Leading finite-size effects on some three-point correlators in *TsT*-deformed $\mathrm{AdS}_5 \times S^5$

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We compute the leading finite-size effects on the normalized structure constants in semiclassical three-point correlation functions of two finite-size giant magnon string states and three different types of "light" states: primary scalar operators, the dilaton operator with nonzero momentum, and singlet scalar operators on higher string levels. This is done for the case of the TsT-transformed, or γ-deformed, $\mathrm{AdS}_5 \times S^5$ string theory background, dual to $\mathcal{N} = 1$ super Yang-Mills theory in four dimensions, arising as an exactly marginal deformation of $\mathcal{N} = 4$ super Yang-Mills theory.

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I. INTRODUCTION

The AdS/CFT duality [1] between string theories on curved spacetimes with anti–de Sitter subspaces and conformal field theories in different dimensions has been actively investigated in the last few years. A lot of impressive progress has been made in this field of research based mainly on the integrability structures discovered on both sides of the correspondence (for a recent overview of the AdS/CFT duality, see Ref. [2]). The most studied example is the correspondence between type IIB string theory on $\mathrm{AdS}_5 \times S^5$ target space and the $\mathcal{N} = 4$ super Yang-Mills theory (SYM) in four spacetime dimensions. However, many other cases are also of interest, and have been investigated intensively.

Different classical string solutions play important roles in checking and understanding the AdS/CFT correspondence [3]. To establish relations with the dual gauge theory, one has to take the semiclassical limit of *large* conserved charges [4]. A crucial example of such a string solution is the so-called "giant magnon," discovered by Hofman and Maldacena in the $R_t \times S^2$ subspace of $\mathrm{AdS}_5 \times S^5$ [5]. It gave a strong support for the conjectured all-loop SU(2) spin chain arising in the dual $\mathcal{N} = 4$ SYM, and made it possible to get a deeper insight into the AdS/CFT duality. A characteristic feature of this solution is that the string energy E and the angular momentum J_1 go to infinity, but the difference $E - J_1$ remains finite and related to the momentum of the magnon excitations in the dual spin chain in $\mathcal{N} = 4$ SYM. This string configuration has been extended to the case of a *dyonic* giant magnon, a solution for a string moving on $R_t \times S^3$ and having a second nonzero angular momentum J_2 [6]. A further extension to $R_t \times S^5$ has been also worked out in Ref. [7]. It was also shown there that such types of string solutions can be obtained by the reduction of the string dynamics to the Neumann-Rosochatius integrable system by using a specific ansatz.

An interesting issue to solve is to find the *finite-size effect*, i.e., a large but finite J_1, related to the wrapping interactions in the dual field theory [8]. For (dyonic) giant magnons living in $\mathrm{AdS}_5 \times S^5$ this was done in Refs. [9,10]. The corresponding string solutions, along with the (leading) finite-size corrections to their dispersion relations, have been found.

Here, we are going to consider the leading finite-size effects on some three-point correlation functions in a γ-deformed [11] or TsT-transformed [12] $\mathrm{AdS}_5 \times S^5$ string theory background. To this end, we will need to use our knowledge about the properties of the finite-size (dyonic) giant magnon solutions on this target space. The corresponding information can be found in Refs. [13,14].

In this paper we will be interested in the case of three-point correlators, when two of the "heavy" string states are finite-size giant magnons, while the third state is a "light" one.[1] We will consider three different types of "light" states: primary scalar operators, a dilaton operator with nonzero momentum, and singlet scalar operators on higher string levels. The finite-size effects on such correlation functions in TsT-transformed $\mathrm{AdS}_5 \times S^5$ were found in Refs. [17,18]. There, the normalized structure constants in these correlators were given in terms of several parameters and hypergeometric functions of two variables depending on them. On the other hand, it is important to know their dependence on the conserved string charges J_1, J_2 and the worldsheet momentum p, because namely these quantities are related to the corresponding operators in the dual $\mathcal{N} = 1$ SYM, and the momentum of the magnon excitations in the dual spin chain. This is why we are going to find this dependence here. Unfortunately, this cannot be done exactly for the finite-size case due to the complicated dependence between the above-mentioned parameters and J_1, J_2, p. Because of this, we will consider only the leading-order finite-size effects on the three-point correlators of this type. Moreover, due to computational complications, we will restrict ourselves to the case of $J_2 = 0$.

This paper is organized as follows. In Sec. II, we give a short review of the finite-size (dyonic) giant magnon's solution on γ-deformed $\mathrm{AdS}_5 \times S^5$. Also, we give the

[1] The first papers in which three-point correlation functions of two "heavy" operators and a "light" operator were computed are Refs. [15,16].

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corresponding exact semiclassical results for the three-point correlators we are interested in, found in Refs. [17,18]. Section III is devoted to the computation of the leading-order finite-size effects on the three-point correlators given in Sec. II in terms of the conserved string angular momentum J_1 and the worldsheet momentum p. In Sec. IV we conclude with some final remarks.

II. FINITE-SIZE GIANT MAGNONS ON *TsT*-DEFORMED $AdS_5 \times S^5$ AND SOME THREE-POINT CORRELATORS

A. Short review of the giant magnon solutions

Investigations on AdS/CFT duality for the cases with reduced or without supersymmetry is of obvious importance and interest. An example of such a correspondence between gauge and string theory models with reduced supersymmetry is provided by an exactly marginal deformation of $\mathcal{N} = 4$ SYM and string theory on a β-deformed $AdS_5 \times S^5$ background suggested by Lunin and Maldacena in Ref. [11]. When $\beta \equiv \gamma$ is real, the deformed background can be obtained from $AdS_5 \times S^5$ by the so-called TsT transformation on S^5. It includes T duality on one angle variable, a shift of another isometry variable, then a second T duality on the first angle [12]. The AdS_5 part of the background is untouched, so the conformal invariance remains.

An essential property of the TsT transformation is that it preserves the classical integrability of string theory on $AdS_5 \times S^5$ [12]. The γ dependence enters only through the *twisted* boundary conditions and the *level-matching* condition. The last one is modified since a closed string in the deformed background corresponds to an open string on $AdS_5 \times S^5$ in general.

The parameter $\tilde{\gamma}$, which appears in the string action, is related to the deformation parameter γ as

$$\tilde{\gamma} = 2\pi T\gamma = \sqrt{\lambda}\gamma,$$

where T is the string tension and λ is the t'Hooft coupling.

The effect of introducing γ on the field-theory side of the duality is to modify the super potential as follows:

$$W \propto \mathrm{Tr}(e^{i\pi\gamma}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\gamma}\Phi_1\Phi_3\Phi_2).$$

This leads to reduction of the supersymmetry of the SYM theory from $\mathcal{N} = 4$ to $\mathcal{N} = 1$.

