Generalized *CP* symmetries in $\Delta(27)$ flavor models

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We classify explicitly all the possible generalized *CP* symmetries that are definable in $\Delta(27)$ flavor models. In total, only 12 transformations are possible. We also show interesting consequences of considering some of them as residual symmetries of the neutrino sector.

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I. INTRODUCTION

Neutrino physics has entered a new era after the discovery of nonzero and relatively large θ_{13} mixing angle [1]. Such a discovery has enabled us to pursue the determination of yet another unknown quantity of the standard three-family description of lepton flavor physics: the Dirac *CP* phase δ_D. For Majorana neutrinos, in addition to this phase, three unknowns remain for the complete description of the Pontecorvo-Maki-Nakagawa-Sataka (PMNS) mixing matrix: the two Majorana *CP* phases and the neutrino mass hierarchy. The determination of δ_D might be possible in the foreseeable future, and hints of a nonzero δ_D are starting to show up in global fits of oscillation parameters [2–4].

On the theoretical side, a relatively large θ_{13} angle discards the exact validity of certain mass-independent textures [5] for the PMNS matrix, the most popular being the tribimaximal (TBM) form [6], which was a good approximation until recently. The great appeal of these mass-independent textures comes from the fact that they can arise naturally as a consequence of non-Abelian discrete flavor symmetries that act on the horizontal space of the three families of leptons. The general scheme consists of assuming a non-Abelian discrete flavor group G_F, which is broken into different subgroups on the charged lepton and neutrino sectors.

If we believe that discrete flavor symmetries govern the observed pattern of the mixing angles and mass hierarchies of leptons, including the observed value of the θ_{13} angle, we need to modify the simple forms, such as the TBM form, by either (i) adding corrections or (ii) considering different symmetries. One route is to consider the minimal amount of residual symmetries on the mass matrices which still remains predictive [7–9]. Here we pursue (ii) by employing generalized *CP* (GCP) transformations [10]. We consider the possibility of reducing the symmetry of the neutrino sector to a *single* GCP symmetry. We will see that such a setting leaves a lot of freedom in the leptonic mixing matrix, but it is compatible with a more general scenario where some other horizontal symmetry is *approximately* valid in the neutrino sector instead of being exactly satisfied at leading order. Obviously, it is possible to account for nonzero θ_{13}, even in the symmetry limit, if we consider more complicated flavor groups [11].

One successful modification of the residual symmetry of the neutrino sector consists of replacing the familiar $\mu\tau$ interchange symmetry [12] (in the flavor basis) with the symmetry called $\mu\tau$ reflection [13], which corresponds to the joint application of $\mu\tau$ interchange together with the complex conjugation of M_ν. This symmetry leads to maximal θ_{23} and maximal Dirac *CP* phase by allowing nonzero but free θ_{13} [14]. This symmetry has been successfully implemented in a number of models [15,16].

The possibility of considering generalized *CP* transformations [10] as symmetries in the leptonic sector has been analyzed recently [17,18]. The work of Ref. [17] focuses on analyzing the consequences of having a residual GCP symmetry in the neutrino sector along with other residual horizontal symmetries in the charged lepton and neutrino sectors. In particular, the flavor groups S_4 and A_4 have been analyzed and implemented recently [19].

On the other hand, the role of GCP symmetries as automorphisms of the (horizontal) flavor group is studied in Ref. [18]. The authors develop the general theory and then analyze the relevant cases from the literature. For example, they show that there is only one possible nontrivial definition for GCP within A_4 models which leads to the $\tilde{S}_4$ flavor symmetry of Ref. [16].

In this work, we further consider the possible GCP symmetries that are definable in $\Delta(27)$ flavor models.[1] Such a flavor group is interesting from the point of view of GCP symmetries because it possesses a large number of automorphisms that can be used as GCP symmetries [18]. At the same time, the group does not possess any order-2 element that could be promoted to a residual symmetry of the neutrino mass matrix. The generators of the automorphism group for $\Delta(27)$ were given in Ref. [18], but we intend here to find all possible GCP symmetries explicitly and consider constraints that were not discussed previously. Furthermore, we analyze the possibility of considering these GCP symmetries as residual symmetries of the neutrino sector.

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[1]The flavor group $\Delta(27)$ was first considered for quarks in Ref. [20] and for leptons in Ref. [21].

The outline of this work is the following: In Sec. II, we review the consequences of having only a single GCP symmetry as a residual symmetry of the neutrino mass matrix. We list all the possible GCP symmetries in $\Delta(27)$ flavor models in Sec. III and extract some interesting features. Section IV reviews the consequences of adding one GCP symmetry in a theory invariant by a discrete flavor symmetry. We justify the list of possible GCP symmetries of $\Delta(27)$ models in Sec. V. The conclusions are shown in Sec. VI.

II. RESIDUAL GCP SYMMETRIES

Let us start our study of the consequences of residual GCP symmetries on the mass matrices by first reviewing here the consequences of the usual (unitary) residual symmetries acting on the neutrino mass matrix as [22]

$$S^{\mathsf{T}}M_\nu S = M_\nu. \tag{1}$$

We assume the charged lepton mass matrix squared, $\bar{M}_l \equiv M_l M_l^\dagger$, is diagonal (flavor basis), which should be ensured by another residual symmetry, G_l.

We first recall that if U is the matrix that diagonalizes M_ν as

$$U^{\mathsf{T}}M_\nu U = \mathrm{diag}(m_i), \quad m_i > 0, \tag{2}$$

with nonzero and nondegenerate masses m_i, then any other matrix U' that also diagonalizes M_ν must be related to U by [22,23]

$$U' = Ud, \tag{3}$$

where d is a diagonal matrix with nonzero entries ± 1.

The symmetry in Eq. (1) dictates that if U diagonalizes M_ν, then SU also diagonalizes it, and then

$$SU = Ud. \tag{4}$$

This means that the eigenvectors of M_ν are also eigenvectors of S with eigenvalues ± 1. Furthermore, the symmetry matrix S fixes one eigenvector of M_ν corresponding to the unique nondegenerate eigenvalue of S; we obviously exclude the cases $S = \mathbb{1}$ or $S = -\mathbb{1}$. On the other hand, the property $d^2 = \mathbb{1}$ implies $S^2 = \mathbb{1}$, and only $\mathbb{Z}_2$ symmetries can be implemented on M_ν, the maximal symmetry being $\mathbb{Z}_2 \times \mathbb{Z}_2$, generated by two matrices S_1, S_2.

Instead of considering the unitary symmetry [Eq. (1)], we can consider the antiunitary symmetry:

$$S^{\mathsf{T}}M_\nu S = M_\nu^*. \tag{5}$$

For example, the choice

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{6}$$

corresponds to $\mu\tau$-reflection symmetry [13,14].

The symmetry in Eq. (5) now implies that SU^* also diagonalizes M_ν if U does, and then Eq. (4) is replaced by

$$SU^* = Ud_\nu. \tag{7}$$

This is equivalent to saying that $S = Ud_\nu U^{\mathsf{T}}$, and then necessarily

$$S^{\mathsf{T}} = S \tag{8}$$

if we require nonzero and nondegenerate neutrino masses. Since S is also unitary, it obeys $S^*S = SS^* = \mathbb{1}$. Note that the symmetry condition [Eq. (8)] is invariant by basis change.

