Contributions of the W-boson propagator to the μ and τ leptonic decay rates

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We derive closed expressions and useful expansions for the contributions of the tree-level W-boson propagator to the muon and τ leptonic decay rates. Calling M and m the masses of the initial and final charged leptons, our results in the limit $m = 0$ are valid to all orders in M^2/M_W^2 . In the terms of $\mathcal{O}(m^2/M^2)$ are with the capacical value $\mathcal{O}(m_j^2/M_W^2)$ $(m_j = M, m)$, our leading corrections, of $\mathcal{O}(M^2/M_W^2)$, agree with the canonical value
(3/5) M^2/M^2 , while the coefficient of our subleading contributions, of $\mathcal{O}(m^2/M^2)$, differs from that $\frac{3}{5}M^2/M_W^2$, while the coefficient of our subleading contributions, of $\mathcal{O}(m^2/M_W^2)$, differs from that reported in the recent literature. A possible explanation of the discrepancy is presented. The numerical reported in the recent literature. A possible explanation of the discrepancy is presented. The numerical effect of the $\mathcal{O}(m_j^2/M_W^2)$ corrections is briefly discussed. A general expression, valid for arbitrary values of $M \sim M \sim m$ is given in the Appendix. The paper also contains a review of M_W , M, and m in the range $M_W > M > m$, is given in the Appendix. The paper also contains a review of the traditional definition and evaluation of the Fermi constant.

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The correction of $O(m_\mu^2/M_W^2)$ to the muon decay rate,
sing from the tree-level W-boson proposator, is well arising from the tree-level W-boson propagator, is well
known in the literature and amounts to a correction factor known in the literature and amounts to a correction factor $1 + (3/5)m_\mu^2/M_W^2$. An analogous result was first derived
by Lee and Yang in the framework of nonlocal extensions by Lee and Yang in the framework of nonlocal extensions of the Fermi theory [[1\]](#page-4-0). Calling M and m the masses of the initial and final leptons, recent papers have included both the leading corrections, of $\mathcal{O}(M^2/M_W^2)$, as well as the subleading contributions of $\mathcal{O}(m^2/M_W^2)$ to the u and τ subleading contributions, of $\mathcal{O}(m^2/M_W^2)$, to the μ and τ
leptonic decay rates [2–4] leptonic decay rates [[2](#page-4-1)[–4](#page-4-2)].

In the present paper, we evaluate the corrections to the μ
d τ leptonic decay rates induced by the W-boson propaand τ leptonic decay rates induced by the W-boson propagator in two cases: (i) in the limit $m = 0$, we derive a closed expression, valid to all orders in M^2/M_W^2 , as well as useful expansion in powers of M^2/M_W^2 . (ii) in the cora useful expansion in powers of M^2/M_W^2 ; (ii) in the cor-
rections of $O(m^2/M^2)$ ($m = M/m$) we evaluate the rections of $\mathcal{O}(m_j^2/M_W^2)$ $(m_j = M, m)$, we evaluate the leading contributions, of $\mathcal{O}(M^2/M_W^2)$, as well as the sub-
leading ones of $\mathcal{O}(m^2/M_W^2)$. In the calculation of the latter leading ones, of $\mathcal{O}(m^2/M_W^2)$. In the calculation of the latter,
it is important to include the contribution of the it is important to include the contribution of the $-q^{\mu}q^{\nu}/M_W^2$ term in the unitary-gauge W-boson propagator or equivalently in other gauges, that of the associated tor or, equivalently, in other gauges, that of the associated Goldstone boson. In fact, this term leads to contributions of $\mathcal{O}(m^2/M_W^2)$. Our result for (ii) is compared with those
reported in the recent literature. In the Appendix we reported in the recent literature. In the Appendix we present expressions valid for arbitrary values of M_W , M , and *m* in the range $M_W > M > m$.