Since we are going to consider three-point correlation functions with two vertices corresponding to giant magnon states, we can restrict ourselves to the subspace $R_t \times S^3_\gamma$ of the $AdS_5 \times S^5_\gamma$ background. Then one can show that by using the ansatz

$$\begin{aligned} &t(\tau,\sigma) = \kappa\tau, \qquad \theta(\tau,\sigma) = \theta(\xi),\\ &\phi_j(\tau,\sigma) = \omega_j\tau + f_j(\xi), \qquad \xi = \alpha\sigma + \beta\tau, \qquad (2.1)\\ &\kappa, \omega_j, \alpha, \beta = \text{constants}, \quad j = 1,2, \end{aligned}$$

the string Lagrangian in conformal gauge, on the γ-deformed three-sphere S^3_γ, can be written as [19] (a prime is used for $d/d\xi$)

$$\begin{aligned} \mathcal{L}_\gamma = (\alpha^2-\beta^2)\Bigg[&\theta'^2 + G\sin^2\theta\left(f_1' - \frac{\beta\omega_1}{\alpha^2-\beta^2}\right)^2 \\ &+ G\cos^2\theta\left(f_2' - \frac{\beta\omega_2}{\alpha^2-\beta^2}\right)^2 \\ &- \frac{\alpha^2}{(\alpha^2-\beta^2)^2}G(\omega_1^2\sin^2\theta + \omega_2^2\cos^2\theta) \\ &+ 2\alpha\tilde{\gamma}G\sin^2\theta\cos^2\theta\frac{\omega_2 f_1' - \omega_1 f_2'}{\alpha^2-\beta^2}\Bigg], \qquad (2.2) \end{aligned}$$

where

$$G = \frac{1}{1+\tilde{\gamma}^2\sin^2\theta\cos^2\theta}.$$

By using Eq. (2.2) and the Virasoro constraints, one can find the following first integrals of the string equations of motion:

$$\begin{aligned} f_1' &= \frac{\Omega_1}{\alpha}\frac{1}{1-v^2}\left[\frac{vW-uK}{1-\chi} - v(1-\tilde{\gamma}K) - \tilde{\gamma}u\chi\right],\\ f_2' &= \frac{\Omega_1}{\alpha}\frac{1}{1-v^2}\left[\frac{K}{\chi} - uv(1-\tilde{\gamma}K) - \tilde{\gamma}v^2W + \tilde{\gamma}(1-\chi)\right],\\ \chi' &= \frac{2\sqrt{1-u^2}}{1-v^2}\sqrt{(\chi_p-\chi)(\chi-\chi_m)(\chi-\chi_n)}, \qquad (2.3) \end{aligned}$$

where

$$\begin{aligned} &\chi = \cos^2\theta, \qquad v = -\beta/\alpha, \qquad u = \frac{\Omega_2}{\Omega_1},\\ &W = \left(\frac{\kappa}{\Omega_1}\right)^2, \qquad K = \frac{C_2}{\alpha\Omega_1}, \qquad \Omega_1 = \omega_1\left(1+\tilde{\gamma}\frac{C_2}{\alpha\omega_1}\right),\\ &\Omega_2 = \omega_2\left(1-\tilde{\gamma}\frac{C_1}{\alpha\omega_2}\right), \qquad C_1, C_2 = \text{constants}. \end{aligned}$$

Also, the following equalities hold:

$$\begin{aligned} \chi_p + \chi_m + \chi_n &= \frac{2-(1+v^2)W-u^2}{1-u^2},\\ \chi_p\chi_m + \chi_p\chi_n + \chi_m\chi_n &= \frac{1-(1+v^2)W+(vW-uK)^2-K^2}{1-u^2}, \qquad (2.4)\\ \chi_p\chi_m\chi_n &= -\frac{K^2}{1-u^2}. \end{aligned}$$

The case of dyonic finite-size giant magnons corresponds to

$$0 < u < 1, \qquad 0 < v < 1, \qquad 0 < W < 1, \qquad 0 < \chi_m < \chi < \chi_p < 1, \qquad \chi_n < 0.$$

The AdS_5 part of the giant magnon solution, in Euclidean Poincare coordinates, can be written as[2] ($t = \sqrt{W}\tau$, $i\tau = \tau_e$)[3]

$$z = \frac{1}{\cosh(\sqrt{W}\tau_e)} x_{0e} = \tanh(\sqrt{W}\tau_e), \qquad x_i = 0, \quad i = 1, 2, 3.$$

Let us also write down the exact expressions for the conserved charges and the angular differences,

$$\mathcal{E} \equiv \frac{2\pi E}{\sqrt{\lambda}} = 2\frac{(1-v^2)\sqrt{W}}{\sqrt{1-u^2}} \frac{\mathbf{K}(1-\epsilon)}{\sqrt{\chi_p - \chi_n}}, \tag{2.5}$$

$$\mathcal{J}_1 \equiv \frac{2\pi J_1}{\sqrt{\lambda}} = \frac{2}{\sqrt{1-u^2}}\left[\frac{1-\chi_n - v(vW - uK)}{\sqrt{\chi_p - \chi_n}}\mathbf{K}(1-\epsilon) - \sqrt{\chi_p - \chi_n}\mathbf{E}(1-\epsilon)\right], \tag{2.6}$$

$$\mathcal{J}_2 \equiv \frac{2\pi J_2}{\sqrt{\lambda}} = \frac{2}{\sqrt{1-u^2}}\left[\frac{u\chi_n - vK}{\sqrt{\chi_p - \chi_n}}\mathbf{K}(1-\epsilon) + u\sqrt{\chi_p - \chi_n}\mathbf{E}(1-\epsilon)\right], \tag{2.7}$$

$$\begin{aligned} p_1 \equiv \Delta\phi_1 = \phi_1(L) - \phi_1(-L) = \frac{2}{\sqrt{1-u^2}}\bigg\{&\frac{vW - uK}{(1-\chi_p)\sqrt{\chi_p - \chi_n}}\Pi\left(-\frac{\chi_p - \chi_m}{1-\chi_p}\,\middle|\, 1-\epsilon\right) \\ &- [v(1-\tilde{\gamma}K) + \tilde{\gamma}u\chi_n]\frac{\mathbf{K}(1-\epsilon)}{\sqrt{\chi_p - \chi_n}} - \tilde{\gamma}u\sqrt{\chi_p - \chi_n}\mathbf{E}(1-\epsilon)\bigg\}, \end{aligned} \tag{2.8}$$