We can show that any unitary and symmetric matrix S can be diagonalized by a real orthogonal matrix R as

$$R^{\mathsf{T}}SR = \eta \equiv \mathrm{diag}(\eta_i), \qquad |\eta_i| = 1. \tag{9}$$

Another way of writing Eq. (9) is

$$SR = R\eta. \tag{10}$$

The proof consists of writing $S = S_1 + iS_2$, where S_1, S_2 are real symmetric matrices. Then $S^*S = \mathbb{1}$ implies that S_1 commutes with S_2, and they can be simultaneously diagonalized by R.

Now let us denote by $\mathbf{u}_i$ the columns of U. The relation in Eq. (7) implies

$$S\mathbf{u}_i^* = \pm\mathbf{u}_i, \tag{11}$$

the sign ± 1 being given by $(d_\nu)_{ii}$. We can choose all $\mathbf{u}_i$ to obey the plus-sign equation of Eq. (11) by conveniently replacing $\mathbf{u}_i$ with $i\mathbf{u}_i$ when $(d_\nu)_{ii} = -1$. This leads to

$$U = (\mathbf{u}_1|\mathbf{u}_2|\mathbf{u}_3)d_\nu^{\frac{1}{2}}; \tag{12}$$

we choose $(d_\nu^{\frac{1}{2}})_{ii} = i$ if $(d_\nu)_{ii} = -1$, or $(d_\nu^{\frac{1}{2}})_{ii} = 1$ if $(d_\nu)_{ii} = 1$.

We now expand $\mathbf{u}_i$ in terms of the real eigenvectors $\mathbf{r}_i$ of S, corresponding to the columns of R,

$$\mathbf{u}_i = \mathbf{r}_j a_i^j. \tag{13}$$

Equation (11) with the plus sign leads to

$$\eta_j(a_i^j)^* = a_i^j. \tag{14}$$

This is solved by

$$\mathbf{u}_i = \mathbf{r}_i\eta_j^{\frac{1}{2}}a_i^j, \quad \text{with real} \quad a_i^j, \tag{15}$$

where we have made the replacement $a_i^j \to \eta_j^{\frac{1}{2}}a_i^j$. Equation (12) is finally

$$U = R\eta^{\frac{1}{2}}O_\nu d_\nu^{\frac{1}{2}}, \tag{16}$$

where O_ν is a real orthogonal matrix defined by $(O_\nu)_{ji} = a_i^j$. Orthogonality of O_ν follows from unitarity of U. The combination

$$U_S = R\eta^{\frac{1}{2}} \tag{17}$$

is uniquely determined by S (except for sign ambiguities) if S is nondegenerate.

In the flavor basis, $V_{\rm MNS} = U_\nu$, and then the lepton mixing matrix,

$$V_{\rm MNS} = R\eta^{\frac{1}{2}}O_\nu d_\nu^{\frac{1}{2}}, \tag{18}$$

is determined by the antiunitary symmetry [Eq. (5)], if S is nondegenerate, up to the three-parameter freedom of choosing O_ν. In particular, the CP properties are completely determined by $\eta^{\frac{1}{2}}$ and $d_\nu^{\frac{1}{2}}$. Therefore, among the six parameters of the PMNS matrix, the three phases are indirectly determined by the symmetry. The presence of another additional unitary symmetry as in Eq. (1) that commutes with the antiunitary symmetry [Eq. (5)] fixes one column of the matrix O_ν [17]. However, compared to Ref. [17], the form in Eq. (18) shows more explicitly the separate dependence of the PMNS matrix on the fixed phases ($\eta^{\frac{1}{2}}$) and real elements (R). In a general basis, the form [Eq. (18)] needs to be adapted to show explicit dependence on the residual symmetry of the charged lepton sector.

The desired setting is the following: if somehow R can be chosen close to the experimental mixing matrix, then O_ν can be close to the identity and treated as a perturbation.

III. POSSIBLE GCP SYMMETRIES IN $\Delta(27)$ MODELS

We seek now some possible GCP symmetries which could be phenomenologically interesting. We choose $\Delta(27)$ as the flavor group because it possesses a large amount of possible nontrivial GCP symmetries [18]. See Ref. [21] for the first applications of the $\Delta(27)$ flavor group in the lepton sector.

The group $\Delta(27) \simeq (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$ is an order-27 non-Abelian finite group which can be defined by using two generators a, b and another auxiliary element a' through the relations [24]

$$\begin{aligned} a^3 = a'^3 = b^3 = e, &\qquad aa' = a'a, \\ bab^{-1} = (aa')^{-1}, &\qquad ba'b^{-1} = a. \end{aligned} \tag{19}$$

Note that a, a' generate the invariant subgroup $\mathbb{Z}_3 \times \mathbb{Z}_3$, and the element

$$z_0 \equiv aa'^{-1} \tag{20}$$

generates the center of the group $Z(\Delta(27)) \simeq \mathbb{Z}_3$.

In three dimensions, we can use the explicit (faithful) representation **3** for $\Delta(27)$:

$$\begin{aligned} D_3(b) &= T \equiv \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ D_3(a) &= \mathrm{diag}(1, \omega, \omega^2), \\ D_3(a') &= \mathrm{diag}(\omega, \omega^2, 1). \end{aligned} \tag{21}$$

This representation differs slightly from that in Ref. [24]. Notice that $Te_i = e_{\sigma(i)}$ where $\sigma = (123)$. This means that, on a vector $x = (x_1, x_2, x_3)^\mathsf{T}$, $x \to Tx$ induces $x_i \to x_{\sigma^{-1}(i)}$, i.e., the permutation (132).

The definable GCP symmetries for any flavor group were studied in Ref. [18] as automorphisms acting on the flavor group. Although the generators of the automorphism group were listed there, the possible GCP symmetries were not listed explicitly. Here we show in Sec. V that there are only 12 possible nonequivalent GCP symmetries that can be defined within a $\Delta(27)$ flavor group for the three families of left-handed leptons L_i transforming as **3** in Eq. (21). They are given by

$$L_i \to (S_k)_{ij}(CL_j^*), \tag{22}$$

where

$$\begin{aligned} S_0 &= \mathbb{1}_3, \\ S_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ S_2 &= \mathrm{diag}(1, 1, \omega), \\ S_3 &= \mathrm{diag}(1, 1, \omega^2) = S_2^2, \\ S_4 &= U_\omega, \\ S_5 &= U_\omega^* = S_4 S_1, \\ S_6 &= \frac{-i\omega}{\sqrt{3}} \begin{pmatrix} \omega^2 & 1 & 1 \\ 1 & 1 & \omega^2 \\ 1 & \omega^2 & 1 \end{pmatrix}, \\ S_7 &= \frac{i\omega^2}{\sqrt{3}} \begin{pmatrix} \omega & 1 & 1 \\ 1 & 1 & \omega \\ 1 & \omega & 1 \end{pmatrix}, \\ S_8 &= \begin{pmatrix} \omega & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ S_9 &= \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ S_{10} &= S_6 S_1, \\ S_{11} &= S_7 S_1. \end{aligned} \tag{23}$$

The GCP symmetries in Eq. (23) are unique up to composition with elements of $\Delta(27)$ itself and multiplication by an overall phase factor. Concerning the first freedom, we choose S_i to be symmetric[2] [Eq. (8)] so that they can be used as residual symmetries of M_ν. We consider only the GCP symmetries that do not enlarge the horizontal flavor group $\Delta(27)$; see discussion in Sec. IV B. The transformation properties for the singlets $\mathbf{1}_{rs}$ are fixed according to the automorphism these matrices induce on $\Delta(27)$ [18]. Sometimes one singlet cannot appear alone but has to be paired up with another singlet.

