

Linear responses of D0-branes via gauge/gravity correspondenceYoshinori Matsuo,^{1,*} Yuya Sasai,^{2,†} and Yasuhiro Sekino^{1,‡}¹*KEK Theory Center, High Energy Accelerator Research Organization (KEK), Tsukuba 305-0801, Japan*²*Institute of Physics, Meiji Gakuin University, Yokohama 244-8539, Japan*

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We study linear responses of D0-branes in the low frequency region by using gauge/gravity correspondence. The dynamics of the D0-branes is described by matrix theory with finite temperature, which is dual to a near extremal D0-brane black hole solution. We analyze the tensor mode and vector modes of a stress tensor and a Ramond-Ramond 1-form current of matrix theory. Then, we show that if a cutoff surface is close to a horizon of the D0-brane black hole, the linear responses take forms similar to the hydrodynamic stress tensor and current on S^8 . By taking a Rindler limit, those linear responses come to obey the hydrodynamics exactly, which is consistent with previous works on a Rindler fluid. We also show that if the cutoff surface is far from the horizon, the linear responses do not take the forms of the hydrodynamic stress tensor and current on S^8 . Especially, we find that the vector modes no longer possess a diffusion pole in the low frequency region, which indicates that the linear responses of the D0-branes cannot be explained by hydrodynamics.

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I. INTRODUCTION

If we apply a time-dependent external field to a black hole, what occurs in the black hole? According to the membrane paradigm [1–3], the response of the black hole can be represented by the degrees of freedom on the stretched horizon. It is expected that these degrees of freedom carry information on the interior of the black hole. This is suggested by two guiding principles of quantum gravity: the holographic principle (which states that the entropy in a spatial region is bounded by the area) and the black hole complementarity (which states that there is a consistent theory in the frame of an observer outside the horizon).

It has long been known that matter on the stretched horizon obeys hydrodynamic laws in the long wavelength limit [1]. On the other hand, it was pointed out that a localized perturbation spreads over the entire horizon in a time logarithmic in the Bekenstein-Hawking entropy. This time scale, which is different from the ones in local quantum field theories, plays a crucial role in forbidding a possible violation of the no-cloning theorem of quantum state [4–6]. These properties have been derived from classical general relativity. To understand what the degrees of freedom on the stretched horizon are, and how they thermalize, we will need a fundamental theory, such as string theory.

In string theory, the Bekenstein-Hawking entropy and the Hawking emission rate of some specific black holes have been correctly reproduced by D-brane systems [7,8]. It is likely that D-branes provide microscopic descriptions for more general black holes, but there are few quantitative results.

Recently, it has been found that transport coefficients in the membrane paradigm agree with those of a highly excited fundamental string at the correspondence point, up to numerical coefficients [9,10]. This can be regarded as a support for string theory as a microscopic description of the stretched horizon, in spite of a few limitations. First, the black hole is realized when the string coupling is larger than the value at the correspondence point [11]. In this situation, one can no longer neglect the excitations of D-branes because the masses of the D-branes are proportional to the inverse of the string coupling [12].¹ In addition, the authors of [9,10] have considered only homogeneous perturbations to obtain the transport coefficients of the fundamental string. To study dynamical processes such as diffusion, one has to apply inhomogeneous perturbations to the system.

According to AdS/CFT correspondence or matrix theory, string theory can be defined nonperturbatively by supersymmetric Yang-Mills theories. These gauge theories should allow us to study the dynamics on the stretched horizon from the first principles, even though it is difficult to solve these theories. In fact, there have been extensive studies of hydrodynamic properties of strongly coupled gauge theories from gravity calculations using AdS/CFT correspondence (or gauge/gravity correspondence, more generally), following the work of Policastro *et al.* [15]. In these studies, transport coefficients of gauge theories have been obtained by studying fluctuations around black brane backgrounds which have momentum along the brane, in

¹In [13], transport coefficients of the D1-D5-P system induced by a few moduli fields have been discussed. However, since the low energy effective theory of the D1-D5-P system does not couple to the bulk metric and gauge field [8,14], we could not discuss the linear responses of the stress tensor and current.

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the limit of small momentum. The computations of gauge/gravity correspondence are closely related to those in the old membrane paradigm [16].

In spite of this development, it is not clear how the transport phenomena in the directions which surround a black hole (or a black brane; the S^5 direction in the case of D3-branes) are represented in gauge theories. This question should be important in the understanding of black holes in the real world, since there are no directions along the brane in this case. Also, answers to this question may shed light on how the space emerges from lower dimensional theories.

Let us consider the case of D0-branes for definiteness. There are no spatial directions along the brane, and the D0-brane black hole is surrounded by S^8 . The low-energy description of D0-branes is given by maximally supersymmetric (0 + 1) dimensional Yang-Mills theory with $U(N)$ gauge symmetry. This theory is called matrix theory and has been proposed to be a description of M theory in a particular large N limit [17].

Black holes should correspond to dynamically realized spherically symmetric configurations of matrix-valued scalar fields. Fluctuations on the stretched horizon should correspond to fluctuations around such a configuration. It is not clear how these fluctuations propagate, and it is not even clear if they can be effectively described by a local field theory on S^8 .²

An important clue is that one knows how the matrices couple to background fields. Kabat and Taylor [20] and Taylor and Van Raamsdonk [21–23] studied one-loop effective potential in matrix theory and found that certain single-trace operators couple to supergravity backgrounds. These operators have definite $SO(9)$ R -charges, meaning that they are in the momentum representation on S^8 . One should be able to find linear responses of matrix theory to external perturbations by computing correlation functions of these operators.

In this paper, we will study transport phenomena along S^8 in matrix theory by using gauge/gravity correspondence. Our aim is to clarify what kind of behavior one should expect from the dynamics of matrices. In particular, we wish to understand to what extent the theory behaves as in field theory on S^8 .

Gauge/gravity correspondence for D0-branes was proposed in [24,25]. Correlation functions at zero temperature have been found [26–30] by applying the Gubser-Klebanov-Polyakov-Witten (GKPW) [31,32] prescription to the near-horizon D0-brane background. It was found that the zero-temperature correlators for operators which couple to supergravity modes obey power law, even though the theory is not conformally invariant. These results have been confirmed by Monte Carlo simulations of matrix theory [33,34].

²In [18,19], fluctuations of matrix theory have been analyzed by a numerical simulation of the classical dynamics to study the thermalization in the high temperature regime.

In this paper, we follow the standard procedure for studying the hydrodynamic limit in gauge/gravity correspondence. We will use the real-time prescription proposed by [15,35]. We evaluate the on-shell action on the near-extremal D0-brane background and obtain correlation functions following the GKPW prescription. We make a series expansion in the frequency and study the low frequency limit. We will study the tensor and vector modes and find shear viscosity and diffusion poles for the stress tensor and Ramond-Ramond (R-R) 1-form current. The scalar modes are deferred to future work.

One should note that the types of operators that we consider are different from the ones familiar in the holographic study of hydrodynamics. The stress-energy tensor on S^8 is represented by scalar operators from the perspective of gauge theory on (0 + 1) dimension. Modes with different momentum on S^8 are represented by different operators. Unlike stress-energy tensors in conformal field theories, these operators are not marginal operators and will have nontrivial wave function renormalization.

We follow the interpretation in [16,36,37] and assume that the position $r = r_c$ of the regulated boundary (or the “cutoff surface”), on which the on-shell action is evaluated, sets the scale of renormalization. We assume that the normalization of the operators is fixed at that scale. Since the gauge theory has only time, the renormalization scale refers to the scale of time separation.

We will consider the two cases: when the cutoff surface is near the horizon and when it is near infinity. In the former case, we obtain the results which can be interpreted as conventional hydrodynamics. This is the limit where the operators are defined at an infrared scale. However, in the latter case, we observe that the theory behaves differently from the usual fluid. This is the limit where the operators are defined at some ultraviolet scale, so that the operators could be sensitive to the short-time behavior of the theory.

This paper is organized as follows. In Sec. II, we review hydrodynamic equations for a charged fluid on S^8 . We consider the tensor and vector modes, and we find the expressions for the stress tensor and current in the presence of external perturbations. In Sec. III, after briefly reviewing matrix theory and gauge/gravity correspondence at zero temperature, we describe gauge/gravity correspondence at finite temperature on which our analysis is based. In Sec. IV, we calculate the linear responses of the stress tensor and R-R 1-form current of matrix theory by using the gauge/gravity correspondence.³ In Secs. IV B and IV C,

³In this paper, we assume that the stress tensor of matrix theory is coupled to the mode of the metric in the bulk and the R-R 1-form current of matrix theory is coupled to the mode of the R-R 1-form field in the bulk. This is different from the correspondence between the operators in matrix theory and the modes of the supergravity fields proposed in [26]. However, we believe that our assumption is more natural to obtain the correct linear responses of matrix theory under the perturbations of the bulk metric and R-R 1-form field.

we calculate the on-shell action for the tensor and vector modes, respectively, at arbitrary r_c . In Sec. **IV D**, we study transport coefficients when the cutoff surface is near the horizon. From the tensor mode, we find that the linear response of the stress tensor takes the form of the hydrodynamic stress tensor on S^8 , and that the shear viscosity to entropy density ratio is equal to $1/4\pi$. From the vector modes, we find that the linear responses of the stress tensor and R-R 1-form current take forms similar to the hydrodynamic stress tensor and current on S^8 . By taking a Rindler limit, the linear responses become the hydrodynamic stress tensor and current on \mathbf{R}^8 , which is consistent with previous work on a Rindler fluid [16,38]. In Sec. **IV E**, we consider the case in which the cutoff surface is far from the horizon. We find that both the tensor mode and vector modes do not follow the hydrodynamics. Especially, there is no diffusion pole in the vector modes in the low frequency region, which indicates that the linear responses of the D0-branes cannot be explained by hydrodynamics. The final section is devoted to the summary and comments. In Appendix **A**, we briefly summarize the definitions and properties of the spherical harmonics on S^8 . In Appendix **B**, we derive the on-shell action of the tensor mode and vector modes.

II. HYDRODYNAMICS ON S^8

In this section, we review a charged fluid on 9-dimensional spacetime whose spatial part is S^8 . We introduce external perturbations of the metric $g_{\mu\nu}$ and gauge field A_μ and consider the linear response [39]. The background metric and gauge field are given by

$$\bar{g}_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & \bar{g}_{ij} \end{pmatrix}, \quad (1)$$

$$\bar{A}_\mu = (\bar{\mu}, 0), \quad (2)$$

where the indices μ, ν run from 0 to 8, and \bar{g}_{ij} is the metric on S^8 with radius R . We introduced the chemical potential μ as the constant mode of A_0 , and $\bar{\mu}$ is its background part.

The hydrodynamic equations of the charged fluid are

$$0 = \nabla_\mu T^{\mu\nu} - F^{\nu\mu} J_\mu, \quad (3)$$

$$0 = \nabla_\mu J^\mu, \quad (4)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength. The constitutive relations of the stress tensor and current are

$$\begin{aligned} T^{\mu\nu} &= \epsilon u^\mu u^\nu + p \Delta^{\mu\nu} \\ &\quad - \eta \Delta^{\mu\alpha} \Delta^{\nu\beta} \left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{1}{4} g_{\alpha\beta} \nabla_\gamma u^\gamma \right) \\ &\quad - \zeta \Delta^{\mu\nu} \nabla_\gamma u^\gamma, \end{aligned} \quad (5)$$

$$J^\mu = n u^\mu + \sigma \Delta^{\mu\lambda} (E_\lambda - T \Delta_{\lambda\rho} \nabla^\rho (\mu/T)), \quad (6)$$

where ϵ is the energy density, p is the pressure, n is the charge density, T is the temperature, μ is the chemical potential, and

$$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu, \quad (7)$$

$$E_\mu = F_{\mu\nu} u^\nu, \quad (8)$$

are the projection to spatial direction and the external electric flux, respectively. The coefficients of second order parts, η , ζ , and σ , are the shear viscosity, bulk viscosity, and conductivity, respectively. The normalization condition of the velocity field u^μ is given by $g_{\mu\nu} u^\mu u^\nu = -1$.

Now, we introduce perturbations for the metric and gauge field and then consider the response of the fluid at the linear order of perturbations. By expanding them in terms of spherical harmonics on S^8 , they can be classified into the tensor, vector, and scalar modes which are associated to the tensor, vector, and scalar harmonics, respectively. We consider the tensor mode and vector modes and introduce no perturbation for the scalar mode. Then, the scalar quantities such as ϵ , p , and n have no response and remain constant. Since in this case, the velocity field satisfies the incompressible condition $\nabla_\mu u^\mu = 0$, the constitutive relations are simplified as

$$T^{\mu\nu} = \bar{\epsilon} u^\mu u^\nu + \bar{p} \Delta^{\mu\nu} - \eta \Delta^{\mu\alpha} \Delta^{\nu\beta} (\nabla_\alpha u_\beta + \nabla_\beta u_\alpha), \quad (9)$$

$$J^\mu = \bar{n} u^\mu + \sigma \Delta^{\mu\nu} E_\nu, \quad (10)$$

where $\bar{\epsilon}$, \bar{p} , and \bar{n} denote the energy density, pressure, and charge density in equilibrium, respectively.

We apply the following external perturbations to the fluid in equilibrium:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad (11)$$

$$A_\mu = \bar{A}_\mu + \delta A_\mu, \quad (12)$$

where

$$h_{ij}(t, x^i) = \sum_I b^I(t) Y_{ij}^I(x^i), \quad (13)$$

$$h_{0i}(t, x^i) = \sum_I b_0^I(t) Y_i^I(x^i), \quad (14)$$

$$\delta A_i(t, x^i) = \sum_I a^I(t) Y_i^I(x^i). \quad (15)$$

Here, Y_i^I and Y_{ij}^I are the vector harmonics and tensor harmonics on S^8 , respectively. The tensor mode is b^I and the vector modes are b_0^I and a^I . Hereafter, we often suppress the angular momentum index I and the sum in the spherical harmonic expansions. The definitions and properties of the spherical harmonics are summarized in Appendix **A**.