We focus our attention on μ decay and later on we tend the results to the τ leptonic decay rates in a straightextend the results to the τ leptonic decay rates in a straightforward manner. Defining

$$
x = \frac{M^2}{M_W^2},\tag{1}
$$

the terms of $\mathcal{O}(x^n)$ ($n \ge 1$) are very small. For this reason
they are evaluated at the tree level i.e. to zeroth order in α they are evaluated at the tree level, i.e. to zeroth order in α . On the other hand, the QED correction δ_{μ} to muon decay
in the V.A. Fermi theory is very important in the term of in the V-A Fermi theory is very important in the term of zeroth order in x . In order to obtain simple expressions, we follow the usual procedure of factorizing out the QED correction $[1 + \delta_\mu]$ in all the terms of order x^n ($n \ge 0$)
(see for example Ref. [51). This factorization induces (see, for example, Ref. [\[5](#page-4-3)]). This factorization induces terms of $\mathcal{O}(\alpha x^n)$ ($n \ge 1$), which are however extremely
small. As a consistency check, we have carried out the small. As a consistency check, we have carried out the calculations of the decay rate in two different ways. The first one is based on a method described in detail by Veltman in Ref. [[6](#page-4-4)]. The method requires working first in the neutrino-pair rest frame, where all scalar products can be written in terms of the energy transferred to the neutrino

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pair and the angle between the muon and $\bar{\nu}_e$ momenta. The integral over the $\bar{\nu}_e$ momentum is then carried out using the three-dimensional delta function. The angular integrals over the ν_{μ} momentum are trivial, while the integral over
its absolute value is implemented by employing the residual its absolute value is implemented by employing the residual one-dimensional δ function. Finally, one can rewrite the energy transferred to the neutrino pair in a Lorentzinvariant way and carry out the integration over the electron momentum in the muon rest frame. In the second, more conventional approach, one works always in the muon rest frame. We integrate first over the $\bar{\nu}_e$ momentum, thus reducing the four-dimensional δ function to a one-dimensional one, which leads to the relation E_2 = $M(E_0 - E) / [M - E + |\vec{p}| \cos \alpha]$, where E_2 is the energy of the muon neutrino, E and \vec{p} are the energy and momentum of the electron, $E_0 = (M^2 + m^2)/2M$ its end-point energy, and α the angle between \vec{p} and the ν_{μ} momentum.
We then integrate over F_{ν} using the one-dimensional δ We then integrate over E_2 using the one-dimensional δ function, over the angle α , and finally over \vec{p} . The two approaches lead to the same results, which we present below. In this paper, we call $\Gamma_{(W)}$ the decay rate when the contributions of the tree lovel W become proposator are contributions of the tree-level W-boson propagator are included.

(i) Integrating over the full W-boson propagator, in the limit $m \rightarrow 0$ we find the closed expression

$$
\Gamma_{(W)} = \Gamma_0 \left\{ \frac{12}{x^3} \left[1 - \frac{x}{2} - \frac{x^2}{6} + \frac{(1-x)}{x} \ln(1-x) \right] \right\},\tag{2}
$$

where

$$
\Gamma_0 = \frac{G_{\mu}^2 M^5}{192 \pi^3} [1 + \delta_{\mu}].
$$
 (3)

Furthermore,

$$
\frac{G_{\mu}}{\sqrt{2}} = \frac{g^2}{8M_W^2} (1 + \Delta r),
$$
 (4)

where g is the SU(2)_L gauge coupling constant, Δr is the electroweak correction introduced in Ref. [[7\]](#page-4-5), and, as mentioned before, δ_{μ} is the QED correction
to muon decay evaluated in the Fermi V-A theory to muon decay evaluated in the Fermi V-A theory. Expanding $\ln(1 - x)$, Eq. ([2](#page-1-0)) leads to a useful and

quickly convergent expression,
\n
$$
\Gamma_{(W)} = \Gamma_0 \sum_{n=0}^{\infty} \frac{12x^n}{(n+3)(n+4)}
$$
\n
$$
= \Gamma_0 \Biggl\{ 1 + \frac{3}{5}x + \frac{2}{5}x^2 + \frac{2}{7}x^3 + \frac{3}{14}x^4 + \frac{x^5}{6} + \cdots \Biggr\}. \tag{5}
$$

We note that the term of $\mathcal{O}(x)$ in Eq. [\(5](#page-1-1)) agrees with the canonical result $(3/5)M^2/M_W^2$ $(3/5)M^2/M_W^2$ $(3/5)M^2/M_W^2$. Equation (5) ex-
tends this result to all orders in x. Since Eq. (2) tends this result to all orders in x . Since Eq. (2) involves sharp cancellations, the expansion in Eq. [\(5](#page-1-1)) is much more useful for numerical calculations.