In principle, we can use all S_i with $i = 0, \ldots, 11$ as residual GCP symmetries for M_ν. However, S_0 corresponds to the usual *CP* transformation and is therefore noninteresting. The GCP symmetry corresponding to S_1 is interesting, but it corresponds to $\mu\tau$ reflection, which was considered previously in the literature, e.g., in Ref. [16]. The matrices S_2, S_3 are diagonal, so they have trivial eigenvectors. Analogously, S_8, S_9 are block diagonal, so they have one trivial eigenvector. The matrices S_{10}, S_{11} should also be discarded, because they have degenerate eigenvalues.[3] The remaining S_4, S_5, S_6, S_7 are potential candidates for further study.

The GCP transformations corresponding to S_4, S_5, S_6, S_7 are potentially interesting, because their matrices [Eq. (17)] are given by

$$U_{S_4} = \begin{pmatrix} \frac{1+\sqrt{3}}{\sqrt{2(3+\sqrt{3})}} & \frac{(1-\sqrt{3})}{\sqrt{6-2\sqrt{3}}} & 0 \\ \frac{1}{\sqrt{2(3+\sqrt{3})}} & \frac{1}{\sqrt{6-2\sqrt{3}}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2(3+\sqrt{3})}} & \frac{1}{\sqrt{6-2\sqrt{3}}} & \frac{1}{\sqrt{2}} \end{pmatrix} \mathrm{diag}(1, i, e^{i\pi/4}),$$
$$U_{S_5} = \begin{pmatrix} \frac{1+\sqrt{3}}{\sqrt{2(3+\sqrt{3})}} & \frac{(1-\sqrt{3})}{\sqrt{6-2\sqrt{3}}} & 0 \\ \frac{1}{\sqrt{2(3+\sqrt{3})}} & \frac{1}{\sqrt{6-2\sqrt{3}}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2(3+\sqrt{3})}} & \frac{1}{\sqrt{6-2\sqrt{3}}} & \frac{1}{\sqrt{2}} \end{pmatrix} \mathrm{diag}(1, i, e^{-i\pi/4}),$$
$$U_{S_6} = U_{\mathrm{TB}}\mathrm{diag}(e^{i2\pi/3}, 1, e^{i\pi/6}),$$
$$U_{S_7} = U_{\mathrm{TB}}\mathrm{diag}(e^{-i2\pi/3}, 1, e^{-i\pi/6}), \quad (24)$$

where U_{TB} is the familiar tribimaximal mixing matrix,

$$U_{\mathrm{TB}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (25)$$

We have chosen the order of the eigenvectors appropriately. Note that the tribimaximal matrix appears for S_6 and S_7. Numerically, however, all matrices are close, as

$$|U_{S_4}| = |U_{S_5}| = \begin{pmatrix} 0.888074 & 0.459701 & 0 \\ 0.325058 & 0.627963 & 0.707107 \\ 0.325058 & 0.627963 & 0.707107 \end{pmatrix},$$
$$|U_{S_6}| = |U_{S_7}| = \begin{pmatrix} 0.816497 & 0.57735 & 0 \\ 0.408248 & 0.57735 & 0.707107 \\ 0.408248 & 0.57735 & 0.707107 \end{pmatrix}. \quad (26)$$

Instead of S_{10} and S_{11}, which we discarded because of degenerate eigenvalues, we could have considered $D_3(a)S_{10}D_3(a)$ and $D_3(a)S_{11}D_3(a)$ or their composition with $D_3(a^2)$. They have nondegenerate eigenvalues and are symmetric, but they lead to either

$$|U_S| = \begin{pmatrix} 0.84403 & 0.449099 & 0.293128 \\ 0.293128 & 0.84403 & 0.449099 \\ 0.449099 & 0.293128 & 0.84403 \end{pmatrix} \quad \text{or}$$
$$\begin{pmatrix} 0.84403 & 0.449099 & 0.293128 \\ 0.449099 & 0.293128 & 0.84403 \\ 0.293128 & 0.84403 & 0.449099 \end{pmatrix}, \quad (27)$$

which inevitably lead to a large θ_{13} angle. This case reminds us that compositions with elements of the horizontal symmetry do lead to different physical predictions if GCP transformations are considered as residual symmetries [17].

IV. INCLUSION OF A GCP TRANSFORMATION

To obtain all the possible GCP symmetries listed in Eq. (23) which are consistent with the flavor group $\Delta(27)$, we need to study how to extend a discrete symmetry group G_H by the inclusion of *one* generalized *CP* transformation acting, e.g., as Eq. (22) for the three families of left-handed leptons. This study was performed in general in Ref. [18]. Here we consider it in more detail and additionally add more constraints not previously considered.

A. *CP* as automorphism

We begin by reviewing how GCP transformations induce an automorphism on other symmetry groups of the theory, especially on discrete symmetries [18].

Let a discrete group, G_H,[4] act on the scalar multiplet of fields ϕ as

$$\phi \to D(g)\phi, \quad (28)$$

where $g \in G_H$ and D is a (possibly reducible) representation of G_H. This setting can be easily extended to other nonscalar fields.

A generalized *CP* transformation acts as

$$\phi \to \tilde{S} \cdot \phi \equiv S\phi^*(\hat{x}), \quad (29)$$

[2]In $\Delta(27)$, this choice is always possible; see Sec. V D.
[3]The unitary version of S_{11} was used in a different context in Ref. [25].

[4]H stands for horizontal.

where $\hat{x} = (x_0, -\mathbf{x})$ if $x = (x_0, \mathbf{x})$. Notice that S should be unitary to preserve the kinetic term. For fermionic fields, it is implicit that we factor $\mathsf{CP}^2 = -1$.

Invariance of the theory by $\tilde{S}$ and G_H leads to an invariance by the composition

$$\phi \xrightarrow{\tilde{S}} S\phi^* \xrightarrow{g} SD(g)^*\phi^* \xrightarrow{\tilde{S}^{-1}} SD(g)^*S^{-1}\phi, \tag{30}$$

which is a horizontal (unitary) transformation. The last transformation in Eq. (30) should be an element of $D(G_H)$, because otherwise we would have to enlarge G_H. Hence, by defining

$$D_{\tilde{S}}(g) \equiv SD(g)^*S^{-1}, \tag{31}$$

it is required that there always exist some $g' \in G_F$ such that

$$D_{\tilde{S}}(g) = D(g'), \quad \text{for all } g \in G_F. \tag{32}$$

We can easily show that D_S is also a representation for G_H [18]. Moreover, $\ker D_S = \ker D$, and then D_S is faithful if D is faithful. Considering that the representation D is *faithful*, the mapping $\tau \equiv D^{-1} \circ D_S$ exists (restricted to the image of D and D_S) and is a homomorphism between G_H and itself:

$$g \to g' = \tau(g). \tag{33}$$

Since D_S is also faithful, τ is invertible, and it is then an *automorphism* between G_H and itself. The possible matrices S in Eq. (31) then realize some element of the automorphism group $\mathrm{Aut}(G_H)$. We can then rewrite the condition in Eq. (32) as

$$D_{\tilde{S}}(g) = D(\tau(g)) \tag{34}$$

for all $g \in G_F$ and some automorphism τ.

