}l_2^{\mathrm{r}} + \frac{25}{576\pi^2}\right] + \frac{1}{24\pi^2}\frac{1}{\xi+2}, \qquad (3.15)$$

$$\Sigma_3(\xi) \underset{\xi\to -2}{\sim} \frac{M^4}{f_\pi^4}\left[-l_1^{\mathrm{r}} + \frac{1}{2}l_2^{\mathrm{r}} + \frac{1}{192\pi^2}\right]. \qquad (3.16)$$

(iv) On the other hand, the leading high energy behaviors of the $F_i(s,0)$ amplitudes are governed by the $\Sigma_i(\xi-2)$ at $\xi \to 1$ which, if the FM bound is saturated for all the $\sigma_{\pi\pi}^{\mathrm{tot}}$ cross sections, have triple poles at the values:

$$\Sigma_i(\xi-2) \underset{\xi\to 1}{\sim} \frac{-2a_i}{(\xi-2+1)^3}. \qquad (3.17)$$

For the amplitudes $\mathrm{Re}F_2(s,0)$ and $\mathrm{Re}F_3(s,0)$ the effect of this triple pole is softened by the fact that

$$\frac{\pi}{\Gamma(\frac{1}{2}+\xi)\Gamma(\frac{1}{2}-\xi)} \underset{\xi\to 1}{\sim} -1 + \frac{\pi^2}{2}(\xi-1)^2 + \mathcal{O}(\xi-1)^4, \qquad (3.18)$$

and there is a cancellation between the two terms in the brackets in the second line at the rhs of Eq. (3.10). Therefore, the leading asymptotic behavior of $F_{2,3}(s,0)$ is then of the type

$$\mathrm{Re}F_{2,3}(s,0) \underset{s\to\infty}{\sim} \mathcal{O}[a_{2,3}|s|\log|s|]. \qquad (3.19)$$

By contrast, if $a_1 \neq 0$, there is no such a cancellation for the $F_1(s,0)$ amplitude and its leading high energy behavior will then be of the type:

$$\mathrm{Re}F_1(s,0) \underset{s\to\infty}{\sim} -\frac{s}{f_\pi^2} + \mathcal{O}[a_1 s\log^3|s|]. \qquad (3.20)$$

From the previous considerations we conclude that the Mellin-Barnes representation of the elastic $\pi\pi$ forward scattering amplitudes $F_i(s,0)$ show explicitly how their asymptotic behaviors for $s\to\infty$ (relevant to the FM bound) and for $s\to 0$ (relevant to the χPT expansion) are governed by the Mellin transforms $\Sigma_i(\xi)$ defined in Eq. (3.2). We find from χPT that the chiral limits ($m_\pi \to 0$) of these Mellin functions exist and are perfectly well defined in QCD at the *left* of the corresponding *fundamental strips*. We have also shown how the high energy behaviors of the $F_i(s,0)$ amplitudes are governed by the same Mellin transforms and, therefore, the FM bound has direct implications on their behaviors. If, as implied by the normalization of the FM bound in Eq. (1.1), and hence Eq. (3.5), the Mellin functions $\Sigma_i(\xi)$ at the *right* of their *fundamental strips* are singular in the chiral limit, it means that they must have a discontinuous behavior with respect to the pion mass in the sense that they exist in the chiral limit at the left of their fundamental strips yet they blow up to infinity, in the same limit, at the right of their fundamental strips. This we find a rather peculiar behavior which, although mathematically possible, questions the presence of a pion mass factor in the denominator of the normalization of the FM bound in QCD.

IV. THE FROISSART-MARTIN BOUND IN THE QCD LARGE-N$_c$ LIMIT

In this section we shall directly work with the $\pi\pi$ scattering amplitudes $\mathrm{Im}T^I(s,0)$ with well-defined isospin ($I = 0,\ 1,\ 2$). They are related to the $\mathrm{Im}F_i(s,0)$ amplitudes which we have considered in the previous section as follows:

$$\begin{aligned}\mathrm{Im}T^0(s,0) &= \frac{1}{2}[-4\,\mathrm{Im}F_1(s,0) \\ &\quad + 5\,\mathrm{Im}F_2(s,0) - 3\,\mathrm{Im}F_3(s,0)],\\ \mathrm{Im}T^1(s,0) &= -\mathrm{Im}F_1(s,0) + \mathrm{Im}F_3(s,0),\\ \mathrm{Im}T^2(s,0) &= \mathrm{Im}F_1(s,0) + \mathrm{Im}F_2(s,0). \end{aligned} \tag{4.1}$$

In the large-N$_c$ limit of QCD, the $\mathrm{Im}T^I(s,0)$ amplitudes are composed of an infinite set of narrow states:

$$\frac{1}{\pi}\mathrm{Im}T^I(s,0) = \sum_{n=0}^{\infty}|F_{I,n}|^2\delta(s-M_{I,n}^2), \quad I=0,1,2. \tag{4.2}$$

The question is then the following: is it possible to find constraints on the couplings $F_{I,n}$ and the masses $M_{I,n}$ of a possible large-N$_c$ ansatz so as to reproduce the FM asymptotic behavior for the $\sigma_{\pi\pi}^{\mathrm{tot}}$ cross sections?