Under the perturbations, the velocity field changes as

$$u^\mu = \bar{u}^\mu + \delta u^\mu, \quad (16)$$

where \bar{u}^μ is the velocity field in the equilibrium and the linear responses can be expanded by the spherical harmonics:

$$\bar{u}^\mu = (1, 0, \dots, 0), \quad (17)$$

$$\delta u^0 = 0, \quad \delta u^i = u(t)Y^i. \quad (18)$$

Since u^0 behaves as a scalar on S^8 , it does not change in this case. Note that $\delta u_i = \delta u^\mu \bar{g}_{\mu i} + \bar{u}^\mu \delta g_{\mu i} = (u - b^0)Y_i$. Thus, the changes of the stress tensor and current are

$$\delta T^{0i} = ((\bar{\epsilon} + \bar{p})u - \bar{p}b^0)Y^i, \quad (19)$$

$$\delta T^{ij} = -(\bar{p}b + \eta\partial_0 b)Y^{ij} - \eta u(\bar{\nabla}^i Y^j + \bar{\nabla}^j Y^i), \quad (20)$$

$$\delta J^i = (\bar{n}u - \sigma\partial_0 a)Y^i. \quad (21)$$

Inserting (19)–(21) into the hydrodynamic equation (3), we find

$$u(\omega) = \frac{i\omega b^0(\omega) - i\omega \frac{\bar{n}}{\bar{\epsilon} + \bar{p}} a(\omega)}{i\omega - D \frac{(l+8)(l-1)}{R^2}}, \quad (22)$$

where we have used the Fourier transformation,

$$u(t) = \int \frac{d\omega}{2\pi} u(\omega) e^{-i\omega t}. \quad (23)$$

In the expression (22), l is the angular momentum (see also Appendix A) and

$$D = \frac{\eta}{\bar{\epsilon} + \bar{p}}, \quad (24)$$

is the diffusion constant. Therefore, the linear response of the stress tensor and current under the external perturbations are

$$\begin{aligned} \delta T^{ij}(\omega, x^i) = & \sum_I \left[-\bar{p}b_I(\omega)Y_I^{ij}(x^i) + i\omega\eta b_I(\omega)Y_I^{ij}(x^i) \right. \\ & + i\omega\eta \frac{-b_I^0(\omega) + \frac{\bar{n}}{\bar{\epsilon} + \bar{p}} a_I(\omega)}{i\omega - D \frac{(l+8)(l-1)}{R^2}} (\bar{\nabla}^i Y_I^j(x^i) \\ & \left. + \bar{\nabla}^j Y_I^i(x^i)) \right], \quad (25) \end{aligned}$$

$$\begin{aligned} \delta T^{0i}(\omega, x^i) = & \sum_I \left[\left(\bar{\epsilon} + \eta \frac{(l+8)(l-1)}{R^2} \right) b_I^0(\omega) Y_I^i(x^i) \right. \\ & \left. - \bar{n} \frac{i\omega}{i\omega - D \frac{(l+8)(l-1)}{R^2}} a_I(\omega) Y_I^i(x^i) \right], \quad (26) \end{aligned}$$

$$\begin{aligned} \delta J^i(\omega, x^i) = & \sum_I \left[\left(i\omega\sigma - \frac{\bar{n}^2}{\bar{\epsilon} + \bar{p}} \frac{i\omega}{i\omega - D \frac{(l+8)(l-1)}{R^2}} \right) a_I(\omega) Y_I^i(x^i) \right. \\ & \left. + \bar{n} \frac{i\omega}{i\omega - D \frac{(l+8)(l-1)}{R^2}} b_I^0(\omega) Y_I^i(x^i) \right]. \quad (27) \end{aligned}$$

Since we are interested in the dissipative behavior of the stress tensor and current, we neglect the nondissipative terms in (25) and (26). If we expand the stress tensor and current in terms of the spherical harmonics as

$$\begin{aligned} \delta T^{ij}(\omega, x^i) = & \sum_I T^I(\omega) Y_I^{ij}(x^i) + \tilde{T}^I(\omega) (\bar{\nabla}^i Y_I^j(x^i) \\ & + \bar{\nabla}^j Y_I^i(x^i)), \quad (28) \end{aligned}$$

$$\delta T^{0i}(\omega, x^i) = \sum_I T_I^0(\omega) Y_I^i(x^i), \quad (29)$$

$$\delta J^i(\omega, x^i) = \sum_I J^I(\omega) Y_I^i(x^i), \quad (30)$$

the coefficients T^I , \tilde{T}^I , T_I^0 , and J^I are

$$T^I(\omega) = i\omega\eta b^I(\omega), \quad (31)$$

$$\tilde{T}^I(\omega) = i\omega\eta \frac{-b_I^0(\omega) + \frac{\bar{n}}{\bar{\epsilon} + \bar{p}} a_I(\omega)}{i\omega - D \frac{(l+8)(l-1)}{R^2}}, \quad (32)$$

$$T_I^0(\omega) = \eta \frac{\frac{(l+8)(l-1)}{R^2}}{i\omega - D \frac{(l+8)(l-1)}{R^2}} b_I^0(\omega) - \bar{n} \frac{i\omega}{i\omega - D \frac{(l+8)(l-1)}{R^2}} a^I(\omega), \quad (33)$$

$$\begin{aligned} J^I(\omega) = & \left(i\omega\sigma - \frac{\bar{n}^2}{\bar{\epsilon} + \bar{p}} \frac{i\omega}{i\omega - D \frac{(l+8)(l-1)}{R^2}} \right) a^I(\omega) \\ & + \bar{n} \frac{i\omega}{i\omega - D \frac{(l+8)(l-1)}{R^2}} b_I^0(\omega). \quad (34) \end{aligned}$$

III. GAUGE/GRAVITY CORRESPONDENCE FOR MATRIX THEORY

In this section, we briefly review matrix theory and the gauge/gravity correspondence for matrix theory in the extremal and the near-extremal case.

A. Matrix theory

Let us consider a system which is composed of N D0-branes on top of one another in 10-dimensional type IIA string theory. In this system, there are open strings whose ends are attached on the D0-branes and closed strings which are propagating in the bulk. Although the closed strings are usually coupled to the D0-branes, we can decouple the closed strings from the D0-branes by taking a near-horizon limit [24]. Since all the massive string modes are also decoupled in this limit, the dynamics of the

D0-branes can be described by the lowest modes of the open strings, namely, matrix theory [17].

Matrix theory is the maximally supersymmetric $U(N)$ Yang-Mills theory in $(0+1)$ dimensions, which can be viewed as matrix quantum mechanics. The action is

$$S = \int dt \text{Tr} \left[\frac{1}{2g_s l_s} \dot{X}^m \dot{X}_m + \frac{1}{4g_s l_s^5} [X^m, X^n]^2 + (\text{fermionic terms}) \right], \quad (35)$$

where g_s is the string coupling constant and l_s is the string length. In this action, we have adopted the gauge condition $A = 0$. The Yang-Mills coupling constant is $g_{\text{YM}}^2 = (2\pi)^{-2} g_s l_s^{-3}$, which has mass dimension 3. The fields $X^m (m = 1, \dots, 9)$, which are $N \times N$ Hermitian matrices, describe the lowest modes of open strings connecting the D0-branes, and the diagonal components represent the positions of the D0-branes in the 9 spatial dimensions.

B. Gauge/gravity correspondence: Extremal case

For $g_s \ll 1$ and $g_s N \gg 1$, the D0-brane system can be treated as a classical solution of type IIA supergravity. The extremal D0-brane black hole solution in string frame is given by⁴

$$ds_s^2 = - \left(1 + \frac{R^7}{r^7} \right)^{-\frac{1}{2}} dt^2 + \left(1 + \frac{R^7}{r^7} \right)^{\frac{1}{2}} (dr^2 + r^2 d\Omega_8^2), \quad (36)$$

$$e^\phi = g_s e^{\tilde{\phi}} = g_s \left(1 + \frac{R^7}{r^7} \right)^{\frac{3}{4}}, \quad (37)$$

$$A_0 = g_s^{-1} \left[\left(1 + \frac{R^7}{r^7} \right)^{-1} - 1 \right], \quad (38)$$

where ϕ and A_μ are the dilaton and R-R 1-form field, respectively. The ‘‘radius’’ R is determined by the number of D0-branes as

$$R = (60\pi^3)^{\frac{1}{4}} (g_s N)^{\frac{1}{4}} l_s. \quad (39)$$

⁴ Although the overall sign of A_0 is different from that in [26], this is just a matter of convention. Let us calculate the total R-R charge in our convention. We apply a homogeneous chemical potential $\mu \equiv \delta A_0$ to the system at the asymptotic boundary, which is coupled to the total R-R charge $q \equiv \int_{S^8} d^8 x \sqrt{g_8} J^0$. The change of the Lagrangian is $\delta L = \mu q$. The variation of the action with respect to A_0 is

$$\begin{aligned} \delta S_{\text{IIA}} &= - \frac{g_s^2}{16\pi G} \int d^{10} x \sqrt{-g} [\nabla_\mu (\delta A_0 F^{\mu 0}) + (e.o.m.)] \\ &= \int_{r=\infty} dt \frac{7g_s V_8}{16\pi G R} \delta A_0, \end{aligned}$$

where we have inserted the solution (36) and (38) in the second line. Therefore, the total R-R charge is $q = + \frac{7g_s V_8}{16\pi G R}$.

By taking the near-horizon limit $R^7/r^7 \gg 1$, the solution becomes

$$ds_s^2 = - \left(\frac{r}{R} \right)^{\frac{7}{2}} dt^2 + \left(\frac{R}{r} \right)^{\frac{7}{2}} (dr^2 + r^2 d\Omega_8^2), \quad (40)$$

$$e^{\tilde{\phi}} = \left(\frac{R}{r} \right)^{\frac{21}{4}}, \quad (41)$$

$$A_0 = \frac{1}{g_s} \frac{r^7}{R^7}. \quad (42)$$

For the classical supergravity description to be reliable, the string coupling e^ϕ must be much smaller than 1 and the curvature radius in string frame must be much longer than l_s . Then, one finds the following condition for r [24,26]:

$$(g_s N)^{\frac{1}{2}} N^{-\frac{4}{21}} \ll \frac{r}{l_s} \ll (g_s N)^{\frac{1}{2}}. \quad (43)$$

The former condition leads to the first inequality and the latter condition leads to the second inequality. In addition, from the near-horizon condition $r \ll R$, we have

$$\frac{r}{l_s} \ll (g_s N)^{\frac{1}{2}}. \quad (44)$$

Therefore, the total region of r becomes [26]

$$g_s^{\frac{4}{21}} (g_s N)^{\frac{1}{2}} \ll \frac{r}{l_s} \ll (g_s N)^{\frac{1}{2}}. \quad (45)$$

The condition (45) is satisfied in a wide range of r if $g_s \ll 1$ and $g_s N \gg 1$. Thus, matrix theory can be described by the classical solution (40)–(42) in this region.

The near-horizon metric (40) is related to the metric on $\text{AdS}_2 \times S^8$ by a Weyl transformation as [26]

$$ds_s^2 = e^{2\tilde{\phi}} ds_w^2, \quad (46)$$

$$ds_w^2 = R^2 \left[\left(\frac{2}{5} \right)^2 \frac{1}{z^2} (-dt^2 + dz^2) + d\Omega_8^2 \right], \quad (47)$$

where

$$z \equiv \frac{2}{5} R^{\frac{7}{2}} r^{-\frac{5}{2}}. \quad (48)$$

In our paper, we call the frame whose metric is given by (47) ‘‘AdS frame.’’

The GKPW relation for this gauge/gravity correspondence is given by [26]

$$e^{iS_{\text{IIA}}[h]} |_{h_i^s(z=z_c)=\tilde{h}_i^s} = \left\langle \exp \left(i \int dt \tilde{h}_i^s(t) \mathcal{O}_i^s(t) \right) \right\rangle, \quad (49)$$

where S_{IIA} is the action of 10-dimensional type IIA supergravity, h_i^s denotes each mode of perturbations of the bulk fields, and z_c is the radial coordinate of the cutoff surface, on which matrix theory is defined. On the left-hand side, h_i^s is a solution of the bulk equations of motion which

satisfies two boundary conditions $h_i^s(z \rightarrow \infty) = 0$ and $h_i^s(z = z_c) = \bar{h}_i^s(t)$. Therefore, the left-hand side is a functional of $\bar{h}_i^s(t)$. The right-hand side represents a generating functional for connected correlation functions of matrix theory operator $\mathcal{O}_i^s(t)$, which couples to the source $\bar{h}_i^s(t)$. By taking the functional derivatives of (49) with respect to $\bar{h}_i^s(t)$ and sending $\bar{h}_i^s(t)$ to zero, we obtain the correlation functions of matrix theory via the on-shell action of type IIA supergravity.

C. Gauge/gravity correspondence: Near-extremal case

Next, we consider the near-extremal D0-branes. If non-extremality is sufficiently small, the horizon remains in the near-horizon region $r \ll R$ and we can take the near-horizon limit in a similar fashion to the extremal case. Then, in the near-horizon limit, the near-extremal D0-brane black hole solution becomes

$$ds_s^2 = e^{\frac{2}{5}\bar{\phi}} ds_w^2, \quad (50)$$

$$ds_w^2 = \tilde{R}^2 \left[z^{-2} (-f dt^2 + f^{-1} dz^2) + \left(\frac{5}{2}\right)^2 d\Omega_8^2 \right], \quad (51)$$

$$f = 1 - \left(\frac{z}{z_0}\right)^{\frac{14}{5}}, \quad e^{\bar{\phi}} = \left(\frac{z}{\tilde{R}}\right)^{\frac{21}{10}}, \quad (52)$$

$$A_0 = \frac{1}{g_s} \left(\frac{\tilde{R}}{z}\right)^{\frac{14}{5}}, \quad (53)$$

where $\tilde{R} \equiv \frac{2}{5}R$ and $z_0 = \frac{2}{5}R^{\frac{7}{5}}r_0^{-\frac{5}{2}}$ denotes the radius of the horizon. The Hawking temperature T_H , the Bekenstein-Hawking entropy S_{BH} and the total D0-brane charge q are⁵

$$T_H = \frac{7}{10\pi z_0}, \quad (54)$$

$$S_{\text{BH}} = \frac{V_8}{4G} \left(\frac{\tilde{R}}{z_0}\right)^{\frac{9}{5}}, \quad (55)$$

$$q = \frac{7g_s}{16\pi GR} V_8, \quad (56)$$

where V_8 is the volume of S^8 with radius R .

For the gauge/gravity correspondence to be valid, r_0 should be in the region of (45). Then, one finds⁶

$$(g_s N)^{\frac{10}{7}} \ll \frac{g_{\text{YM}}^2 N}{T_H^3} \ll N^{\frac{10}{7}}. \quad (57)$$

⁵The D0-brane charge q has mass dimension 1 because the 1-form field A_μ is defined to be dimensionless.

⁶The condition (57) does not hold if we take the large N limit when other parameters are fixed because $\frac{g_{\text{YM}}^2 N}{T_H^3} \sim (g_s N)^{\frac{5}{2}} \left(\frac{L_s}{r_0}\right)^{\frac{15}{2}}$. It is satisfied if g_s scales as N^α ($-1 < \alpha < -\frac{3}{7}$) [5].

Thus, we can study the strongly coupled D0-brane system by using the gauge/gravity correspondence.

