PHYSICAL REVIEW D **88,** 026014 (2013)

Open string self-energy on the lightcone worldsheet lattice

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(Received 3 June 2013; published 19 July 2013)

We continue our study of open-string perturbation theory on the lightcone worldsheet lattice, which is an $M \times N$ rectangular grid. Here M is the number of P^+ units, and N is the number of ix^+ units. We extend our previous analysis to the bosonic open-string one-planar-loop self-energy. We find that, when all open-string coordinates satisfy Neumann conditions, the ultraviolet worldsheet divergences associated with the closed-string tachyon and boundary effects can be canceled by renormalization of bulk (AM^1) and boundary (BM^0) worldsheet "cosmological constants." The bulk divergence for the open string matches that for the closed string. The open-string tachyon mass shift displays the dilaton logarithmic divergence with the correct coefficient for its consistent absorption by renormalization of the string tension. The ultraviolet contribution to the open-string gluon mass shift vanishes, in accord with its interpretation as a gauge particle. We also find that when the bosonic string ends on a D-brane, additional negative powers of $\ln M$ multiply the bulk and boundary divergences. These can no longer be canceled by the "cosmological constants," perhaps pointing to the need, in the presence of D-branes, for the cancellations of divergences provided by supersymmetry.

DOI: 10.1103/PhysRevD.88.026014 PACS numbers: 11.25.−w, 11.25.Db, 11.25.Uv

I. INTRODUCTION

The lightcone parameterization of the string worldsheet [1,2] provides a framework for the description of multiloop interacting string diagrams [3]. The definition of the lightcone worldsheet path integrals on a worldsheet lattice [4] then provides a concrete nonperturbative method to study this multiloop expansion numerically. Monte Carlo methods should be particularly apt when the diagrams are restricted to planar open-string multiloop diagrams, for which string interactions decorate the worldsheet lattice in a local manner. This restricted sum of diagrams defines the 't Hooft large-N limit [5] of the interacting string theory, where N is the size of the Chan-Paton matrices associated with constraining the ends of each open string to move on a stack of N D-branes. In the case where these D-branes are coincident D3-branes, the open-string spectrum contains a massless $U(N)$ gauge particle in four spacetime dimensions. This then indicates that the zero-slope limit $\alpha' \to 0$ [6] of this sum of diagrams could describe large-N QCD [7]. In this article we restrict our worldsheet lattice studies to the bosonic string. We should keep in mind that the bosonic open-string tachyon could make applications to QCD problematic, either through a failure to stabilize the vacuum or through a stabilization that breaks the $U(N)$ gauge invariance. If so, these problems might be cured by replacing the bosonic open string with the even G-parity bosonic sector of the Ramond-Neveu-Schwarz model [8–10].

Given that we will focus on the bosonic string, which has open-string tachyons and has not been shown to stabilize, it is necessary, for our studies, to impose an infrared cutoff that temporarily stabilizes the theory. As we will describe shortly, there is a nice way to do this in the context of the worldsheet lattice. In effect, we can naturally impose an energy cost to the existence of each open-string end such that virtual open strings can only exist for relatively short times. Note that closed-string tachyons are not affected by this infrared cutoff. But closed strings do not propagate within the planar open-string diagrams: in fact, their existence is only felt in their disappearance into the vacuum as described by the holes in the multiloop worldsheet. Indeed, if we interpret the holes as closed-string emission/absorption by the vacuum, each planar multiloop diagram can be interpreted as a closed-string tree in a closed-string condensate. Thus, the 't Hooft limit just provides us with the subset of diagrams which might stabilize the vacuum via closed-string condensation. The divergences, which one would normally think of as infrared properties of the closed-string amplitudes, are actually ultraviolet divergences on the open-string worldsheet, which are regulated by the worldsheet lattice itself.

Over the last two years, we have been critically analyzing the continuum limit of the lattice path integrals for the simplest one-loop open-string worldsheets [11,12], and this article is a continuation of these studies. Our motivation is to clearly understand the UV divergence structure emerging from the continuum limit of the lattice and to determine whether all UV divergences can be consistently dealt with, either through cancellation against naturally defined worldsheet counterterms or through renormalization of the physical parameters of the theory, the string tension T_0 or the 't Hooft coupling Ng^2. Our previous articles [11,12] discussed these issues in the context of the one-loop open-string corrections to the closed-string

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propagator. Because the only boundary was that of the slit describing the open-string loop, the UV divergences, in this case, arise only from the limit that the slit length vanishes. For the open-string propagator analyzed in the present article, there are additional UV divergences arising from the collision of the slit with the boundaries of the open-string worldsheet. The method of Ref. [11], which took a string field theory approach to the construction of the one-loop propagator proved insufficiently precise to deal with these boundary divergences. But here we successfully apply the worldsheet methods introduced in Ref. [12] to this problem. The key is to represent the lattice worldsheet propagator in terms of normal modes in discrete time rather than normal modes in discrete space as was done in Ref. [12]. This makes the discrete space dependence explicit, so that the boundary contributions to the UV divergence structure can be efficiently analyzed.

The Giles-Thorn (GT) discretization of the worldsheet [4] begins with a representation of the free closed- or open-string propagator as a lightcone worldsheet path integral defined on a lattice. The lattice replaces the transverse coordinates of the string $\boldsymbol{x}(\sigma, \tau)$, living on a rectangular $P^+ \times T$ domain, with discretely labeled coordinates $\boldsymbol{x}_k^j = \boldsymbol{x}(kaT_0, ja)$, living on an $M \times N$ grid with spacing a, where $P^+ = MaT_0$ and $T = a(N+1)$. The free-string propagator is then simply a Gaussian integral

$$\begin{aligned}\mathcal{D}_0 &= \int \prod_{kj} d\boldsymbol{x}_k^j e^{-S},\\ S &= \frac{T_0}{2}\sum_{kj}[(\boldsymbol{x}_k{}^{j+1} - \boldsymbol{x}_k^j)^2 + (\boldsymbol{x}_{k+1}{}^j - \boldsymbol{x}_k{}^j)^2]\\ &\equiv \frac{T_0}{2}\boldsymbol{x}^T \cdot \Delta^{-1}\boldsymbol{x},\end{aligned} \tag{1}$$

where the $MN \times MN$ matrix Δ is the lattice worldsheet propagator. Then, up to an overall normalization factor, $\mathcal{D}_0 = \det^{-(D-2)}\Delta^{-1}$, where D is the spacetime dimension ($D = 26$ for the bosonic string).

At zero loops, the UV divergences arising in the continuum limit of the GT lattice representation of the open- and closed-string propagators reside in bulk and boundary contributions to the ground-state energies. The lightcone energy is $P^- = (P^0 - P^1)/\sqrt{2}$, and one finds in the continuum limit [4]

$$aP^-_{\rm closed,G} \sim (D-2)\left[\alpha_0 M - \frac{\pi}{6M} + \mathcal{O}(M^{-2})\right], \tag{2}$$

$$aP^-_{\rm open,G} \sim (D-2)\left[\alpha_0 M - \beta_0 - \frac{\pi}{24M} + \mathcal{O}(M^{-2})\right]. \tag{3}$$

For the GT lattice, one has specifically $\alpha_0 = 2C/\pi$ and $\beta_0 = \ln(1+\sqrt{2})$, where C is Catalan's constant. Remembering that $P^+ = aMT_0$, we see that the $1/M$ terms precisely account for the tachyonic masses of the free closed and open strings. As explained in Ref. [4], the $\alpha_0 M$ term enters time evolution as an exponential of the combination $TP^- = (N+1)M\alpha_0$ which is simply proportional to the discretized area $(N+1)M$ of the lattice: α_0 is just a contribution to the worldsheet bulk "cosmological constant" expected in any quantum field theory. Because the interactions preserve this discrete area, one can harmlessly introduce a bare bulk cosmological constant A which is ultimately chosen to cancel all bulk contributions to the string energies. Similarly, the β_0 can be associated with the free ends of the open string because it enters the evolution as an exponential of the combination $-\beta_0(N+1)$ proportional to the length of the worldsheet boundary. Then we can consistently introduce a bare worldsheet-boundary cosmological constant B chosen to cancel all these boundary contributions to the string energies. Unlike the bulk cosmological constant, this boundary term alters the dynamics. It is this parameter that provides the infrared cutoff we alluded to earlier. It is naturally nonzero: even at zero loops it is necessary to absorb boundary divergences. For the purposes of our lattice studies, we are free to choose it large enough to suppress the open-string tachyonic instability.

On the GT lattice, the sum of all open-string multiloop planar diagrams can be obtained by summing over all patterns of missing spatial bonds. Formally, this is achieved by introducing Ising-like variables $S_k^j = 0, 1$ and taking the worldsheet action to be

$$\begin{aligned}S_{\rm Planar} = \frac{T_0}{2}\sum_{ij}&[(\boldsymbol{x}_i{}^{j+1} - \boldsymbol{x}_i{}^j)^2 + S_i{}^j(\boldsymbol{x}_{i+1}{}^j - \boldsymbol{x}_i{}^j)^2]\\ &+ (D-2)B\sum_{kj}(1 - S_k^j) - \sum_{ij}[S_i{}^j(1 - S_i{}^{j+1})\\ &+ S_i{}^{j+1}(1 - S_i{}^j)]\ln g\end{aligned} \tag{4}$$

$$\equiv \frac{T_0}{2}\boldsymbol{x}^T \cdot [\Delta^{-1} + V(S)]\boldsymbol{x} + A(\{S\}). \tag{5}$$

The terms in $A(\{S\})$ insert the coupling constant g in the appropriate way and allow for an open-string self-energy counterterm B. Then we have

$$\mathcal{D} = \mathcal{D}_0 \sum_{\{S\}} \det^{-12}(I + V\Delta)e^{-A(\{S\})}. \tag{6}$$

When V is a sparse matrix, i.e., when there is a relatively small number of missing bonds [e.g., $\sum_{kj}(1 - S_k^j) \ll M$, which can be arranged by taking $B \gg 1$], this will be a particularly efficient way to evaluate the terms of perturbation theory. Holding B sufficiently large serves as a physical and convenient infrared regulator in our studies of the properties of the planar diagrams.

The planar open-string loop expansion organizes the sum over spins in Eq. (6) as a power series in g^2, with the number of loops equal to the number of "holes" in the lattice. Orient the worldsheet so that the time axis (τ) is horizontal and the space axis (σ) vertical. Then each hole is a horizontal row of contiguous missing links. The number of missing links is the number of time steps the broken string lasts. In this article we study exclusively one-loop corrections to the open-string self-energy, or, in this language, a single row of contiguous missing links, as we can see in Fig. 1. Since we are concerned here with energy shifts to the free-string spectrum, the initial and final states are energy eigenstates with the same energy, so we can (and do) take the total number of time steps $N \to \infty$, keeping the slit's size finite and its location in the vicinity of $N/2$. Then a given diagram is characterized by the total number of steps in space (i.e., the number of string bits) M, the length of the slit in lattice units (or number of missing links) $(K-1)$, and the number of spatial steps M_1 between the slit and one of the open-string boundaries. The worldsheet path integral will depend on M, K, M_1, and in principle K should be summed from 1 to ∞ and M_1 should be summed from 1 to $M-1$. Of course, M is just proportional to the fixed P^+ of the string state whose energy shift is being calculated. The presence of the open-string tachyon renders the K sum exponentially divergent.

The nature of this tachyonic divergence is easy to see. In the one-loop correction to the open-string propagator, the slit represents the propagation of two open strings as an intermediate state. The initial and final state is a single open string, say with $(\text{mass})^2 = 2\pi(n-1)T_0$ with $n = 0, 1, 2, \ldots$ the mode number of the state. The intermediate state is two open strings with $(\text{mass})^2 = 2\pi(n_1-1)T_0$, $2\pi(n_2-1)T_0$. If the two open strings last for a time $(K-1)a$, then the amplitude acquires a factor $\exp\{-a(K-1)\Delta P^-\}$, where

$$a\Delta P^- = \frac{\pi(n_1-1)}{M_1} + \frac{\pi(n_2-1)}{M-M_1} - \frac{\pi(n-1)}{M}. \quad (7)$$

If $n_1 = 0$ (or $n_2 = 0$), ΔP^- becomes negative for small enough M_1 (or $M - M_1$). If M_1, $M - M_1$, and M are all of order M in the continuum limit, the coefficient of $(K-1)$ is of order M^{-1} so as long as $K \ll M$, which is the ultraviolet region we study here, the exponent stays small. On the other hand, either M_1 or $M - M_1$ can be as small as 1, in which case the coefficient of $(K-1)$ in the exponential growth is of order 1, even when $K \ll M$. These large exponential factors cause practical difficulties with numerical studies, but we will show that they are absent from the order-M and order-M^{-1} contributions to the self-energy.