Suppose now that there is a matrix $S = S(\tau)$ which solves Eq. (34) for some automorphism τ. We can see that the matrix $S'S$, where $S' = D(g')$ corresponds to a group element, also solves Eq. (34) for the automorphism $c_{g'} \circ \tau$, since

$$\begin{aligned} D_{S'\tilde{S}}(g) &= (S'S)D(g)^*(S'S)^{-1} = S'D(\tau(g))S'^{-1} \\ &= D(g'\tau(g)g'^{-1}) = D(c_{g'} \circ \tau(g)); \end{aligned} \tag{35}$$

we have defined the conjugation by the element g' as

$$c_{g'}(g) \equiv g'gg'^{-1}. \tag{36}$$

The automorphism generated by conjugation as in Eq. (36) is denoted as *inner*, whereas the automorphism that is not inner is called *outer*. All the inner automorphisms compose the inner autormophism group $\mathrm{Inn}(G_H)$, an invariant subgroup of $\mathrm{Aut}(G_H)$. Given that conjugation by group elements trivially corresponds to an automorphism, we only need to consider the outer automorphism group defined by

$$\mathrm{Out}(G_H) \equiv (G_H)/\mathrm{Inn}(G_H). \tag{37}$$

At the Lagrangian level, inner automorphisms do not introduce any restriction when we extend G_H to $G_H \rtimes \langle \mathrm{GCP} \rangle$.

Suppose now that there are two matrices S_0 and $S = S_1 S_0$ which satisfy Eq. (34) for a common automorphism τ. The relation between S and S_0 is

$$S_1 = \bigoplus_\alpha (s_\alpha \otimes \mathbb{1}_{d_\alpha}), \tag{38}$$

where s_α is an $m_\alpha \times m_\alpha$ unitary matrix acting on the *horizontal space* of m_α copies of the irreducible representation (irrep) α. We are using the decomposition

$$D(g) = \bigoplus_\alpha (\mathbb{1}_{m_\alpha} \otimes D_\alpha(g)). \tag{39}$$

The proof of Eq. (38) follows from the Schur lemma and is analogous to Theorem 1 of Ref. [26]. Therefore, two matrices that satisfy Eq. (34) for the same automorphism τ differ only by the unitary change of basis on the horizontal space of replicated irreps of G_H.

Let us analyze the case of trivial automorphism, i.e., $\tau = \mathrm{id}$. Equation (34) implies

$$SD(g)^*S^{-1} = D(g), \quad \text{for all } g \in G_F. \tag{40}$$

If D is an irreducible representation, we can distinguish three cases: real $[D^{(r)}]$, pseudoreal $[D^{(p)}]$, or complex $[D^{(c)}]$ representation. For real $D^{(r)}$, Eq. (40) can be satisfied with $S = \mathbb{1}$. For pseudoreal $D^{(p)}$, by definition, there is also a unitary (antisymmetric) W such that

$$WD^{(p)}(g)^*W^{-1} = D^{(p)}(g), \quad \text{for all } g \in G_F. \tag{41}$$

For a single complex $D^{(c)}$, it is not possible to satisfy Eq. (40), because $D^{(c)}$ and $D^{(c)*}$ are inequivalent. However, for a reducible representation $D^{(c)} \oplus D^{(c)*}$, Eq. (40) can be satisfied as

$$\begin{aligned} &\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} D^{(c)}(g) & 0 \\ 0 & D^{(c)}(g)^* \end{pmatrix}^* \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \\ &\quad = \begin{pmatrix} D^{(c)}(g) & 0 \\ 0 & D^{(c)}(g)^* \end{pmatrix}. \end{aligned} \tag{42}$$

Therefore, in a G_H-invariant theory, invariance by the CP transformation associated with the identity automorphism demands the presence of a multiplet ψ' transforming as $D^{(c)*}$ if the theory contains a multiplet ψ transforming as $D^{(c)}$. We have to keep in mind that a GCP transformation also induces an automorphism on other groups involved, such as gauge groups or the Lorentz group. Thus, ψ' should have the same quantum numbers of ψ with respect to these other groups, because the GCP transformation that leads to Eq. (42) is

$$\psi(x) \xrightarrow{\mathrm{GCP}} \psi'^*(\hat{x}), \qquad \psi'(x) \xrightarrow{\mathrm{GCP}} \psi^*(\hat{x}). \tag{43}$$

In contrast, for gauge groups, the automorphism that customarily meets the expectation of reversing gauge quantum numbers is the contragradient automorphism ψ^Δ [26], which can be defined for the fundamental representation of $SU(n)$ by $S = \mathbb{1}$ and

$$D^{(c)*}(g) = D^{(c)}(\psi^\Delta(g)). \tag{44}$$

We note that ψ^Δ is outer for $SU(n)$, $n \geq 3$, and $U(1)$. By associating CP with ψ^Δ, any gauge theory with scalars or fermions interacting only by gauge interactions is always CP invariant [26].

Assume now that τ has finite order[5] m, i.e., $\tau^m = \mathrm{id}$. Let us study the composition of Eq. (34). If we apply it twice, we obtain

$$D(\tau^2(g)) = (SS^*)D(g)(SS^*)^{-1}. \tag{45}$$

More generally, we obtain

$$D(\tau^{2n}(g)) = (SS^*)^n D(g)(SS^*)^{-n} \tag{46}$$

if we apply it an even number of times, or

$$D(\tau^{2n+1}(g)) = ((SS^*)^n S)D(g)^*((SS^*)^n S)^{-1}, \tag{47}$$

if we apply it an odd number of times. We will see in Sec. IV B that the order of the automorphism τ associated with a GCP transformation should be even. Therefore, from the Schur lemma (and unitarity of S), for $m = 2n$, we need

$$(SS^*)^n = \mathbb{1} \tag{48}$$

within all irrep sectors.

B. Composition of GCP transformations

We should analyze now the conditions imposed by the composition of the GCP transformation $\tilde{S}$ itself.

If we apply the transformation in Eq. (29) twice, we would obtain

$$\phi(x) \xrightarrow{\tilde{S}} S\phi^*(\hat{x}) \xrightarrow{\tilde{S}} (SS^*)\phi(x). \tag{49}$$

This is just the statement that usual CP has order 2 (order 4 for fermions). However, Eq. (49) also implies that

$$SS^* = D(s), \quad \text{for some } s \in G_H, \tag{50}$$

because otherwise G_H would be larger by the symmetry represented by SS^*.

The requirement of Eq. (50) applied to Eq. (45) implies

$$D(\tau^2(g)) = D(sgs^{-1}), \quad \text{for all } g \text{ in } G_H. \tag{51}$$

This means that any automorphism τ associated with a GCP transformation should have order 2, modulo inner automorphisms, i.e., $\tau^2 = c_s$. This requirement was not considered in Ref. [18]. We should emphasize that the consistency condition [Eq. (50)] is indeed *independent* from the automorphism condition [Eq. (34)] when $\tau^2 = c_s$ is not automatic, i.e., when $\mathrm{Out}(G_H)$ has elements of order greater than 2.[6] This is the case with $G_H = \Delta(27)$; see a specific example in Eq. (106). On the other hand, if $\mathrm{Out}(G_H)$ has only elements of order at most 2, the Schur lemma applied to Eq. (45) implies that the condition in Eq. (50) is automatically satisfied. This is the case, e.g., of $G_H = A_4$.