(ii) For $m \neq 0$, to zeroth order in x, the decay rate is given by the well-known expression

$$
\Gamma_{(W)}^{(0)} = \Gamma_0 F(y),\tag{6}
$$

where

$$
y = \frac{m^2}{M^2},\tag{7}
$$

and

$$
F(y) = 1 - 8y - 12y^2 \ln y + 8y^3 - y^4
$$
 (8)

is a phase-space factor (see, for example, Ref. [\[8](#page-4-6)]). Equations ([6\)](#page-1-2)–[\(8](#page-1-3)) correspond to the $M_W \rightarrow \infty$ limit and are the usual result in the V-A theory. In order to evaluate the terms of $\mathcal{O}(x = M^2/M_W^2)$ with $m \neq 0$,
in the calculation we include the correction factor in the calculation we include the correction factor $(1 + 2q^2/M_W^2)$ arising from the expansion of the W-boson propagator as well as the contribution of W-boson propagator, as well as the contribution of the $-q^{\mu}q^{\nu}/M_W^2$ term in the propagator. This leads
to the simple and compact result to the simple and compact result

$$
\Gamma_{(W)}^{(1)} = \Gamma_0 \frac{3}{5} x (1 - y)^5.
$$
 (9)

An interesting theoretical feature of Eq. ([9\)](#page-1-4) is that logarithmic terms proportional to ln y cancel between the contributions of the $(1 + 2q^2/M_W^2)$ cor-
rection factor and the $-a^{\mu} a^{\nu}/M_W^2$ term in the rection factor and the $-q^{\mu}q^{\nu}/M_W^2$ term in the proposator. We also observe that the y dependence propagator. We also observe that the y dependence in Eq. (9) (9) (9) is very different from that in Eqs. (6) and ([8](#page-1-3)), so that in their sum $F(y)$ does not factorize in a simple manner. Neglecting terms of $\mathcal{O}(xy^2 =$ $m^4/(M_W^2M^2)$ and higher, Eq. ([9\)](#page-1-4) reduces to

$$
\Gamma_{(W)}^{(1)} = \Gamma_0 \left\{ \frac{3}{5} \frac{M^2}{M_W^2} - \frac{3m^2}{M_W^2} \right\}.
$$
 (10)

The leading correction, $\left(3/5\right)M^2/M_W^2$, agrees once more
th the canonical result. In the subleading correction of with the canonical result. In the subleading correction of $\mathcal{O}(m^2/M_W^2)$, $-2m^2/M_W^2$ arises from the contribution of the $-a^{\mu}a^{\nu}/M_W^2$, jestingly and additional $-m^2/M_W^2$ is in- $-q^{\mu}q^{\nu}/M_W^2$ term, while an additional $-m^2/M_W^2$ is in-
duced by the $(1+2a^2/M^2)$ correction factor duced by the $(1 + 2q^2/M_W^2)$ correction factor.
In the muon decay case M and m are identify

In the muon decay case, M and m are identified with m_{μ} and m_e . The extension of our results to the τ leptonic decay rates is straightforward: M is identified with m_{τ} , while $m = m_{\mu}$ in $\Gamma_{(W)}(\tau \to \nu_{\tau} + \mu + \bar{\nu}_{\mu})$ and $m = m_{e}$ in $\Gamma_{(W)}(\tau \to \nu_{\tau} + e + \bar{\nu}_{e})$. Furthermore, δ_{μ} should be changed into δ , namely the appropriate OED corrections changed into δ_{τ} , namely the appropriate QED corrections in τ decays.