In the near-extremal case, the definition of the GKPW relation (49) is subtle if we discuss the real time correlation functions [40]. Since the regularity condition on the horizon is not well-defined in this case, we impose the ingoing boundary condition [15,35]. Up to the quadratic order of the perturbations, the on-shell action takes the following form:

$$S_{\text{on-shell}} = \sum_I A_I \int \frac{d\omega}{2\pi} \bar{h}_i^s(-\omega) \mathcal{F}_I(\omega, z) \bar{h}_i^{s'}(\omega)|_{z=z_c}, \quad (58)$$

where

$$A_I = \begin{cases} D_2^I \equiv \frac{1}{2} \int d^8x \sqrt{g_8} Y_I^{ij} Y_{ij}^I, & \text{(for tensor mode),} \\ D_1^I \equiv \int d^8x \sqrt{g_8} Y_I^i Y_i^I, & \text{(for vector modes),} \end{cases} \quad (59)$$

and $\sqrt{g_8}$ is the square root of the determinant of the metric on S^8 with radius R . Hereafter, we suppress the angular momentum index I and the sum over I . Then, the retarded Green function of matrix theory is given by

$$G_R^{ss'}(\omega) = \begin{cases} -2\mathcal{F}(\omega, z)|_{z=z_c}, & \text{(for } s = s'), \\ -\mathcal{F}(\omega, z)|_{z=z_c}, & \text{(for } s \neq s'), \end{cases} \quad (60)$$

where the retarded Green function of operators \mathcal{O}^s is defined by

$$G_R^{ss'}(\omega) = -i \int_{-\infty}^{\infty} dt e^{i\omega t} \theta(t) \langle [\mathcal{O}^s(t), \mathcal{O}^{s'}(0)] \rangle_0. \quad (61)$$

Here, $\langle \rangle_0$ denotes the ensemble average in equilibrium. Thus, according to the linear response theory [41], the linear response of the matrix theory operator is

$$\delta \langle \mathcal{O}^s(\omega) \rangle \equiv \langle \mathcal{O}^s(\omega) \rangle - \langle \mathcal{O}^s(\omega) \rangle_0 = -G_R^{ss'}(\omega) \bar{h}^{s'}(\omega), \quad (62)$$

where $\langle \rangle$ denotes the ensemble average when the source fields $\bar{h}^s(\omega)$ are turned on.

IV. LINEAR RESPONSES OF D0-BRANES

In this section, we investigate fluid in the gauge/gravity correspondence for matrix theory. We introduce perturbations in tensor and vector modes and calculate the linear responses of the stress tensor and R-R 1-form current. Our strategy for the calculation is the following:

- (i) We put the cutoff surface at $z = z_c$.
- (ii) We solve the bulk equations of motion for the perturbations of the metric and the R-R 1-form. Then, we impose the ingoing boundary condition at $z = z_0$ and the Dirichlet boundary condition at $z = z_c$ on the solutions.
- (iii) We evaluate the on-shell action and calculate the linear responses of the operators which are coupled to those perturbations.
- (iv) We compare the results with the hydrodynamic stress tensor and current on S^8 (or \mathbf{R}^8) when

- (a) the cutoff surface is near the horizon.
 (b) the cutoff surface is far from the horizon.

The bosonic action of the 10-dimensional type IIA supergravity in string frame is⁷

$$S_{\text{IIA}} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left[e^{-2\phi} (R + 4\partial_\mu \phi \partial^\mu \phi) - \frac{g_s^2}{4} F_{\mu\nu} F^{\mu\nu} \right], \quad (63)$$

where $\mu, \nu = 0, \dots, 9$; $2\kappa^2 = 16\pi G$; and G is the 10-dimensional Newton constant. Because Neveu-Schwarz–Neveu-Schwarz 2-form and R-R 3-form have no nontrivial backgrounds in the D0-brane solution (50)–(53), they are decoupled from the metric and R-R 1-form at the linear order. Hence, we have omitted them.

To reduce the calculations, it is convenient to use the AdS frame (51). Then, the action becomes [26]

$$S'_{\text{IIA}} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} e^{-\frac{6}{7}\phi} \left(R + \frac{16}{49} \partial_\mu \phi \partial^\mu \phi - \frac{g_s^2}{4} e^{\frac{12}{7}\phi} F_{\mu\nu} F^{\mu\nu} \right). \quad (64)$$

To obtain the correct on-shell action, we need to add the Gibbons-Hawking term [42],

$$S_{\text{GH}} = -\frac{1}{\kappa^2} \int_{z=z_c} d^9x \sqrt{-\gamma} e^{-\frac{6}{7}\phi} K, \quad (65)$$

on the boundary, where $\gamma_{\mu\nu}$ is the induced metric on the cutoff surface and K is the trace of the extrinsic curvature $K_{\mu\nu}$. Therefore, the total action is

$$S_{\text{total}} = S'_{\text{IIA}} + S_{\text{GH}}. \quad (66)$$

By varying the action (64) with respect to the metric, 1-form and dilaton, one finds the equations of motion

$$0 = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{4}{7} \left(g_{\mu\nu} \partial_\rho \phi \partial^\rho \phi - \frac{5}{7} \partial_\mu \phi \partial_\nu \phi \right) + \frac{6}{7} (\nabla_\mu \partial_\nu \phi - g_{\mu\nu} \nabla_\rho \partial^\rho \phi) + \frac{g_s^2}{8} e^{\frac{12}{7}\phi} (g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} - 4F_{\mu\rho} F_\nu^\rho), \quad (67)$$

$$0 = \nabla_\mu (e^{\frac{6}{7}\phi} F^{\mu\nu}), \quad (68)$$

$$0 = R - \frac{16}{49} \partial_\mu \phi \partial^\mu \phi + \frac{16}{21} \nabla_\mu \nabla^\mu \phi + \frac{g_s^2}{4} e^{\frac{12}{7}\phi} F_{\mu\nu} F^{\mu\nu}. \quad (69)$$

We denote the linear perturbations of the metric, R-R 1-form and dilaton around the background fields (51)–(53) as $h_{\mu\nu}$, \hat{A}_μ , and $\hat{\phi}$, respectively. At the linear order of the perturbations, the equations of motion become

$$0 = \nabla_\mu \partial_\nu h + (\nabla^\rho \nabla_\rho h_{\mu\nu} - \nabla_\rho \nabla_\mu h_\nu^\rho - \nabla_\rho \nabla_\nu h_\mu^\rho) + h_{\mu\nu} R - g_{\mu\nu} (h^\rho{}_\sigma R^\sigma{}_\rho + \nabla^\rho \partial_\rho h - \nabla_\rho \nabla^\sigma h^\rho{}_\sigma) - \frac{8}{7} \left[h_{\mu\nu} \partial_\rho \phi \partial^\rho \phi - g_{\mu\nu} (h^\rho{}_\sigma \partial_\rho \phi \partial^\sigma \phi - 2\partial_\rho \hat{\phi} \partial^\rho \phi) - \frac{5}{7} (\partial_\mu \hat{\phi} \partial_\nu \phi + \partial_\mu \phi \partial_\nu \hat{\phi}) \right] - \frac{12}{7} \left[\nabla_\mu \partial_\nu \hat{\phi} - g_{\mu\nu} \nabla_\rho \partial^\rho \hat{\phi} - \frac{1}{2} (\nabla_\nu h^\rho{}_\mu + \nabla_\mu h^\rho{}_\nu - \nabla^\rho h_{\mu\nu}) \partial_\rho \phi - h_{\mu\nu} \nabla_\rho \partial^\rho \phi + g_{\mu\nu} (\nabla_\rho h^\rho{}_\sigma \partial^\sigma \phi + h^\rho{}_\sigma \nabla_\rho \partial^\sigma \phi - \frac{1}{2} \partial_\rho h \partial^\rho \phi) \right] - \frac{g_s^2}{4} e^{\frac{12}{7}\phi} \left[h_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + 4h^\rho{}_\sigma F_{\mu\rho} F_\nu{}^\sigma - 2g_{\mu\nu} h^\rho{}_\lambda F_{\rho\sigma} F^{\lambda\sigma} + 2g_{\mu\nu} \hat{F}_{\rho\sigma} F^{\rho\sigma} - 4(\hat{F}_{\mu\rho} F_\nu{}^\rho + F_{\mu\rho} \hat{F}_\nu{}^\rho) + \frac{12}{7} \hat{\phi} (g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} - 4F_{\mu\rho} F_\nu{}^\rho) \right], \quad (70)$$

$$0 = \frac{6}{7} (\partial_\nu \hat{\phi} F^{\nu\mu} + \partial_\nu \phi \hat{F}^{\nu\mu} - \partial_\nu \phi h^\nu{}_\rho F^{\rho\mu}) + \nabla_\nu \hat{F}^{\nu\mu} + \frac{1}{2} \partial_\nu h F^{\nu\mu} - \nabla_\nu h^\nu{}_\rho F^{\rho\mu} - h^\nu{}_\rho \nabla_\nu F^{\rho\mu} - \nabla_\nu h^\mu{}_\rho F^{\nu\rho}, \quad (71)$$

$$0 = -\nabla_\mu \partial^\mu h + \nabla_\mu \nabla^\nu h^\mu{}_\nu - h^\mu{}_\nu R^\nu{}_\mu - \frac{16}{49} (2\partial_\mu \hat{\phi} \partial^\mu \phi - h^\mu{}_\nu \partial_\mu \phi \partial^\nu \phi) + \frac{16}{21} \left(\frac{1}{2} \partial_\mu h \partial^\mu \phi - \nabla_\mu h^\mu{}_\nu \partial^\nu \phi - h^\mu{}_\nu \nabla_\mu \partial^\nu \phi + \nabla_\mu \partial^\mu \hat{\phi} \right) + \frac{g_s^2}{2} e^{\frac{12}{7}\phi} \left(\frac{6}{7} \hat{\phi} F_{\mu\nu} F^{\mu\nu} - h^\mu{}_\rho F_{\mu\nu} F^{\rho\nu} + \hat{F}_{\mu\nu} F^{\mu\nu} \right), \quad (72)$$

where $\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu$.

⁷Hereafter, we denote $\hat{\phi}$ as ϕ for simplicity.

We adopt the following gauge conditions [26]:

$$\nabla_i \left(h^i_j - \frac{1}{8} \delta_j^i h^k_k \right) = 0, \quad (73)$$

$$\nabla^i h_i^0 = \nabla^i h_i^z = 0, \quad (74)$$

$$\nabla^i \hat{A}_i = 0, \quad (75)$$

where $i = 1, \dots, 8$. Then, using the spherical harmonic expansions on S^8 , we can classify the perturbations into the scalar modes,

$$\begin{aligned} h_0^0(x^\mu) &= \sum b_0^0(t, z) Y(x^i), & h_z^0(x^\mu) &= \sum b_z^0(t, z) Y(x^i), \\ h_z^z(x^\mu) &= \sum b_z^z(t, z) Y(x^i), & h_i^i(x^\mu) &= \sum b_i^i(t, z) Y(x^i), \\ \hat{A}_0(x^\mu) &= \sum a_0(t, z) Y(x^i), & \hat{A}_z(x^\mu) &= \sum a_z(t, z) Y(x^i), \\ \hat{\phi}(x^\mu) &= \sum \varphi(t, z) Y(x^i), \end{aligned} \quad (76)$$

vector modes,

$$\begin{aligned} h_i^0(x^\mu) &= \sum b^0(t, z) Y_i(x^i), & h_i^z(x^\mu) &= \sum b^z(t, z) Y_i(x^i), \\ \hat{A}_i(x^\mu) &= \sum a(t, z) Y_i(x^i), \end{aligned} \quad (77)$$

and tensor mode,

$$h^i_j(x^\mu) - \frac{1}{8} \delta^i_j h^k_k(x^\mu) = \sum b(t, z) Y^i_j(x^i), \quad (78)$$

where Y , Y_i , and Y_{ij} are the scalar, vector, and tensor harmonics on S^8 , respectively. Here, we have suppressed the angular momentum indices. Since these modes are decoupled from each other, we can analyze each mode independently.

A. Solutions of equations of motion

As we will see later, the equations of motion can be reduced into the differential equations which generally take the following form:

$$\begin{aligned} 0 &= f^{-1} u^{-p} (u^p f \chi')' + \tilde{\omega}^2 f^{-2} u^{-\frac{9}{2}} \chi - k^2 f^{-1} u^{-2} \chi \\ &= \chi'' + \left(\frac{p}{u} - \frac{1}{1-u} \right) \chi' + \frac{\tilde{\omega}^2}{u^{\frac{9}{2}} (1-u)^2} \chi - \frac{k^2}{u^2 (1-u)} \chi. \end{aligned} \quad (79)$$

In this section, we will discuss the solutions and boundary conditions for this differential equation. The function χ is related to perturbations of the metric and R-R 1-form. The parameters p and k take

$$p = 0, \quad k^2 = \frac{l(l+7)}{49}, \quad (80)$$

for the tensor mode, and

$$p = \frac{9}{7}, \quad k^2 = \frac{(l+1)(l-1)}{49}, \frac{(l+6)(l+8)}{49}, \quad (81)$$

for two independent modes of the vector modes. The variable u is related to the radial coordinate z as

$$u = \left(\frac{z}{z_0} \right)^{\frac{14}{5}}, \quad (82)$$

and $\tilde{\omega}$ is the dimensionless frequency,

$$\tilde{\omega} = \frac{\omega}{4\pi T_H}. \quad (83)$$

Although it is difficult to solve the differential equation (79) for an arbitrary $\tilde{\omega}$, we can obtain the solution in the hydrodynamic regime, $\tilde{\omega} \ll 1$.

Near the horizon $u \simeq 1$, the leading contributions of the differential equation (79) are

$$0 = \chi'' - \frac{1}{1-u} \chi' + \frac{\tilde{\omega}^2}{(1-u)^2} \chi. \quad (84)$$

Then, the leading terms of two independent solutions of (79) are

$$\begin{aligned} \chi(\tilde{\omega}, u) &= C_1 (1-u)^{-i\tilde{\omega}} + C_2 (1-u)^{i\tilde{\omega}} \\ &\stackrel{\tilde{\omega} \ll 1}{\simeq} C_1 (1 - i\tilde{\omega} \ln(1-u)) + C_2 (1 + i\tilde{\omega} \ln(1-u)), \end{aligned} \quad (85)$$

where C_1 and C_2 are the integration constants. Imposing the ingoing boundary condition, C_2 must vanish [15,35].