In order to answer this question we shall proceed as follows. The Mellin transforms of $\frac{1}{\pi}\mathrm{Im}T^I(s,0)$ in the large-N$_c$ limit are given by Dirichlet-like series[5]:

$$\Sigma^I(\xi) = \sum_{n=0}^{\infty}\frac{|F_{I,n}|^2}{M^2}\left(\frac{M^2}{M_{I,n}^2}\right)^{-\xi+1}, \tag{4.3}$$

and the corresponding inverse Mellin transforms are

$$\begin{aligned}\frac{1}{\pi}\,\mathrm{Im}T^I(s,0) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}d\xi\left(\frac{s}{M^2}\right)^{-\xi}\\ &\quad\times\sum_{n=0}^{\infty}\frac{|F_{I,n}|^2}{M^2}\left(\frac{M^2}{M_{I,n}^2}\right)^{-\xi+1}. \end{aligned}\tag{4.4}$$

As discussed in the previous section, a FM-like asymptotic behavior for the $\sigma_{\pi\pi}^{\mathrm{tot}}$ cross sections fixes the leading singularity of the Mellin transforms of the $\frac{1}{\pi}\,\mathrm{Im}F_i(s,0)$ amplitudes as given in Eqs. (3.6) and (3.7) and therefore, from Eq. (4.1), there follows that

$$\Sigma^I(\xi) \underset{\xi\to-1}{\sim} \frac{-2A^I}{(\xi+1)^3}, \tag{4.5}$$

where

$$A^0 = \frac{1}{2}[-4a_1 + 5a_2 - 3a_3], \qquad A^1 = -a_1 + a_3, \qquad A^2 = a_1 + a_2. \tag{4.6}$$

[5]For a recent discussion of QCD large-N$_c$ properties in connection with the asymptotic behaviors of two-point functions see Ref. [13].

In order to construct a simple large-N$_c$ ansatz with the required properties we shall assume a Regge growth for the masses of the narrow states with $I = 1$:

$$M_{I=1,n}^2 = M_\rho^2 + n\Lambda^2, \tag{4.7}$$

and the absence of exotic trajectories i.e., no poles with $I = 2$. We are then assuming that the $I = 1$ channel fully dominates the physical $\mathcal{A}[\pi^\pm\pi^0 \to \pi^\pm\pi^0]$ amplitude and focus our attention on this amplitude in the limit where

$$\begin{aligned}\mathcal{A}_{\pi^\pm\pi^0\to\pi^\pm\pi^0}(s,0) &= \frac{1}{2}[F_2(s,0) + F_3(s,0)]\\ &\simeq \frac{1}{2}T^{I=1}(s,0). \end{aligned}\tag{4.8}$$

Saturation of the FM bound for the corresponding total cross section $\sigma_{\pi^\pm\pi^0}^{\mathrm{tot}}$ requires the couplings $\frac{|F_{I=1,n}|^2}{M^2}$ in Eq. (4.3) to grow like $n\log^2 n$ as $n\to\infty$. The simplest form of a large-N$_c$ $\mathrm{Im}T^{I=1}(s,0)$ amplitude satisfying these requirements is then

$$\begin{aligned}\frac{1}{\pi}\,\mathrm{Im}T^{I=1}(s,0) &= \mathrm{C}\sum_{n=0}^{\infty}(M_\rho^2 + n\Lambda^2)\log^2\left(\frac{M_\rho^2}{\Lambda^2}+n\right)\\ &\quad\times\delta(s - M_\rho^2 - n\Lambda^2), \end{aligned}\tag{4.9}$$

where C denotes a dimensionless constant. Here we have fixed the arbitrary scale M to $M = M_\rho$, and Λ is the mass scale which as shown in Eq. (4.7) fixes the equally spaced Regge-like spectra. Quite remarkably, the Mellin transform of this large-N$_c$ ansatz has a close analytic form,

$$\Sigma^{I=1}(\xi) = \mathrm{C}\left(\frac{M_\rho^2}{\Lambda^2}\right)^{-\xi}\frac{d^2}{d\xi^2}\zeta\left(-\xi,\frac{M_\rho^2}{\Lambda^2}\right), \tag{4.10}$$

where $\zeta(-\xi,\frac{M_\rho^2}{\Lambda^2})$ is the Hurwitz function, a generalization of the Riemann zeta function, defined by the series,