As far as we know, our calculation (i), valid to all orders in M^2/M_W^2 in the $m \to 0$ limit, has not been carried out in the literature. In order to compare our calculation (ii) with the literature. In order to compare our calculation (ii) with existing results, we combine Eqs. (6) and (10) (10) (10) ,

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$$
\Gamma_{(W)}^{(0)} + \Gamma_{(W)}^{(1)} = \Gamma_0 \bigg[F(y) + \frac{3}{5} \frac{M^2}{M_W^2} - \frac{3m^2}{M_W^2} + \mathcal{O} \bigg(\frac{m^4}{M_W^2 M^2} \bigg) \bigg]. \tag{11}
$$

In the literature, the phase-space factor $F(y)$ is often factorized. Performing such a factorization, Eq. ([11](#page-2-0)) becomes

 $-$ (1)

$$
\Gamma_{(W)}^{(0)} + \Gamma_{(W)}^{(1)}
$$
\n
$$
= \Gamma_0 F(y) \left[1 + \frac{3}{5} \frac{M^2}{M_W^2} + \frac{9}{5} \frac{m^2}{M_W^2} + \mathcal{O}\left(\frac{m^4}{M_W^2 M^2}\right) \right].
$$
\n(12)

Thus, the factorization of $F(y)$ induces a large change in the coefficient of the subleading correction of $\mathcal{O}(m^2/M_W^2)$.
This is easy to understand recalling Eqs. (7) and (8) This is easy to understand recalling Eqs. [\(7](#page-1-6)) and [\(8\)](#page-1-3): through terms of $\mathcal{O}(m^2/M^2)$ the factorization of $F(y)$ effectively leads to the change

$$
\frac{3}{5} \frac{M^2}{M_W^2} \rightarrow \frac{3}{5} \frac{M^2}{M_W^2} \left(1 + \frac{8m^2}{M^2} \right) = \frac{3}{5} \frac{M^2}{M_W^2} + \frac{24}{5} \frac{m^2}{M_W^2}.
$$
 (13)

As a consequence, the factorization of $F(y)$ induces a new subleading correction $(24/5)m^2/M_W^2$ in the expression
between square brackets in Eq. (12) Combining this with between square brackets in Eq. [\(12\)](#page-2-1). Combining this with $-3m^2/M_W^2$ in Eq. ([11](#page-2-0)), one obtains the subleading correc-
tion (9/5) m^2/M_{\odot}^2 , reported in Eq. (12) tion $\left(\frac{9}{5}\right)m^2/M_W^2$ reported in Eq. ([12](#page-2-1)).
Our result in Eq. (12) can be compared

Our result in Eq. (12) (12) (12) can be compared with expressions published in the recent literature. For example, Refs. [[2](#page-4-1)[–4\]](#page-4-2) consider the decay $\tau \to l + \bar{\nu}_l + \nu_\tau$ ($l = \mu$, e). Modulo
OED corrections, the result for the leptonic decay rates QED corrections, the result for the leptonic decay rates presented in those papers is

$$
\Gamma^l = \frac{G_{\mu}^2 m_{\tau}^5}{192 \pi^3} f \left(\frac{m_l^2}{m_{\tau}^2} \right) \left(1 + \frac{3}{5} \frac{m_{\tau}^2}{M_W^2} - \frac{2m_l^2}{M_W^2} \right),\tag{14}
$$

while, in this case, our Eq. (12) (12) (12) becomes

$$
\Gamma_{(W)}^{(0)l} + \Gamma_{(W)}^{(1)l} = \frac{G_{\mu}^2 m_{\tau}^5}{192\pi^3} F\left(\frac{m_l^2}{m_{\tau}^2}\right) \left[1 + \frac{3}{5} \frac{m_{\tau}^2}{M_W^2} + \frac{9}{5} \frac{m_l^2}{M_W^2}\right] + O\left(\frac{m_l^4}{M_W^2 m_{\tau}^2}\right).
$$
\n(15)