For $\tilde{\omega} \ll 1$, χ can be expanded as a series of $\tilde{\omega}^2$ as

$$\chi(\tilde{\omega}, u) = \chi_0(u) + \tilde{\omega}^2 \chi_2(u) + \tilde{\omega}^4 \chi_4(u) + \dots, \quad (86)$$

and the coefficients $\chi_n(u)$ can be solved recursively. The differential equation for χ_0 is

$$0 = \chi_0'' + \left(\frac{p}{u} - \frac{1}{1-u} \right) \chi_0' - \frac{k^2}{u^2 (1-u)} \chi_0, \quad (87)$$

and the solution is⁸

$$\chi_0 = \tilde{C}_1 u^\alpha {}_2F_1(\alpha, \beta; \alpha + \beta; u) + \tilde{C}_2 u^\gamma {}_2F_1(\gamma, \delta; \gamma + \delta; u), \quad (88)$$

where \tilde{C}_1 and \tilde{C}_2 are the integration constants, ${}_2F_1$ is the Gauss' hypergeometric function and

$$\alpha = \frac{1}{2} (1 - p - \sqrt{(1-p)^2 + 4k^2}), \quad (89)$$

⁸When $\alpha + \beta \in \mathbf{Z}$ ($l \in 7\mathbf{Z}$), the solution of (87) is not given by (88) [43]. In this paper, we do not deal with this exceptional case.

$$\beta = \frac{1}{2}(1 + p - \sqrt{(1-p)^2 + 4k^2}), \quad (90)$$

$$\gamma = \frac{1}{2}(1 - p + \sqrt{(1-p)^2 + 4k^2}), \quad (91)$$

$$\delta = \frac{1}{2}(1 + p + \sqrt{(1-p)^2 + 4k^2}). \quad (92)$$

The expansion of the Gauss' hypergeometric function ${}_2F_1(a, b; c; u)$ around $u = 1$ is special when $a + b = c$. It is given by [43]

$$\begin{aligned} & {}_2F_1(a, b; a + b; u) \\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} [2\psi(n+1) - \psi(a+n) \\ &\quad - \psi(b+n) - \ln(1-u)] (1-u)^n, \end{aligned} \quad (93)$$

where $\psi(n)$ is the digamma function and

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1), \quad (a)_0 = 1, \quad (94)$$

$$\psi(1) = -\gamma_e = -0.57721\cdots, \quad (\gamma_e: \text{Euler constant}). \quad (95)$$

Thus, near the horizon $u \simeq 1$, χ_0 becomes

$$\begin{aligned} \chi_0 \stackrel{u \rightarrow 1}{\simeq} & \tilde{C}_1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} [-2\gamma_e - \psi(\alpha) - \psi(\beta) - \ln(1-u)] \\ & + \tilde{C}_2 \frac{\Gamma(\gamma+\delta)}{\Gamma(\gamma)\Gamma(\delta)} [-2\gamma_e - \psi(\gamma) - \psi(\delta) - \ln(1-u)]. \end{aligned} \quad (96)$$

This can be summarized in the form of (85). Since $C_2 = 0$ from the ingoing boundary condition, we find

$$\begin{aligned} \tilde{C}_2 = & -\tilde{C}_1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)} \frac{1 + i\tilde{\omega}(2\gamma_e + \psi(\alpha) + \psi(\beta))}{1 + i\tilde{\omega}(2\gamma_e + \psi(\gamma) + \psi(\delta))} \\ & \stackrel{\tilde{\omega} \rightarrow 0}{\simeq} -\tilde{C}_1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)} (1 + i\tilde{\omega}(\psi(\alpha) + \psi(\beta) \\ & \quad - \psi(\gamma) - \psi(\delta))). \end{aligned} \quad (97)$$

Imposing the Dirichlet boundary condition at the cutoff surface,

$$\chi(\tilde{\omega}, u_c) = \bar{\chi}(\tilde{\omega}), \quad (98)$$

the solution is

$$\begin{aligned} \chi(\tilde{\omega}, u) = & \frac{\bar{\chi}(\tilde{\omega})}{\mathcal{F}} \left[u^\alpha {}_2F_1(\alpha, \beta; \alpha + \beta; u) \right. \\ & \left. - \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)} (1 + i\tilde{\omega}(\psi(\alpha) + \psi(\beta) \right. \\ & \quad \left. - \psi(\gamma) - \psi(\delta))) u^\gamma {}_2F_1(\gamma, \delta; \gamma + \delta; u) \right] \\ & + \mathcal{O}(\tilde{\omega}^2), \end{aligned} \quad (99)$$

where

$$\begin{aligned} \mathcal{F} = & u_c^\alpha {}_2F_1(\alpha, \beta; \alpha + \beta; u_c) - \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)} \\ & \times (1 + i\tilde{\omega}(\psi(\alpha) + \psi(\beta) - \psi(\gamma) - \psi(\delta))) \\ & \times u_c^\gamma {}_2F_1(\gamma, \delta; \gamma + \delta; u_c). \end{aligned} \quad (100)$$

B. Tensor mode

Now, we solve the equations of motion, and calculate the on-shell action. We first consider the tensor mode. The equation of motion for the tensor mode b is

$$0 = f^{-1} z^{\frac{9}{5}} \partial_z (f z^{-\frac{9}{5}} \partial_z b) - f^{-2} \partial_0^2 b - \frac{4}{25} l(l+7) f^{-1} z^{-2} b. \quad (101)$$

For the tensor mode, the angular momentum must satisfy $l \geq 2$. Using the Fourier transformation,

$$b(t, u) = \int \frac{d\omega}{2\pi} b(\omega, u) e^{-i\omega t}, \quad (102)$$

and the coordinate u which is defined by (82), the equation of motion (101) becomes

$$0 = f^{-1} (fb')' + \tilde{\omega}^2 u^{-\frac{9}{5}} f^{-2} b - \frac{l(l+7)}{49} u^{-2} f^{-1} b, \quad (103)$$

where $\tilde{\omega}$ is the dimensionless frequency,

$$\tilde{\omega} = \frac{5z_0}{14} \omega = \frac{\omega}{4\pi T_H}. \quad (104)$$

The prime ' denotes the derivative with respect to u . From the result of Sec. IV A, the solution of (103) for $\tilde{\omega} \ll 1$ is

$$b(\tilde{\omega}, u) = \bar{b}(\tilde{\omega}) \frac{u^{-\frac{1}{7}} F(u) - X(\tilde{\omega}) u^{1+\frac{1}{7}} \tilde{F}(u)}{u_c^{-\frac{1}{7}} F(u_c) - X(\tilde{\omega}) u_c^{1+\frac{1}{7}} \tilde{F}(u_c)}, \quad (105)$$

where $u_c = (z_c/z_0)^{14/5}$ and

$$\bar{b}(\tilde{\omega}) \equiv b(\tilde{\omega}, u_c), \quad (106)$$

$$\begin{aligned} F(u) & \equiv {}_2F_1\left(-\frac{l}{7}, -\frac{l}{7}; -\frac{2l}{7}; u\right), \\ \tilde{F}(u) & \equiv {}_2F_1\left(1 + \frac{l}{7}, 1 + \frac{l}{7}; 2 + \frac{2l}{7}; u\right), \end{aligned} \quad (107)$$

$$X(\tilde{\omega}) \equiv \frac{\Gamma(-\frac{2l}{7})}{\Gamma(-\frac{l}{7})^2} \frac{\Gamma(1 + \frac{l}{7})^2}{\Gamma(2 + \frac{2l}{7})} \left[1 + 2\pi i \tilde{\omega} \cot\left(\frac{l\pi}{7}\right) \right]. \quad (108)$$

The on-shell action for b is

$$2\kappa^2 S_{\text{on-shell}} = \frac{7}{5} \tilde{R}^{\frac{9}{5}} z_0^{-\frac{14}{5}} D_2 \int_{u=u_c} dt [fbb' - u^{-1}(1+f)b^2]. \quad (109)$$

Derivation of the on-shell action is given in Appendix B 1. Hereafter, we do not consider the contact term, which does not contribute to the dissipative behavior. Inserting (105) into the on-shell action, we find

$$2\kappa^2 S_{\text{on-shell}} = \frac{7}{5} \tilde{R}^9 z_0^{-\frac{14}{5}} D_2 \int \frac{d\omega}{2\pi} (1-u_c) \bar{b}(-\omega) \bar{b}(\omega) \cdot \frac{-\frac{1}{7} u_c^{-1} F(u_c) + F'(u_c) - X u_c^{\frac{2}{7}} ((1+\frac{1}{7}) \tilde{F}(u_c) + u_c \tilde{F}'(u_c))}{F(u_c) - X u_c^{1+\frac{2}{7}} \tilde{F}(u_c)}. \quad (110)$$

C. Vector modes

Next, we consider the vector modes. In this case, the vector modes consist of metric components b^0 , b^z and R-R 1-form a . There are four equations of motion for these fields,

$$\begin{aligned} 0 = & \partial_z^2 b^0 + f^{-2} \partial_0 \partial_z b^z - \left(\frac{19}{5} z^{-1} - 2f^{-1} \partial_z f \right) \partial_z b^0 \\ & - \left(\frac{9}{5} z^{-1} f^{-2} + f^{-3} \partial_z f \right) \partial_0 b^z \\ & - \frac{4}{25} ((l+1)(l+6) - 49) z^{-2} f^{-1} b^0 \\ & + \frac{14}{5} g_s \tilde{R}^{-\frac{14}{5}} z^{\frac{9}{5}} f^{-1} \partial_z a, \quad (l \geq 1), \end{aligned} \quad (111)$$

$$\begin{aligned} 0 = & -f^{-2} \partial_0^2 b^z + (2z^{-1} - f^{-1} \partial_z f) \partial_0 b^0 - \partial_0 \partial_z b^0 \\ & - \frac{4}{25} (l+8)(l-1) f^{-1} z^{-2} b^z - \frac{14}{5} g_s z^{\frac{9}{5}} \tilde{R}^{-\frac{14}{5}} f^{-1} \partial_0 a, \\ & (l \geq 1), \end{aligned} \quad (112)$$

$$\begin{aligned} 0 = & \partial_z (f \partial_z a) + \frac{9}{5} z^{-1} f \partial_z a - f^{-1} \partial_0^2 a - \frac{4}{25} (l+1)(l+6) z^{-2} a \\ & + \frac{14}{5} g_s^{-1} \tilde{R}^{\frac{14}{5}} z^{-\frac{10}{5}} [\partial_z (f b^0) + f^{-1} \partial_0 b^z - 2z^{-1} f b^0], \\ & (l \geq 1), \end{aligned} \quad (113)$$

$$0 = \partial_0 b^0 + \partial_z b^z - \frac{19}{5} z^{-1} b^z, \quad (l \geq 2). \quad (114)$$

The Eq. (114) can be derived from (111) and (112). Since two Eqs. (112) and (113) contain the second order time derivative, there are two physical degrees of freedom in the vector modes. The other Eq. (114), or equivalently (111), gives a constraint on the boundary conditions. Therefore, these equations of motion yield two second order differential equations and one first order equation. The solution has five integration constants which can be fixed by two incoming boundary conditions at the horizon and three Dirichlet boundary conditions on the cutoff surface for b^0 , b^z , and a .

Let us set [26]

$$\hat{a} = -g_s a, \quad (115)$$

$$\hat{b} = -\frac{5}{14} \tilde{R}^{\frac{14}{5}} z^{-\frac{9}{5}} (z^2 \partial_z (z^{-2} f b^0) + f^{-1} \partial_0 b^z). \quad (116)$$

Then, the Eqs. (111)–(113) become

$$0 = \partial_z \hat{b} + \partial_z \hat{a} + \frac{2}{35} (l+8)(l-1) \tilde{R}^{\frac{14}{5}} z^{-\frac{19}{5}} b^0, \quad (117)$$

$$0 = \partial_0 \hat{b} + \partial_0 \hat{a} - \frac{2}{35} (l+8)(l-1) \tilde{R}^{\frac{14}{5}} z^{-\frac{19}{5}} b^z, \quad (118)$$

$$0 = z^{\frac{1}{5}} \partial_z (z^{\frac{9}{5}} f \partial_z \hat{a}) - z^2 f^{-1} \partial_0^2 \hat{a} - \frac{4}{25} (l+1)(l+6) \hat{a} + \frac{196}{25} \hat{b}. \quad (119)$$

Inserting (117) and (118) into (116), we find

$$\begin{aligned} 0 = & z^{\frac{1}{5}} \partial_z (z^{\frac{9}{5}} f \partial_z (\hat{a} + \hat{b})) - z^2 f^{-1} \partial_0^2 (\hat{a} + \hat{b}) \\ & - \frac{4}{25} (l+8)(l-1) \hat{b}. \end{aligned} \quad (120)$$

Using (119), the Eq. (120) becomes

$$\begin{aligned} 0 = & z^{\frac{1}{5}} \partial_z (z^{\frac{9}{5}} f \partial_z \hat{b}) - z^2 f^{-1} \partial_0^2 \hat{b} + \frac{4}{25} (l+1)(l+6) \hat{a} \\ & - \frac{4}{25} ((l+8)(l-1) + 49) \hat{b}. \end{aligned} \quad (121)$$

Thus, we have obtained two Eqs. (119) and (121) for \hat{a} and \hat{b} , which represent the two physical degrees of freedom of the vector modes.

To solve the equations, let us set

$$\mathbf{b} = \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix}, \quad (122)$$

$$\mathbf{M} = \begin{pmatrix} (l+1)(l+6) & -49 \\ -(l+1)(l+6) & (l+8)(l-1) + 49 \end{pmatrix}. \quad (123)$$

Then, the Eqs. (119) and (121) can be summarized in the following form:

$$0 = z^{\frac{1}{5}} \partial_z (z^{\frac{9}{5}} f \partial_z \mathbf{b}) - z^2 f^{-1} \partial_0^2 \mathbf{b} - \frac{4}{25} \mathbf{M} \cdot \mathbf{b}. \quad (124)$$

Since the eigenmatrix of \mathbf{M} is

$$\mathbf{A} = \mathbf{U}^{-1} \mathbf{M} \mathbf{U}, \quad = \begin{pmatrix} (l+1)(l-1) & 0 \\ 0 & (l+6)(l+8) \end{pmatrix}, \quad (125)$$

where

$$\mathbf{U} = \begin{pmatrix} 1 & 1 \\ \frac{l+1}{7} & -\frac{l+6}{7} \end{pmatrix}, \quad (126)$$

the Eq. (124) can be diagonalized as

$$0 = z^{\frac{1}{2}} \partial_z (z^{\frac{9}{2}} f \partial_z \mathbf{a}) - z^2 f^{-1} \partial_0^2 \mathbf{a} - \frac{4}{25} \boldsymbol{\Lambda} \cdot \mathbf{a}, \quad (127)$$

where

$$\mathbf{a} \equiv \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} = \mathbf{U}^{-1} \mathbf{b}. \quad (128)$$

Therefore, we obtain the diagonalized equations of motion,

$$0 = f^{-1} u^{-\frac{9}{2}} (u^{\frac{9}{2}} f \hat{a}'_1)' + f^{-2} u^{-\frac{9}{2}} \tilde{\omega}^2 \hat{a}_1 - \frac{(l+1)(l-1)}{49} f^{-1} u^{-2} \hat{a}_1, \quad (129)$$

$$0 = f^{-1} u^{-\frac{9}{2}} (u^{\frac{9}{2}} f \hat{a}'_2)' + f^{-2} u^{-\frac{9}{2}} \tilde{\omega}^2 \hat{a}_2 - \frac{(l+6)(l+8)}{49} f^{-1} u^{-2} \hat{a}_2, \quad (130)$$

where we have used the Fourier transformations of \hat{a}_1 and \hat{a}_2 .