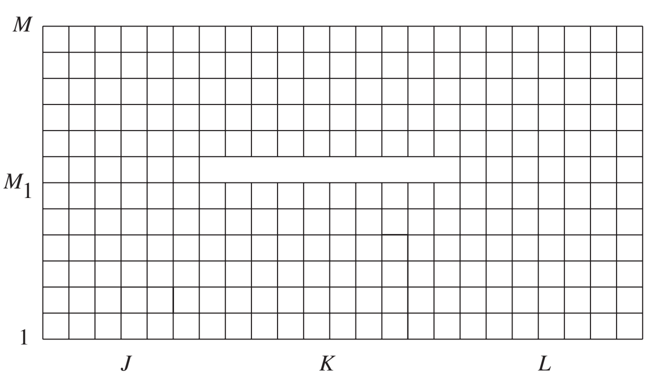

FIG. 1. One-loop open-string self-energy diagram on the lattice worldsheet. A single open string splits at time J and rejoins at time $J+K$, with total time $N+1 = J+K+L \to \infty$ and J, $L \sim N/2$. Thus the diagram is characterized by the number of missing links $K-1$, their position M_1, and total string length M.

Because the K sum is divergent, we suspend the sum over K, keeping it fixed while we study the large-M behavior. The ultraviolet structure that we wish to analyze is defined by slits much shorter than P^+, or in lattice units $K \ll M$. The continuum limit is $M \to \infty$, so we focus on obtaining the limit of our calculations in the regime $1 \ll K \ll M$. As we contemplate numerical studies of multiloop diagrams, it is natural to restrict the hole size summations by simply taking B sufficiently large, rather than by literally suspending them. With B large enough, the tachyonic instability is stabilized, at the expense of losing Lorentz invariance. Thus, our conclusions strictly apply to this Lorentz-violating cutoff model.

We close this introduction with a brief summary of the results of our previous work and an outline of the rest of this paper. In Refs. [11,12] we analyzed the one-loop correction to the closed-string energy. In this case, the sum over M_1 is trivial: it just supplies a factor of M. Then the self-energy correction has the form of a single sum over the slit length K,

$$\Delta P^- = \sum_{K=2}^{\infty} \delta P_K^-, \quad (8)$$

and we found for M large at fixed K,

$$a\delta P_K^- \sim \alpha(K)M + \frac{c(K)}{M} + \frac{d(K)}{M^3} + \cdots. \quad (9)$$

We determined the large-K dependence of the coefficients numerically to be $\alpha(K) \sim K^{-3}$ and $c(K) \sim K^{-1}$. Thus, the coefficient of M summed over K is finite. This term in the energy is a quadratic divergence, corresponding to the closed-string tachyon. Here we see that it is in fact harmless.[1] The K^{-1} behavior of the $1/M$ term signals the UV logarithmic divergence due to the closed-string dilaton. This divergence is, of course, real but can be absorbed in the slope parameter $\alpha' = 1/(2\pi T_0)$. To prove this, it is important that the divergence be universal for all states. Our work on the closed-string self-energy showed that $c(K) = 0$ for the graviton and had the appropriate value for selected massive closed-string states to be absorbable into T_0.

In this article, we deal with the extra complications of the open-string boundaries. In this case, the large-M expansion at fixed K has many more terms:

[1]The harmlessness of the tachyon divergence in the continuum amplitudes is usually argued by analytic continuation [13].

$$a\delta P_K^- \sim \alpha'(K)M + b(K) + \frac{c'(K)}{M} + \frac{d'(K)}{M^2} + \cdots. \qquad (10)$$

Here we will show that $\alpha'(K) = \alpha(K)$, necessary to show the harmlessness of the bulk divergences. We also show that $c'(K) = c(K)/4$, as required for the consistent absorption of the logarithmic divergences into the Regge slope parameter. Moreover, we are able to obtain the large-K behavior of the coefficients analytically, using the Fisher-Hartwig formula for the asymptotic behavior of Toeplitz determinants [14]. In obtaining these results it is crucial to show that the exponential divergences, due to an intermediate open-string tachyon with $P^+/(aT_0) = \mathcal{O}(1)$, do not contribute to either the M term or the $1/M$ term. This happens because, before the M_1 sum, the expansion has the form $a + b/M^2$, and the order-M term only arises by summing M_1 over a range of order M, and similarly for the $1/M$ term, which comes from summing the $1/M^2$ term over a similar range. We show that the dangerous exponential divergences contribute only to the $\mathcal{O}(M^0)$ and higher orders in $1/M$, starting at $\mathcal{O}(M^{-4})$. The constant term can be absorbed in a renormalization of B, but the exponential factors multiplying the $\mathcal{O}(M^{-4})$ and higher powers of $1/M$ raise practical obstacles to purely numerical efforts to extract the physically relevant coefficient of M^{-1}. Since these obstacles are directly associated with the open-string tachyon instability, there is at least some hope that if a stabilizing mechanism can be identified, the numerical difficulties would be surmounted.

In Sec. II we review and generalize the representations for the worldsheet propagator given in Ref. [12]. We then use these results to analyze the open-string self-energy for the tachyon (Sec. III) and the gluon and selected excited states (Sec. IV). Then, in Sec. V we obtain the large-K behavior of the coefficients in the $1/M$ expansion of the self-energies. Section VI is devoted to numerical analysis of our results. Our final Sec. VII gives a preliminary discussion of the problems arising when we try to describe open strings ending on D-branes, and the possibility that the superstring alleviates them, as indicated by the discretization of the continuum self-energy expressions for the latter.

II. WORLDSHEET PROPAGATORS

We gather in this section the expressions for the propagator on the closed, open, and Dirichlet worldsheets found in Ref. [12] (see also Ref. [15]). In that reference, the worldsheet propagator was represented as a spatial normal mode expansion. But representations based on temporal normal modes are also useful, so we include them in our presentation.

Of central interest are the worldsheet correlators of the coordinates on the $M \times N$ lattice corresponding to the free closed or open string:

$$\Delta_{ij,kl} = \langle x_i^j x_k^l \rangle = \frac{\int \mathcal{D}x x_i^j x_k^l e^{-S}}{\int \mathcal{D}x e^{-S}}, \qquad (11)$$

where the worldsheet action S is appropriate to the type of string coordinates (closed, open, or Dirichlet) being described. Because the expectations are taken with Gaussian weight, the two-point correlator in a single dimension captures all of the relevant information. A straightforward evaluation is to use closure to write the numerator as the product of three string propagators (see Appendix D): one from time 0 to j, one from time j to l, and the last from time l to $+(N+1)$. We choose Dirichlet boundary conditions in time: $x_i^0 = x_i^{N+1} = 0$. We can resolve x_i^j, x_k^l into spatial normal modes q_m^j, q_n^l respectively. Then, because each normal mode integral is independent, $\langle q_m^j q_n^l \rangle = \delta_{mn} \langle q_m^j q_m^l \rangle$, one ends up with a simple two-variable Gaussian:

$$\int dq_m^j dq_m^l q_m^j q_m^l \exp\left\{-\frac{1}{2}[A_1 q_m^{j2} + A_2 q_m^{l2}) + 2B q_m^j q_m^l]\right\}$$
$$= -\frac{B}{A_1A_2 - B^2} \det{}^{-1/2}\begin{pmatrix} A_1 & B \\ B & A_2 \end{pmatrix},$$
$$\langle q_m^j q_n^l \rangle = -\frac{B}{A_1A_2 - B^2}\delta_{mn}. \qquad (12)$$

Here A_1, A_2 and B are read off from the formulas of Appendix D. We set the q's at the initial and final times to zero.

Then for nonzero modes they are

$$A_1 = T_0 \sinh\lambda[\coth j\lambda + \coth(l-j)\lambda], \qquad (13)$$

$$A_2 = T_0 \sinh\lambda[\coth(N+1-l)\lambda + \coth(l-j)\lambda],$$
$$B = \frac{-T_0\sinh\lambda}{\sinh(l-j)\lambda}, \qquad (14)$$

$$A_1A_2 - B^2 = T_0^2 \sinh^2\lambda[1 + \coth j\lambda \coth(N+1-l)\lambda + (\coth j\lambda + \coth(N+1-l)\lambda)\coth(l-j)\lambda]$$
$$= T_0^2\sinh^2\lambda\left[\frac{\sinh(N+1)\lambda}{\sinh j\lambda \sinh(N+1-l)\lambda \sinh(l-j)\lambda}\right],$$
$$\frac{-B}{A_1A_2 - B^2} = \frac{1}{T_0\sinh\lambda}\frac{\sinh j\lambda \sinh(N+1-l)\lambda}{\sinh(N+1)\lambda}, \qquad (15)$$

where λ is λ_m^o or λ_m^c for the open or closed string, respectively. For the zero modes,

$$A_{10} = T_0 \frac{l}{j(l-j)}, \qquad A_{20} = T_0 \frac{(N+1-j)}{(N+1-l)(l-j)}, \\ B_0 = -\frac{T_0}{l-j}, \qquad \frac{-B_0}{A_{10}A_{20} - B_0^2} = \frac{j(N+1-l)}{T_0(N+1)}. \tag{16}$$

Then the worldsheet propagator for the open-string worldsheet is given by

$$\Delta^o_{ij,kl} = \frac{j(N+1-l)}{(N+1)M} + \frac{2}{M}\sum_{m=1}^{M-1} \frac{1}{\sinh \lambda_m^o} \times \frac{\sinh j\lambda_m^o \sinh (N+1-l)\lambda_m^o}{\sinh (N+1)\lambda_m^o} \cos\frac{m(i-1/2)\pi}{M} \times \cos\frac{m(k-1/2)\pi}{M}, \qquad l > j. \tag{17}$$

We must keep in mind that this formula applies when $l > j$. In the opposite case, we switch the roles of j and l. In this formula we have chosen to expand in the normal modes of the spatial coordinates i, k. But we could equally well have chosen to expand in normal modes in the time coordinates j, l. In that case, the propagator takes the form

$$\Delta^o_{ij,kl} = \frac{2}{N+1}\sum_{n=1}^{N} \frac{1}{\sinh \lambda_n^o} \times \frac{\cosh (i-1/2)\lambda_n^o \cosh (M-k+1/2)\lambda_n^o}{\sinh M\lambda_n^o} \times \sin\frac{nj\pi}{N+1} \sin\frac{nl\pi}{N+1}, \qquad k > i. \tag{18}$$

In this case, the formula applies when $k > i$. In the opposite case, we switch the roles of k and i.