$$\begin{aligned} p_2 \equiv \Delta\phi_2 = \phi_2(L) - \phi_2(-L) = \frac{2}{\sqrt{1-u^2}}\bigg\{&\frac{K}{\chi_p\sqrt{\chi_p - \chi_n}}\Pi\left(1 - \frac{\chi_m}{\chi_p}\,\middle|\, 1-\epsilon\right) \\ &- [uv + \tilde{\gamma}v(vW - uK) - \tilde{\gamma}(1-\chi_n)]\frac{\mathbf{K}(1-\epsilon)}{\sqrt{\chi_p - \chi_n}} - \tilde{\gamma}\sqrt{\chi_p - \chi_n}\,\mathbf{E}(1-\epsilon)\bigg\}, \end{aligned} \tag{2.9}$$

where the following notation has been introduced:

$$\epsilon = \frac{\chi_m - \chi_n}{\chi_p - \chi_n}. \tag{2.10}$$

Here, E, $J_{1,2}$ are the string energy and angular momenta. $\mathbf{K}(1-\epsilon)$, $\mathbf{E}(1-\epsilon)$, and $\Pi(1-\frac{\chi_m}{\chi_p}|1-\epsilon)$ are the complete elliptic integrals of the first, second, and third kind. The parameter L appearing above is related to the size of the giant magnons. For finite-size giant magnons L is finite, while for infinite-size giant magnons $L \to \infty$.

Let us also point out that for the γ-deformed case even the giant magnon with $J_2 = 0$ lives on S^3_γ. This happens because that is the smallest consistent reduction due to the *twisted* boundary conditions [13].

The dyonic giant magnon dispersion relation, including the leading finite-size correction, can be written as [14]

$$\mathcal{E} - \mathcal{J}_1 = \sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)} - \frac{\sin^4(p/2)}{\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}}\cos(\Phi)\epsilon, \tag{2.11}$$

where

$$\epsilon = 16\exp\left[-\frac{2(\mathcal{J}_1 + \sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)})\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}\sin^2(p/2)}{\mathcal{J}_2^2 + 4\sin^4(p/2)}\right]. \tag{2.12}$$

[2]The Euclidean continuation of the time-like directions to $t_e = it$, $x_{0e} = ix_0$ will allow the classical trajectories to approach the AdS_5 boundary $z = 0$ when $\tau_e \to \pm\infty$, and to compute the corresponding correlation functions.

[3]We set $\alpha = \Omega_1 = 1$ for simplicity.

The second term in Eq. (2.11) represents the leading finite-size effect on the energy-charge relation, which disappears for $\epsilon \to 0$, or equivalently $\mathcal{J}_1 \to \infty$. It is nonzero only when $\mathcal{J}_1$ is finite. The γ-deformation effect is represented by $\cos(\Phi)$.

In the next section, we will restrict our considerations to the case $\mathcal{J}_2 = 0$. Then Eq. (2.11) simplifies to

$$\mathcal{E} - \mathcal{J}_1 = 2\sin(p/2) - \frac{1}{2}\sin^3(p/2)\cos(\Phi)\epsilon, \quad (2.13)$$

where

$$\epsilon = 16\exp\left[-\frac{\mathcal{J}_1}{\sin(p/2)} - 2\right],$$
$$\Phi = 2\pi\left(n_2 - \frac{\tilde{\gamma}}{2\pi}\mathcal{J}_1\right), \quad n_2 \in \mathbb{Z}. \quad (2.14)$$

B. Semiclassical three-point correlation functions

It is known that the correlation functions of any conformal field theory can be determined in principle in terms of the basic conformal data $\{\Delta_i, C_{ijk}\}$, where Δ_i are the conformal dimensions defined by the two-point correlation functions,

$$\langle \mathcal{O}_i^\dagger(x_1)\mathcal{O}_j(x_2)\rangle = \frac{C_{12}\delta_{ij}}{|x_1 - x_2|^{2\Delta_i}},$$

and C_{ijk} are the structure constants in the operator product expansion,

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \frac{C_{ijk}}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_1 - x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1}}.$$

Therefore, the determination of the initial conformal data for a given conformal field theory is the most important step in the conformal bootstrap approach.

The three-point functions of two "heavy" operators and a "light" operator can be approximated by a supergravity vertex operator evaluated at the "heavy" classical string configuration [20,21],

$$\langle V_H(x_1)V_H(x_2)V_L(x_3)\rangle = V_L(x_3)_{\text{classical}}.$$

For $|x_1| = |x_2| = 1$, $x_3 = 0$, the correlation function reduces to

$$\langle V_H(x_1)V_H(x_2)V_L(0)\rangle = \frac{C_{123}}{|x_1 - x_2|^{2\Delta_H}}.$$

Then, the normalized structure constants

$$\mathcal{C} = \frac{C_{123}}{C_{12}}$$

can be found from

$$\mathcal{C} = c_\Delta V_L(0)_{\text{classical}}, \quad (2.15)$$

were c_Δ is the normalized constant of the corresponding "light" vertex operator.

Until now, investigations of the *finite-size* effects in the three-point correlators have been performed in Refs. [17–19,22–24]. This was done for the cases when the "heavy" string states are *finite-size* giant magnons, with one or two angular momenta,[4] and for three different "light" states:

(1) Primary scalar operators: $V_L = V_j^{\text{pr}}$.
(2) Dilaton operator: $V_L = V_j^d$.
(3) Singlet scalar operators on higher string levels: $V_L = V^q$.

According to Ref. [20], the corresponding (unintegrated) vertices are given by

$$V_j^{\text{pr}} = (Y_4 + Y_5)^{-\Delta_{\text{pr}}}(X_1 + iX_2)^j \times [z^{-2}(\partial x_m\bar{\partial}x^m - \partial z\bar{\partial}z) - \partial X_k\bar{\partial}X_k], \quad (2.16)$$

where the scaling dimension is $\Delta_{\text{pr}} = j$. The corresponding operator in the dual gauge theory is $\text{Tr}(Z^j)$,[5]

$$V_j^d = (Y_4 + Y_5)^{-\Delta_d}(X_1 + iX_2)^j \times [z^{-2}(\partial x_m\bar{\partial}x^m + \partial z\bar{\partial}z) + \partial X_k\bar{\partial}X_k], \quad (2.17)$$

where now the scaling dimension $\Delta_d = 4 + j$ to the leading order in the large-$\sqrt{\lambda}$ expansion. The corresponding operator in the dual gauge theory is proportional to $\text{Tr}(F_{\mu\nu}^2 Z^j + \cdots)$, or for $j = 0$, just to the SYM Lagrangian,

$$V^q = (Y_4 + Y_5)^{-\Delta_q}(\partial X_k\bar{\partial}X_k)^q. \quad (2.18)$$

This operator corresponds to a scalar *string* state at level $n = q - 1$, and to the leading order in the $\frac{1}{\sqrt{\lambda}}$ expansion

$$\Delta_q = 2\left(\sqrt{(q-1)\sqrt{\lambda} + 1 - \frac{1}{2}q(q-1)} + 1\right). \quad (2.19)$$

The value $n = 1(q = 2)$ corresponds to a massive string state on the first exited level and the corresponding operator in the dual gauge theory is an operator contained within

[4]See also Ref. [25], where the finite-size correction to a three-point correlation function was found when the "heavy" state is not a giant magnon one.