From (117), (118), and (128), the original modes \hat{a} , b^0 , and b^z are

$$\hat{a} = \hat{a}_1 + \hat{a}_2, \quad (131)$$

$$b^0 = -7 \left(\frac{z_0}{\tilde{R}} \right)^{\frac{14}{5}} u^2 \left(\frac{\hat{a}'_1}{l-1} - \frac{\hat{a}'_2}{l+8} \right), \quad (132)$$

$$b^z = -7i \tilde{\omega} \left(\frac{z_0}{\tilde{R}} \right)^{\frac{14}{5}} u^{\frac{19}{14}} \left(\frac{\hat{a}_1}{l-1} - \frac{\hat{a}_2}{l+8} \right). \quad (133)$$

From the result of Sec. IV A, the solutions of the diagonalized equations of motion for $\tilde{\omega} \ll 1$ are

$$\hat{a}_1(\tilde{\omega}, u) = \bar{a}_1(\tilde{\omega}) \frac{u^{-\frac{1}{2}-\frac{l}{2}} F_1(u) - X_1(\tilde{\omega}) u^{-\frac{1}{2}+\frac{l}{2}} \tilde{F}_1(u)}{u_c^{-\frac{1}{2}-\frac{l}{2}} F_1(u_c) - X_1(\tilde{\omega}) u_c^{-\frac{1}{2}+\frac{l}{2}} \tilde{F}_1(u_c)}, \quad (134)$$

$$\hat{a}_2(\tilde{\omega}, u) = \bar{a}_2(\tilde{\omega}) \frac{u^{-\frac{8}{2}-\frac{l}{2}} F_2(u) - X_2(\tilde{\omega}) u^{\frac{6}{2}+\frac{l}{2}} \tilde{F}_2(u)}{u_c^{-\frac{8}{2}-\frac{l}{2}} F_2(u_c) - X_2(\tilde{\omega}) u_c^{\frac{6}{2}+\frac{l}{2}} \tilde{F}_2(u_c)}, \quad (135)$$

where $\bar{a}_{1,2}(\tilde{\omega}) \equiv \hat{a}_{1,2}(\tilde{\omega}, u_c)$ and

$$F_1 \equiv {}_2F_1 \left(-\frac{1}{7} - \frac{l}{7}, \frac{8}{7} - \frac{l}{7}; 1 - \frac{2l}{7}; u \right), \quad (136)$$

$$\tilde{F}_1 \equiv {}_2F_1 \left(-\frac{1}{7} + \frac{l}{7}, \frac{8}{7} + \frac{l}{7}; 1 + \frac{2l}{7}; u \right),$$

$$F_2 \equiv {}_2F_1 \left(-\frac{8}{7} - \frac{l}{7}, \frac{1}{7} - \frac{l}{7}; -1 - \frac{2l}{7}; u \right), \quad (137)$$

$$\tilde{F}_2 \equiv {}_2F_1 \left(\frac{6}{7} + \frac{l}{7}, \frac{15}{7} + \frac{l}{7}; 3 + \frac{2l}{7}; u \right),$$

$$X_1 \equiv \frac{\Gamma(1 - \frac{2l}{7})}{\Gamma(-\frac{1}{7} - \frac{l}{7}) \Gamma(\frac{8}{7} - \frac{l}{7})} \frac{\Gamma(-\frac{1}{7} + \frac{l}{7}) \Gamma(\frac{8}{7} + \frac{l}{7})}{\Gamma(1 + \frac{2l}{7})} (1 + i \tilde{\omega} S_l), \quad (138)$$

$$X_2 \equiv \frac{\Gamma(-1 - \frac{2l}{7})}{\Gamma(-\frac{8}{7} - \frac{l}{7}) \Gamma(\frac{1}{7} - \frac{l}{7})} \frac{\Gamma(\frac{6}{7} + \frac{l}{7}) \Gamma(\frac{15}{7} + \frac{l}{7})}{\Gamma(3 + \frac{2l}{7})} (1 + i \tilde{\omega} S_l), \quad (139)$$

$$S_l \equiv \frac{\pi \sin(\frac{2l}{7} \pi)}{\sin(\frac{l-1}{7} \pi) \sin(\frac{l+1}{7} \pi)}. \quad (140)$$

The on-shell action for the vector modes is

$$2\kappa^2 S_{\text{on-shell}} = \tilde{R}^{\frac{19}{5}} z_0^{-\frac{24}{5}} D_1 \int_{u=u_c} dt \left[\frac{1}{2} z_0 u^{-\frac{19}{14}} b^z \partial_0 b^0 + \left(\frac{2}{5} - \frac{7}{5} f^{-1} \right) u^{-\frac{12}{2}} (b^z)^2 + \frac{7}{5} u^{-\frac{12}{2}} f ((1+f)(b^0)^2 - f u b^0 (b^0)') \right] - \frac{7}{5} \tilde{R} z_0^{-2} D_1 \int_{u=u_c} dt u^{-\frac{5}{2}} f b^0 \hat{a} + \frac{7}{5} \tilde{R}^{-\frac{9}{5}} z_0^{\frac{4}{5}} D_1 \int_{u=u_c} dt f u^{\frac{9}{2}} \hat{a} \hat{a}'. \quad (141)$$

Derivation of the on-shell action is given in Appendix B 2. The first line of (141) is of the order of $\tilde{\omega}^2$ because b^z is proportional to $\tilde{\omega}$ from (133). Since we only consider the solutions of the equations of motion to the linear order of $\tilde{\omega}$, we neglect these terms. Suppressing the contact terms of (141), the on-shell action which we should analyze is

$$2\kappa^2 S_{\text{on-shell}} = -\frac{7}{5} \tilde{R}^{\frac{19}{5}} z_0^{-\frac{24}{5}} D_1 \int_{u=u_c} dt f^2 u^{-\frac{5}{2}} b^0 (b^0)' + \frac{7}{5} \tilde{R}^{-\frac{9}{5}} z_0^{\frac{4}{5}} D_1 \int_{u=u_c} dt f u^{\frac{9}{2}} \hat{a} \hat{a}'. \quad (142)$$

To evaluate the on-shell action, we need to express $(b^0)'$ and \hat{a}' in terms of $\bar{b}^0 \equiv b^0(u_c)$ and $\bar{a} \equiv a(u_c)$. In order to do this, we define the following functions:

$$G_1 = u^{\frac{8}{2}+\frac{l}{2}} (u^{-\frac{1}{2}-\frac{l}{2}} F_1)', \quad (143)$$

$$\tilde{G}_1 = u^{\frac{8}{2}-\frac{l}{2}} (u^{-\frac{1}{2}+\frac{l}{2}} \tilde{F}_1)', \quad (144)$$

$$G_2 = u^{\frac{15}{2}+\frac{l}{2}} (u^{-\frac{8}{2}-\frac{l}{2}} F_2)', \quad (145)$$

$$\tilde{G}_2 = u^{\frac{1}{2}-\frac{l}{2}} (u^{\frac{6}{2}+\frac{l}{2}} \tilde{F}_2)', \quad (146)$$

$$H_1 = u^{\frac{15}{2}+\frac{l}{2}} (u^{-\frac{8}{2}-\frac{l}{2}} G_1)', \quad (147)$$

$$\tilde{H}_1 = u^{\frac{15}{2}-\frac{l}{2}} (u^{-\frac{8}{2}+\frac{l}{2}} \tilde{G}_1)', \quad (148)$$

$$H_2 = u^{\frac{22}{2}+\frac{l}{2}} (u^{-\frac{15}{2}-\frac{l}{2}} G_2)', \quad (149)$$

$$\tilde{H}_2 = u^{\frac{8}{7}-\frac{l}{7}}(u^{-\frac{1}{7}+\frac{l}{7}}\tilde{G}_2)', \quad (150)$$

and

$$\mathcal{F}_1 = u_c^{-\frac{1}{7}-\frac{l}{7}}(F_1(u_c) - X_1 u_c^{\frac{2l}{7}}\tilde{F}_1(u_c)), \quad (151)$$

$$\mathcal{F}_2 = u_c^{-\frac{8}{7}-\frac{l}{7}}(F_2(u_c) - X_2 u_c^{2+\frac{2l}{7}}\tilde{F}_2(u_c)), \quad (152)$$

$$\mathcal{G}_1 = u_c^{-\frac{8}{7}-\frac{l}{7}}(G_1(u_c) - X_1 u_c^{\frac{2l}{7}}\tilde{G}_1(u_c)), \quad (153)$$

$$\mathcal{G}_2 = u_c^{-\frac{15}{7}-\frac{l}{7}}(G_2(u_c) - X_2 u_c^{2+\frac{2l}{7}}\tilde{G}_2(u_c)), \quad (154)$$

$$\mathcal{H}_1 = u_c^{-\frac{15}{7}-\frac{l}{7}}(H_1(u_c) - X_1 u_c^{\frac{2l}{7}}\tilde{H}_1(u_c)), \quad (155)$$

$$\mathcal{H}_2 = u_c^{-\frac{22}{7}-\frac{l}{7}}(H_2(u_c) - X_2 u_c^{2+\frac{2l}{7}}\tilde{H}_2(u_c)). \quad (156)$$

Then, one finds

$$\hat{a}'_{1,2}(u_c) = \bar{a}_{1,2} \frac{\mathcal{G}_{1,2}}{\mathcal{F}_{1,2}}, \quad (157)$$

$$\hat{a}''_{1,2}(u_c) = \bar{a}_{1,2} \frac{\mathcal{H}_{1,2}}{\mathcal{F}_{1,2}}. \quad (158)$$

From (131) and (132), \bar{a}_1 and \bar{a}_2 are

$$\bar{a}_1 = \frac{\mathcal{F}_1}{Q} \left(-\frac{\mathcal{G}_2}{l+8} g_s \bar{a} - \frac{1}{7} \left(\frac{\tilde{R}}{z_0} \right)^{\frac{14}{5}} u_c^{-2} \mathcal{F}_2 \bar{b}^0 \right), \quad (159)$$

$$\bar{a}_2 = \frac{\mathcal{F}_2}{Q} \left(-\frac{\mathcal{G}_1}{l-1} g_s \bar{a} + \frac{1}{7} \left(\frac{\tilde{R}}{z_0} \right)^{\frac{14}{5}} u_c^{-2} \mathcal{F}_1 \bar{b}^0 \right), \quad (160)$$

where

$$Q = \frac{\mathcal{F}_1 \mathcal{G}_2}{l+8} + \frac{\mathcal{F}_2 \mathcal{G}_1}{l-1}. \quad (161)$$

Taking the derivative of (131) and (132) with respect to u and using (157)–(160), we find

$$\begin{aligned} (b^0)'|_{u=u_c} &= \frac{2}{u_c} \bar{b}^0 + \frac{\bar{b}^0}{Q} \left(\frac{\mathcal{F}_2 \mathcal{H}_1}{l-1} + \frac{\mathcal{F}_1 \mathcal{H}_2}{l+8} \right) \\ &\quad - \frac{7u_c^2 g_s \bar{a}}{(l-1)(l+8)Q} \left(\frac{z_0}{\tilde{R}} \right)^{\frac{14}{5}} (\mathcal{G}_1 \mathcal{H}_2 - \mathcal{H}_1 \mathcal{G}_2), \end{aligned} \quad (162)$$

$$\hat{a}'|_{u=u_c} = \frac{\mathcal{F}_1 \mathcal{G}_2 - \mathcal{F}_2 \mathcal{G}_1}{7u_c^2 Q} \left(\frac{\tilde{R}}{z_0} \right)^{\frac{14}{5}} \bar{b}^0 - \frac{2l+7}{(l+8)(l-1)} \frac{\mathcal{G}_1 \mathcal{G}_2}{Q} g_s \bar{a}. \quad (163)$$

Since the first term of (162) becomes a contact term in the on-shell action, we will suppress it. In terms of $\mathcal{F}_{1,2}$, $\mathcal{G}_{1,2}$, $\mathcal{H}_{1,2}$, and Q , the on-shell action (142) is expressed as

$$\begin{aligned} 2\kappa^2 S_{\text{on-shell}} &= -\frac{7}{5} \tilde{R}^{\frac{19}{5}} z_0^{-\frac{24}{5}} D_1 \int_{u=u_c} \frac{d\omega}{2\pi} u^{-\frac{5}{7}} (1-u)^2 \frac{1}{Q} \left(\frac{\mathcal{F}_2 \mathcal{H}_1}{l-1} + \frac{\mathcal{F}_1 \mathcal{H}_2}{l+8} \right) \bar{b}^0(-\omega) \bar{b}^0(\omega) \\ &\quad + \frac{49}{5} g_s \tilde{R} z_0^{-2} D_1 \int_{u=u_c} \frac{d\omega}{2\pi} u^{\frac{9}{7}} (1-u)^2 \frac{\mathcal{G}_1 \mathcal{H}_2 - \mathcal{H}_1 \mathcal{G}_2}{(l-1)(l+8)Q} \bar{b}^0(-\omega) \bar{a}(\omega) \\ &\quad - \frac{1}{5} g_s \tilde{R} z_0^{-2} D_1 \int_{u=u_c} \frac{d\omega}{2\pi} u^{-\frac{5}{7}} (1-u) \frac{\mathcal{F}_1 \mathcal{G}_2 - \mathcal{F}_2 \mathcal{G}_1}{Q} \bar{b}^0(-\omega) \bar{a}(\omega) \\ &\quad + \frac{7}{5} g_s^2 \tilde{R}^{-\frac{9}{5}} z_0^{\frac{4}{5}} D_1 \int_{u=u_c} \frac{d\omega}{2\pi} u^{\frac{9}{7}} (1-u) \frac{2l+7}{(l-1)(l+8)} \frac{\mathcal{G}_1 \mathcal{G}_2}{Q} \bar{a}(-\omega) \bar{a}(\omega). \end{aligned} \quad (164)$$

D. The case of $u_c \simeq 1$

In the previous section, we have calculated the on-shell action on the cutoff surface at u_c . Since the solutions are expressed in terms of the hypergeometric functions, it is difficult to discuss properties of the linear response for arbitrary u_c . Hence, we focus on two regions, $u_c \simeq 1$ and $u_c \simeq 0$. We first consider the case of $u_c \simeq 1$ in which the cutoff surface is near the horizon. This corresponds to putting a cutoff at the low energy scale in the matrix theory side. Following [16,38], we evaluate the linear responses of matrix theory in terms of the proper quantities on the cutoff surface. According to [16,38], the stress tensor and R-R 1-form current are given by

$$\mathcal{T}^{\mu\nu} = \frac{\delta S_{\text{on-shell}}}{\sqrt{-\gamma} \delta \gamma_{\mu\nu}}, \quad (165)$$

$$\mathcal{J}^\mu = \frac{\delta S_{\text{on-shell}}}{\sqrt{-\gamma} \delta \bar{A}_\mu}, \quad (166)$$

where $\gamma_{\mu\nu}$ and \bar{A}_μ are the induced metric and R-R 1-form on the cutoff surface, respectively. These expressions can be understood in terms of the quasilocal charges. In fact, the expression (165) is the same as the definition of the Brown-York stress tensor [44].

1. Tensor mode

Here, we consider the tensor mode. Expanding (110) around $u_c = 1$, the on-shell action becomes

$$S_{\text{on-shell}} = \frac{1}{16\pi G} \frac{1}{2} \left(\frac{\tilde{R}}{z_0}\right)^{\frac{9}{5}} D_2 \int_{u_c \simeq 1} \frac{d\omega}{2\pi} i\omega \bar{b}(-\omega) \bar{b}(\omega), \quad (167)$$

up to the linear order of ω . Here, we have suppressed the contact terms in the action.

Let us define the proper frequency as

$$\text{w} = \frac{\omega}{\sqrt{-g_{00}}}. \quad (168)$$

Then, the on-shell action can be written as

$$S_{\text{on-shell}} = \frac{1}{16\pi G} \frac{1}{2} \left(\frac{\tilde{R}}{z_0}\right)^{\frac{9}{5}} D_2 \int_{u_c \simeq 1} \frac{d\omega}{2\pi} \sqrt{-g_{00}} i\text{w} \bar{b}(-\omega) \bar{b}(\omega). \quad (169)$$

The tensor mode of the metric perturbation $\bar{b}(\omega)$ is coupled to the tensor mode of the stress tensor $\mathcal{T}(\omega)$ in matrix theory.⁹ According to Sec. III C and (165), the linear response of the stress tensor is

$$\mathcal{T}(\omega) = \frac{1}{16\pi G} \left(\frac{r_0}{R}\right)^{\frac{9}{2}} i\text{w} \bar{b}(\omega), \quad (170)$$

which is the same as (31) with

$$\eta = \frac{1}{16\pi G} \left(\frac{r_0}{R}\right)^{\frac{9}{2}}. \quad (171)$$

Therefore, the linear response obeys the hydrodynamics on S^8 when $u_c \simeq 1$. Since the entropy density on the horizon in AdS frame is

$$s = \frac{S_{\text{BH}}}{V_8} = \frac{1}{4G} \left(\frac{r_0}{R}\right)^{\frac{9}{2}}, \quad (172)$$

we find¹⁰

$$\frac{\eta}{s} = \frac{1}{4\pi}, \quad (173)$$

which is the same as the shear viscosity to entropy density ratio in the membrane paradigm [2] or the AdS/CFT correspondence [15].