$$\zeta(\xi,v) = \sum_{n=0}^{\infty}\frac{1}{(n+v)^\xi}, \qquad \mathrm{Re}\,\xi > 1, \quad \text{with} \quad 0 < v \leq 1, \tag{4.11}$$

and its analytic continuation. For $v = 1$ it reduces to the Riemann zeta function. The asymptotic behavior of $\frac{1}{\pi}\,\mathrm{Im}T^{I=1}(s,0)$ for $s\to\infty$ and hence of $\sigma_{\pi^+\pi^0}^{\mathrm{tot}}$ is governed by the residue of the triple pole of $\Sigma^{I=1}(\xi)$ at $\xi = -1$ [see Eq. (4.5)]. This relates the overall constant C in Eq. (4.9) to the coefficient $A_{\pi^+\pi^0}$ in Eq. (3.4) and hence:

$$\sigma_{\pi^+\pi^0}^{\mathrm{tot}}(s) \sim_{s\to\infty} \frac{\mathrm{C}}{2}\,\frac{M_\rho^2}{\Lambda^2}\,\frac{\pi}{M_\rho^2}\log^2\frac{s}{M_\rho^2}. \tag{4.12}$$

Let us next discuss the low energy constraint that we can impose to the simple large-N$_c$ ansatz in Eq. (4.9) so as to fix the value of the overall constant C. The isospin $I = 1$ dominance assumption of the χPT expressions in Eq. (3.15), when restricted to the large-N$_c$ limit, fixes them to the values

$$\Sigma_2(\xi) \underset{\xi\to -2}{\sim} \frac{M^4}{f_\pi^4}\left[l_1 + \frac{3}{2} l_2\right] \sim \frac{1}{4}\frac{M_\rho^2}{f_\pi^2}, \tag{4.13}$$

$$\Sigma_3(\xi) \underset{\xi\to -2}{\sim} \frac{M^4}{f_\pi^4}\left[-l_1 + \frac{1}{2} l_2\right] \sim \frac{3}{4}\frac{M_\rho^2}{f_\pi^2}, \tag{4.14}$$

where in the rhs we have used the ρ-dominance approximation for the l_i constants [14,15] and, as before, we have fixed $M = M_\rho$. Matching the large-N_c ansatz in Eq. (4.10) to these considerations using Eq. (4.8) fixes the C constant to

$$C \sim \frac{1}{\zeta''(2, \frac{M_\rho^2}{\Lambda^2})}\frac{\Lambda^4}{f_\pi^2 M_\rho^2}, \tag{4.15}$$

and, therefore, the leading asymptotic growth of the total $\pi^+\pi^0$ cross section to

$$\sigma^{\mathrm{tot}}_{\pi^\pm\pi^0}(s) \sim_{s\to\infty} \frac{\frac{\Lambda^2}{M_\rho^2}}{2\zeta''(2, \frac{M_\rho^2}{\Lambda^2})}\frac{\pi}{f_\pi^2}\log^2\frac{s}{M_\rho^2}. \tag{4.16}$$

Like the usual FM bound, it grows as $\log^2 s$, but it is finite in the chiral limit and of $\mathcal{O}(1/N_c)$ in the large-N_c counting. Numerically, for $\Lambda = M_\rho$, the rhs of Eq. (4.16) becomes

$$\sigma^{\mathrm{tot}}_{\pi^\pm\pi^0}(s) \underset{s\to\infty}{\sim} 0.25\frac{\pi}{f_\pi^2}\log^2\frac{s}{M_\rho^2}, \tag{4.17}$$

but we should stress that this is only a large-N_c model estimate with many simplifications.

V. CONCLUSIONS

We have shown that it is possible to construct a large-N_c QCD ansatz compatible with the Froissart-Martin bound. The bound, however, is finite in the chiral limit and it is of $\mathcal{O}(1/N_c)$ in the large-N_c counting rules. In fact, it seems very likely that these two features should be generic to full QCD because of the fact that QCD has spontaneous chiral symmetry breaking. This implies the existence of a mass gap between the Nambu-Goldstone states (the pions) and the other hadronic states. It is this property which, very likely, forces the presence of characteristic scales like f_π in the normalization of the FM bound and which at the same time provides the correct $\mathcal{O}(1/N_c)$ large-N_c counting.

The usual derivation of the FM bound does not take into account the fact that the underlying dynamics of the strong interactions has the property of spontaneous chiral symmetry breaking. In fact, it implicitly assumes a realization of the hadronic spectrum *a la* Wigner-Weyl without Nambu-Goldstone particles, in which case, the normalization in Eq. (1.1) is not surprising.

Finally, we wish to emphasize that the discussion above does not answer the third question in the Introduction. We have only shown that, in the large-N_c limit of QCD, it is possible to construct models which *a priori* show no obstruction for the asymptotic behavior of the total $\pi\pi$ cross sections to saturate a $\log^2 s$-like behavior.

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