For string self-energy calculations, we want to take $N \to \infty$, but with j, l well away $[\mathcal{O}(N)]$ from 0, $N+1$. So, to study this limit we put $j = (N+1)/2 + \hat{j}$, $l = (N+1)/2 + \hat{l}$, and take the limit with $\hat{j}$, $\hat{l}$ fixed. Then the two representations take qualitatively different forms. In the first case, we find

$$\Delta^o_{ij,kl} \sim \frac{N+1}{4M} - \frac{|l-j|}{2M} + \frac{1}{M}\sum_{m=1}^{M-1} \frac{e^{-|l-j|\lambda_m^o}}{\sinh \lambda_m^o} \times \cos\frac{m(i-1/2)\pi}{M} \cos\frac{m(k-1/2)\pi}{M}, \tag{19}$$

and we see the zero-mode divergence in the first term linear in N. In the second case, the sum over n turns into an integral and we find

$$\Delta^o_{ij,kl} = \int_0^1 dx \frac{\cosh(i-1/2)\lambda^o(x)\cosh(M-k+1/2)\lambda^o(x)}{\sinh \lambda^o(x) \sinh M\lambda^o(x)} \times \cos x(l-j)\pi. \tag{20}$$

In this case, the zero-mode divergence shows up as a divergence in the integral at the lower limit. In obtaining this formula, we used

$$\sin\frac{nj\pi}{N+1}\sin\frac{nl\pi}{N+1} = \sin^2\frac{n\pi}{2}\cos\frac{n\hat{j}\pi}{N+1}\cos\frac{n\hat{l}\pi}{N+1} + \cos^2\frac{n\pi}{2}\sin\frac{n\hat{j}\pi}{N+1}\sin\frac{n\hat{l}\pi}{N+1}. \tag{21}$$

The first term contributes only for odd n, and the second term only for even n. But in the limit $N \to \infty$ where the sum over n becomes an integral, the right side can be replaced by

$$\sin\frac{nj\pi}{N+1}\sin\frac{nl\pi}{N+1} \to \frac{1}{2}\cos x\hat{j}\pi \cos x\hat{l}\pi + \frac{1}{2}\sin x\hat{j}\pi \sin x\hat{l}\pi = \frac{1}{2}\cos x(\hat{l}-\hat{j})\pi = \frac{1}{2}\cos x(l-j)\pi. \tag{22}$$

If the open-string coordinate satisfies Dirichlet boundary conditions, the analogs of Eqs. (17) and (18) are

$$\Delta^D_{ij,kl} = \frac{2}{M}\sum_{m=1}^{M-1} \frac{1}{\sinh \lambda_m^o} \frac{\sinh j\lambda_m^o \sinh (N+1-l)\lambda_m^o}{\sinh (N+1)\lambda_m^o} \times \sin\frac{mi\pi}{M}\sin\frac{mk\pi}{M}, \qquad l > j \tag{23}$$

and

$$\Delta^D_{ij,kl} = \frac{2}{N+1}\sum_{n=1}^{N} \frac{1}{\sinh \lambda_n^o} \frac{\sinh i\lambda_n^o \sinh (M-k)\lambda_n^o}{\sinh M\lambda_n^o} \times \sin\frac{nj\pi}{N+1}\sin\frac{nl\pi}{N+1}, \qquad k > i. \tag{24}$$

Correspondingly, the analogs of the $N \to \infty$ formulas [Eqs. (19) and (20)] are

$$\Delta^D_{ij,kl} = \frac{1}{M}\sum_{m=1}^{M-1} \frac{e^{-|l-j|\lambda_m^o}}{\sinh \lambda_m^o} \sin\frac{mi\pi}{M}\sin\frac{mk\pi}{M} \tag{25}$$

and

$$\Delta^D_{ij,kl} = \int_0^1 dx \frac{\sinh i\lambda^o(x) \sinh (M-k)\lambda^o(x)}{\sinh \lambda^o(x) \sinh M\lambda^o(x)} \cos x(l-j)\pi. \tag{26}$$

In this case, the lower end of the integral shows no divergence, because zero modes are absent.

For completeness, we also mention the two alternative forms for the worldsheet propagator on the closed-string worldsheet. Expansion in spatial normal modes gives

$$\Delta^c_{ij,kl} = \frac{j(N+1-l)}{(N+1)M} + \frac{1}{M}\sum_{m=1}^{M-1}\frac{1}{\sinh\lambda^c_m} \times \frac{\sinh j\lambda^c_m \sinh(N+1-l)\lambda^c_m}{\sinh(N+1)\lambda^c_m}\exp\frac{2m(i-k)\mathrm{i}\pi}{M}, \quad l > j, \tag{27}$$

whereas the expansion in temporal normal modes gives

$$\Delta^c_{ij,kl} = \frac{1}{N+1}\sum_{n=1}^{N}\frac{1}{\sinh\lambda^o_n}\frac{\cosh(M/2-|i-k|)\lambda^o_n}{\sinh(M/2)\lambda^o_n} \times \sin\frac{nj\pi}{N+1}\sin\frac{nl\pi}{N+1}. \tag{28}$$

In the first formula we have used Roman $\mathrm{i} = \sqrt{-1}$ to distinguish it from the index i. Then taking the $N \to \infty$ limit as before leads to, respectively,

$$\Delta^c_{ij,kl} \sim \frac{N+1}{4M} - \frac{|l-j|}{2M} + \frac{1}{2M}\sum_{m=1}^{M-1}\frac{e^{-|l-j|\lambda^c_m}}{\sinh\lambda^c_m} \times \exp\frac{2m(i-k)\mathrm{i}\pi}{M}, \tag{29}$$

$$\Delta^c_{ij,kl} = \frac{1}{2}\int_0^1 dx\frac{1}{\sinh\lambda^o_n}\frac{\cosh(M/2-|i-k|)\lambda^o(x)}{\sinh(M/2)\lambda^o(x)} \times \cos x(l-j)\pi. \tag{30}$$

III. OPEN-STRING TACHYON SELF-ENERGY

The one-loop self-energy of the ground string state (the tachyon) can be extracted from the string field propagator [Eq. (6)] by limiting the Ising spin configurations to those of a single hole of length K (i.e., $K-1$ missing contiguous missing links) and evaluating the $N \to \infty$ limit at fixed spin configuration, with the missing links in the vicinity of time $N/2$. Excited initial and final string states are suppressed exponentially, so one is left with an amplitude proportional to the ground string expectation of the interaction, i.e., the tachyon self-energy times N. The proportionality constant is removed by simply deleting the factor $\mathcal{D}_0$ from the expression. The overall factor of N is removed by fixing the initial time step of the hole at say, $N/2$, so the Ising spin sum is just the sum over the number of missing links and over the spatial location of the hole. For the closed string, that second sum just provides a factor of M by spatial translation invariance. But it is nontrivial for the open string. After all these steps, we arrive at the formula

$$-\Delta P^- = g^2\sum_{K=2}^{\infty}\sum_{M_1=1}^{M-1}\det{}^{-12}(I+V(M_1,K)\Delta)e^{-24B(K-1)}. \tag{31}$$

The formula for the closed-string tachyon self-energy simplifies because the summand is then independent of M_1, so the M_1 sum is trivial, leaving only the single sum over K. If B is set equal to its free-string value, the sum over K is badly divergent, because the two-string intermediate states include tachyon contributions which are lower in energy than the single-string tachyon. Thus, in our studies we are forced to choose B large enough to regularize this sum. If one could get the answer as an explicit function of B, one could in principle try to continue back to the free-string value. But the perturbation expansion really does not make sense unless the tachyon instability is resolved. What one can do is study the sum over Ising spins nonperturbatively, holding B sufficiently large so that the sums over S are convergent, and then scan the results as a function of B in search of a meaningful (i.e., Lorentz-invariant) result.

A. Self-energy formulas on the continuous worldsheet

Before doing the worldsheet lattice analysis, we recall the formal continuum expressions for the open-string tachyon and gluon self-energy in cylinder coordinates, following the notations of Ref. [7]:

$$\begin{aligned}\Delta P^-_{\mathrm{Tach}} &= \frac{C_o}{2P^+}\int_0^1\frac{dq}{q^3}\frac{1}{\prod_n(1-q^{2n})^{24}}\int_0^{2\pi}d\theta\frac{1}{4\sin^2(\theta/2)} \times \prod_{n=1}^{\infty}\left(\frac{(1-q^{2n}e^{i\theta})(1-q^{2n}e^{-i\theta})}{(1-q^{2n})^2}\right)^{-2}\\ &= \frac{C_o}{2P^+}\int_0^1\frac{dq}{q^3}\int_0^{2\pi}d\theta\left[\frac{1+24q^2}{4\sin^2(\theta/2)} - 2q^2 + \mathcal{O}(q^4)\right],\end{aligned} \tag{32}$$

$$\begin{aligned}\Delta P^-_{\mathrm{Gluon}} &= \frac{C_o}{2P^+}\int_0^1\frac{dq}{q^3}\frac{1}{\prod_n(1-q^{2n})^{24}}\int_0^{2\pi}d\theta\left[\frac{1}{4\sin^2(\theta/2)} - \sum_{n=1}^{\infty}\frac{2nq^{2n}}{1-q^{2n}}\cos n\theta\right]\\ &= \frac{C_o}{2P^+}\int_0^1\frac{dq}{q^3}\int_0^{2\pi}d\theta\left[\frac{1+24q^2}{4\sin^2(\theta/2)} - 2q^2\cos\theta + \mathcal{O}(q^4)\right],\end{aligned} \tag{33}$$

where in each case we display the UV behavior $q \sim 0$ of the integrand. The conformal mapping to the lightcone diagram, found in Ref. [16], determines the relation of q, θ to the length T and height σ_1 of the slit. Interestingly,

θ is simply proportional to σ_1, $\theta = 2\pi\sigma_1/P^+$ exactly. The relation of q to T is an implicit one involving elliptic functions, which we give only in the UV limit $q \sim 0$:

$$q = \frac{\pi T T_0}{8P^+ \sin \pi\sigma_1/P^+} - \frac{5 + \cos 2\pi\sigma_1/P^+}{3} \times \left(\frac{\pi T T_0}{8P^+ \sin \pi\sigma_1/P^+}\right)^3 + \mathcal{O}(T^5) \tag{34}$$

$$\to \frac{\pi K}{8M \sin \pi M_1/M} - \frac{5 + \cos 2\pi M_1/M}{3} \times \left(\frac{\pi K}{8M \sin \pi M_1/M}\right)^3 + \mathcal{O}(K^5), \tag{35}$$

where the second line shows q in the discretized variables of the lattice, $T = Ka$, $\sigma_1 = M_1 T_0 a$.

It is now easy to discretize the self-energy shift in the UV regime using

$$\frac{C_o}{2P^+}\int d\theta \int \frac{dq}{q^3} \to \frac{C_o}{a\pi T_0 M^2}\sum_{M_1,K}(1 + \mathcal{O}(K^4))\frac{64M^2}{K^3}\sin^2\frac{\pi M_1}{M}. \tag{36}$$

Then, for the gluon mass shift, we have

$$a\Delta P^-_{\text{Gluon}} \to \frac{16\pi C_o}{T_0}\sum_{M_1,K}\left(\frac{1}{\pi^2 K^3} + \frac{3}{8KM^2\sin^2\pi M_1/M} - \frac{1}{8KM^2}\cos\frac{2\pi M_1}{M} + \mathcal{O}(K^4)\right).$$

As discussed above, we deal with the severe IR divergences by suspending the K sum as we study the large-M limit:

$$a\delta P^-_{\text{Gluon},K} \to \frac{16\pi C_o}{T_0}\left(\frac{M-1}{\pi^2 K^3} + \frac{1}{4K}\sum_{M_1=1}^{(M-1)/2}\left[\frac{3}{M^2\sin^2\pi M_1/M} - \frac{1}{M^2}\cos\frac{2\pi M_1}{M}\right] + \mathcal{O}(K^4)\right). \tag{37}$$

The first term is just the familiar bulk term, which we also encountered for the closed string, and it can be absorbed in the worldsheet cosmological constant. The first term in square brackets formally can contribute a physically significant $1/M$ term, but also an order-M^0 term.[2] This can be seen as follows:

$$\begin{aligned}\sum_{M_1=1}^{(M-1)/2}\frac{1}{M^2\sin^2\pi M_1/M} &= \sum_{M_1=1}^{(M-1)/2}\left[\frac{1}{M^2\sin^2\pi M_1/M} - \frac{1}{\pi^2 M_1^2}\right] + \sum_{M_1=1}^{(M-1)/2}\frac{1}{\pi^2 M_1^2} \\ &\sim \frac{1}{6} - \sum_{m_1=(M+1)/2}^{\infty}\frac{1}{\pi^2 M_1^2} + \frac{1}{M}\int_0^{1/2}dx\left[\frac{1}{\sin^2\pi x} - \frac{1}{\pi^2 x^2}\right] \\ &\sim \frac{1}{6} - \frac{2}{\pi^2(M+1)} + \frac{2}{\pi^2 M} \sim \frac{1}{6} + \mathcal{O}(M^{-2}).\end{aligned} \tag{39}$$

In this case, the coefficient of the $1/M$ term is zero. The contribution of the second term in square brackets involves

$$\sum_{M_1=1}^{(M-1)/2}\frac{1}{M^2}\cos\frac{2\pi M_1}{M} \sim \frac{1}{M}\int_0^{1/2}dx\cos 2\pi x + \mathcal{O}(M^{-2}) = \mathcal{O}(M^{-2}). \tag{40}$$

So, in fact, there is no $1/M$ contribution to the gluon self-energy,

$$a\delta P^-_{\text{Gluon},K} \to \frac{16\pi C_o}{T_0}\left(\frac{M-1}{\pi^2 K^3} + \frac{1}{8K} + \mathcal{O}(M^{-2})\right), \tag{41}$$

consistent with zero mass shift for the gluon.