2. Vector modes

Next, we consider the vector mode for $u_c \simeq 1$. Expanding $\mathcal{F}_{1,2}$, $\mathcal{G}_{1,2}$, $\mathcal{H}_{1,2}$, and Q around $u_c = 1$ and suppressing the contact terms, the on-shell action becomes¹¹

$$\begin{aligned} 2\kappa^2 S_{\text{on-shell}} = & \frac{1}{2} \left(\frac{\tilde{R}}{z_0}\right)^{\frac{9}{5}} D_1 \int_{u_c \simeq 1} \frac{d\omega}{2\pi} \sqrt{-g_{00}} \frac{R^{(l+8)(l-1)}}{i\text{w} - \mathcal{D} \left(\frac{(l+8)(l-1)}{R^2} - \frac{14(2l^2+14l-7)}{(2l+7)S_l R^2} \right)} \bar{b}^{\tilde{0}}(-\omega) \bar{b}_{\tilde{0}}(\omega), \\ & - \frac{7g_s}{R} D_1 \int_{u_c \simeq 1} \frac{d\omega}{2\pi} \sqrt{-g_{00}} \frac{i\text{w} + \mathcal{D} \frac{14(l^2+7l+1)}{(2l+7)S_l R^2}}{i\text{w} - \mathcal{D} \left(\frac{(l+8)(l-1)}{R^2} - \frac{14(2l^2+14l-7)}{(2l+7)S_l R^2} \right)} \bar{b}_{\tilde{0}}(-\omega) \bar{a}(\omega) \\ & + \frac{g_s^2}{2} \left(\frac{z_0}{\tilde{R}}\right)^{\frac{9}{5}} D_1 \int_{u_c \simeq 1} \frac{d\omega}{2\pi} \sqrt{-g_{00}} \bar{a}(-\omega) \bar{a}(\omega) \\ & \cdot \frac{\mathcal{D}^2 \left(\frac{(l-1)(l+1)}{R^2} - \frac{14l}{S_l R^2} \right) \left(\frac{(l+6)(l+8)}{R^2} - \frac{14(l+7)}{S_l R^2} \right) - i\text{w} \mathcal{D} \left(\frac{2l^2+14l+47}{R^2} - \frac{14(2l+7)}{S_l R^2} \right) - \text{w}^2}{i\text{w} - \mathcal{D} \left(\frac{(l+8)(l-1)}{R^2} - \frac{14(2l^2+14l-7)}{(2l+7)S_l R^2} \right)}, \end{aligned} \quad (174)$$

where $\tilde{0}$ is the proper time index (namely, $\bar{b}_{\tilde{0}} = \bar{b}_0 / \sqrt{-g_{00}}$) and

$$\mathcal{D} \equiv \frac{1}{4\pi T} = \frac{\sqrt{-g_{00}}}{4\pi T_H}. \quad (175)$$

Here, T is the proper temperature.

The vector modes of the source fields $\bar{b}^{\tilde{0}}(\omega)$ and $\bar{a}(\omega)$ are coupled to the vector modes of the stress tensor and

R-R 1-form current in matrix theory, respectively. According to Sec. III C and (165) and (166), the linear responses of the vector modes of the stress tensor and R-R 1-form current are

⁹The tensor mode of the stress tensor is the Fourier coefficient of the tensor harmonics Y^{ij} in the spherical harmonic expansion of the stress tensor \mathcal{T}^{ij} .

¹⁰The dimensionless transport coefficients such as η/s do not depend on a choice of the frame.

¹¹Strictly speaking, we should neglect the $\mathcal{O}(\text{w}^2)$ term in the numerator of the last term in (174), since we have calculated only to linear order of w . Although this term could possibly receive corrections if we calculate $\mathcal{O}(\text{w}^2)$ contributions, we can see that this term is consistent with charged fluid.

$$\mathcal{T}^{\bar{0}} = \frac{1}{16\pi G} \left(\frac{R}{r_0}\right)^{\frac{9}{2}} \frac{\frac{(l+8)(l-1)}{R^2}}{i\omega - \mathcal{D}\left(\frac{(l+8)(l-1)}{R^2} - \frac{14(2l^2+14l-7)}{(2l+7)S_l R^2}\right)} \bar{b}^{\bar{0}} - \frac{7g_s}{16\pi GR} \frac{i\omega + \mathcal{D}\frac{14(l^2+7l+1)}{(2l+7)S_l R^2}}{i\omega - \mathcal{D}\left(\frac{(l+8)(l-1)}{R^2} - \frac{14(2l^2+14l-7)}{(2l+7)S_l R^2}\right)} \bar{a}, \quad (176)$$

$$\mathcal{J} = \frac{g_s^2}{16\pi G} \left(\frac{R}{r_0}\right)^{\frac{9}{2}} i\omega \bar{a} - \frac{49\mathcal{D}g_s^2}{16\pi GR^2} \left(\frac{R}{r_0}\right)^{\frac{9}{2}} \times \frac{i\omega - V}{i\omega - \mathcal{D}\left(\frac{(l+8)(l-1)}{R^2} - \frac{14(2l^2+14l-7)}{(2l+7)S_l R^2}\right)} \bar{a} + \frac{7g_s}{16\pi GR} \frac{i\omega + \mathcal{D}\frac{14(l^2+7l+1)}{(2l+7)S_l R^2}}{i\omega - \mathcal{D}\left(\frac{(l+8)(l-1)}{R^2} - \frac{14(2l^2+14l-7)}{(2l+7)S_l R^2}\right)} \bar{b}^{\bar{0}}, \quad (177)$$

where we have suppressed the contact terms and

$$V = \frac{18}{(2l+7)S_l} \left(i\omega - \mathcal{D} \frac{l^2 + 7l - 1}{R^2} \right) + \frac{28\mathcal{D}}{(2l+7)^2 S_l^2} \frac{11l^2 + 77l - 7}{R^2}. \quad (178)$$

Comparing the linear responses to the hydrodynamic stress tensor and current on S^8 with radius R , which are given by (33) and (34), we find the following:

- (i) Compared to the diffusion pole in (33) and (34), the denominators in (176) and (177) possess an extra term, $\frac{14(2l^2+14l-7)}{(2l+7)S_l R^2}$. However, the quantity \mathcal{D} matches with the diffusion constant (24). In fact, using the thermodynamic relation, $\bar{\epsilon} + \bar{p} = Ts$, and $\eta/s = 1/4\pi$, the diffusion constant becomes

$$D = \frac{\eta}{\bar{\epsilon} + \bar{p}} = \frac{\eta}{s} \cdot \frac{1}{T} = \frac{1}{4\pi T} = \mathcal{D}. \quad (179)$$

- (ii) Except for the extra term in the denominator, the first term of (176) agrees with the first term of (33) because the shear viscosity is given by (171).
- (iii) Except for the extra terms in the denominator and numerator, the second term of (176) and the third term of (177) agree with the second term of (33) and the third term of (34), respectively, because from (56), the charge density on the horizon in AdS frame is

$$\bar{n} = \frac{q}{V_8} = \frac{7g_s}{16\pi GR}. \quad (180)$$

- (iv) Except for the extra terms in the denominator and numerator, the second term of (177) agrees with the second term of (34) because

$$\frac{\bar{n}^2}{\bar{\epsilon} + \bar{p}} = \left(\frac{7g_s}{16\pi GR}\right)^2 \frac{1}{Ts} = \frac{49\mathcal{D}g_s^2}{16\pi GR^2} \left(\frac{R}{r_0}\right)^{\frac{9}{2}}. \quad (181)$$

- (v) The first term of (177) agrees with the first term of (34) if

$$\sigma = \frac{g_s^2}{16\pi G} \left(\frac{R}{r_0}\right)^{\frac{9}{2}}. \quad (182)$$

- (vi) The extra terms which appear in (176) and (177) are decoupled if we take l as large with l/R fixed (S_l^{-1} is of the order of 1). This means that the linear responses of the vector modes locally obey the hydrodynamics.

Although we have obtained charged fluid, the fluid should have universal structure near the horizon. Such universal structures appear if we take the Rindler limit. In the next subsection, we see that by taking a Rindler limit, (176) and (177) become the hydrodynamic stress tensor and current with $\bar{n} = 0$ in the 8-dimensional flat space.

3. Rindler limit

Let us look at a local region of S^8 , which can be approximated by \mathbf{R}^8 . Then, the metric of the 8-dimensional space is replaced by

$$R^2 d\Omega_8^2 \rightarrow dx^i dx_i. \quad (183)$$

The magnitude of the momentum in the flat 8-dimensional space is

$$k = \frac{l}{R}. \quad (184)$$

The Rindler limit is defined as follows: setting

$$1 - u = \left(\frac{7}{5\hat{R}}\right)^2 \varepsilon^2 \hat{r}^2, \quad (185)$$

$$x^i = \varepsilon \hat{x}^i, \quad (186)$$

and sending $\varepsilon \rightarrow 0$, the metric (50) becomes conformal to the Rindler metric,

$$ds_s^2 = \varepsilon^2 d\hat{s}^2, \quad (187)$$

$$d\hat{s}^2 = -\kappa^2 \hat{r}^2 dt^2 + d\hat{r}^2 + d\hat{x}^i d\hat{x}_i, \quad (188)$$

where $\kappa = 2\pi T_H$ gives the Hawking temperature of the Rindler spacetime, T_H . Since we have magnified a small region in S^8 , the proper frequency, momentum, and proper temperature are also rescaled. Those in the Rindler spacetime are related to the original ones as

$$\hat{\omega} = \varepsilon \omega, \quad (189)$$

$$\hat{k}^i = \varepsilon k^i, \quad (190)$$

$$\hat{T} = \varepsilon T. \quad (191)$$

The $(0, i)$ component of the metric perturbation in the Rindler spacetime is related to the original one as

$$h_{0i} = \varepsilon \hat{h}_{0i}, \quad (192)$$

because t is not rescaled by ε [38]. The Newton constant $G \sim g_s^2 l_s^8$ is also rescaled as $G = \varepsilon^8 \hat{G}$, where \hat{G} is the Newton constant in the Rindler spacetime because the string length in the Rindler spacetime is $\hat{l}_s = l_s/\varepsilon$. In this limit, the stress tensor (176) and the R-R 1-form current (177) become

$$\begin{aligned} \mathcal{T}^{\hat{0}i} &= \frac{1}{16\pi\hat{G}\varepsilon^9} \left(\frac{r_0}{R}\right)^{\frac{9}{2}} \frac{\hat{k}^2}{i\hat{\omega} - \hat{D}\hat{k}^2} \hat{h}^{\hat{0}i} \\ &\quad - \frac{7g_s}{16\pi\hat{G}R\varepsilon^8} \frac{i\hat{\omega}}{i\hat{\omega} - \hat{D}\hat{k}^2} \delta A^i, \end{aligned} \quad (193)$$

$$\begin{aligned} \mathcal{J}^i &= \frac{g_s^2}{16\pi\hat{G}\varepsilon^9} \left(\frac{R}{r_0}\right)^{\frac{9}{2}} i\hat{\omega} \delta A^i \\ &\quad - \frac{49\hat{D}g_s^2}{16\pi\hat{G}R^2\varepsilon^7} \left(\frac{R}{r_0}\right)^{\frac{9}{2}} \frac{i\hat{\omega}}{i\hat{\omega} - \hat{D}\hat{k}^2} \delta A^i \\ &\quad + \frac{7g_s}{16\pi\hat{G}R\varepsilon^8} \frac{i\hat{\omega}}{i\hat{\omega} - \hat{D}\hat{k}^2} \hat{h}^{\hat{0}i}, \end{aligned} \quad (194)$$

where $\hat{h}^{\hat{0}i} = \kappa \hat{r} \hat{h}^{0i}$ and $\hat{D} \equiv \frac{1}{4\pi\hat{r}}$. We have omitted the bar $(\bar{})$ which denotes the perturbations on the cutoff surface. Since the stress tensor and current in the Rindler spacetime are related to the original ones as

$$\hat{\mathcal{T}}^{\hat{0}i} = \varepsilon^9 \mathcal{T}^{\hat{0}i}, \quad (195)$$

$$\hat{\mathcal{J}}^i = \varepsilon^9 \mathcal{J}^i, \quad (196)$$

we find

$$\hat{\mathcal{T}}^{\hat{0}i} = \frac{1}{16\pi\hat{G}} \left(\frac{r_0}{R}\right)^{\frac{9}{2}} \frac{\hat{k}^2}{i\hat{\omega} - \hat{D}\hat{k}^2} \hat{h}^{\hat{0}i}, \quad (197)$$

$$\hat{\mathcal{J}}^i = \frac{g_s^2}{16\pi\hat{G}} \left(\frac{R}{r_0}\right)^{\frac{9}{2}} i\hat{\omega} \delta A^i. \quad (198)$$

Note that the second term in (193) and the second and third terms in (194) are decoupled in the limit of $\varepsilon \rightarrow 0$. Therefore, in the Rindler limit, (176) and (177) exactly match with the hydrodynamic stress tensor and current on \mathbf{R}^8 with no charge density. This result is consistent with the previous works on a Rindler fluid [16,38]. In a Rindler fluid, there is no charge density because the Rindler metric is a solution of the vacuum Einstein equation.

E. The case of $u_c \simeq 0$

We consider the case in which the cutoff surface is far from the horizon. At first, we calculate the linear responses

in terms of the proper quantities on the cutoff surface as in the previous section.

When we do not put the cutoff surface but consider the asymptotic boundary at $r \rightarrow \infty$, or equivalently, in the limit of $u_c = 0$, the divergent warp factor in the gravity side should be excluded from the correspondence [31,32]. In our case, since the dual geometry of matrix theory is essentially AdS₂ (S^8 is interpreted as the internal space in matrix theory), we have to care about the time-time component of the metric. We take into account the metric of matrix theory and obtain the linear responses of the stress tensor and R-R 1-form current. Then, we compare the linear responses with the hydrodynamic stress tensor and current on S^8 and discuss the differences between them.