[5]Z is a complex scalar.

the Konishi multiplet. Higher values of n label higher string levels.

In Eqs. (2.16), (2.17), and (2.18) we denoted with Y, X the coordinates in AdS and the sphere parts of the $AdS_5 \times S^5_\gamma$ background,

$$Y_1 + iY_2 = \sinh\rho \sin\eta e^{i\varphi_1},$$
$$Y_3 + iY_4 = \sinh\rho \cos\eta e^{i\varphi_2},$$
$$Y_5 + iY_0 = \cosh\rho e^{it}.$$

The coordinates Y are related to the Poincare coordinates by

$$Y_m = \frac{x_m}{z}, \qquad Y_4 = \frac{1}{2z}(x^m x_m + z^2 - 1),$$
$$Y_5 = \frac{1}{2z}(x^m x_m + z^2 + 1),$$

where $x^m x_m = -x_0^2 + x_i x_i$, with $m = 0, 1, 2, 3$ and $i = 1, 2, 3$.

The semiclassical results found in Refs. [17,18] for the normalized structure constants (2.15), in the case of finite-size giant magnons on the γ-deformed $AdS_5 \times S^5_\gamma$ and the above three vertices, are given by

$$\mathcal{C}^{\rm pr}_{j\tilde\gamma} = \pi^{3/2} c^{\rm pr}_j \frac{\Gamma(\frac{j}{2})}{\Gamma(\frac{1+j}{2})} \frac{(1-v^2)\chi_p^{j/2}}{\sqrt{(1-u^2)(\chi_p-\chi_n)}} \left\{\left[\sqrt{W}\frac{j-1}{j+1} + \frac{1}{\sqrt{W}(1-v^2)}(2-(1+v^2)W - 2\tilde\gamma K)\right]\right.$$
$$\times F_1\left(1/2, 1/2, -j/2; 1; 1-\epsilon, 1-\frac{\chi_m}{\chi_p}\right) - \frac{2}{\sqrt{W}(1-v^2)}[1-\tilde\gamma K - u(u-\tilde\gamma u K + \tilde\gamma v W)]\chi_p$$
$$\left.\times F_1\left(1/2, 1/2, -1-j/2; 1; 1-\epsilon, 1-\frac{\chi_m}{\chi_p}\right)\right\}, \tag{2.20}$$

$$\mathcal{C}^{d}_{j\tilde\gamma} = 2\pi^{3/2} c^{d}_{4+j} \frac{\Gamma(\frac{4+j}{2})}{\Gamma(\frac{5+j}{2})} \frac{\chi_p^{j/2}}{\sqrt{(1-u^2)W(\chi_p-\chi_n)}} \left\{[1-\tilde\gamma K - u(u+\tilde\gamma(vW - uK))]\chi_p\right.$$
$$\left.\times F_1\left(1/2, 1/2, -1-j/2; 1; 1-\epsilon, 1-\frac{\chi_m}{\chi_p}\right) - (1-W-\tilde\gamma K) F_1\left(1/2, 1/2, -j/2; 1; 1-\epsilon, 1-\frac{\chi_m}{\chi_p}\right)\right\}, \tag{2.21}$$

$$\mathcal{C}^{q}_{\tilde\gamma} = c_{\Delta_q}\pi^{3/2} \frac{\Gamma(\frac{\Delta_q}{2})}{\Gamma(\frac{\Delta_q+1}{2})} \frac{(-2A)^q}{(1-v^2)^{q-1}\sqrt{(1-u^2)W(\chi_p-\chi_n)}} \sum_{k=0}^{q} \frac{q!}{k!(q-k)!}\left(-\frac{B}{A}\right)^k \chi_p^k F_1\left(\frac{1}{2},\frac{1}{2},-k;1;1-\epsilon,1-\frac{\chi_m}{\chi_p}\right), \tag{2.22}$$

where

$$A = 1 - \frac{1}{2}(1+v^2)W - \tilde\gamma K, \qquad B = 1 - \tilde\gamma K - u[u - \tilde\gamma(Ku - vW)], \tag{2.23}$$

and $F_1(a, b_1, b_2; c; z_1, z_2)$ is one of the hypergeometric functions of two variables (Appell F_1).

III. LEADING-ORDER FINITE-SIZE CORRECTIONS

From now on, we restrict ourselves to the case $\mathcal{J}_2 = 0$ and with $\mathcal{J}_1$ large but finite, i.e., $J_1 \gg \sqrt{\lambda}$. This means that the problem reduces to considering the limit $\epsilon \to 0$, since $\epsilon = 0$ corresponds to the infinite-size case, i.e., $\mathcal{J}_1 = \infty$ [see Eq. (2.14)]. To this end, we introduce the expansions

$$\chi_p = \chi_{p0} + (\chi_{p1} + \chi_{p2}\log(\epsilon))\epsilon, \qquad \chi_m = \chi_{m0} + (\chi_{m1} + \chi_{m2}\log(\epsilon))\epsilon, \qquad \chi_n = \chi_{n0} + (\chi_{n1} + \chi_{n2}\log(\epsilon))\epsilon,$$
$$v = v_0 + (v_1 + v_2\log(\epsilon))\epsilon, \qquad u = u_0 + (u_1 + u_2\log(\epsilon))\epsilon,$$
$$W = W_0 + (W_1 + W_2\log(\epsilon))\epsilon, \qquad K = K_0 + (K_1 + K_2\log(\epsilon))\epsilon. \tag{3.1}$$

To be able to reproduce the dispersion relation for the infinite-size giant magnons, we set

$$\chi_{m0} = \chi_{n0} = K_0 = 0, \qquad W_0 = 1. \tag{3.2}$$

Also, it can be proved that if we keep the coefficients χ_{m2}, χ_{n2}, W_2, and K_2 nonzero, the known leading correction to the giant magnon energy-charge relation in Eq. (2.13) will be modified by a term proportional to $\mathcal{J}_1^2$. This is why we choose

$$\chi_{m2} = \chi_{n2} = W_2 = K_2 = 0. \tag{3.3}$$

In addition, since we are considering for simplicity giant magnons with one angular momentum ($\mathcal{J}_2 = 0$), we also set

$$u_0 = 0, \tag{3.4}$$

because the leading term in the ϵ expansion of $\mathcal{J}_2$ is proportional to u_0. Thus, Eq. (3.1) simplifies to

$$\begin{gathered}\chi_p = \chi_{p0} + (\chi_{p1} + \chi_{p2}\log(\epsilon))\epsilon, \qquad \chi_m = \chi_{m1}\epsilon, \qquad \chi_n = \chi_{n1}\epsilon, \qquad v = v_0 + (v_1 + v_2\log(\epsilon))\epsilon, \\ u = (u_1 + u_2\log(\epsilon))\epsilon, \qquad W = 1 + W_1\epsilon, \qquad K = K_1\epsilon.\end{gathered} \tag{3.5}$$