For the tachyon self-energy, the first term in square brackets is the same as in the gluon self-energy and so gives no contribution to a $1/M$ term. The second

[2]Contributions to this constant order-M^0 term also come from higher terms in the q expansion of the integrand. In general, one encounters M_1 sums of the form

$$\begin{aligned}\sum_{M_1=1}^{(M-1)/2}\frac{1}{M^{2n}\sin^{2n}\pi M_1/M} &\sim \frac{1}{M^{2n-1}}\int_0^{1/2}dx\left[\frac{1}{\sin^{2n}\pi x} - \sum_{k=1}^{n}\frac{c_k}{x^{2k}}\right] \\ &\quad + \sum_{k=1}^{n}\sum_{M_1=1}^{(M-1)/2}\frac{c_k}{M^{2(n-k)}M_1^{2k}} \\ &\sim \sum_{k=1}^{n}\frac{c_k\zeta(2k)}{M^{2(n-k)}} + \mathcal{O}(M^{-2n+1}) \\ &= \frac{\zeta(2n)}{\pi^{2n}} + \mathcal{O}(M^{-2}), \qquad n > 1,\end{aligned} \tag{38}$$

where the c_k are chosen to make the integral over x finite.

term in square brackets, however, when discretized becomes

$$-\frac{C_o a}{P^+}\int\frac{dq}{q}\int d\theta \to -\frac{C_o}{MT_0}\sum_K\sum_{M_1=1}^{M-1}\frac{2\pi}{MK}\sim -\frac{2\pi C_o}{MT_0}\sum_K\frac{1}{K}, \tag{42}$$

giving the expected logarithmically divergent tachyon mass shift. As we shall see in the remainder of this article, the analysis of the lattice worldsheet is in qualitative accord with these results.

B. Lattice self-energy, single missing link, $K = 2$

In coordinate space, the matrix V for a single missing link at time j and between spatial positions k and $k+1$ in the open-string worldsheet is

$$V_{ml;m'l'} = -\delta_{lj}\delta_{l'j}(\delta_{m,k+1}\delta_{m',k+1} + \delta_{m,k}\delta_{m',k} - \delta_{m,k+1}\delta_{m'k} - \delta_{m',k+1}\delta_{mk}), \tag{43}$$

exactly as in the closed-string worldsheet. Keeping the propagator Δ in coordinate space, the necessary determinant of the contributing 2×2 matrix can be taken over from the closed-string case:

$$\det(I+V\Delta) = \det\begin{pmatrix} 1+\Delta_{(k+1)j,kj}-\Delta_{kj,kj} & \Delta_{(k+1)j,(k+1)j}-\Delta_{kj,(k+1)j} \\ -\Delta_{(k+1)j,kj}+\Delta_{kj,kj} & 1-\Delta_{(k+1)j,(k+1)j}+\Delta_{kj,(k+1)j}\end{pmatrix} = 1-\Delta_{(k+1)j,(k+1)j}+\Delta_{kj,(k+1)j}+\Delta_{(k+1)j,kj}-\Delta_{kj,kj}. \tag{44}$$

We now substitute for Δ the representation of Eq. (20) for the open-string worldsheet propagator, which for $l = j$ reduces to

$$\Delta^o_{ij,kj} = \int_0^1 dx\frac{\cosh(i-1/2)\lambda^o(x)\cosh(M-k+1/2)\lambda^o(x)}{\sinh\lambda^o(x)\sinh M\lambda^o(x)}, \quad k>i, \tag{45}$$

and we remind the reader that for $k<i$ we switch the roles of i and k. It is helpful to rewrite the numerator in the integrand as

$$\cosh(i-1/2)\lambda^o\cosh(M-k+1/2)\lambda^o = \frac{1}{2}[\cosh\lambda^o(M+i-k)+\cosh\lambda^o(M-k-i+1)], \tag{46}$$

$$=\frac{1}{2}[\cosh\lambda^o M+\cosh\lambda^o(M-2i+1)], \quad k=i, \tag{47}$$

$$=\frac{1}{2}[\cosh\lambda^o(M-1)+\cosh\lambda^o(M-2i)], \quad k=i+1. \tag{48}$$

Inserting these results into Eq. (44), and relabeling $k\to M_1$ to more suitably describe the position of the missing link, leads to

$$\det(I+V\Delta^o) = \int_0^1 dx\frac{\sinh\lambda^o(x)(M-M_1)\sinh\lambda^o(x)M_1}{\sinh(\lambda^o(x)M/2)\cosh(\lambda^o(x)M/2)} \times\tanh\frac{\lambda^o(x)}{2}. \tag{49}$$

Now $\lambda^o(x) = 2\sinh^{-1}\sin(\pi x/2)$, and changing integration variables to $\lambda=\lambda^o$ requires

$$\frac{d\lambda}{dx} = \frac{\pi\sqrt{1-\sinh^2(\lambda/2)}}{\cosh(\lambda/2)}. \tag{50}$$

Then we can write

$$\mathrm{D}_{M_1} \equiv \det(I+V\Delta^o) = \frac{1}{\pi}\int_0^{\lambda_0} d\lambda\frac{\sinh\lambda(M-M_1)\sinh\lambda M_1}{\sinh(\lambda M/2)\cosh(\lambda M/2)} \times\frac{\sinh(\lambda/2)}{\sqrt{1-\sinh^2(\lambda/2)}}, \tag{51}$$

where $\lambda_0 = 2\sinh^{-1}1$. It is the value of λ where the argument of the square root in the denominator of the integrand vanishes.

We are interested in the limit $M\to\infty$ of the quantity

$$-\delta P_2^- = \sum_{M_1=1}^{M-1}\mathrm{D}_{M_1}^{-(D-2)} \to \sum_{M_1=1}^{M-1}\mathrm{D}_{M_1}^{-12} = 2\sum_{M_1<M/2}\mathrm{D}_{M_1}^{-12}+\mathrm{D}_{M/2}\delta_{M,\mathrm{even}}. \tag{52}$$

We begin with a study of the large-M behavior of D_{M_1} itself. The explicit M dependence of the integrand is buried in the ratio of sinh and cosh factors, which for fixed $\lambda>0$ has the behavior

$$\frac{\sinh\lambda(M-M_1)\sinh\lambda M_1}{\sinh(\lambda M/2)\cosh(\lambda M/2)} \sim\begin{cases}1-e^{-2\lambda M_1}+\mathcal{O}(e^{-\lambda M}) & \text{for } M_1\le\frac{M}{2},\\ 1-e^{-2\lambda(M-M_1)}+\mathcal{O}(e^{-\lambda M}) & \text{for } M_1\ge\frac{M}{2}.\end{cases} \tag{53}$$

Here the exponential terms are included to accurately account for the cases $M_1=\mathcal{O}(1)$, $M-M_1=\mathcal{O}(1)$. These terms are as small as the neglected terms when M_1 and $M-M_1$ are of order M. Next we use $D_{M-M_1}=D_{M_1}$ to write the sum over M_1 in terms of a sum over

$M_1 \le M/2$. (If M is odd, it is precisely twice the sum over $M_1 < M/2$.) Then we break up

$$\frac{\sinh\lambda(M-M_1)\sinh\lambda M_1}{\sinh(\lambda M/2)\cosh(\lambda M/2)} = 1 - e^{-2\lambda M_1} + \left[-\frac{2e^{-M\lambda}\sinh^2 M_1\lambda}{\sinh M\lambda}\right] \quad (54)$$

and evaluate the integral separately for the first two terms and the term in square brackets:

$$\frac{1}{\pi}\int_0^{\lambda_0} d\lambda (1-e^{-2\lambda M_1})\frac{\sinh(\lambda/2)}{\sqrt{1-\sinh^2(\lambda/2)}} = \frac{1}{2} - I_{M_1}, \quad (55)$$

$$I_{M_1} = \frac{1}{\pi}\int_0^{\lambda_0} d\lambda e^{-2\lambda M_1}\frac{\sinh(\lambda/2)}{\sqrt{1-\sinh^2(\lambda/2)}}. \quad (56)$$

We leave the integral defining I_{M_1} unevaluated, but we will need its explicit behavior at large M_1, which can be obtained by expanding the coefficient of $e^{-2\lambda M_1}$ in a power series:

$$I_{M_1} = \frac{1}{\pi}\int_0^{\infty} d\lambda e^{-2\lambda M_1}\left[\frac{\lambda}{2} + \frac{\lambda^3}{12} + \cdots\right] + \mathcal{O}(e^{-2M_1\lambda_0}) = \frac{1}{8\pi M_1{}^2} + \frac{1}{32\pi M_1{}^4} + \mathcal{O}(M_1{}^{-6}) + \mathcal{O}(e^{-2M_1\lambda_0}). \quad (57)$$

The exponentially small corrections to this asymptotic expansion come from the extension of the upper limit from λ_0 to ∞ used to evaluate the power corrections.

Finally, we turn to the contribution of the terms enclosed in square brackets to D_{M_1}. By construction it is exponentially small, as $M \to \infty$ at fixed λ. Thus, in a manner similar to our asymptotic analysis of I_{M_1}, we can find its power-behaved large-M behavior by expanding its coefficient in a power series in λ and extending the upper limit of integration to ∞. The errors in these steps are exponentially small:

$$\frac{1}{\pi}\int_0^{\lambda_0} d\lambda \Big[\Big]\frac{\sinh(\lambda/2)}{\sqrt{1-\sinh^2(\lambda/2)}} \sim \frac{1}{\pi}\int_0^{\infty} d\lambda\left(\frac{\lambda}{2M^2} + \frac{\lambda^3}{12M^4} + \cdots\right)\left[-\frac{2e^{-\lambda}\sinh^2(x\lambda)}{\sinh\lambda}\right] \quad (58)$$

$$\equiv \frac{f_2(x)}{2\pi M^2} + \frac{f_4(x)}{12\pi M^4} + \cdots, \quad (59)$$

where $x \equiv M_1/M$. Putting everything together, we have

$$\mathrm{D}_{M_1} = \frac{1}{2} - I_{M_1} + \frac{f_2(x)}{2\pi M^2} + \frac{f_4(x)}{12\pi M^4} + \cdots. \quad (60)$$

We are interested in the large-M behavior of $-\delta P^- \sim aM + b + c/M + \cdots$ through order $1/M$. The sum over M_1 ranges over $M-1$ values and can thus add up to a power of M to the explicit $1/M$ dependence of the summand. Thus, it is sufficient to keep only up to order $1/M^2$ in the summand.

Inserting these results into Eq. (52) and expanding to the desired order gives for M odd[3]

$$-\delta P_2^- = 2\sum_{M_1=1}^{(M-1)/2}\left[\left(\frac{1}{2} - I_{M_1}\right)^{-12} - \frac{12 f_2(M_1/M)}{M^2}\left(\frac{1}{2} - I_{M_1}\right)^{-13}\right]. \quad (61)$$

Now, I_{M_1} is only small at large M_1, so it is not safe to expand in powers of I_{M_1}. However, we can write

$$\left(\frac{1}{2} - I_{M_1}\right)^{-p} = 2^p + 2^p[(1-2I_{M_1})^{-p} - 1], \qquad p = 12, 13, \quad (62)$$

where the second term behaves as $1/M_1^2$ at large M_1. Because of that extra convergence, the sum over M_1 does not add a factor of M to the explicit $1/M$ dependence. So, for the first term in square brackets, the first term is of order M and the second of order 1:

$$2\sum_{M_1=1}^{(M-1)/2}\left(\frac{1}{2} - I_{M_1}\right)^{-12} = (M-1)2^{12} + 2^{13}\sum_{M_1=1}^{\infty}[(1-2I_{M_1})^{-12} - 1] - 2^{13}\sum_{M_1=(M+1)/2}^{\infty}[(1-2I_{M_1})^{-12} - 1] \quad (63)$$

$$\sim 2^{12}\left[(M-1) + 2\sum_{M_1=1}^{\infty}[(1-2I_{M_1})^{-12} - 1] - \frac{12}{\pi M}\right]. \quad (64)$$

Similarly, for the $1/M^2$ term in square brackets, the first term contributes order $1/M$, but the second term stays of order $1/M^2$:

$$-2^{13}\frac{12}{\pi M^2}\sum_{M_1=1}^{(M-1)/2} f_2(M_1/M)(1-2I_{M_1})^{-13} \sim -2^{13}\frac{12}{\pi M}\int_0^{1/2} dx f_2(x). \quad (65)$$

So, we evaluate

$$\int_0^{1/2} dx f_2(x) = \int_0^{\infty} d\lambda\lambda\int_0^{1/2} dx\left[-\frac{2e^{-\lambda}\sinh^2(x\lambda)}{\sinh\lambda}\right] = \frac{\pi^2}{24} - \frac{1}{2}. \quad (66)$$

[3]When M is even, the upper limit is $(M-2)/2$, and there is an additional term for $M_1 = M/2$.