Another condition coming from the finiteness of s requires

$$(SS^*)^n = \mathbb{1} \tag{52}$$

for $s^n = e$, e being the identity element of G_H. This relation is identical to Eq. (48). Thus, the GCP transformation $\tilde{S}$ always has an even order $2n$, and it induces an automorphism of the same even order.

Let us now rewrite Eq. (50) as

$$SS^* = SD(s)^* S^{-1} = D_{\tilde{S}}(s) = D(\tau(s)). \tag{53}$$

Therefore, the automorphism τ induced by S should leave the element s invariant.

If we want S to generate a residual GCP symmetry on the neutrino mass matrix, S needs to be symmetric [Eq. (8)], and then $SS^* = \mathbb{1}_3$. Hence, Eqs. (50) and (51) imply

$$s = e, \qquad \tau^2 = \mathrm{id}. \tag{54}$$

This is the case of usual CP symmetry $\tilde{S} = \mathsf{CP}$.

However, even if S is nonsymmetric, the addition of the GCP transformation $\tilde{S}$ might be equivalent to the addition of another transformation $\tilde{S}'$ with symmetric S'. Let us define

$$S' \equiv D(g)S \tag{55}$$

and calculate

$$S'S'^* = D(g\tau(g)s). \tag{56}$$

We have used Eq. (34). Thus, we obtain $S'S'^* = \mathbb{1}$ if we can find g in G_H such that

$$g\tau(g)s = e. \tag{57}$$

If s has an odd order $2m+1$, this condition is automatically satisfied by $g = s^m$, since

$$g\tau(g)s = s^m s^m s = s^{2m+1} = e \tag{58}$$

where Eq. (53) was applied.

Additionally, we can see from Eq. (52) that for each matrix S that satisfies Eq. (34) and then corresponds to a consistent GCP transformation, there are equally consistent choices of S related by rephasing:

$$S \to e^{i\alpha} S. \tag{59}$$

[5] A finite group G_H always has a finite $\mathrm{Aut}(G_H)$.

[6] The author is thankful to G.-J. Ding for raising this question during FLASY2013.

All these matrices induce the same automorphism τ on the group G_H.

Now, suppose we have at our disposal two GCP transformations [Eq. (29)] defined by two unitary matrices S_1 and S_2. If we apply them in succession, we obtain

$$\phi(x) \to S_1\phi^*(\hat{x}) \to S_1 S_2^* \phi(x). \tag{60}$$

This means that two different GCP transformations induce a (unitary) horizontal transformation

$$\phi(x) \to U\phi(x), \tag{61}$$

with $U = S_1 S_2^*$.

We distinguish two cases: (i) If this new horizontal U transformation corresponds to a representation $D(g)$ of an element g in G_H, then only one of the two GCP transformations $\tilde{S}_1, \tilde{S}_2$ has to be included as an additional transformation subjected to the constraints of Eqs. (34) and (50). (ii) If the horizontal transformation U does not correspond to an element of G_H, then such a group has to be extended to a larger group G'_H. The simplest way to extend G_H to G'_H is by split extension of the form $G_H \rtimes \langle U \rangle$. In this case, U also induces an automorphism τ, by a unitary version of the transformation in Eq. (34), as

$$D_U(g) \equiv UD(g)U^{-1} = D(\tau(g)). \tag{62}$$

We denote this unitary transformation by D_U without the tilde symbol. We can also understand this requirement by the successive application of $\tilde{S}_1, \tilde{S}_2$ as

$$\begin{aligned} D(g) \to D_{\tilde{S}_1}(g) &= S_1 D(g)^* S_1^{-1} \to D_{\tilde{S}_2\tilde{S}_1}(g) \\ &= S_2 D_{\tilde{S}_1}(g)^* S_2^{-1} = S_2 S_1^* D(g) S_1^{\mathsf{T}} S_2^{\dagger}. \end{aligned} \tag{63}$$

We identify $U = S_2 S_1^*$ and $\tau = \tau_2 \circ \tau_1$ in Eq. (62) if τ_k is induced by $\tilde{S}_k$, $k = 1, 2$. In these cases, the action of the antiunitary transformations $\tilde{S}_1$ and $\tilde{S}_2$, inducing automorphisms τ_1, τ_2, is equivalent to the action of the unitary transformation U which induces the combined automorphism $\tau_2 \circ \tau_1$. We can compose unitary automorphisms with antiunitary automorphisms, as well as unitary ones with another unitary transformation. The set of all matrices S in Eq. (31) and U in Eq. (62) represents the automorphism group $\mathrm{Aut}(G_H)$.

V. THE CASE OF $G_H = \Delta(27)$

The automorphism group of $\Delta(27)$ was discussed in Ref. [18]. The structure of the group is

$$\begin{aligned} &\mathrm{Aut}(\Delta(27)) \simeq (((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2, \\ &Z \equiv \mathcal{Z}(\Delta(27)) \simeq \mathbb{Z}_3, \\ &\mathrm{Inn}(\Delta(27)) \simeq \Delta(27)/\mathcal{Z}(\Delta(27)) \simeq \mathbb{Z}_3 \times \mathbb{Z}_3, \\ &\mathrm{Out}(\Delta(27)) \simeq GL_2(\mathbb{F}_3) \simeq (Q_8 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2. \end{aligned} \tag{64}$$

Recall that the outer automorphism group is defined by Eq. (37). The possible nontrivial GCP transformations have to be associated with one of the 48 elements of $\mathrm{Out}(\Delta(27))$.

We can study $\mathrm{Aut}(\Delta(27))$ by using the explicit representation in Eq. (21) for the generators a, b, a' in Eq. (19). But instead of using a', we can use Eq. (20) as an auxiliary generator. For the representation in Eq. (21), we have

$$z_0 \sim \omega^2 \mathbb{1}_3. \tag{65}$$

We can replace the presentation in Eq. (19) with

$$\begin{aligned} a^3 = b^3 = z_0^3 = e, \qquad az_0 = z_0 a, \\ bab^{-1} = az_0, \qquad bz_0 b^{-1} = z_0. \end{aligned} \tag{66}$$

We can write all 27 elements of $\Delta(27)$ as

$$g = b^{n_1} a^{n_2} z_0^{n_3}, \tag{67}$$

where n_1, n_2, n_3 runs from 0 to 2.

A. Auxiliary result

Let us show that for $G = \Delta(27)$, the following is true:

$$\mathrm{Out}(G) \equiv \mathrm{Aut}(G)/\mathrm{Inn}(G) \simeq \mathrm{Aut}(G/\mathcal{Z}(G)). \tag{68}$$

This means that to study the outer automorphism group of $\Delta(27)$, all we need to know is the automorphism group of the smaller group $\Delta(27)/Z \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$. This property is very particular to $\Delta(27)$, and it is not satisfied, for example, for $G = A_4$ or a cyclic group.

To establish Eq. (68), it is useful to define a homomorphism from $\mathrm{Aut}(G)$ to $\mathrm{Aut}(G/Z)$ by mapping $\tau \in \mathrm{Aut}(G)$ to $\tau' \in \mathrm{Aut}(G/Z)$ by

$$\tau'(xZ) = \tau(x)Z, \tag{69}$$

where $xZ \in G/Z$ is a coset of Z in G, containing x in G. By the homomorphism theorem of group theory, we prove Eq. (68) by showing that the kernel of this homomorphism is $\mathrm{Inn}(G)$.