By replacing Eq. (3.5) in Eqs. (2.4) and (2.10), one finds

$$\begin{gathered}\chi_{p0} = 1 - v_0^2, \qquad \chi_{p1} = \frac{v_0}{1 - v_0^2}\left[v_0\sqrt{(1 - v_0^2)^4 - 4K_1^2(1 - v_0^2)} - 2(1 - v_0^2)v_1\right], \qquad \chi_{p2} = -2v_0v_2, \\ \chi_{m1} = \frac{(1 - v_0^2)^2 + \sqrt{(1 - v_0^2)^4 - 4K_1^2(1 - v_0^2)}}{2(1 - v_0^2)}, \qquad \chi_{n1} = -\frac{(1 - v_0^2)^2 - \sqrt{(1 - v_0^2)^4 - 4K_1^2(1 - v_0^2)}}{2(1 - v_0^2)}, \\ W_1 = -\frac{\sqrt{(1 - v_0^2)^4 - 4K_1^2(1 - v_0^2)}}{1 - v_0^2}.\end{gathered} \tag{3.6}$$

The expressions for the other parameters in Eqs. (3.5) and (3.6) can be derived in the following way. First, we impose the conditions $\mathcal{J}_2 = 0$ and p_1 to be independent of ϵ. This leads to four equations,

$$\begin{gathered}v_1 = \frac{v_0\sqrt{(1 - v_0^2)^4 - 4K_1^2(1 - v_0^2)}(1 - \log 16)}{4(1 - v_0^2)}, \qquad v_2 = \frac{v_0\sqrt{(1 - v_0^2)^4 - 4K_1^2(1 - v_0^2)}}{4(1 - v_0^2)}, \\ u_1 = \frac{K_1 v_0 \log 4}{1 - v_0^2}, \qquad u_2 = -\frac{K_1 v_0}{2(1 - v_0^2)},\end{gathered} \tag{3.7}$$

where

$$v_0 = \cos\frac{p_1}{2}. \tag{3.8}$$

Second, expanding $\mathcal{J}_1$ and $p_2 = 2\pi n_2$ ($n_2 \in \mathbb{Z}$)[6] to the leading order in ϵ, we obtain [compare with Eq. (2.14)]

$$\epsilon = 16\exp\left(-\frac{\mathcal{J}_1}{\sin\frac{p_1}{2}} - 2\right), \qquad K_1 = \frac{1}{2}\sin^3\frac{p_1}{2}\sin\Phi, \qquad \Phi = 2\pi\left(n_2 - \frac{\tilde{\gamma}}{2\pi}\mathcal{J}_1\right). \tag{3.9}$$

Now, we are going to use the above results to find the leading-order finite-size effects on the normalized structure constants in terms of $\mathcal{J}_1 \equiv \mathcal{J}$, $p_1 \equiv p$, and Φ.

A. Giant magnons on $AdS_5 \times S^5_\gamma$ and primary scalar operators

As was pointed out in Ref. [24], where the undeformed case was considered, $j = 1$ and $j = 2$ are special values. This is why the corresponding normalized structure constants can be found in the Appendix. Here, we will deal with $j \geq 3$, when we can use the following representation of $F_1(a, b_1, b_2; c; z_1, z_2)$ [26]:

$$F_1(a, b_1, b_2; c; z_1, z_2) = \sum_{k=0}^{\infty}\frac{(a)_k(b_2)_k}{(c)_k}\,{}_2F_1(a + k, b_1; c + k; z_1)\frac{z_2^k}{k!}. \tag{3.10}$$

For all cases we are going to consider—primary, dilaton, and higher string level vertices—only b_2 is different, while the other parameters and arguments of F_1 are the same [see Eqs. (2.20), (2.21), and (2.22)],

[6] This follows from the periodicity condition on ϕ_2.

$$a = \frac{1}{2}, \qquad b_1 = \frac{1}{2}, \qquad c = 1, \qquad z_1 = 1 - \epsilon, \qquad z_2 = 1 - \frac{\chi_m}{\chi_p}. \tag{3.11}$$

Then, by expanding ${}_2F_1(\frac{1}{2} + k, \frac{1}{2}; 1 + k; 1 - \epsilon)(1 - \chi_m/\chi_p)^k$ around $\epsilon = 0$ one finds

$$\begin{aligned} {}_2F_1\left(\frac{1}{2}+k,\frac{1}{2};1+k;1-\epsilon\right)\left(1-\frac{\chi_m}{\chi_p}\right)^k \approx \frac{\Gamma(1+k)}{\sqrt{\pi}\Gamma(\frac{1}{2}+k)}\Big\{&\log(4)-H_{k-\frac{1}{2}}-\frac{1}{4\chi_{p0}}[2\chi_{p0}+(4k\chi_{m1}-(1+2k)\chi_{p0})(\log(4)-H_{k-\frac{1}{2}})]\epsilon \\ &-\log(\epsilon)-\frac{\chi_{p0}+2k(\chi_{p0}-2\chi_{m1})}{4\chi_{p0}}\epsilon\log(\epsilon)\Big\}, \end{aligned} \tag{3.12}$$

where H_z is defined as [26]

$$H_z = \psi(z + 1) + \gamma.$$

The replacement of Eq. (3.12) in Eq. (3.10), and taking into account Eq. (3.11), gives

$$F_1\left(\frac{1}{2}, \frac{1}{2}, b_2; 1; 1 - \epsilon, 1 - \frac{\chi_m}{\chi_p}\right) \approx C_0 + C_1\epsilon + C_2\epsilon\log(\epsilon) + C_3\log(\epsilon), \tag{3.13}$$

where

$$\begin{aligned} C_0 &= \frac{\Gamma(-b_2)}{\sqrt{\pi}\Gamma(\frac{1}{2} - b_2)} + \frac{\log(16)}{\pi}\,{}_1F_0(b_2, 1), \\ C_1 &= \frac{1}{4\pi}\left\{\frac{1}{\chi_{p0}}\left[-\frac{\sqrt{\pi}\Gamma(-1 - b_2)}{\Gamma(\frac{1}{2} - b_2)}(\chi_{p0} + 2b_2\chi_{m1}) + 8\log(2)b_2(\chi_{p0} - 2\chi_{m1})\,{}_1F_0(1 + b_2, 1)\right] - 2(1 - \log(4))\,{}_1F_0(b_2, 1)\right\}, \\ C_2 &= -\frac{1}{4\pi\chi_{p0}}[\chi_{p0}\,{}_1F_0(b_2, 1) + 2b_2(\chi_{p0} - 2\chi_{m1})\,{}_1F_0(1 + b_2, 1)], \qquad C_3 = -\frac{1}{\pi}\,{}_1F_0(b_2, 1). \end{aligned} \tag{3.14}$$

Here, ${}_1F_0(b, z)$ is one of the hypergeometric functions.