Since the function f in Eq. (14) (14) (14) is identical to F, we see that the two results agree on the leading correction $(3/5)m_{\tau}^2/M_W^2$, but sharply disagree on the coefficient of
the subleading term of $\mathcal{O}(m^2/M_{\tau}^2)$. In particular, the signs the subleading term of $\mathcal{O}(m_i^2/M_W^2)$. In particular, the signs
of the $\mathcal{O}(m_i^2/M^2)$ correction are opposite of the $\mathcal{O}(m_l^2/M_W^2)$ correction are opposite.
A possible explanation for this different

A possible explanation for this difference could be that (1) in the derivation of Eq. (14) , only the subleading $-2m_l^2/M_W^2$ contribution from the $-q^{\mu}q^{\nu}/M_W^2$ term in
the proposator has been retained (thus neglecting the the propagator has been retained (thus neglecting the additional $-m_l^2/M_W^2$ contribution arising from the
(1+2 a^2/M^2) correction factor) and (2) the additional $(1 + 2q^2/M_W^2)$ correction factor), and (2) the additional $(24/5)m^2/M_W^2$ contribution induced by the factorization of $\left(\frac{24}{5}\right)m_l^2/M_W^2$ contribution induced by the factorization of $F(y)$ has not been taken into account $F(y)$ has not been taken into account.

Numerically, the corrections of $\mathcal{O}(m_j^2/M_W^2)(m_j = M, m)$
wery small. Their largest values are attained in the decay are very small. Their largest values are attained in the decay $\tau \rightarrow \mu + \bar{\nu}_{\mu} + \nu_{\tau}$. In this case, the correction factor is

$$
1 + \frac{3}{5} \frac{m_{\tau}^2}{M_W^2} + \frac{9}{5} \frac{m_{\mu}^2}{M_W^2} = 1 + 2.9315 \times 10^{-4} + 3.11 \times 10^{-6},
$$
\n(16)

where we employed $M_W = 80.385$ GeV. Since the current relative error in the measurement of the τ lifetime is 3.44×10^{-3} , in order to be sensitive to the leading correction in Eq. (16) (16) , it would be necessary to decrease the experimental error by more than a factor 10.

In the case of muon decay,

$$
1 + \frac{3}{5} \frac{m_{\mu}^{2}}{M_{W}^{2}} + \frac{9}{5} \frac{m_{e}^{2}}{M_{W}^{2}} = 1 + 1.0366 \times 10^{-6} + 7.3 \times 10^{-11}.
$$
\n(17)

The current relative error in the measured muon lifetime is 1.00×10^{-6} [[9](#page-4-7)]. Thus, the leading correction in Eq. (17) is very close to the experimental error; it is also very close to the two-loop QED correction [see, for example, Eq. (36) in Ref. [[8](#page-4-6)]]. Thus, at present, the $\mathcal{O}(m_\mu^2/M_W^2)$ correction has a marginal effect in muon
decay. On the other hand, in the foreseeable future the decay. On the other hand, in the foreseeable future the subleading corrections of $O(m^2/M_W^2)$ are out of experi-
mental reach in both u and τ decays mental reach in both μ and τ decays.
We remind the reader that in the 1

We remind the reader that, in the traditional approach, the Fermi constant G_F is defined from the muon lifetime, as evaluated in the Fermi V-A theory to first order in the weak-interaction coupling constant. Specifically, G_F is defined by the relation

$$
\frac{1}{\tau_{\mu}} = \frac{G_F^2 m_{\mu}^5}{192\pi^3} F(y)[1 + \delta_{\mu}],
$$
 (18)

where τ_{μ} is the muon lifetime and δ_{μ} the QED correction.
This approach has several important advantages (see for This approach has several important advantages (see, for example Ref. [[8](#page-4-6)]): (1) the muon lifetime has been measured with great accuracy; (2) to first order in G_F , but all orders in α , the very important QED correction to muon decay in the Fermi V-A theory is known to be finite after charge and mass renormalization $[10]$ $[10]$ $[10]$; (3) at present, its contribution to the muon lifetime has been evaluated through two-loop order $[11-13]$ $[11-13]$ $[11-13]$ and estimated at three loops [[14](#page-4-11)]; (4) very importantly, in the traditional definition, G_F is a true constant of nature, like the electric charge: it does not need to be redefined and numerically changed every time a new particle contributing to muon decay is discovered; (5) the relation of G_F to the fundamental constants of the Standard Model of particle physics involves the electroweak radiative correction Δr and has been explained in Ref. [\[7\]](#page-4-5). A detailed description of the current evaluation of G_F is provided in Section II D of Ref. [\[8](#page-4-6)]. It includes one- and two-loop QED corrections treated in two alternative ways, very small contributions of $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha^2)$ proportional to powers of y, and an estimate of the theoretical error due to truncation of the QED perturbative series. The current value is [[9\]](#page-4-7)