So we finally arrive at the large-M behavior

$$-\delta P_2^- \sim 2^{12}\Big[M - 1 + 2\sum_{M_1=1}^{\infty}[(1-2I_{M_1})^{-12} - 1] - \frac{\pi}{M}\Big] + \mathcal{O}(M^{-2}). \tag{67}$$

Although for simplicity we assumed that M was odd, it is not difficult to see that the same result holds for M even.

C. Single slit with $K-1$ missing links

As we showed in Ref. [12], in the case of a single slit with $K-1$ missing links between spatial positions k and $k+1$, the path integral [Eq. (6)] involves a determinant of the form

$$\det(I + V\Delta) = \det(h_{lp}), \qquad l, p = 1, 2, \ldots K-1, \tag{68}$$

where

$$h_{lp} = \delta_{lp} + \Delta_{(k+1)l,kp} - \Delta_{kl,kp} + \Delta_{kl,(k+1)p} - \Delta_{(k+1)l,(k+1)p}. \tag{69}$$

Focusing on the open string, we insert the two equivalent representations in Eqs. (19) and (20) for the propagators, and again switch to a more distinct notation for the slit position $k \to M_1$, obtaining, respectively,

$$h_{lp} = \delta_{lp} - \frac{2}{M}\sum_{m=1}^{M-1}\frac{\sin\frac{m\pi}{2M}\sin^2\frac{m\pi M_1}{M}}{\sqrt{1+\sin^2\frac{m\pi}{2M}}} \times\left(\sin\frac{m\pi}{2M} + \sqrt{1+\sin^2\frac{m\pi}{2M}}\right)^{-2|l-p|} \tag{70}$$

$$= \int_0^{\lambda_0} d\lambda \frac{\sinh\frac{\lambda}{2}\cos[2(l-p)\sin^{-1}(\sinh\frac{\lambda}{2})]}{\pi\sqrt{1-\sinh^2\frac{\lambda}{2}}} \times \frac{\sinh\lambda(M-M_1)\sinh\lambda M_1}{\sinh(\lambda M/2)\cosh(\lambda M/2)}. \tag{71}$$

In what follows, we will use the integral form [Eq. (71)]. Clearly the only difference from Eq. (51) is the additional cosine factor, which carries the dependence on $|l-p|$. The small-λ behavior of the first fraction in the integrand of Eq. (71) is given by

$$\frac{\sinh\frac{\lambda}{2}\cos[2(l-p)\sin^{-1}(\sinh\frac{\lambda}{2})]}{\pi\sqrt{1-\sinh^2\frac{\lambda}{2}}} = \frac{\lambda}{2\pi} + \frac{[1-3(l-p)^2]\lambda^3}{12\pi} + \mathcal{O}(\lambda^5). \tag{72}$$

Hence, we see that contributions to the asymptotic expansion of h_{lp} coming from the $\mathcal{O}(\lambda)$ term will be the same for one or many missing links.

Separating the second fraction in the integrand of Eq. (71) according to Eq. (54), we similarly obtain

$$h_{lp} = \tilde{c}_{lp} + \epsilon_{lp} = (c_{lp} - I_{lp}) + \epsilon_{lp}, \tag{73}$$

where

$$c_{lp} = \int_0^{\lambda_0} d\lambda \frac{\sinh\frac{\lambda}{2}\cos[2(l-p)\sin^{-1}(\sinh\frac{\lambda}{2})]}{\pi\sqrt{1-\sinh^2\frac{\lambda}{2}}} = \int_0^1 dx \frac{\sin\frac{\pi x}{2}\cos[(l-p)\pi x]}{\sqrt{1+\sin^2\frac{\pi x}{2}}}, \tag{74}$$

$$I_{lp} = \int_0^{\lambda_0} d\lambda \frac{\sinh\frac{\lambda}{2}\cos[2(l-p)\sin^{-1}(\sinh\frac{\lambda}{2})]}{\pi\sqrt{1-\sinh^2\frac{\lambda}{2}}} e^{-2M_1\lambda}, \tag{75}$$

$$\epsilon_{lp} = \int_0^{\lambda_0} d\lambda \frac{\sinh\frac{\lambda}{2}\cos[2(l-p)\sin^{-1}(\sinh\frac{\lambda}{2})]}{\pi\sqrt{1-\sinh^2\frac{\lambda}{2}}} \times \frac{2 - e^{-2M_1\lambda} - e^{2M_1\lambda}}{-1 + e^{2M\lambda}}. \tag{76}$$

The first part of the matrix element [Eq. (74)] can be shown to coincide with the M-independent part of h_{lp} for the closed string. The $\tilde{c}_{ij}$ combination, for $M_1 \le (M-1)/2$, encodes the leading behavior of the integrand for M large.

Expanding as in Eq. (72), we may formally take the I_{lp} integral term by term, using

$$\int_0^{\lambda_0} \lambda^{s-1}e^{-2M_1\lambda} = \frac{1}{(2M_1)^s}\int_0^{2M_1\lambda_0} \lambda^{s-1}e^{-\lambda} = \gamma(s, 2M_1\lambda), \tag{77}$$

where $\gamma(s,x)$ is the lower incomplete gamma function, which for a positive integer s is

$$\gamma(s,x) = (s-1)! - (s-1)!e^{-x}\sum_{k=0}^{s-1}\frac{x^k}{k!}. \tag{78}$$

This expression is particularly useful for extracting the large-M_1 behavior of the integral, since up to exponentially suppressed terms we can write

$$I_{lp} = \frac{1}{8\pi M_1^2} + \frac{1-3(l-p)^2}{32\pi M_1^4} + \mathcal{O}(M_1^{-6}) + \mathcal{O}(e^{-2M_1\lambda}). \tag{79}$$

In a similar fashion, we obtain a large-M expansion for the third integral, with $x = M_1/M$ arbitrary:

$$\epsilon_{lp} = \frac{1}{2\pi M^2}f_2(x) + \frac{1-3(l-p)^2}{12\pi M^4}f_4(x) + \mathcal{O}(M^{-6}) + \mathcal{O}(e^{-2M\lambda}), \tag{80}$$

where $f_i(x)$ are the same functions that appeared in the single-link case.[4] Keeping terms of $\mathcal{O}(M^{-2})$ for the matrix elements will of course yield the determinant to the same accuracy, and in particular

$$\det(h_{lp}) = \det\left(\tilde{c}_{lp} + \frac{f_2(x)}{2\pi M^2} + \mathcal{O}(M^{-4})\right) = \det(\tilde{c}_{lp})\left(1 + \frac{f_2(x)}{2\pi M^2}\sum_{l,p=1}^{K-1}(\tilde{c}^{-1})_{lp}\right) + \mathcal{O}(M^{-4}). \tag{81}$$

This will be sufficient for obtaining the tachyon self-energy [Eq. (31)] for fixed K and M_1 summed up to the physically relevant $\mathcal{O}(M^{-1})$ term, as the latter sum can contribute an extra factor of M at most.[5]

Finally, another procedure to evaluate the large-M expansion of the sum in question is to add and subtract the value of the summand for large M_1, as we did for the single missing link. In particular, the analogues of Eq. (75) for the quantities appearing in Eq. (81) are

$$\det(\tilde{c}_{lp}) = \det(c_{lp})\left(1 - \frac{1}{8\pi M_1^2}\sum_{l,p=1}^{K-1}(c^{-1})_{lp}\right) + \mathcal{O}(M_1^{-4}) + \mathcal{O}(e^{-2M_1\lambda}), \tag{82}$$

$$\sum_{l,p=1}^{K-1}(\tilde{c}^{-1})_{lp} = \sum_{l,p=1}^{K-1}(c^{-1})_{lp}\left(1 + \frac{1}{8\pi M_1^2}\sum_{l,p=1}^{K-1}(c^{-1})_{lp}\right) + \mathcal{O}(M_1^{-4}) + \mathcal{O}(e^{-2M_1\lambda}). \tag{83}$$

Thus, focusing on M odd, we can calculate the self-energy of a tachyon due to a single slit of $K-1$ time steps as follows [in all steps we keep terms up to $\mathcal{O}(M^{-2})$ in the summand or equivalently $\mathcal{O}(M^{-1})$ for the full sum]:

$$\begin{aligned}
-\delta P_K^- &= \sum_{M_1=1}^{M-1}\det(h_{lp})^{-12} = 2\sum_{M_1=1}^{(M-1)/2}\det(\tilde{c}_{lp}+\epsilon_{lp})^{-12} \\
&\simeq 2\sum_{M_1=1}^{(M-1)/2}\det(\tilde{c}_{lp})^{-12}\left(1 + \frac{f_2(x)}{2\pi M^2}\sum_{l,p=1}^{K-1}(\tilde{c}^{-1})_{lp}\right)^{-12} \\
&\simeq 2\sum_{M_1=1}^{(M-1)/2}\left\{\det(c_{lp})^{-12} + (\det(\tilde{c}_{lp})^{-12} - \det(c_{lp})^{-12}) - 12\frac{f_2(x)}{2\pi M^2}\det(\tilde{c}_{lp})^{-12}\sum_{l,p=1}^{K-1}(\tilde{c}^{-1})_{lp}\right\} \\
&\simeq (M-1)\det(c_{lp})^{-12} + 2\sum_{M_1=1}^{\infty}(\det(\tilde{c}_{lp})^{-12} - \det(c_{lp})^{-12}) - 2\sum_{M_1=\frac{M+1}{2}}^{\infty}(\det(\tilde{c}_{lp})^{-12} - \det(c_{lp})^{-12}) \\
&\quad - 24\sum_{M_1=1}^{(M-1)/2}\frac{f_2(x)}{2\pi M^2}\det(\tilde{c}_{lp})^{-12}\sum_{l,p=1}^{K-1}(\tilde{c}^{-1})_{lp} \\
&\simeq M\det(c_{lp})^{-12} + \left[2\sum_{M_1=1}^{\infty}(\det(\tilde{c}_{lp})^{-12} - \det(c_{lp})^{-12}) - \det(c_{lp})^{-12}\right] \\
&\quad - 24\det(c_{lp})^{-12}\sum_{l,p=1}^{K-1}(c^{-1})_{lp}\left(\sum_{M_1=\frac{M+1}{2}}^{\infty}\frac{1}{8\pi M_1^2} + \sum_{M_1=1}^{(M-1)/2}\frac{f_2(x)}{2\pi M^2}\right).
\end{aligned}$$

For clarity, we mention that in the last step we dropped the term containing the difference between $\det(\tilde{c}_{lp})^{-12}\sum(\tilde{c}^{-1})_{lp}$ and its asymptotic value in M_1, as it will only contribute at order $\mathcal{O}(M^{-2})$ in the final answer. Employing the asymptotics

$$\sum_{M_1=\frac{M+1}{2}}^{\infty}\frac{1}{8\pi M_1^2} = \frac{1}{4\pi M} + \mathcal{O}(M^{-2}), \qquad \sum_{M_1=1}^{(M-1)/2}\frac{f_2(x)}{2\pi M^2} = -\frac{1}{4M\pi} + \frac{\pi}{48M} + \mathcal{O}(M^{-3}), \tag{84}$$

we finally obtain

[4]Neglecting exponentially suppressed factors, it is not difficult to calculate these functions explicitly. For example, $f_2(x) = \frac{\pi^2}{12M^2} + \frac{1}{4M^2x^2} - \frac{\pi^2\csc[\pi x]^2}{4M^2}$.

[5]This can be seen, for example, with the help of the Euler-Maclaurin formula.

$$-\delta P_K^- = M\det(c_{lp})^{-12} + \Big[2\sum_{M_1=1}^{\infty}(\det(\tilde{c}_{lp})^{-12} - \det(c_{lp})^{-12}) - \det(c_{lp})^{-12}\Big] - \frac{\pi}{2M}\det(c_{lp})^{-12}\sum_{l,p=1}^{K-1}(c^{-1})_{lp} + \mathcal{O}(M^{-2}). \quad (85)$$

Comparing with the respective summand for the closed string, see Eqs. (38) and (59) in Ref. [12], we note that the leading term in the two expressions is the same, and the $\mathcal{O}(M^{-1})$ is four times larger in the closed string. The same proportionality holds between the tachyon masses of the free closed and open strings.