In the normalized structure constants (2.20), there are two hypergeometric functions $F_1(\frac{1}{2}, \frac{1}{2}, b_2; 1; 1 - \epsilon, 1 - \frac{\chi_m}{\chi_p})$ with $b_2 = -j/2$ and $b_2 = -1 - j/2$. By using Eqs. (3.13) and (3.14) in Eq. (2.20) and expanding it about $\epsilon = 0$, we can write down the following approximate equality for $j \geq 3$:

$$\mathcal{C}^{\text{pr}}_{j\tilde{\gamma}} \approx A_0 + A_1\epsilon + A_2\epsilon\log(\epsilon), \tag{3.15}$$

where the coefficients are given by

$$\begin{aligned} A_0 &= c^{\text{pr}}_j\pi\frac{\Gamma(\frac{j}{2})^2}{\Gamma(\frac{1+j}{2})\Gamma(\frac{3+j}{2})}j\chi_{p0}^{\frac{1}{2}(j-1)}(1 - v_0^2 - \chi_{p0}), \\ A_1 &= c^{\text{pr}}_j\frac{\pi}{4}\frac{\Gamma(\frac{j}{2})\Gamma(\frac{j}{2} - 1)}{\Gamma(\frac{1+j}{2})\Gamma(\frac{3+j}{2})}\chi_{p0}^{\frac{1}{2}(j-3)}\{4(W_1 + \chi_{m1})\chi_{p0} - 2\chi_{p0}^2 - [2\chi_{n1}(1 - v_0^2 - \chi_{p0}) + \chi_{p0}(1 - v_0(v_0 + 8v_1 + 2v_0W_1) \\ &\quad - \chi_{p0}(1 - 2W_1)) - 2(1 - v_0^2 + \chi_{p0})\chi_{p1}]j + [\chi_{n1} - 4v_0v_1\chi_{p0} + \chi_{m1}(1 - v_0^2 - \chi_{p0}) \\ &\quad - v_0^2(\chi_{n1} + W_1\chi_{p0} - 3\chi_{p1}) - 3\chi_{p1} + \chi_{p0}(-\chi_{n1} + W_1(-1 + \chi_{p0}) + \chi_{p1})]j^2 + (1 - v_0^2 - \chi_{p0})\chi_{p1}j^3 \\ &\quad + \tilde{\gamma}[4K_1\chi_{p0} + (2\chi_{p0}(K_1 - 2(K_1 + v_0u_1)\chi_{p0}))j + (2\chi_{p0}(v_0u_1\chi_{p0} - K_1(1 - \chi_{p0})))j^2]\}, \\ A_2 &= -c^{\text{pr}}_j\frac{\pi}{2}\frac{\Gamma(\frac{j}{2})^2}{\Gamma(\frac{1+j}{2})\Gamma(\frac{3+j}{2})}j\chi_{p0}^{\frac{1}{2}(j-3)}[4v_0v_2\chi_{p0} + (1 - v_0^2 + \chi_{p0})\chi_{p2} - (1 - v_0^2 - \chi_{p0})\chi_{p2}j - 2\tilde{\gamma}v_0u_2\chi_{p0}^2]. \end{aligned} \tag{3.16}$$

Now, our goal is to express Eq. (3.15) in terms of $\mathcal{J}$, p, and Φ. To this end, we replace Eqs. (3.6), (3.7), (3.8), and (3.9) in Eq. (3.16). This leads to the following final result for the normalized structure constants $\mathcal{C}^{\text{pr}}_{j\tilde{\gamma}}$ for $j \geq 3$:

$$\mathcal{C}^{\rm pr}_{j\tilde{\gamma}} \approx c^{\rm pr}_j \frac{\pi}{8}\frac{\Gamma(\frac{j}{2})}{\Gamma(\frac{1+j}{2})\Gamma(\frac{3+j}{2})}\sin\left(\frac{p}{2}\right)^{1+j}\left[4(j-1)\Gamma\left(\frac{j}{2}-1\right)\cos(\Phi)-\tilde{\gamma}\Gamma\left(\frac{j}{2}\right)\left(4\sin\left(\frac{p}{2}\right)-j(1+\cos(p))\mathcal{J}\right)\sin(\Phi)\right]\epsilon. \tag{3.17}$$

Let us point out that Eq. (3.17) reduces exactly to the result found for the undeformed case in Ref. [24] when $\tilde{\gamma}=0$, $\Phi=0$. Moreover, it generalizes it for any $j\geq 3$. The cases $j=1$ and $j=2$ will be considered separately in the Appendix.

B. Giant magnons on $AdS_5\times S^5_\gamma$ and the dilaton operator

Since the hypergeometric functions in Eq. (2.21) are the same as in Eq. (2.20) (only the coefficients in front of them are different), we can use Eqs. (3.13) and (3.14) for the case under consideration, with $b_2=-j/2$ and $b_2=-1-j/2$. Thus, by expanding Eq. (2.21) to the leading order in ϵ, one finds ($j\geq 1$)

$$\begin{aligned}\mathcal{C}^{d}_{j\tilde{\gamma}} \approx{}& c^d_{4+j}\frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{4+j}{2})}{\Gamma(\frac{5+j}{2})}\chi_{p0}^{\frac{1}{2}(j-1)}\left\{\epsilon[4W_1+(2+j)(2\chi_{m1}-\chi_{p0})+4\tilde{\gamma}K_1]\log\frac{16}{\epsilon}\,{}_1F_0\left(-\frac{j}{2},1\right)\right.\\ &+\sqrt{\pi}\frac{\Gamma(\frac{j}{2})}{\Gamma(\frac{3+j}{2})}[2j\chi_{p0}+(2\chi_{m1}-\chi_{p0}+W_1(2+j(2-\chi_{p0}))1+j(\chi_{m1}+\chi_{n1}+(1+j)\chi_{p1})\\ &\left.+2\tilde{\gamma}(K_1+j(K_1-(K_1+v_0u_1)\chi_{p0})))\epsilon+(j(1+j)\chi_{p2}-2\tilde{\gamma}jv_0u_2\chi_{p0})\epsilon\log\epsilon]\right\}.\end{aligned} \tag{3.18}$$