By definition, the kernel of the homomorphism defined by Eq. (69) is given by automorphisms τ of G mapped to $\tau' = \mathrm{id}$. This means

$$\tau'(xZ) = \tau(x)Z = xZ, \quad \text{for all } xZ \text{ in } G/Z. \tag{70}$$

Then $\tau(x)z' = xz''$ for some z', z'' in Z. Finally, τ in the kernel of the homomorphism [Eq. (69)] is required to obey

$$\tau(x) = xz, \quad \text{for some } z \text{ in } Z \text{ and for all } x \text{ in } G. \tag{71}$$

The remaining task is to show that any automorphism τ of G that obeys Eq. (71) is an inner automorphism. We do it explicitly for $G = \Delta(27)$ by considering its generators a, b. Any automorphism τ that obeys Eq. (71) should be entirely determined by how it acts on the generators a, b, i.e.,

$$\tau(a) = az_0^n, \qquad \tau(b) = bz_0^m. \tag{72}$$

Let us now show that any automorphism of the type in Eq, (72) is an inner automorphism. We begin by confirming that the validity of the property $bab^{-1} = az_0$ in Eq. (66) through the automorphism τ implies $\tau(z_0) = z_0$. Then we check the action through the conjugation of

$$c_a(b) = bz_0^2, \qquad c_b(a) = az_0, \tag{73}$$

where we use the notation $c_g(x) = gxg^{-1}$ for conjugation. Conjugation by a, b in Eq. (73) allows us to compute the conjugation by a general $g = b^{n_1}a^{n_2}z_0^{n_3}$ as

$$c_g(a) = az_0^{n_1}, \qquad c_g(b) = bz_0^{-n_2}. \tag{74}$$

We made use of the property $b^n a^m = a^m b^n z_0^{mn}$. We can then conclude that any automorphism as in Eq. (72) corresponds to a conjugation by some g in G, and conversely any inner automorphism will have the form of Eq. (72). This result establishes Eq. (68) for $\Delta(27)$.

B. Automorphism group of $\Delta(27)/\mathbb{Z}_3$

Let us study the automorphism group of $\Delta(27)/Z \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$, where $Z \equiv \mathcal{Z}(\Delta(27)) \simeq \mathbb{Z}_3$ is the center of $\Delta(27)$. The group $\Delta(27)/Z$ is generated by the cosets $\bar{a} = aZ$ and $\bar{b} = bZ$, while Z is generated by z_0 in Eq. (20). An automorphism τ in $\Delta(27)/Z$ can be defined by knowing the mapping of the generators $(\bar{a}, \bar{b}) \mapsto (\tau(\bar{a}), \tau(\bar{b}))$. Part of this discussion can be also found in Ref. [27].

Next we know $\mathrm{Aut}(\mathbb{Z}_3 \times \mathbb{Z}_3) \simeq GL_2(\mathbb{F}_3)$, the group of 2×2 invertible matrices with entries in the finite field $\mathbb{F}_3 = \{-1, 1, 0\}$. We identify $\Delta(27)/Z$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$ as follows: for each element $\bar{x} = \bar{a}^n\bar{b}^m$ in $\Delta(27)/Z$ we define a vector in $\mathbb{F}_3^2 = \mathbb{F}_3 \times \mathbb{F}_3$ as

$$\bar{x} = \bar{a}^n\bar{b}^m \to \mathbf{p} = (n, m)^\mathsf{T}, \tag{75}$$

where $n, m = -1, 0, 1$. For example,

$$\begin{aligned} \bar{a} &\to (1, 0), \\ \bar{b} &\to (0, 1), \quad \text{and} \\ \bar{a}\bar{b}^2 &\to (1, -1) = (1, 0) + (0, -1). \end{aligned} \tag{76}$$

Therefore, we trade group multiplication in $\mathbb{Z}_3 \times \mathbb{Z}_3 = \langle\bar{a}\rangle \times \langle\bar{b}\rangle$ for vector addition in $\mathbb{F}_3^2$. Now a matrix A in $GL_2(\mathbb{F}_3)$ induces an autormophism in $\mathbb{F}_3^2$ by

$$\mathbf{p} \to A\mathbf{p}. \tag{77}$$

The automorphism on $\Delta(27)/Z$ can be read off from Eq. (75). For example,

$$\tau \sim A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \text{ is equivalent to } \begin{cases} \tau(\bar{a}) = \bar{a}\bar{b}^2 \\ \tau(\bar{b}) = \bar{a} \end{cases}. \tag{78}$$

We can split $GL_2(\mathbb{F}_3)$ into $SL_2(\mathbb{F}_3) \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by a 2×2 matrix of determinant -1, associated with the automorphism σ. Let us choose

$$\sigma \sim d \equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{79}$$

so that $\mathrm{Aut}(\Delta(27)/Z) \simeq SL_2(\mathbb{F}_3) \rtimes \langle\sigma\rangle$. Now we only need to study the subgroup isomorphic to $SL_2(\mathbb{F}_3)$.

Let us show that $SL_2(\mathbb{F}_3) \simeq Q_8 \rtimes \mathbb{Z}_3$ by picking up some elements of $SL_2(\mathbb{F}_3)$,

$$\begin{aligned} e_1 &\equiv \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad e_2 \equiv \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \\ e_3 &\equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad c \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{80}$$

We can show that

$$\langle e_1, e_2, e_3\rangle \simeq Q_8, \qquad \langle c\rangle \simeq \mathbb{Z}_3, \tag{81}$$

and that $\langle e_1, e_2, e_2\rangle\langle c\rangle$ generates an order-24 group which exhausts $SL_2(\mathbb{F}_3)$.

Firstly, we can directly show that the following properties hold:

$$\begin{aligned} e_1^2 = e_2^2 = e_3^2 &= -\mathbb{1}_2, \\ e_1e_2 = -e_2e_1 &= e_3, \\ c^3 &= \mathbb{1}_2. \end{aligned} \tag{82}$$

These properties establish Eq. (81). The semidirect product $Q_8 \rtimes \mathbb{Z}_3$ is confirmed from the automorphism on Q_8 generated by c as

$$ce_1c^{-1} = e_2, \qquad ce_2c^{-1} = e_3, \qquad ce_3c^{-1} = e_1. \tag{83}$$

Finally, we can check that $\langle e_1, e_2, e_2\rangle$ and $\langle c\rangle$ have trivial intersection and that $\langle e_1, e_2, e_2\rangle\langle c\rangle$ has 24 elements.

For completeness, we can add the element d in Eq. (79) to generate $GL_2(\mathbb{F}_3)$. The element d induces an automorphism on $SL_2(\mathbb{F}_3)$ as

$$\begin{aligned} de_1d^{-1} &= e_2^{-1}, \qquad de_2d^{-1} = e_1^{-1}, \\ de_3d^{-1} &= e_3^{-1}, \qquad dcd^{-1} = c^{-1}. \end{aligned} \tag{84}$$

We can write all elements of $GL_2(\mathbb{F}_3)$ as a product of an element in $SL_2(\mathbb{F}_3)$ and $\mathbb{1}_2$ or d.