$$
G_F = 1.1663788(7) \times 10^{-5} \text{ GeV}, \tag{19}
$$

an important determination at the 0.6 ppm level.

It is then clear that, in the traditional approach, the corrections from the W-boson propagator we discuss in this paper do not affect the definition or the value of G_F . Rather, they are interpreted as additional, albeit very small, corrections to the μ and τ leptonic decay rates that emerge
in the Standard Model of particle physics. In fact, by in the Standard Model of particle physics. In fact, by writing $\Gamma_{(W)}$ in the form

$$
\Gamma_{(W)} = \frac{G_{\mu}^2 m_{\mu}^5}{192 \pi^3} F(y) [1 + \delta_{\mu}] (1 + \delta_{(W)}), \tag{20}
$$

and comparing Eqs. [\(18\)](#page-2-5) and [\(20\)](#page-3-0), one finds that the relation between G_F^2 and G_μ^2 is given by -

$$
G_F^2 = G_\mu^2 (1 + \delta_{(W)}) = \frac{g^4}{32M_W^4} (1 + \Delta r)^2 (1 + \delta_{(W)}), \quad (21)
$$

where, in the last step, we employed Eq. ([4](#page-1-7)). The last factor in Eqs. (20) (20) (20) and (21) represents the additional tree-level correction induced by the W-boson propagator in the Standard Model.

In summary, we have re-examined the contribution of the W-boson propagator to the μ and τ leptonic decay rates.
Calling M and m the masses of the initial and final charged Calling M and m the masses of the initial and final charged leptons, in the limit $m \rightarrow 0$ we have derived a closed expression, Eq. [\(2](#page-1-0)), and a useful expansion, Eq. [\(5\)](#page-1-1), valid to all orders in $x = M^2/M_W^2$. They extend the canonical result $(3/5)M^2/M_W^2$, to all orders in x. In the terms of $\frac{(3/5)M^2}{M_W^2}$ to all orders in x. In the terms of $O(m^2/M_\odot^2)$ (m, = M m) we have evaluated the leading $\mathcal{O}(m_j^2/M_W^2)$ ($m_j = M$, *m*), we have evaluated the leading
corrections of $\mathcal{O}(M^2/M^2)$ as well as the subleading ones. corrections, of $\mathcal{O}(M^2/M_W^2)$, as well as the subleading ones,
of $\mathcal{O}(m^2/M^2)$ [Eq. (12)]. While our leading corrections of $\mathcal{O}(m^2/M_W^2)$ [Eq. [\(12](#page-2-1))]. While our leading corrections
agree with the canonical result the coefficient of our subagree with the canonical result, the coefficient of our subleading corrections differs sharply from the one reported in the recent literature. A possible explanation of this discrepancy was presented. The numerical effect of the $\mathcal{O}(m_j^2/M_W^2)$
corrections was briefly discussed. In the Annealy we have corrections was briefly discussed. In the Appendix we have presented an expression for the leptonic decay rates that includes the contribution of the W-boson propagator for arbitrary values of M_W , M, and m in the range $M_W > M >$ m [Eq. [\(A3](#page-3-2))]. This paper also contains a review of the traditional definition and evaluation of the Fermi constant.

We would like to thank M. Passera for calling our attention to Ref. [[2\]](#page-4-1) and for useful discussions. The work of A. F. and Z. Z. was supported in part by the National Science Foundation Grant No. PHY-1068317. The work of A. S. was supported in part by the National Science Foundation Grant No. PHY-0758032. The work of C. G. was partially supported by the Swiss National Science Foundation.