By taking into account Eqs. (3.6), (3.7), (3.8), and (3.9) in Eq. (3.18) one finally derives

$$\begin{aligned}\mathcal{C}^{d}_{j\tilde{\gamma}} \approx{}& c^d_{4+j}\pi\frac{\Gamma(\frac{j}{2})\Gamma(\frac{4+j}{2})}{\Gamma(\frac{3+j}{2})\Gamma(\frac{5+j}{2})}\sin^{1+j}(p/2)\left\{j-\frac{1}{8}[(4-j(1+3j)(1+\cos p)\right.\\ &\left.-j(1+j)(1+\cos p)\csc(p/2))\mathcal{J}\cos\Phi-\tilde{\gamma}(4\sin(p/2)-j(1+\cos p)\mathcal{J})\sin\Phi]\epsilon\right\}.\end{aligned}$$

C. Giant magnons on $AdS_5\times S^5_\gamma$ and singlet scalar operators on higher string levels

For this case, we were not able to obtain a general formula for the leading finite-size corrections to the three-point correlation functions in terms of $\mathcal{J}$, p, and Φ, for any $q\geq 1$. This is why we are going to present here the results for $q=1,\ldots,5$ (string levels $n=0,1,2,3,4$).

Let us first point out that the hypergeometric functions $F_1(\frac{1}{2},\frac{1}{2},-k;1;1-\epsilon,1-\frac{\chi_m}{\chi_p})$ entering Eq. (2.22) can be expressed in terms of the complete elliptic integrals $\mathbf{K}(1-\epsilon)$, $\mathbf{E}(1-\epsilon)$ of the first and second kind. For example,

$$\begin{aligned}F_1\left(\frac{1}{2},\frac{1}{2},0;1;1-\epsilon,1-\frac{\chi_m}{\chi_p}\right)&=\frac{2}{\pi}\mathbf{K}(1-\epsilon),\\ F_1\left(\frac{1}{2},\frac{1}{2},-1;1;1-\epsilon,1-\frac{\chi_m}{\chi_p}\right)&=\frac{2}{\pi}\frac{(\chi_m-\epsilon\chi_p)\mathbf{K}(1-\epsilon)-(\chi_m-\chi_p)\mathbf{E}(1-\epsilon)}{(1-\epsilon)\chi_p},\\ F_1\left(\frac{1}{2},\frac{1}{2},-2;1;1-\epsilon,1-\frac{\chi_m}{\chi_p}\right)&=\frac{1}{3\pi(1-\epsilon)^2\chi_p^2}[2((3-\epsilon)\chi_m^2-4\epsilon\chi_m\chi_p\\ &\quad-(1-3\epsilon)\epsilon\chi_p^2)\mathbf{K}(1-\epsilon)-4(\chi_m-\chi_p)((2-\epsilon)\chi_m+(1-2\epsilon)\chi_p)\mathbf{E}(1-\epsilon)].\end{aligned}$$

Then, Eq. (2.22) can be written in terms of hypergeometric functions of the type ${}_pF_q$ with the argument $1-\epsilon$. However, this is just a much more complicated representation of the semiclassically exact result.

Here, we are interested in the small-ϵ (or, equivalently, large-$\mathcal{J}$) limit. So, we will expand everything in ϵ. Since the computations are similar to the previously considered cases, we will write down the final results only. They are given by the following approximate equalities:

$$\mathcal{C}^1_{\tilde{\gamma}} \approx c_{\Delta_1} \frac{\sqrt{\pi}}{8} \frac{\Gamma(\frac{\Delta_1}{2})}{\Gamma(\frac{1+\Delta_1}{2})} \sin(p/2)\{16 - 8\mathcal{J}\csc(p/2) + [4 - (2(1-\cos p + \mathcal{J}^2\cot^2(p/2)) + \mathcal{J}(5-\cos p)\csc(p/2))\cos\Phi + 8\tilde{\gamma}\mathcal{J}\sin^2(p/2)\sin\Phi]\epsilon\},$$

$$\mathcal{C}^2_{\tilde{\gamma}} \approx -c_{\Delta_2} \frac{\sqrt{\pi}}{24} \frac{\Gamma(\frac{\Delta_2}{2})}{\Gamma(\frac{1+\Delta_2}{2})} \{8(2\sin(p/2) - 3\mathcal{J}) + [12\sin(p/2) + (2(27+5\cos p)\sin(p/2) - \mathcal{J}(31 + 13\cos p + 3\mathcal{J}(1+\cos p)\csc(p/2)))\cos\Phi - 8\tilde{\gamma}\sin(p/2)(8\sin(p/2) - \mathcal{J}(7+\cos p))\sin\Phi]\epsilon\},$$

$$\mathcal{C}^3_{\tilde{\gamma}} \approx c_{\Delta_3} \frac{\sqrt{\pi}}{120} \frac{\Gamma(\frac{\Delta_3}{2})}{\Gamma(\frac{1+\Delta_3}{2})} \{8(38\sin(p/2) - 15\mathcal{J}) + [60\sin(p/2) + (18(13+19\cos p)\sin(p/2) - \mathcal{J}(187 + 97\cos p + 15\mathcal{J}(1+\cos p)\csc(p/2)))\cos\Phi - 12\tilde{\gamma}\sin(p/2)(48\sin(p/2) - \mathcal{J}(23 - 7\cos p))\sin\Phi]\epsilon\},$$

$$\mathcal{C}^4_{\tilde{\gamma}} \approx -c_{\Delta_4} \frac{\sqrt{\pi}}{840} \frac{\Gamma(\frac{\Delta_4}{2})}{\Gamma(\frac{1+\Delta_4}{2})} \{1264\sin(p/2) - 840\mathcal{J} + [\sin(p/2)(420 + (4730 + 2054\cos p - \mathcal{J}(1837 + 1207\cos p)\csc(p/2) - 210\mathcal{J}^2\cot^2(p/2))\cos\Phi - 16\tilde{\gamma}(424\sin(p/2) - 3\mathcal{J}(79 + 9\cos p))\sin\Phi)]\epsilon\},$$

$$\mathcal{C}^5_{\tilde{\gamma}} \approx c_{\Delta_5} \frac{\sqrt{\pi}}{2520} \frac{\Gamma(\frac{\Delta_5}{2})}{\Gamma(\frac{1+\Delta_5}{2})} \{8(902\sin(p/2) - 315\mathcal{J}) + [1260\sin(p/2) + (2(6093 + 7667\cos p)\sin(p/2) - \mathcal{J}(6343 + 4453\cos p + 315\mathcal{J}(1+\cos p)\csc(p/2)))\cos\Phi - 20\tilde{\gamma}\sin(p/2)(1376\sin(p/2) - \mathcal{J}(523 - 107\cos p))\sin\Phi]\epsilon\}.$$

IV. CONCLUDING REMARKS

In this article, we have derived the leading finite-size effects on the normalized structure constants in some semiclassical three-point correlation functions in $AdS_5 \times S^5_\gamma$—dual to $\mathcal{N} = 1$ SYM theory in four dimensions—arising as an exactly marginal deformation of $\mathcal{N} = 4$ SYM, expressed in terms of the conserved string angular momentum $\mathcal{J}$, and the worldsheet momentum p, identified with the momentum p of the magnon excitations in the dual spin chain. More precisely, we found the leading finite-size effects on the structure constants in the three-point correlators of two "heavy" giant magnons' string states and the following three "light" states:

(1) Primary scalar operators;
(2) Dilaton operator with nonzero momentum ($j \geq 1$);
(3) Singlet scalar operators on higher string levels.