IV. OPEN-STRING GLUON SELF-ENERGY

To extract energy shifts for excited states, we examine the propagator on a lattice worldsheet with some pattern of missing links described by V:

$$\Delta^V = (\Delta^{-1} + V)^{-1} = \Delta(I + V\Delta)^{-1} = \Delta - \Delta(I + V\Delta)^{-1}V\Delta \equiv \Delta - \Delta\mathcal{V}\Delta. \quad (86)$$

Inserting the normal mode expansion [Eq. (19)] for Δ in the rightmost side of this equation, we can write Δ^V:

$$\Delta^V_{ij,kl} = \sum_{m,m'} e^{j\lambda_m^o - l\lambda_{m'}^o}\frac{\tilde{\Delta}^V_{mm'}}{M\sqrt{\sinh\lambda_m^o\sinh\lambda_{m'}^o}}\cos\frac{m(i-1/2)\pi}{M} \times\cos\frac{m'(k-1/2)\pi}{M}, \quad (87)$$

$$\tilde{\Delta}^V_{mm'} = \delta_{mm'} - \frac{\tilde{\mathcal{V}}_{mm'}}{M\sqrt{\sinh\lambda_m^o\sinh\lambda_{m'}^o}}, \quad (88)$$

$$\tilde{\mathcal{V}}_{mm'} = \sum_{pq,rs}\mathcal{V}_{pq,rs}e^{-q\lambda_m^o + s\lambda_{m'}^o}\cos\frac{m(p-1/2)\pi}{M} \times\cos\frac{m'(r-1/2)\pi}{M}. \quad (89)$$

Then the contribution of this diagram to the one-loop gluon self-energy is

$$-\tilde{\Delta}^V_{11}\det{}^{-12}(I + V\Delta), \quad (90)$$

where V corresponds to the missing link patterns of a single hole in the worldsheet.

A. Single missing link, $K = 2$

Working in the 2×2 subspace selected by V for a single missing link between positions k, $k+1$ at time j, we have, setting $A = \Delta_{(k+1)j,kj} - \Delta_{kj,kj}$ and $A' = \Delta_{(k+1)j,kj} - \Delta_{(k+1)j,(k+1)j}$,

$$V = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad I + V\Delta = \begin{pmatrix} 1+A & -A \\ -A' & 1+A' \end{pmatrix},$$
$$\mathcal{V} = (I + V\Delta)^{-1}V = \frac{1}{1+A+A'}\begin{pmatrix} 1+A' & A \\ A' & 1+A \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{V}{1+A+A'} = V\det{}^{-1}(1 + V\Delta). \quad (91)$$

Then

$$\tilde{\mathcal{V}}_{mm'} = \Big[\cos\frac{m(k-1/2)\pi}{M}\cos\frac{m'(k+1/2)\pi}{M} + \cos\frac{m(k+1/2)\pi}{M}\cos\frac{m'(k-1/2)\pi}{M} - \cos\frac{m(k-1/2)\pi}{M}\cos\frac{m'(k-1/2)\pi}{M} - \cos\frac{m(k+1/2)\pi}{M}\cos\frac{m'(k+1/2)\pi}{M}\Big]\frac{e^{-j(\lambda_m^o-\lambda_{m'}^o)}}{\det(1+V\Delta)}$$
$$= -4\sin\frac{m\pi}{2M}\sin\frac{m'\pi}{2M}\sin\frac{mk\pi}{M}\sin\frac{m'k\pi}{M}\frac{e^{-j(\lambda_m^o-\lambda_{m'}^o)}}{\det(1+V\Delta)}, \quad (92)$$

$$\tilde{\Delta}^V_{mm'} = \delta_{mm'} + 4\frac{\sin\frac{m\pi}{2M}\sin\frac{m'\pi}{2M}\sin\frac{mk\pi}{M}\sin\frac{m'k\pi}{M}}{M\sqrt{\sinh\lambda_m^o\sinh\lambda_{m'}^o}}\frac{e^{-j(\lambda_m^o-\lambda_{m'}^o)}}{\det(1+V\Delta)}. \quad (93)$$

Then the contribution to the self-energy of the excited string state $a_{-m}|0\rangle$ from this diagram (setting $m'=m$) is

$$-\delta P_2^-(m) = \sum_{k=1}^{M-1}\Big(1 + 4\frac{\sin^2(\frac{m\pi}{2M})\sin^2(\frac{mk\pi}{M})}{M\sinh\lambda_m^o}\frac{1}{\det(1+V\Delta)}\Big) \times\det{}^{-12}(1+V\Delta) \quad (94)$$

$$\approx\sum_{k=1}^{M-1}\Big(1 + \frac{m\pi}{M^2}\frac{\sin^2(\frac{mk\pi}{M})}{\det(1+V\Delta)} + \mathcal{O}(M^{-4})\Big)\det{}^{-12}(1+V\Delta)$$
$$\sim 2^{12}\Big[M - 1 + 2\sum_{k=1}^{\infty}[(1-2I_k)^{-12} - 1] + (m-1)\frac{\pi}{M}\Big] + \mathcal{O}(M^{-2}). \quad (95)$$

It is, of course, significant that the coefficient of $1/M$ vanishes for $m=1$, reflecting the fact that perturbative corrections to the gluon mass should be 0.

B. Single slit with $K-1$ missing links

As we have shown in Ref. [12], and can directly verify from the definition

$$\mathcal{V} \equiv (I + V\Delta)^{-1}V \Rightarrow (I + V\Delta)\mathcal{V} = V, \quad (96)$$

the elements of $\mathcal{V}$ are given by

$$\mathcal{V}_{kl,ks} = \mathcal{V}_{(k+1)l,(k+1)s} = -\mathcal{V}_{(k+1)l,ks} = -\mathcal{V}_{kl,(k+1)s} = -h^{-1}_{ls}, \tag{97}$$

where k denotes the spatial position of the slit, and the matrix h is defined in Eqs. (70) and (71). In what follows, we will again redefine $k \to M_1$ so as to label the slit position in a more distinctive manner. In this notation, and with the help of Eq. (97), the Fourier transform of $\mathcal{V}$ with the open-string wave functions [Eq. (89)] becomes

$$\tilde{\mathcal{V}}_{mm'} = -4\sin\frac{m\pi}{2M}\sin\frac{m'\pi}{2M}\sin\frac{mM_1\pi}{M} \times \sin\frac{m'M_1\pi}{M}\sum_{q,s=1}^{K-1} e^{-s\lambda^o_m + q\lambda^o_{m'}} h^{-1}_{qs}. \tag{98}$$

Then the analogue of Eq. (95) for many missing links will be

$$\begin{aligned}-\delta P^-_K(m) &= \sum_{M_1=1}^{M-1} \tilde{\Delta}^V_{mm}\det{}^{-12}(I + V\Delta) \\ &= \sum_{M_1=1}^{M-1}\left(1 - \frac{\tilde{\mathcal{V}}_{mm}}{M\sinh\lambda^o_m}\right)\det{}^{-12}(h_{lp}) \\ &\approx \sum_{M_1=1}^{M-1}\left(1 + \frac{m\pi}{M^2}\sin^2(m\pi M_1/M)\sum_{q,s} e^{(q-s)\lambda^o_m} h^{-1}_{qs} + \mathcal{O}(M^{-4})\right)\det{}^{-12}(h_{lp}).\end{aligned} \tag{99}$$

The additional contribution for the gluon as compared to the tachyon comes from the second term in the parenthesis in Eq. (99). We are interested in the asymptotic expansion of the latter equation only up to $\mathcal{O}(M^{-1})$, and hence we only need the leading term of the additional contribution. As we discussed in the case of the tachyon, this may be obtained by replacing all quantities in the sum in M_1 with their asymptotic form for large M_1, which for the case at hand implies

$$\left(\sum_{q,s} e^{(s-q)\lambda^o_m} h^{-1}_{qs}\right)\det{}^{-12}(h_{lp}) \to \left(\sum_{q,s} c^{-1}_{qs}\right)\det{}^{-12}(c_{lp}). \tag{100}$$

Then, the sum in M_1 can be done exactly in terms of geometric series, and together with the contribution which is identical to the tachyon self-energy [Eq. (85)], we obtain the final formula:

$$\begin{aligned}-\delta P^-_K(m) &= M\det(c_{lp})^{-12} + \left[2\sum_{M_1=1}^{\infty}(\det(\tilde{c}_{lp})^{-12} - \det(c_{lp})^{-12}) - \det(c_{lp})^{-12}\right] \\ &\quad + (m-1)\frac{\pi}{2M}\det(c_{lp})^{-12}\sum_{l,p=1}^{K-1}(c^{-1})_{lp} + \mathcal{O}(M^{-2}).\end{aligned} \tag{101}$$

V. DEPENDENCE OF STRING SELF-ENERGY ON SLIT SIZE

A. Leading term in the M expansion via Fisher-Hartwig formula

We would like to know the dependence of the coefficients of Eq. (101) on the slit size $K-1$, for $M \gg K \gg 1$. At first this seems quite challenging, as the dependence on $n = K-1$ enters primarily via the size of the $\det(c_{lp})$ determinant.

Fortunately, this can be achieved by exploiting the fact that the latter is the determinant of a Toeplitz matrix, meaning that $c_{lp} = c(l-p)$, or in other words, that all elements in a left-to-right descending diagonal are the same. In this case, there exists a formula for the asymptotic behavior of the determinant due to Fisher and Hartwig [14]; see also Ref. [17] for a more recent treatment. We will rely on the notations of the latter paper, and in particular we will rewrite the matrix elements as Fourier transforms of the same function $f(z)$,

$$c_{lp} = \int_0^1 dx \frac{\sin(\pi x/2)\cos[(l-p)\pi x]}{\sqrt{1+\sin^2(\pi x/2)}} = \frac{1}{2\pi}\int_0^{2\pi} d\theta f(e^{i\theta})e^{-i(l-p)\theta}, \quad l, p = 1, \ldots, n, \tag{102}$$

where

$$f(e^{i\theta}) = \frac{\sin(\theta/2)}{\sqrt{1+\sin^2(\theta/2)}} \tag{103}$$

or equivalently, for $z = e^{i\theta}$,

$$f(z) = \frac{|z-1|}{2\sqrt{1+(z-1)(1/z-1)/4}}. \tag{104}$$

This implies that $f(z)$ is a special case of the function considered in Ref. [17], with the following values for the parameters according to their conventions:

$$z_0 = 1, \qquad \alpha_0 = \frac{1}{2}, \qquad \beta_0 = 0, \qquad V(e^{i\theta}) = -\log\left(2\sqrt{1+\sin^2\frac{\theta}{2}}\right). \tag{105}$$

Consequently, the asymptotic behavior of the n-dimensional determinant will be given by

$$\det(c_{lp}) = n^{\frac{1}{4}} \exp\left(nV_0 + \sum_{k=1}^{\infty} kV_k V_{-k} - \frac{1}{2}\sum_{k=1}^{\infty} V_k - \frac{1}{2}\sum_{k=-\infty}^{-1} V_k\right) \frac{G(\frac{3}{2})^2}{G(2)}(1 + \mathcal{O}(n^{-1})), \quad (106)$$

where $G(x)$ is the Barnes G function and V_k are the Fourier modes of $V(e^{i\theta})$,

$$V_k = \frac{1}{2\pi}\int_0^{2\pi} d\theta V(e^{i\theta})e^{-ik\theta}, \qquad V(z) = \sum_{k=-\infty}^{\infty} V_k z^k. \quad (107)$$

Our function $V(e^{i\theta})$ is simple enough that $V_{-k} = V_k$, and in particular we can calculate them exactly,

$$V_0 = -\log(1+\sqrt{2}), \qquad V_k = \frac{1}{2k}(1-\sqrt{2})^{2k}, \quad (108)$$

from which we can in turn obtain

$$\sum_{k=1}^{\infty} kV_k V_{-k} = -\frac{1}{4}\log[4(-4+3\sqrt{2})],$$
$$\sum_{k=1}^{\infty} V_k = \sum_{k=1}^{\infty} V_{-k} = -\frac{1}{2}\log[2(\sqrt{2}-1)]. \quad (109)$$

Substituting back into Eq. (106), we obtain the final formula

$$\begin{aligned}\det(c_{lp}) &= n^{\frac{1}{4}}\exp\left(-\log(1+\sqrt{2})n - \frac{1}{8}\log 2\right) \times \frac{G(\frac{3}{2})^2}{G(2)}(1+\mathcal{O}(n^{-1})) \\ &\simeq \exp(0.25\log n - 0.881n + 0.0472)(1+\mathcal{O}(n^{-1})) \\ &\simeq 1.048n^{\frac{1}{4}}\exp(-0.881n)(1+\mathcal{O}(n^{-1})). \end{aligned} \quad (110)$$

As a consistency check, we can compare the asymptotic formula above with fits for the value of the determinant over a range of different n. For this purpose, it becomes more efficient to fit the logarithm of the determinant, and we choose the range $n \in [100, 200]$ in steps on 1. We find that

$$\log\det(c_{lp}) \simeq 0.2499\log n - 0.88137n + 0.0472 + \frac{0.17}{n}, \quad (111)$$

where the errors in the coefficients are at the order of the last digit, and we also included a term $\log(1+c/n) \simeq c/n$ to account for the subleading asymptotic term in Eq. (110). Evidently, the coefficients of the fit are in excellent agreement with the Fisher-Hartwig formula.