1. Tensor mode

Here, we consider the tensor mode. Expanding (110) around $u_c = 0$, the on-shell action becomes

$$\begin{aligned} S_{\text{on-shell}} &= \frac{1}{16\pi G} \frac{1}{2} \left(\frac{r_0}{R}\right)^{\frac{9}{2}} D_2 \int \frac{d\omega}{2\pi} i\omega \frac{\Gamma(1 + \frac{1}{7})^4}{\Gamma(1 + \frac{2}{7})^2} \\ &\quad \times u_c^{\frac{2}{7}} \bar{b}(-\omega) \bar{b}(\omega), \end{aligned} \quad (199)$$

to the linear order of ω . Expressing this in terms of the proper quantities, we find

$$\begin{aligned} S_{\text{on-shell}} &= \frac{1}{16\pi G} \frac{1}{2} \left(\frac{r_0}{R}\right)^{\frac{9}{2}} D_2 \int \frac{d\omega}{2\pi} \sqrt{-g_{00}} i\omega \frac{\Gamma(1 + \frac{1}{7})^4}{\Gamma(1 + \frac{2}{7})^2} \\ &\quad \times u_c^{\frac{2}{7}} \bar{b}(-\omega) \bar{b}(\omega). \end{aligned} \quad (200)$$

Although the factor $u_c^{\frac{2}{7}}$ is usually absorbed into the field redefinition $\bar{b} \rightarrow u_c^{-\frac{4}{7}} \bar{b}$, we do not consider such a wave function renormalization because it does not change our conclusion. Namely, \bar{b} is a bare field in an energy scale which is determined by u_c .

Therefore, the linear response of the tensor mode of the stress tensor in terms of the proper quantities is

$$\mathcal{T}(\omega) = \frac{1}{16\pi G} \left(\frac{r_0}{R}\right)^{\frac{9}{2}} i\omega \frac{\Gamma(1 + \frac{1}{7})^4}{\Gamma(1 + \frac{2}{7})^2} u_c^{\frac{2}{7}} \bar{b}(\omega). \quad (201)$$

Comparing this with (170), we find the extra factor $\Gamma(1 + \frac{1}{7})^4/\Gamma(1 + \frac{2}{7})^2$ except for the factor $u_c^{\frac{2}{7}}$, which could be absorbed into the field redefinition of \bar{b} .

Taking into account the metric of matrix theory, the linear response becomes

$$\mathcal{T}(\omega) = \frac{1}{16\pi G} \left(\frac{r_0}{R}\right)^{\frac{9}{2}} i\omega \frac{\Gamma(1 + \frac{1}{7})^4}{\Gamma(1 + \frac{2}{7})^2} u_c^{\frac{2}{7}} \bar{b}(\omega). \quad (202)$$

This is different from the hydrodynamic stress tensor (31) even if we absorb the factor $u_c^{\frac{2l}{7}}$ into the field redefinition, because the shear viscosity η does not depend on l . Therefore, for the tensor mode, the linear response of the D0-branes cannot be explained by the hydrodynamics when the cutoff surface is far from the horizon.

2. Vector modes

Here, we consider the vector modes. Expanding (164) around $u_c = 0$, the on-shell action becomes

$$\begin{aligned}
2\kappa^2 S_{\text{on-shell}} = & -\frac{7}{5} \tilde{R}^{\frac{19}{5}} z_0^{-\frac{24}{5}} D_1 \int \frac{d\omega}{2\pi} \bar{b}^0(-\omega) \bar{b}^0(\omega) u_c^{-\frac{19}{7}-\frac{2l}{7}} \frac{2l+7}{18l(l-1)B^2} \cdot \frac{2l^2+23l-7+\frac{2}{7}l(2l+7)(l-1)B^2 u_c^{\frac{2l}{7}} i\tilde{\omega}}{i\tilde{\omega} - \frac{7(2l+7)}{9(l-1)B^2} u_c^{-1-\frac{2l}{7}}}, \\
& -\frac{7}{5} g_s \tilde{R} z_0^{-2} D_1 \int \frac{d\omega}{2\pi} \bar{b}^0(-\omega) \bar{a}(\omega) u_c^{-\frac{12}{7}-\frac{2l}{7}} \frac{2l+7}{9l(l-1)B^2} \cdot \frac{7-\frac{2}{7}l(2l+7)(l-1)B^2 u_c^{\frac{2l}{7}} i\tilde{\omega}}{i\tilde{\omega} - \frac{7(2l+7)}{9(l-1)B^2} u_c^{-1-\frac{2l}{7}}}, \\
& +\frac{7}{5} g_s^2 \tilde{R}^{-\frac{9}{5}} z_0^{\frac{4}{5}} D_1 \int \frac{d\omega}{2\pi} \bar{a}(-\omega) \bar{a}(\omega) u_c^{-\frac{5}{7}-\frac{2l}{7}} \frac{(2l+7)^2}{18l(l-1)B^2} \cdot \frac{l+1-\frac{2}{7}l(l-1)B^2 u_c^{\frac{2l}{7}} i\tilde{\omega}}{i\tilde{\omega} - \frac{7(2l+7)}{9(l-1)B^2} u_c^{-1-\frac{2l}{7}}}, \tag{203}
\end{aligned}$$

to the linear order of $\tilde{\omega}$, where

$$B \equiv \frac{\Gamma(-\frac{1}{7} + \frac{l}{7}) \Gamma(\frac{8}{7} + \frac{l}{7})}{\Gamma(1 + \frac{2l}{7})}. \tag{204}$$

According to Sec. III C and (165) and (166), the linear responses of the vector modes of the stress tensor and R-R 1-form current in terms of the proper quantities are

$$\begin{aligned}
\mathcal{T}^{\bar{0}} = & \frac{1}{16\pi G} \left(\frac{r_0}{R}\right)^{\frac{9}{5}} u_c^{-\frac{9}{7}-\frac{2l}{7}} \frac{2l+7}{18l(l-1)B^2} \\
& \cdot \frac{\frac{49}{R^2}(2l^2+23l-7) + \frac{2}{R}l(2l+7)(l-1)B^2 i\nu u_c^{-\frac{5}{14}+\frac{2l}{7}}}{i\nu - \frac{49(2l+7)}{9(l-1)B^2 R} u_c^{-\frac{9}{14}-\frac{2l}{7}}} \bar{b}^{\bar{0}} \\
& - \frac{7g_s}{16\pi GR} \frac{(2l+7)^2}{63u_c} \frac{i\nu - \frac{343}{2l(2l+7)(l-1)B^2 R} u_c^{\frac{5}{14}-\frac{2l}{7}}}{i\nu - \frac{49(2l+7)}{9(l-1)B^2 R} u_c^{-\frac{9}{14}-\frac{2l}{7}}} \bar{a}, \tag{205}
\end{aligned}$$

$$\begin{aligned}
\mathcal{J} = & \frac{g_s^2}{16\pi G} \left(\frac{R}{r_0}\right)^{\frac{9}{5}} \frac{(2l+7)^2}{9} u_c^{-\frac{5}{14}} \frac{\frac{49(l+1)}{2l(l-1)B^2 R^2} u_c^{\frac{5}{14}-\frac{2l}{7}} - \frac{i\nu}{R}}{i\nu - \frac{49(2l+7)}{9(l-1)B^2 R} u_c^{-\frac{9}{14}-\frac{2l}{7}}} \bar{a} \\
& + \frac{7g_s}{16\pi GR} \frac{(2l+7)^2}{63u_c} \frac{i\nu - \frac{343}{2l(2l+7)(l-1)B^2 R} u_c^{\frac{5}{14}-\frac{2l}{7}}}{i\nu - \frac{49(2l+7)}{9(l-1)B^2 R} u_c^{-\frac{9}{14}-\frac{2l}{7}}} \bar{b}^{\bar{0}}. \tag{206}
\end{aligned}$$

Taking into account the metric of matrix theory, the linear responses become

$$\begin{aligned}
\mathcal{T}^0 = & \frac{1}{16\pi G} \left(\frac{r_0}{R}\right)^{\frac{9}{5}} u_c^{-\frac{9}{7}-\frac{2l}{7}} \frac{2l+7}{18l(l-1)B^2} \\
& \cdot \frac{\frac{49}{R^2}(2l^2+23l-7) + \frac{5z_0}{R^2}l(2l+7)(l-1)B^2 u_c^{\frac{2l}{7}} i\omega}{i\omega - \frac{98(2l+7)}{45z_0(l-1)B^2} u_c^{-1-\frac{2l}{7}}} \bar{b}^0 \\
& - \frac{7g_s}{16\pi GR} \frac{(2l+7)^2}{63u_c} \frac{i\omega - \frac{343}{5z_0l(2l+7)(l-1)B^2} u_c^{-\frac{2l}{7}}}{i\omega - \frac{98(2l+7)}{45z_0(l-1)B^2} u_c^{-1-\frac{2l}{7}}} \bar{a}, \tag{207}
\end{aligned}$$

$$\begin{aligned}
\mathcal{J} = & \frac{g_s^2}{16\pi G} \left(\frac{R}{r_0}\right)^{\frac{9}{5}} u_c^{-\frac{5}{7}} \frac{2(2l+7)^2}{45} \frac{\frac{49(l+1)}{5l(l-1)z_0^4 B^2} u_c^{-\frac{2l}{7}} - \frac{i\omega}{z_0}}{i\omega - \frac{98(2l+7)}{45z_0(l-1)B^2} u_c^{-1-\frac{2l}{7}}} \bar{a} \\
& + \frac{7g_s}{16\pi GR} \frac{(2l+7)^2}{63u_c} \frac{i\omega - \frac{343}{5z_0l(2l+7)(l-1)B^2} u_c^{-\frac{2l}{7}}}{i\omega - \frac{98(2l+7)}{45z_0(l-1)B^2} u_c^{-1-\frac{2l}{7}}} \bar{b}^0. \tag{208}
\end{aligned}$$

These expressions are quite different from the linear responses in the case of $u_c \simeq 1$ or the hydrodynamic stress tensor and current on S^8 . Especially, there is no pole structure in (207) and (208) [or (205) and (206)] in the low frequency region because in any $l \geq 1$, the factor $u_c^{-1-\frac{2l}{7}}$ (or $u_c^{-\frac{9}{14}-\frac{2l}{7}}$) is very large when $u_c \simeq 0$. It is important to note that this fact is independent of the field redefinitions of \bar{b}^0 and \bar{a} . On the other hand, in hydrodynamics, there is a diffusion pole in the low frequency region.

In order to look at the pole structure in the vector modes, let us consider the denominators in the above expressions for arbitrary u_c . In general u_c , the denominators of the linear responses in the vector modes vanish when (161) equals zero. Namely, it is when

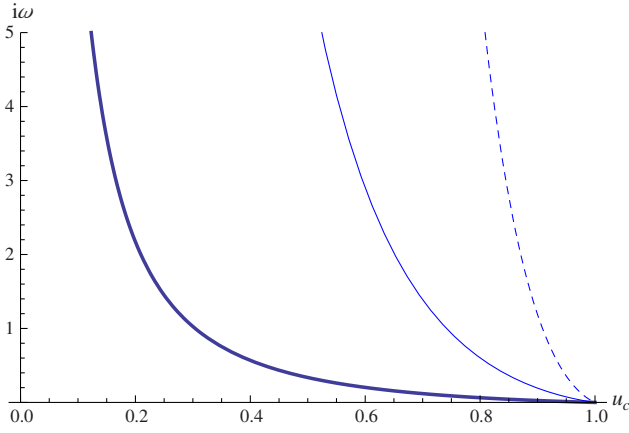


FIG. 1 (color online). The right-hand side of (209) is plotted against u_c when $l = 2$ (bold line), $l = 5$ (normal line), and $l = 10$ (dashed line).

$$i\tilde{\omega} = \frac{K_1 - B_1^{-1}\tilde{B}_1K_2 - B_2^{-1}\tilde{B}_2K_3 + B_1^{-1}\tilde{B}_1B_2^{-1}\tilde{B}_2K_4}{S_l(B_1^{-1}\tilde{B}_1K_2 + B_2^{-1}\tilde{B}_2K_3 - 2B_1^{-1}\tilde{B}_1B_2^{-1}\tilde{B}_2K_4)}, \quad (209)$$

where

$$K_1 = \frac{F_1G_2}{l+8} + \frac{F_2G_1}{l-1}, \quad (210)$$

$$K_2 = u^{\frac{2l}{7}}\left(\frac{\tilde{F}_1\tilde{G}_2}{l+8} + \frac{F_2\tilde{G}_1}{l-1}\right), \quad (211)$$

$$K_3 = u^{2+\frac{2l}{7}}\left(\frac{F_1\tilde{G}_2}{l+8} + \frac{\tilde{F}_2\tilde{G}_1}{l-1}\right), \quad (212)$$

$$K_4 = u^{2+\frac{4l}{7}}\left(\frac{\tilde{F}_1\tilde{G}_2}{l+8} + \frac{\tilde{F}_2\tilde{G}_1}{l-1}\right), \quad (213)$$

and

$$B_1 = \frac{\Gamma(-\frac{l+1}{7})\Gamma(\frac{8-l}{7})}{\Gamma(1-\frac{2l}{7})}, \quad (214)$$

$$\tilde{B}_1 = \frac{\Gamma(\frac{l-1}{7})\Gamma(\frac{l+8}{7})}{\Gamma(1+\frac{2l}{7})}, \quad (215)$$

$$B_2 = \frac{\Gamma(-\frac{l+8}{7})\Gamma(\frac{l-1}{7})}{\Gamma(-1-\frac{2l}{7})}, \quad (216)$$

$$\tilde{B}_2 = \frac{\Gamma(\frac{l+6}{7})\Gamma(\frac{l+15}{7})}{\Gamma(3+\frac{2l}{7})}. \quad (217)$$

Figure 1 shows the value of the right-hand side of (209) against u_c . Since $\tilde{\omega} \simeq \omega/T_H = \omega/T$, the left-hand side of (209) does not depend on the redshift. When $u_c \simeq 1$, the value of the right-hand side of (209) is close to zero for

any l . Therefore, we can find a pole structure in the low frequency region when the cutoff surface is close to the horizon. However, as u_c approaches zero, the value of the right-hand side of (209) becomes large and exceeds 1. Therefore, within the low frequency approximation, we cannot find the pole structure when the cutoff surface is far from the horizon.

V. SUMMARY AND COMMENTS

We have studied the linear responses of the near-extremal D0-branes in the low frequency region by using the gauge/gravity correspondence. We have analyzed the tensor mode and vector modes. We have found that when the cutoff surface, on which matrix theory is defined, is close to the horizon, the linear responses of the stress tensor and R-R 1-form current in matrix theory take forms similar to the hydrodynamic stress tensor and current on S^8 with radius R . By taking the Rindler limit, the linear responses of matrix theory exactly agree with the hydrodynamic stress tensor and current on \mathbf{R}^8 , which is consistent with the previous result on a Rindler fluid [16,38]. This is the limit in which the fluid takes the universal form for many black holes, but our results show that without taking the Rindler limit, the fluid keeps properties of D0-branes, such as the correct background charge. We have also found that when the cutoff surface is far from the horizon, the linear responses of matrix theory do not correspond to the hydrodynamic stress tensor and current on S^8 . Especially, we have found that in the low frequency region, the vector modes of the linear responses do not possess the pole structure, although the vector modes of the hydrodynamic stress tensor and current possess the diffusion pole. This fact does not depend on the field redefinitions of the source fields. From our results, we conclude that the linear responses of the D0-branes cannot be explained by hydrodynamics.

Three comments are in order. Firstly, we have analyzed the linear responses of matrix theory in the AdS frame, which is not the conventional frame, such as the Einstein frame or string frame. However, the choice of the frame does not change the dimensionless transport coefficients such as η/s , which are the physically sensible quantities.