It would be interesting to investigate other cases for which the finite-size corrections to the giant magnon's dispersion relations are known, like $AdS_4 \times CP^3$, $AdS_4 \times CP^3_\gamma$, $AdS_5 \times T^{1,1}$, or $AdS_5 \times T^{1,1}_\gamma$.

APPENDIX: GIANT MAGNONS ON $AdS_5 \times S^5_\gamma$ AND PRIMARY SCALAR OPERATORS WITH $j = 1$ AND $j = 2$

Let us start with the case $j = 1$. Expanding the coefficients in $\mathcal{C}^{pr}_{1\tilde{\gamma}}$ according to Eq. (3.5), one can rewrite it in the following form:

$$\begin{aligned}\mathcal{C}^{pr}_{1\tilde{\gamma}} \approx c^{pr}_1 \frac{\pi^2}{2} \Big\{ &\frac{1}{\chi_{p0}} F_1\Big(1/2, 1/2, -1/2; 1; 1-\epsilon, 1 - \frac{\chi_{m1}}{\chi_{p0}}\epsilon\Big) \\ &\times [(1 - v_0^2)\chi_{n1}\epsilon + (2 - (4v_0 v_1 + 3W_1 + 4\tilde{\gamma}K_1)\epsilon - v_0^2(2 + W_1\epsilon))\chi_{p0} - 4v_0 v_2 \chi_{p0}\epsilon \log(\epsilon)] \\ &- F_1\Big(1/2, 1/2, -3/2; 1; 1-\epsilon, 1 - \frac{\chi_{m1}}{\chi_{p0}}\epsilon\Big)[4\chi_{p0} + 2(\chi_{n1} - (W_1 + 2\tilde{\gamma}(K_1 + v_0 u_1))\chi_{p0} + 2\chi_{p1})\epsilon \\ &+ 4(\chi_{p2} - \tilde{\gamma} v_0 u_2 \chi_{p0})\epsilon \log(\epsilon)]\Big\}. \end{aligned} \tag{A1}$$

In order to represent $\mathcal{C}^{pr}_{1\tilde{\gamma}}$ as a function of $\mathcal{J}$, p, and Φ, one would have to use Eqs. (3.6), (3.7), (3.8), and (3.9) in Eq. (A1). This leads to

$$\mathcal{C}^{\rm pr}_{1\tilde{\gamma}} \approx -c^{\rm pr}_1 \frac{\pi^2}{4}\sin^2(p/2)\Big\{8F_1\Big(1/2,1/2,-3/2;1;1-\epsilon,1-\frac{1}{2}(1+\cos\Phi)\epsilon\Big) - 4F_1\Big(1/2,1/2,-1/2;1;1-\epsilon,1-\frac{1}{2}(1+\cos\Phi)\epsilon\Big) + \Big[F_1\Big(1/2,1/2,-1/2;1;1-\epsilon,1-\frac{1}{2}(1+\cos\Phi)\epsilon\Big)(1-\cos\Phi(9+2\cos p+\mathcal{J}(1+\cos p)\csc(p/2)) + 4\tilde{\gamma}\sin(p/2)\sin\Phi) - F_1\Big(1/2,1/2,-3/2;1;1-\epsilon,1-\frac{1}{2}(1+\cos\Phi)\epsilon\Big)(2-2\cos\Phi(5+2\cos p + \mathcal{J}(1+\cos p)\csc(p/2)) + \tilde{\gamma}(\mathcal{J}(1+\cos p)+4\sin(p/2))\sin\Phi)\Big]\epsilon\Big\}. \tag{A2}$$

For the undeformed case, when $\tilde{\gamma}=0$, $\Phi=0$, Eq. (A2) simplifies to

$$\mathcal{C}^{\rm pr}_1 \approx -c^{\rm pr}_1 \frac{\pi^2}{4}\sin(p/2)[3\sin(p/2)+\sin(3p/2)+\mathcal{J}(1+\cos p)]\epsilon^2. \tag{A3}$$

This is in accordance with the result $\mathcal{C}^{\rm pr}_1 \approx 0$ found in Ref. [24], where only the leading order in ϵ was taken into account.

Now, let us consider the case $j=2$, when Eq. (2.20) reduces to

$$\mathcal{C}^{\rm pr}_{2\tilde{\gamma}} = -\frac{8}{3}c^{\rm pr}_2 \frac{1}{(1-\epsilon)^2\sqrt{(1-u^2)W(\chi_p-\chi_n)}}\{[3-(1+2v^2)W-3\tilde{\gamma}K](1-\epsilon)[(\chi_m-\chi_p)\mathbf{E}(1-\epsilon) - (\chi_m-\chi_p\epsilon)\mathbf{K}(1-\epsilon)] + (1-u(u-\tilde{\gamma}(Ku-vW))-\tilde{\gamma}K)(2(\chi_p-\chi_m)((2-\epsilon)\chi_m+(1-2\epsilon)\chi_p)\mathbf{E}(1-\epsilon) + ((3-\epsilon)\chi_m^2-4\chi_m\chi_p\epsilon-\chi_p^2(1-3\epsilon)\epsilon)\mathbf{K}(1-\epsilon)]\}. \tag{A4}$$

By expanding Eq. (A4) in ϵ, and taking into account Eqs. (3.6), (3.7), (3.8), and (3.9), one finds

$$\mathcal{C}^{\rm pr}_{2\tilde{\gamma}} \approx \frac{2}{3}c^{\rm pr}_2\sin^2(p/2)[2\mathcal{J}\cos\Phi-\tilde{\gamma}(2\sin(p/2)-\mathcal{J}(1+\cos p))\sin(p/2)\sin\Phi]\epsilon. \tag{A5}$$

Obviously, the result for the undeformed case [24] is properly reproduced by the above formula.

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