In order to obtain the dependence of the leading term in the M expansion of the tachyon and gluon self-energy summand[6] on the duration of the self-interaction $n = K - 1$, we simply have to raise Eq. (110) to the (-12) power. In this manner, we obtain a power dependence of n^{-3}, which is a rigorous confirmation of the rough estimate we had obtained in Ref. [11].

B. $\mathcal{O}(M^{-1})$ term

In order to find the dependence on slit size for the $\mathcal{O}(M^{-1})$ term in Eq. (101), we will have to additionally analyze $\sum c_{lp}^{-1}$. To this end, we will be using asymptotic expansions for the inverses of Toeplitz matrices in the same category with c_{lp}, which have relatively recently appeared in the literature [18,19].

In more detail, these papers focus on Toeplitz matrices of the form in Eq. (102), where

$$f(z) = |z-1|^{2\alpha} f_1(z), \quad (112)$$

so that our matrix of interest, c_{lp}, is a special case with $\alpha = 1/2$ and

$$f_1(z) = \frac{1}{2\sqrt{1+(z-1)(1/z-1)/4}}. \quad (113)$$

In order to stay as close as possible to the notations of Refs. [18,19], let us call the dimensionality of the matrix $n \equiv N+1$. Then, for $0<x<1$, $0<y<1$, $x \neq y$, the asymptotic forms of the inverse matrix element will be

$$c^{-1}_{[Nx]+1,1} = \frac{1}{g_1(1)\sqrt{\pi N}}\frac{\sqrt{1-x}}{\sqrt{x}} + o(N^{-1/2}), \qquad f_1 = g_1\bar{g}_1, \quad (114)$$

$$c^{-1}_{[Nx]+1,[Nx]+1} = \frac{1}{f_1(1)\pi}\log N + o(\log N), \quad (115)$$

$$c^{-1}_{[Nx]+1,[Ny]+1} = \frac{1}{f_1(1)\pi}G_{\frac{1}{2}}(x,y) + o(1), \quad (116)$$

where $[a]$ denotes the integer part of a,

$$\begin{aligned} G_{\frac{1}{2}}(x,y) &= \sqrt{x}\sqrt{y}\int_{\max(x,y)}^{1}\frac{dt}{t\sqrt{t-x}\sqrt{t-y}} \\ &= 2\operatorname{arctanh}\frac{\sqrt{y}}{\sqrt{1-y}}\frac{\sqrt{1-x}}{\sqrt{x}}, \end{aligned} \quad (117)$$

and the last equality in the above equation holds if $x > y$; otherwise, we simply exchange $x \leftrightarrow y$.

According to the Euler-Maclaurin formula, the leading contribution to the sum over the $[Nx]$ or $[Ny]$ indices in each of the formulas above will be N times the integral

[6] In fact, this term is universal for all states of the open and closed string.

over x or y. This implies contributions of $\mathcal{O}(N^{1/2})$, $\mathcal{O}(N \log N)$, and $\mathcal{O}(N^2)$ to the sum over all indices from Eqs. (114)–(116), respectively. So up to leading order, we may write

$$\begin{aligned}\sum_{l,p=1}^{N+1}(c^{-1})_{lp} &\simeq \sum_{l\neq p=1}^{N+1}(c^{-1})_{lp} = 2\sum_{l>p=1}^{N+1}(c^{-1})_{lp}\\ &\simeq \frac{2N^2}{f_1(1)\pi}\int_0^1 dx\int_0^x dy G_{\frac{1}{2}}(x,y)\\ &\simeq \frac{4N^2}{\pi}\int_0^1 dx\Big[2\sqrt{x(1-x)}\arcsin\sqrt{y}\\ &\quad + 2(y-x)\operatorname{arctanh}\frac{\sqrt{1-x}\sqrt{y}}{\sqrt{x}\sqrt{1-y}}\Big]_{y=0}^{x}\\ &\simeq \frac{8N^2}{\pi}\int_0^1 dx\sqrt{x(1-x)}\arcsin\sqrt{x}, \end{aligned} \tag{118}$$

and given that the x integral above yields $\pi^2/32$, we finally obtain, after we restore $n = N + 1$,

$$\sum_{l,p=1}^{n}(c^{-1})_{lp} \simeq \frac{\pi}{4}n^2 \simeq 0.78539816 n^2. \tag{119}$$

Another way to arrive at this result is to notice that only the value of $f_1(z)$ at $z = 1$ matters for the leading term in the expansions of Eqs. (114)–(116). In order to extract the term in question, we can thus examine the determinant where we have replaced $f_1(z)$ with its constant value $f_1(1)$, namely

$$d_{lp} = \frac{1}{2\pi}\int_0^{2\pi} d\theta \sin\frac{\theta}{2} e^{-i(l-p)\theta} = \frac{2}{\pi}\frac{1}{1-4(l-p)^2}. \tag{120}$$

Evidently, the virtue of this replacement is that it allows us to compute the integral explicitly. Then, by analytically inverting the matrix and summing its elements for $n \in [1, 10]$, we experimentally find that the sum of all elements of the inverse matrix is given by the following simple formula:

$$\sum_{l,p=1}^{n}(d^{-1})_{lp} = \frac{\pi}{4}n(n+1). \tag{121}$$

As a final test of our result in Eq. (119), we may compare it to fits of the quantity for varying n. In particular, we choose $n \in [100, 200]$ in steps of 5, and determine the coefficients of a polynomial fit of degree 2, as potentially existing logarithms at orders lower than $\mathcal{O}(n^2)$ can be well approximated by constants within this range. The results are depicted in Fig. 2 and leave no doubt that the leading dependence of the sum of all elements of matrix c^{-1} on its size is given by Eq. (119).

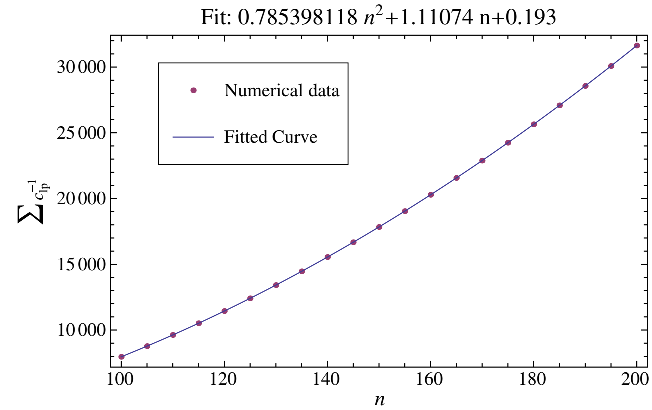

FIG. 2 (color online). Polynomial fit of degree 2 for the quantity $\sum_{l,p}^{n}(c^{-1})_{lp}$, appearing in the $\mathcal{O}(M^{-1})$ term in the asymptotic expansion for one-loop tachyon self-energy [Eq. (85)]. The coefficient of the leading $\mathcal{O}(n^2)$ term agrees excellently with the analytically derived value $\pi/4$.

VI. NUMERICAL ANALYSIS

In this section, we numerically evaluate the tachyon and gluon self-energy δP_K^- due to an interaction lasting $K - 1$ time steps, investigate its behavior for different values of M and K, and obtain fits that we compare to the analytic asymptotic formulas [Eqs. (85) and (101)]. We are interested in the ultraviolet $M \gg K$ behavior of the self-energy, so we choose $K \in [2, 30]$ and $M \in [495, 1995]$ in steps of 10. We perform the evaluation using the sum expression for the matrix elements of the determinant [Eq. (70)], as it turns out to be numerically more stable than the integral expression [Eq. (71)].

Starting with the tachyon, we observe that, for the first few values of fixed K, the leading behavior of δP_K^- is indeed linear in M within the range we have chosen, and performing fits of the form $\delta P_K^- = \sum c_i M^i$ with three free parameters $c_{\pm 1}$, c_0, we find excellent agreement with the values predicted by Eq. (85). The results of the fits and the comparison with the asymptotic prediction are depicted in Table I; see also Fig. 3(a).

As it is evident in Fig. 3(b), however, starting at $K = 12$ and higher, $-\delta P_K^-$ becomes convex, and subleading terms in the M expansion begin to dominate over the linearly increasing term in the lower end of our range. In particular, this behavior at lower M cannot be due to the $\mathcal{O}(M^{-1})$ term, which could only cause a deviation below the straight line because of its negative sign. Therefore, it must come from higher terms in the expansion, and experimentation with different fitting functions suggests that it is in fact due to the $\mathcal{O}(M^{-4})$ term.

This can be seen in more detail in Fig. 4, where we compare δP_K^- against a fit with a constant and an $\mathcal{O}(M^{-4})$ term for $K = 14, 30$, finding very good agreement. The fit suggests that its two parameters always have comparable sizes and grow very fast with K. We already know from the

TABLE I. Tachyon self-energy for an interaction lasting $K-1$ time steps, $K=2,\ldots,5$. Coefficients of asymptotic expansion in M up to $\mathcal{O}(M^{-1})$, as obtained by numerically evaluating and fitting the quantity in question for $M\in[1005,1995]$ in steps of 10, and compared to formula [Eq. (85)].

K	$-\delta P^-_{\text{Tachyon}}$ Fit	$-\delta P^-_{\text{Tachyon}}$ Asymptotic Formula
2	$4096M+19804-12867.90/M$	$4096M+19803-12867.96/M$
3	$4.2569937517\times10^7M+1.48720\times10^9-3.68032\times10^8/M$	$4.2569937516\times10^7M+1.48717\times10^9-3.68037\times10^8/M$
4	$6.602641227\times10^{11}M+1.63308\times10^{14}-1.09497\times10^{13}/M$	$6.602641228\times10^{11}M+1.63307\times10^{14}-1.09501\times10^{13}/M$
5	$1.2725545528\times10^{16}M+2.743318\times10^{19}-3.4330\times10^{17}/M$	$1.2725545522\times10^{16}M+2.743315\times10^{19}-3.4332\times10^{17}/M$

analysis of Sec. V that the $\mathcal{O}(M)$ and $\mathcal{O}(M^{-1})$ coefficients also have comparable sizes (due to the same exponential factor), and to give a measure of comparison, they range between orders 10^{56} and 10^{58} for $K=14$, and between orders 10^{128} and 10^{131} for $K=30$. This in turn implies that for the range of M we are examining, already at $K=14$ the $\mathcal{O}(M^{-4})$ is 1 order of magnitude larger than the $\mathcal{O}(M)$ term, and their ratio grows to 24 orders of magnitude for $K=30$.

Given that the $\mathcal{O}(M^{-1})$ term is a few orders of magnitude smaller than the $\mathcal{O}(M)$ term, the considerations of the previous paragraph justify why we do not need to include them in order to obtain good fits for the δP^-_K depicted in Fig. 4. More importantly, they imply that as K increases, it becomes very challenging to extract the $\mathcal{O}(M^{-1})$ dependence by purely numerical analysis. Taking the first difference in M does not improve the resolution substantially, as it removes the large $\mathcal{O}(1)$, but not the $\mathcal{O}(M^{-4})$

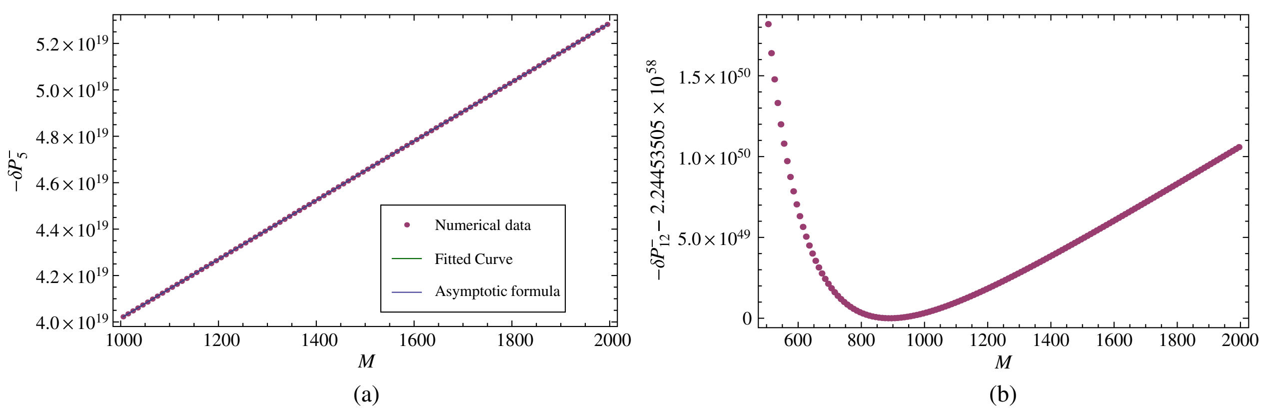

FIG. 3 (color online). Plots of tachyon self-energy for an interaction lasting $K-1$ time steps, for $K=5$ in (a) and $K=12$ in (b). Whereas for (a) the $\mathcal{O}(M)$ term dominates and the data, fit, and asymptotic formula up to $\mathcal{O}(M^{-1})$ are indistinguishable, the same does not hold for the lower end of M values in (b).