Secondly, if we were able to absorb the extra factor $\Gamma(1+\frac{l}{7})^4/\Gamma(1+\frac{2l}{7})^2$ in (200) into the field redefinition of \tilde{b} , the tensor mode of the linear responses for $u_c \simeq 0$ would take the same form as the hydrodynamic stress tensor. However, since the discussion of the pole structure in the vector modes is independent of the field redefinitions, our conclusion does not change.

Finally, to understand what occurs in the D0-branes in the time-dependent external sources, we also need to analyze the linear responses in the high frequency region (or full frequency region). If we obtain the linear responses of the D0-branes in the high frequency region, we might be able to discuss the fast scrambling time via the

gauge/gravity correspondence [5,6]. This should be investigated in a future work.

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APPENDIX A: SPHERICAL HARMONICS ON S^8

We briefly summarize the definitions and properties of the spherical harmonics on S^8 according to [26].

A scalar field $\hat{\phi}$ on S^8 with radius R can be expanded as

$$\hat{\phi} = \sum_I \varphi^I(R) Y^I(x^i), \quad (\text{A1})$$

where I denotes the angular momentum indices and $x^i (i = 1, \dots, 8)$ are the spherical coordinates on the sphere. The function Y^I is called the scalar harmonic. In terms of the normalized Cartesian coordinates $\{x^m | m = 1, \dots, 9, x^m x_m = 1\}$, the explicit form of Y^I is given by

$$Y^I = C_{m_1 \dots m_l}^I x^{m_1} \dots x^{m_l}, \quad (l = 0, 1, \dots), \quad (\text{A2})$$

where $C_{m_1 \dots m_l}^I$ are totally symmetric and traceless in the indices (m_1, \dots, m_l) . The scalar harmonic satisfies

$$\nabla^i \nabla_i Y^I = -\frac{l(l+7)}{R^2} Y^I, \quad (l \geq 0), \quad (\text{A3})$$

where ∇_i is the covariant derivative on the sphere.

A vector field \hat{A}_i on the sphere can be expanded as

$$\hat{A}_i = \sum_I a^I(R) Y_i^I(x^i) + \sum_I \bar{a}^I(R) \nabla_i Y^I(x^i). \quad (\text{A4})$$

The function Y_i^I , which is divergentless $\nabla_i Y_i^I = 0$, is called the vector harmonic. In terms of the normalized Cartesian coordinates, the explicit form is given by

$$Y_n^I = C_{nm_1 \dots m_l}^I x^{m_1} \dots x^{m_l}, \quad (l = 1, 2, \dots), \quad (\text{A5})$$

where the coefficients $C_{nm_1 \dots m_l}^I$ are antisymmetric under the exchange of n and m_1 and totally symmetric and traceless with respect to the indices (m_1, \dots, m_l) . The vector harmonic satisfies

$$\nabla_i \nabla^i Y_j^I = \frac{-l(l+7)+1}{R^2} Y_j^I, \quad (l \geq 1). \quad (\text{A6})$$

If we impose the gauge condition $\nabla_i \hat{A}^i = 0$, the second term of (A4) vanishes. Then, the harmonic expansion is simplified as follows:

$$\hat{A}_i = \sum_I a^I Y_i^I. \quad (\text{A7})$$

A symmetric traceless tensor on the sphere can be expanded as

$$\begin{aligned} h_{ij} - \frac{1}{8} g_{ij} h_k^k &= \sum_I b^I(R) Y_{ij}^I(x^i) \\ &+ \sum_I \bar{b}^I(R) (\nabla_i Y_j^I + \nabla_j Y_i^I)(x^i) \\ &+ \sum_I \bar{\bar{b}}^I(R) \left(\nabla_i \nabla_j - \frac{1}{8} g_{ij} \nabla^k \nabla_k \right) Y^I(x^i). \end{aligned} \quad (\text{A8})$$

The function Y_{ij}^I , which is symmetric, traceless, and divergentless, is called the tensor harmonic. In terms of the normalized Cartesian coordinates, the explicit form is given by

$$Y_{n_1 n_2}^I = C_{n_1 n_2 m_1 \dots m_l}^I x^{m_1} \dots x^{m_l}, \quad (l = 2, 3, \dots), \quad (\text{A9})$$

where the coefficients $C_{n_1 n_2 m_1 \dots m_l}^I$ are antisymmetric under the exchange of (n_1, m_1) , symmetric under the exchange of (n_1, n_2) , and totally symmetric and traceless with respect to m_1, \dots, m_l . The tensor harmonic satisfies

$$\nabla_i \nabla^i Y_{jk}^I = \frac{-l(l+7)+2}{R^2} Y_{jk}^I, \quad (l \geq 2). \quad (\text{A10})$$

If we impose the gauge condition $\nabla^i (h_{ij} - \frac{1}{8} g_{ij} h_k^k) = 0$, the harmonics expansion is simplified as follows:

$$h_{ij} - \frac{1}{8} g_{ij} h_k^k = \sum_I b^I Y_{ij}^I. \quad (\text{A11})$$

APPENDIX B: DERIVATION OF ON-SHELL ACTION

1. Tensor mode

We insert

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad (\text{B1})$$

$$A_\mu = \bar{A}_\mu + \hat{A}_\mu, \quad (\text{B2})$$

$$\phi = \bar{\phi} + \hat{\phi}, \quad (\text{B3})$$

into the action (64) and expand the action around the background fields up to the quadratic order of the perturbations. Using the mode expansions (76)–(78) and the formulas for the spherical harmonics in Appendix A, we obtain for the tensor mode,

$$\begin{aligned} 2\kappa^2 S'_{IIA} &= 2D_2 \int d^2x \sqrt{-\bar{g}_2} \Phi \left[\left(-\frac{63}{25} - \frac{1}{25} l(l+7) \right) \tilde{R}^{-2} b^2 \right. \\ &\quad \left. + b \nabla_a \nabla^a b + \frac{3}{4} \nabla_a b \nabla^a b \right], \end{aligned} \quad (\text{B4})$$

where $a, b = 0, z$; $\Phi = e^{-\frac{6}{5}\bar{\phi}} = (\tilde{R}/z)^{\frac{6}{5}}$; $\sqrt{-\bar{g}_2}$ is the square root of the determinant of \bar{g}_{ab} ; and n^a is the unit

normal to the cutoff surface. By varying the action (B4) with respect to b , we find the covariant form of the equation of motion (101),

$$0 = \Phi \nabla_a \nabla^a b + \nabla_a \Phi \nabla^a b + 2 \nabla^a \nabla_a \Phi b - \frac{4}{25} \tilde{R}^{-2} \Phi (63 + l(l+7)) b. \quad (\text{B5})$$

Inserting the equation of motion (B5) into the action (B4), we find

$$\begin{aligned} 2\kappa^2 S'_{\text{IIA}} &= 2D_2 \int_{z=z_c} dt \sqrt{-\bar{g}_{00}} \left[-\frac{3}{4} n^a \Phi b \nabla_a b \right. \\ &\quad \left. + \frac{1}{2} n^a \nabla_a \Phi b^2 \right] \\ &= -\frac{3}{2} \tilde{R}^{\frac{9}{5}} D_2 \int_{z=z_c} dt z^{-\frac{9}{5}} f b \partial_z b \\ &\quad - \frac{9}{5} \tilde{R}^{\frac{9}{5}} D_2 \int_{z=z_c} dt z^{-\frac{14}{5}} f b^2. \end{aligned} \quad (\text{B6})$$

Besides, we have to add the Gibbons-Hawking term (65) to the action. For the tensor mode, the Gibbons-Hawking term is

$$\begin{aligned} 2\kappa^2 S'_{\text{IIA}} &= D_1 \int d^2 x \sqrt{-\bar{g}_2} e^{-\frac{6}{7}\phi} \left[\left(-\frac{18}{25} f - \frac{47}{25} - \frac{2}{25} l(l+7) \right) \tilde{R}^{-2} b^a b_a - b^a \nabla_a \nabla_b b^b - (\nabla_a b^a)^2 - b^a \nabla_b \nabla_a b^b \right. \\ &\quad \left. + 2b^a \nabla_b \nabla^b b_a - \frac{1}{2} \nabla_b b_a \nabla^a b^b + \frac{3}{2} \nabla_b b_a \nabla^b b^a + \frac{16}{49} (\partial_a \bar{\phi} \partial_b \bar{\phi}) b^a b^b - \frac{14}{5} g_s \tilde{R}^{-\frac{14}{5}} z^{\frac{9}{5}} \epsilon^{ab} b_a \nabla_b a \right. \\ &\quad \left. - \frac{g_s^2}{2} \tilde{R}^{-\frac{18}{5}} z^{\frac{18}{5}} \nabla_a a \nabla^a a - \frac{2}{25} g_s^2 z^{\frac{18}{5}} \tilde{R}^{-\frac{28}{5}} (l+1)(l+6) a^2 \right], \end{aligned} \quad (\text{B9})$$

where $\epsilon^{0z} = -(-\bar{g}_2)^{-1/2} = -\tilde{R}^{-2} z^2$, $\epsilon_{0z} = (-\bar{g}_2)^{1/2} = \tilde{R}^2 z^{-2}$, and $\nabla_c \epsilon^{ab} = 0$. By varying the action with respect to b^a and a , we obtain the covariant forms of the equations of motion (111)–(113),

$$\begin{aligned} 0 &= \left(-\frac{94}{25} - \frac{36}{25} f - \frac{4}{25} l(l+7) \right) \tilde{R}^{-2} \Phi b^a \\ &\quad + \frac{32}{49} (\partial^a \bar{\phi} \partial_b \bar{\phi}) \Phi b^b - \nabla^a \nabla_b \Phi b^b - \nabla_b \nabla^a \Phi b^b \\ &\quad + 2 \nabla_b \nabla^b \Phi b^a - \nabla_b \Phi \nabla^a b^b + \nabla_b \Phi \nabla^b b^a - \Phi \nabla_b \nabla^a b^b \\ &\quad + \Phi \nabla_b \nabla^b b^a - \frac{14}{5} g_s \tilde{R}^{-1} \epsilon^{ab} \nabla_b a, \end{aligned} \quad (\text{B10})$$

$$\begin{aligned} 0 &= \frac{14}{5} g_s^{-1} \tilde{R}^{\frac{14}{5}} z^{-\frac{19}{5}} \epsilon^{ab} \nabla_b b_a + \tilde{R}^2 z^{-\frac{19}{5}} \nabla_a (z^{\frac{9}{5}} \nabla^a a) \\ &\quad - \frac{4}{25} (l+1)(l+6) z^{-2} a. \end{aligned} \quad (\text{B11})$$

Inserting these equations into the action (B9), we find

$$\begin{aligned} 2\kappa^2 S_{\text{GB}} &= -\tilde{R}^{\frac{9}{5}} D_2 \int_{z=z_c} dt z^{-\frac{14}{5}} f b^2 \\ &\quad + \frac{1}{2} \tilde{R}^{\frac{9}{5}} D_2 \int_{z=z_c} dt z^{-\frac{9}{5}} f' b^2 \\ &\quad + 2 \tilde{R}^{\frac{9}{5}} D_2 \int_{z=z_c} dt z^{-\frac{9}{5}} f b \partial_z b. \end{aligned} \quad (\text{B7})$$

Therefore, the total on-shell action is

$$\begin{aligned} 2\kappa^2 S_{\text{on-shell}} &= 2\kappa^2 (S'_{\text{IIA}} + S_{\text{GB}}) \\ &= \frac{1}{2} \tilde{R}^{\frac{9}{5}} D_2 \int_{z=z_c} dt z^{-\frac{9}{5}} f b \partial_z b \\ &\quad - \frac{7}{5} \tilde{R}^{\frac{9}{5}} D_2 \int_{z=z_c} dt z^{-\frac{14}{5}} (1+f) b^2, \end{aligned} \quad (\text{B8})$$

which is the same as (109).

2. Vector modes

In the same way, we calculate the on-shell action for the vector modes. Up to the quadratic order of the vector modes, the action is

$$\begin{aligned} 2\kappa^2 S'_{\text{IIA}} &= D_1 \int_{z=z_c} dt \sqrt{-g_{00}} n_a \left[\Phi b^a \nabla_b b^b + \frac{1}{2} \Phi b^b \nabla_b b^a \right. \\ &\quad \left. - \frac{3}{2} \Phi b_b \nabla^a b^b - \frac{7}{5} g_s \tilde{R}^{-1} \epsilon^{ab} b_b a \right. \\ &\quad \left. + \frac{g_s^2}{2} \tilde{R}^{-\frac{9}{5}} z^{\frac{9}{5}} a \nabla^a a + \nabla^a \Phi b_b b^b - \nabla_b \Phi b^b b^a \right], \end{aligned} \quad (\text{B12})$$

$$\begin{aligned} &= \tilde{R}^{\frac{19}{5}} D_1 \int_{z=z_c} dt \left[z^{-\frac{19}{5}} b^z \partial_0 b^0 - z^{-\frac{24}{5}} (b^z)^2 + \frac{1}{2} f^{-1} f' z^{-\frac{19}{5}} (b^z)^2 \right. \\ &\quad \left. + \frac{1}{2} z^{-\frac{19}{5}} b^0 \partial_0 b^z - \frac{1}{5} z^{-\frac{24}{5}} f^2 (b^0)^2 + z^{-\frac{19}{5}} f f' (b^0)^2 \right. \\ &\quad \left. + \frac{3}{2} z^{-\frac{19}{5}} f^2 b^0 \partial_z b^0 \right] + \frac{7}{5} g_s \tilde{R} D_1 \int_{z=z_c} dt z^{-2} f b^0 a \\ &\quad + \frac{g_s^2}{2} \tilde{R}^{-\frac{9}{5}} D_1 \int_{z=z_c} dt f z^{\frac{9}{5}} a \partial_z a. \end{aligned} \quad (\text{B13})$$

Since the Gibbons-Hawking term for the vector modes is

$$2\kappa^2 S_{\text{GB}} = \tilde{R}^{\frac{19}{5}} D_1 \int_{z=z_c} dt \left[\left(3z^{-\frac{24}{5}} f^2 - \frac{3}{2} z^{-\frac{19}{5}} f f' \right) (b^0)^2 - 2z^{-\frac{19}{5}} f^2 b^0 \partial_z b^0 \right], \quad (\text{B14})$$

the total on-shell action for the vector modes is

$$2\kappa^2 S_{\text{on-shell}} = \tilde{R}^{\frac{19}{5}} D_1 \int_{z=z_c} dt \left[\frac{1}{2} z^{-\frac{19}{5}} b^z \partial_0 b^0 + \frac{2}{5} z^{-\frac{24}{5}} (b^z)^2 - \frac{7}{5} f^{-1} z^{-\frac{24}{5}} (b^z)^2 + \frac{7}{5} z^{-\frac{24}{5}} f^2 (b^0)^2 + \frac{7}{5} z^{-\frac{24}{5}} f (b^0)^2 - \frac{1}{2} z^{-\frac{19}{5}} f^2 b^0 \partial_z b^0 \right] \\ + \frac{7}{5} g_s \tilde{R} D_1 \int_{z=z_c} dt z^{-2} f b^0 a + \frac{g_s^2}{2} \tilde{R}^{-\frac{9}{5}} D_1 \int_{z=z_c} dt f z^{\frac{9}{5}} a \partial_z a. \quad (\text{B15})$$

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