C. Unitary and antiunitary automorphisms

We show here the following result: antiunitary transformations [Eq. (31)] induce automorphisms in $\mathrm{Out}(\Delta(27))$ corresponding to matrices A in $GL_2(\mathbb{F}_3)$ with $\det A = -1$, and unitary transformations [Eq. (62)] realize automorphisms corresponding to elements of $SL_2(\mathbb{F}_3)$.

Given that the center of a group is always mapped into itself by any automorphism, we can firstly distinguish two types of automorphisms in $\mathrm{Aut}(\Delta(27))$:

$$(\mathrm{I})\tau(z_0) = z_0, \qquad (\mathrm{II})\tau(z_0) = z_0^{-1}. \tag{85}$$

We can easily see that automorphisms of type I form a normal subgroup of $\mathrm{Aut}(\Delta(27))$ with half of the elements.

In general, we would call a *CP*-type transformation a type-II transformation which sends $z_0 \to z_0^{-1}$. We can see that by noting that only the triplet and antitriplet representations, $\mathbf{3}$ and $\bar{\mathbf{3}}$, represent the element z_0 of the center nontrivially. For example, for $\mathbf{3}$, with the choice of Eq. (21), we obtain Eq. (65) for z_0. Since z_0 is in the center, its representation $\mathbf{3}$ is proportional to the identity, and we can immediately see that the automorphisms induced by a unitary transformation [Eq. (62)] are of type I, whereas the automorphisms induced by Eq. (31) are of type II.

Our task is to show that the subgroup of type-I automorphisms coincides with the subgroup $SL_2(\mathbb{F}_3)$ of $\mathrm{Aut}(\Delta(27))$ modulo inner automorphisms. We follow Ref. [27], Sec. 7.1. The first step is to define the commutator of two elements x, y of the group

$$[x, y] \equiv xyx^{-1}y^{-1}. \tag{86}$$

This operation has the properties

$$[y, x] = ([x, y])^{-1} \quad \text{and} \quad [xx', y] = [x, y][x', y], \tag{87}$$

where the last relation is already specialized to $\Delta(27)$ where all commutators lie in the center Z. For example,

$$[b, a] = bab^{-1}a^{-1} = z_0. \tag{88}$$

We can also identify the commutator in $\Delta(27)$ and $\Delta(27)/Z$ as

$$[x, y] = [\bar{x}, \bar{y}], \tag{89}$$

since the commutator is invariant if we replace x with xz, where $z \in Z$. The same is true for y.

The next step is to use the mapping [Eq. (75)] to define a bilinear d function of $\mathbb{F}_3^2$ to $\mathbb{F}_3$ by

$$d(\mathbf{p}, \mathbf{q}) = n, \quad \text{from } [\bar{x}, \bar{y}] = z_0^n, \quad \text{if } \mathbf{p} \to \bar{x}, \qquad \mathbf{q} \to \bar{y}. \tag{90}$$

The integer $n = -1, 0, 1$ belongs to $\mathbb{F}_3$. The properties in Eq. (87) translate to the following properties of d:

$$\begin{aligned} d(\mathbf{q}, \mathbf{p}) &= -d(\mathbf{p}, \mathbf{q}), \\ d(\mathbf{p} + \mathbf{p}', \mathbf{q}) &= d(\mathbf{p}, \mathbf{q}) + d(\mathbf{p}', \mathbf{q}). \end{aligned} \tag{91}$$

Thus, d is a bilinear function. Since $[\bar{a}, \bar{b}] = z_0^2 \sim d((1,0),(0,1)) = -1$, $d(\mathbf{p}, \mathbf{q})$ corresponds to (-1) times the determinant of the matrix formed by columns $\mathbf{p}$, $\mathbf{q}$ (from the uniqueness of the determinant function). This observation leads to

$$d(A\mathbf{p}, A\mathbf{q}) = \det(A)\, d(\mathbf{p}, \mathbf{q}), \tag{92}$$

where $A \in GL_2(\mathbb{F}_3)$. Finally, we can see how an automorphism τ associated with a matrix A acts on $z_0 = [\bar{b}, \bar{a}] \sim d((0,1),(1,0)) = 1$:

$$\begin{aligned} z_0 &\to d(A(0,1), A(1,0)) = \det(A) \sim [\tau(\bar{b}), \tau(\bar{a})] \\ &= \tau([\bar{b}, \bar{a}]) = \tau(z_0) = (z_0)^{\det(A)}. \end{aligned} \tag{93}$$

Hence, elements of $SL_2(\mathbb{F}_3)$ $[GL_2(\mathbb{F}_3) - SL_2(\mathbb{F}_3)]$ act as type-I [type-II] automorphisms.

D. Obtaining the matrices S_i

We are now in a position to calculate the matrices S that induce the automorphisms in Eq. (34) for the triplet representation $\mathbf{3}$ in Eq. (21). These matrices will define the GCP transformations [Eq. (22)] for the lepton doublets.

The relation in Eq. (68) allows us to associate, in a one-to-one fashion, a matrix $A \in GL_2(\mathbb{F}_3)$ with each automorphism τ of $\mathrm{Aut}(\Delta(27))$, modulo inner automorphisms. For GCP transformations, we only need the elements of $GL_2(\mathbb{F}_3) - SL_2(\mathbb{F}_3)$ with determinant (-1), which can be written as

$$A = A'd, \tag{94}$$

where $A' \in SL_2(\mathbb{F}_3)$, and d was defined in Eq. (79); see Sec. V B. In turn, all the elements A' of $SL_2(\mathbb{F}_3)$ can be recovered from the structure $\langle e_1, e_2, e_3\rangle \rtimes \langle c\rangle \simeq Q_8 \rtimes \mathbb{Z}_3$ whose generators were defined in Eq. (80).

We can calculate all the possible matrices S that define GCP transformations by using Eq. (34) for elements of $GL_2(\mathbb{F}_3)$ of the type in Eq. (94). Let us begin with the simplest $A = d$ case. We need a matrix $S(d)$ that induces automorphism $(a, b) \to (a^2, b)$; i.e.,

$$\begin{aligned} S(d)D_3(a)^* S(d)^\dagger &= D_3(a^2), \\ S(d)D_3(b)^* S(d)^\dagger &= D_3(b). \end{aligned} \tag{95}$$

We use the triplet representation in Eq. (21). Since Eq. (95) requires that $S(d)$ commute with both $D_3(a)$ and $D_3(b)$, the only solution is

$$S(d) = \mathbb{1}_3, \tag{96}$$

neglecting a possible phase factor. Thus, the GCP associated with the automorphism d is just the usual *CP* transformation.

We can obtain all other matrices S by composition from Eq. (94), since

$$\begin{aligned} D((A'd)(g)) &= U(A')D(d(g))U(A')^\dagger \\ &= U(A')D(g)^* U(A')^\dagger, \end{aligned} \tag{97}$$

where we have denoted as $(A'd)(g)$ the element mapped by automorphism from g by the matrix $A'd$, using some conventions explained below. For the generators a, b, we need

$$\begin{aligned} U(A')D_3(a)U(A')^\dagger &= D_3(A'(a)), \\ U(A')D_3(b)U(A')^\dagger &= D_3(A'(b)). \end{aligned} \tag{98}$$

We seek only matrices that can be associated with one single GCP transformation, which requires Eqs. (50), (51), and (53). In particular, Eq. (51) implies that we only need to consider order-2 automorphisms in $\mathrm{Out}(\Delta(27))$.