APPENDIX

In this appendix we present expressions for $\Gamma_{(W)}$, valid for arbitrary values of M_W , M , and m , in the range $M_W > M > m$. We find

$$
\Gamma_{(W)} = \Gamma_0 3 \left\{ \frac{4(1-y)}{x^3} - \frac{2(1-y^2)}{x^2} - \frac{1}{x} \left(\frac{2}{3} - y + y^2 - \frac{2}{3} y^3 \right) - \frac{5}{2} y (1-y^2) + xy^2 (1-y) - y^2 (4+x^2y) \ln y + \frac{1}{x^4} \ln \left(\frac{1-x}{1-xy} \right) [4(1-x) - xy(4-3x-x^3(1+y^2(1-x^2)))] \right\},
$$
\n(A1)

where Γ_0 is given in Eq. ([3](#page-1-8)). By setting $y = 0$ in Eq. [\(A1\)](#page-3-3),
one immediately finds the result in Eq. (2). Equation (6) one immediately finds the result in Eq. [\(2](#page-1-0)). Equation [\(6\)](#page-1-2) can be recovered by taking the $x \rightarrow 0$ limit in Eq. [\(A1\)](#page-3-3). Although the decay rate is obviously well behaved in the limit $x \to 0$, several of the terms in Eq. [\(A1\)](#page-3-3) are singular in that limit. This fact gives rise to large cancellations among different terms, which in turn lead to a loss of significant digits in numerical evaluations. This problem can be avoided by rewriting the second logarithm in Eq. ([A1\)](#page-3-3) as an infinite sum,

$$
\ln\left(\frac{1-x}{1-xy}\right) = -\sum_{n=1}^{\infty} (1-y^n) \frac{x^n}{n}.
$$
 (A2)

The first three terms in this series, when inserted into Eq. $(A1)$ $(A1)$ $(A1)$, lead to the cancellation of all terms which are singular in the $x \rightarrow 0$ limit. After a few manipulations of the residual series, $\Gamma_{(W)}$ can be written in the form

$$
\Gamma_{(W)} = \Gamma_0 \Big\{ F(y) + \frac{3}{5} x (1 - y)^5 + \frac{x^2}{20} (8 - 27y + 27y^5 - 8y^6 - 60y^3 \ln y) + 3 \sum_{n=3}^{\infty} x^n H_n(y) \Big\},
$$
\n(A3)

where $F(y)$ is defined in Eq. ([8\)](#page-1-3) and

$$
H_n(y) = \frac{y^3(1 - y^{n-2})}{n-2} - \frac{y(1 + y^2)(1 - y^n)}{n} - \frac{3y(1 - y^{n+2})}{n+2} + \frac{4(1 + y)(1 - y^{n+3})}{n+3} - \frac{4(1 - y^{n+4})}{n+4}.
$$
 (A4)

The contributions of $\mathcal{O}(1, x, x^2)$ are shown explicitly in the first three terms of Eq. (A3), while those of $\mathcal{O}(x^n)$ ($n \ge 3$) first three terms of Eq. ([A3](#page-3-2)), while those of $\mathcal{O}(x^n)$ ($n \ge 3$)
are given in the series presented at the end of the equation are given in the series presented at the end of the equation. An interesting property of the functions $H_n(y)$ ($n \ge 3$) is
that they are proportional to $(1 - y)^5$. This is due to the fact that they are proportional to $(1 - y)^5$. This is due to the fact
that H (y) and its first four derivatives vanish at $y = 1$. The that $H_n(y)$ and its first four derivatives vanish at $y = 1$. The same property holds for the contribution of $\mathcal{O}(x)$, as explicitly shown in the second term of Eq. $(A3)$. It is also interesting to observe that only the contributions of $\mathcal{O}(x^0)$ and of $\mathcal{O}(x^2)$ contain terms proportional to ln y.

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reference there is a misprint: $3m_\mu^2/M_W^2$ should be changed
to $(3/5)m^2/M_\infty^2$. \mathbf{r} to $\left(\frac{3}{5}\right)m_\mu^2/M_W^2$.
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