All the order-2 automorphisms of $\mathrm{Out}(\Delta(27))$ are in the conjugacy class of d. Such a conjugacy class is composed of the 12 elements

$$\left\{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}\{\mathbb{1}, -\mathbb{1}_2\}. \tag{99}$$

We can denote the elements within the first set of braces by $\{d, cd, c^2d, -e_2cd, e_1c^2d, -e_3d\}$, respectively.

To find all the matrices S corresponding to the automorphisms in Eq. (99), we only need to find the unitary matrices U satisfying Eq. (98) for the automorphisms $\{\mathbb{1}_2, c, c^2, -e_2c, e_1c^2, -e_3\}\{\mathbb{1}_2, -\mathbb{1}_2\}$, which correspond to the set in Eq. (99) multiplied by d from the right. To construct all of them, we only need $\{-\mathbb{1}_2, c, -e_2c, e_3\}$, i.e.,

$$\left\{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}, \tag{100}$$

respectively; the rest can be obtained from their inverses.

Note that automorphism A' is unique in $\mathrm{Out}(\Delta(27))$. In $\mathrm{Aut}(\Delta(27))$, they are defined up to inner automorphisms. To define an automorphism in $\mathrm{Aut}(\Delta(27))$ from $GL_2(\mathbb{F}_3)$ we need a convention, i.e., a recipe to extract one representative element from the coset. We adopt the following: we drop the bar in Eq. (78) and define the mapping on a, b. One only needs to define an ordering for terms with products of a and b. We use the ordering $b^{n_2}a^{n_1}$. For example, for the automorphism c above, we seek a $U(c)$ that induces

$$c = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}: (a, b) \mapsto (a, ba). \tag{101}$$

Thus, we conveniently write $c(a) = a$, $c(b) = ba$ as in Eq. (97).

Imposing Eq. (98), we find

$$\begin{aligned} U(-\mathbb{1}_2) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ U(e_3) &= \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \equiv U_\omega, \\ U(c) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \\ U(-e_2c) &= \frac{-i\omega}{\sqrt{3}}\begin{pmatrix} 1 & \omega^2 & 1 \\ 1 & 1 & \omega^2 \\ \omega^2 & 1 & 1 \end{pmatrix} \equiv U_3'. \end{aligned} \tag{102}$$

Note that for $-e_2c$, we have used $e_3ce_3^{-1} = -e_2c$, so that the convention in Eq. (101) is not respected. To obtain a symmetric matrix, we redefine

$$U(-e_2c) = D_3(b)U_3' = \frac{-i\omega}{\sqrt{3}}\begin{pmatrix} \omega^2 & 1 & 1 \\ 1 & \omega^2 & 1 \\ 1 & 1 & \omega^2 \end{pmatrix} \equiv U_3. \tag{103}$$

Finally, the list of matrices [Eq. (23)] is obtained from Eq. (102) as follows:

$$\begin{aligned} &S_0 = \mathbb{1}_3, \quad S_1 = U(-\mathbb{1}_2), \quad S_2 = U(c), \\ &S_3 = U(c)^{-1}, \quad S_4 = U(e_3), \quad S_5 = U(e_3)S_1, \\ &S_6 = U_3S_1, \quad S_7 = U_3^{-1}S_1, \quad S_8 = TS_2T^{-1}S_1, \\ &S_9 = TS_3T^{-1}S_1, \quad S_{10} = U_3, \quad S_{11} = U_3^{-1}. \end{aligned} \tag{104}$$

To define symmetric S_8, S_9, we have conveniently included the inner automorphism $T = D_3(b)$. All GCP transformations in $\Delta(27)$ models are then defined by the matrices S_i, $i = 0, \ldots, 11$, up to inner automorphisms. The choice of symmetric S_i implies that the associated automorphisms τ_i have order 2 in $\mathrm{Aut}(\Delta(27))$, as in Eq. (54), and not only in $\mathrm{Out}(\Delta(27))$. Therefore, in $\Delta(27)$, we can define all GCP transformations in terms of symmetric S_i, which are the symmetries relevant for residual GCP symmetries on the mass matrix M_ν. However, this does not mean that we cannot define GCP transformation with nonsymmetric S. For example, defining $S_{10} = U_3'$ in Eq. (102) instead of U_3 leads to

$$S_{10}S_{10}^* = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = D_3(b). \tag{105}$$

In this case, Eq. (50) is valid with nontrivial $s = b$.

We should also emphasize that the matrices in Eq. (102) are defined up to phases. If we are only interested in GCP transformations, Eq. (59) ensures that phases are unimportant. Instead, if we want to enlarge the $\Delta(27)$ group (identifying it with its triplet representation) by the inclusion of some of the elements above, then the phase factors should be compatible with the order of the element. For example, $U(c)$ is defined up to factors 1, ω, ω^2, while $U(e_3)$ can be multiplied only by ± 1, $\pm i$. For completeness, the order of $-e_2c$ is also 3.

Let us finish the study of GCP in $\Delta(27)$ flavor models by giving an explicit example showing that, for $\Delta(27)$, the consistency condition [Eq. (50)] is *additional* to the automorphism condition [Eq. (34)]. If we take the automorphism associated with $\tau(a) = ba$, $\tau(b) = a$, we have $\tau^8 = \mathrm{id}$ modulo conjugation. If we solve the condition in Eq. (34) for a, b, we find

$$S = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ \omega & 1 & \omega^2 \end{pmatrix} \tag{106}$$

up to rephasing. For the consistency condition in Eq. (50), we find

$$SS^* = \frac{i}{\sqrt{3}}\begin{pmatrix} \omega^2 & \omega^2 & 1 \\ 1 & \omega^2 & \omega^2 \\ 1 & \omega & 1 \end{pmatrix}. \tag{107}$$

This element is not part of Δ(27), so the horizontal group needs to be enlarged by including it. One can check that $(SS^*)^4 = \mathbb{1}_3$.

VI. CONCLUSIONS

We have found all GCP transformations that can be defined in Δ(27) flavor models, up to composition with elements of Δ(27) or multiplication by a phase factor. The list is shown in Eq. (23). The inclusion of any other GCP transformation leads to the enlargement of the flavor group Δ(27). Moreover, the extension of the flavor group by any GCP transformation is equivalent to the addition of an antiunitary transformation [Eq. (22)] for which the unitary part S that connects different families is symmetric.

We have also discussed the consequences of having a *single* GCP symmetry as a residual symmetry of the neutrino mass matrix M_ν. In the flavor basis, the presence of a *CP*-type residual symmetry of M_ν fixes three out of the six parameters of the leptonic mixing matrix—more precisely, three complex phases that lead indirectly to the Dirac *CP* phase and the two Majorana phases. Although the mixing angles are unconstrained, there is an intrinsic part of the mixing matrix that is completely determined by the GCP residual symmetry. If other symmetries are able to ensure that the unconstrained part is near the identity matrix, then the residual symmetry is capable of fixing the approximate features of the leptonic mixing matrix.

Specifically for Δ(27) flavor models, we have identified some potential GCP symmetries that lead to interesting patterns for the parts that are determined by the residual symmetry. In particular, two GCP symmetries lead to the tribimaximal form for the intrinsic part of the PMNS matrix. Another two GCP symmetries lead to a new pattern numerically close to but distinct from the tribimaximal form. These patterns could be further employed in flavor model building to explain the observed mixing patterns of leptons, including the nonzero θ_{13} angle.

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