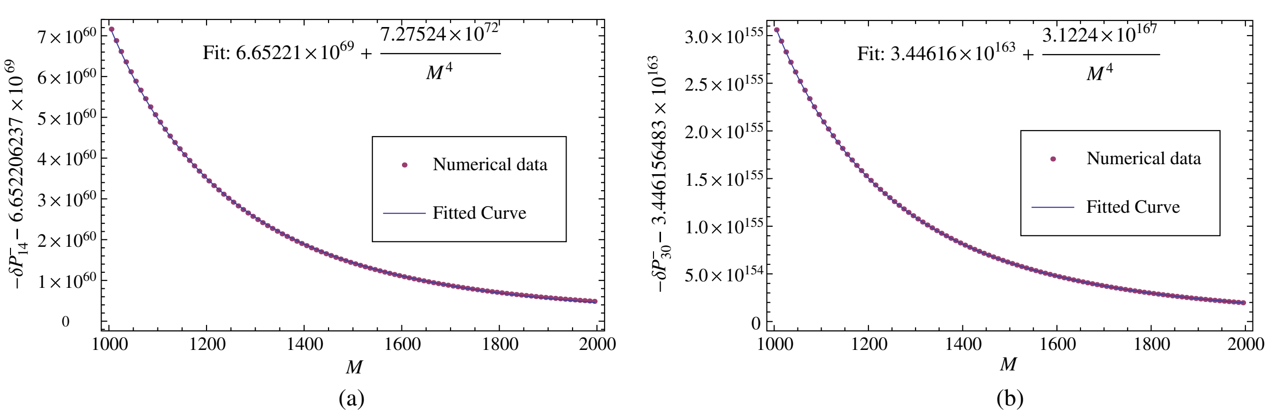

FIG. 4 (color online). Plots of tachyon self-energy for an interaction lasting $K-1$ time steps, for $K=14$ in (a) and $K=30$ in (b). We have shifted the vertical axis by a constant so as to depict the much smaller variation of δP^-_K more clearly. The fits suggest that it is the $\mathcal{O}(M^{-4})$ term which is responsible for the deviation from the $\mathcal{O}(M)$ behavior.

term. Hence, the only remaining possibilities are to either choose a range of much higher values of M so that the two sets of terms become comparable in size, or drastically increase the precision of the numerics, so as to be able to resolve their difference in size. However, since the two sets of terms have different exponential behaviors in K, employing any of the two aforementioned options is also expected to increase computation time exponentially.

In more detail, we can verify that the $\mathcal{O}(1)$ term has an exponential behavior in K of roughly e^{13K} by plotting the logarithm of its fitted value against $K \in [2, 30]$; see Fig. 5. As we discuss in the Introduction, the exponential increase in K of the $\mathcal{O}(1)$ and $\mathcal{O}(M^{-4})$ terms is due to the tachyonic divergence, which appears when one of the two intermediate strings becomes very short; namely, it is a boundary effect. On the contrary, in the regime $1 \ll K \ll M$ we are examining, the tachyonic divergence does not affect the $\mathcal{O}(M)$ and $\mathcal{O}(M^{-1})$ terms, whose exponential dependence $e^{12\beta_0 K} \simeq e^{10.6K}$ is precisely canceled by the tree-level boundary counterterm.

Moving on to a numerical evaluation of the gluon self-energy, we shall aim to compare our fits with the corresponding asymptotic formula, Eq. (101) with $m = 1$. In particular, we will investigate whether our numerical analysis agrees with the $\mathcal{O}(M^{-1})$ term vanishing, and for that reason it will be more advantageous to examine the quantity

$$\delta P^-_{K,\text{gluon}} - \delta P^-_{K,\text{tachyon}} = -4\sin^2\frac{\pi}{2M}\sum_{M_1=1}^{M-1}\sin^2\frac{\pi M_1}{M}\sum_{q,s=1}^{K-1} e^{(q-s)\lambda_1^o} h^{-1}_{qs}, \qquad (122)$$

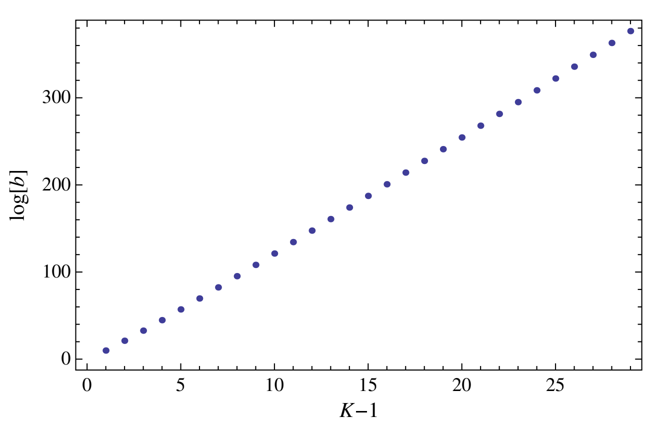

FIG. 5 (color online). Logarithm of the constant term b in the asymptotic expansion in M for the tachyon δP^-_K as a function of the interaction time $K - 1$. The leading behavior is clearly linear, from which we can infer that $b \propto e^{13K}$.

TABLE II. Fit of the difference of tachyon and gluon self-energy for an interaction lasting $K - 1$ time steps (error estimates at the order of the last digit). Comparing with the last row of Table I, we see that the $\mathcal{O}(M^{-1})$ terms are identical, thereby supporting that the corresponding term is zero for $\delta P^-_{K,\text{gluon}}$.

K	$\delta P^-_{K,\text{gluon}} - \delta P^-_{K,\text{tachyon}}$ Fit
2	$-12868.96/M - 2.0 \times 10^5/M^3$
3	$-3.68037 \times 10^8/M - 1.7 \times 10^{10}/M^3$
4	$-1.09501 \times 10^{13}/M - 1.1 \times 10^{15}/M^3$
5	$-3.4332 \times 10^{17}/M - 7 \times 10^{19}/M^3$

which has the large $\mathcal{O}(M)$ and $\mathcal{O}(1)$ dependence removed, thereby providing more accurate fits. As for the tachyon, we choose $K \in [2, 30]$ and $M \in [495, 1995]$, and for the first few values of K, the fits we obtain are depicted in Table II; see also Fig. 6(a).

For small K, the numerics suggest that the $\mathcal{O}(M^{-1})$ term is identical here and for the tachyon, implying it should be zero for the gluon. Furthermore, the numerics suggest that the subleading term in the asymptotic expansion of (122) is $\mathcal{O}(M^{-3})$. Similarly to the tachyon case, however, as K increases, the $\mathcal{O}(M^{-4})$ term dominates the expansion to an extent that does not allow the extraction of the $\mathcal{O}(M^{-1})$ dependence by numerical means [see Fig. 6(b)]. Finally, the $\mathcal{O}(M^{-4})$ term in Eq. (122) appears to be different from the corresponding term for the tachyon alone, in particular about an order of magnitude larger.

VII. OPEN STRINGS ENDING ON D-BRANES

So far, we have assumed that all open-string transverse coordinates obey Neumann conditions. But it is also interesting to impose Dirichlet conditions on a subset of the coordinates [20], denoted by $\boldsymbol{y}(\sigma, \tau)$ to distinguish them from the coordinates $\boldsymbol{x}$, which continue to satisfy Neumann boundary conditions.[7] If $\boldsymbol{x}$ has $p - 1$ components, one says that there is a Dp-brane at the location specified by the (fixed) value of $\boldsymbol{y}$ at the end of each open string. Here we restrict attention to a single Dp-brane location at $\boldsymbol{y} = 0$. In the continuum, the expressions for the one-loop self-energy of an open string with $25 - p$ Dirichlet coordinates differs from the expressions in Eqs. (32) and (33) simply by an insertion of the factors $(-2\pi/\ln q)^{(25-p)/2}$ in the integrands. While these factors marginally soften the leading UV divergence from the integration range near $q \sim 0$, they do so in a way that introduces a logarithmic

[7]We do not consider here the mixed case of Neumann and Dirichlet conditions on opposite ends of an open string.

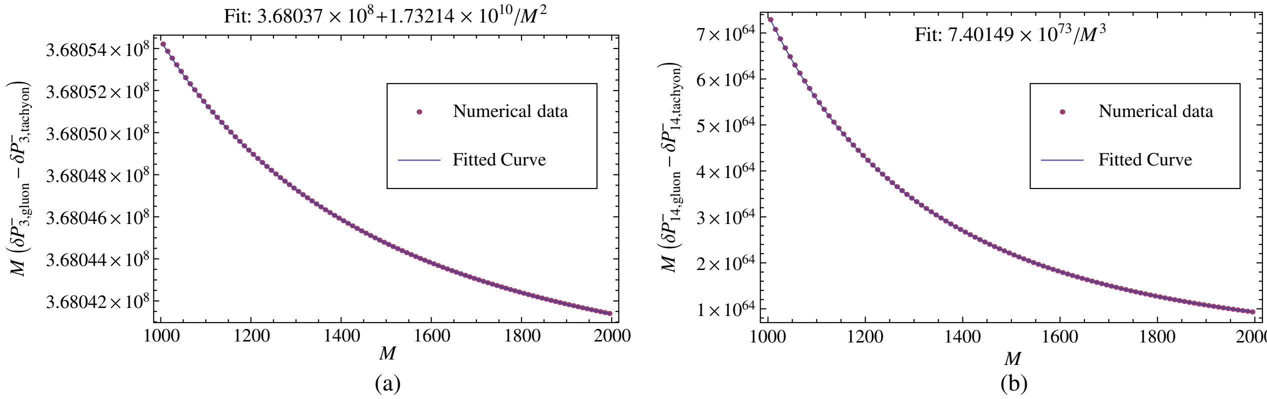

FIG. 6 (color online). Plots of the self-energy difference between the gluon and the tachyon for $K = 3$ in (a) and $K = 14$ in (b). We have rescaled the self-energy by M so as to have only one independent variable. As with the tachyon, subleading terms in the M expansion become dominant with increasing K.

branch point at $q = 0$.[8] After discretization, the presence of these factors leads to an expected leading large-M behavior

$$\delta P_K^- \sim \frac{\alpha M}{K^3(\ln(M/K))^{(25-p)/2}}, \tag{123}$$

which can no longer be canceled by the bulk worldsheet cosmological constant. In particular, the leading singularity must be accepted as a real divergence, at least in the perturbative loop expansion of bosonic string theory.[9] The worldsheet lattice provides a physical cutoff, but there is no consistent way to define a finite continuum limit in perturbation theory.

A. D-branes and the GT lattice

We turn to a detailed analysis of the self-energy on the GT lattice which will confirm these expectations. The Dirichlet worldsheet propagator on the free-string worldsheet takes either of the forms of Eqs. (23) or (24). The main new feature of these formulas is the absence of zero modes in the open-string spectrum because the Dirichlet conditions break translation invariance. Of course, on the lattice loop corrections will involve a different choice for the matrix V describing the breaking and joining of strings, which we shall denote as $\tilde{V}$ for clarity. A broken Dirichlet string coordinate y involves the replacement [24]

$$(y_{k+1}^j - y_k^j)^2 + (y_k^j - y_{k-1}^j)^2 \to (y_{k+1}^j)^2 + (y_{k-1}^j)^2 + 2\kappa(y_k^j)^2, \tag{124}$$

where the parameter κ gives us some flexibility in specifying the Dirichlet condition on the lattice. The matrix $\tilde{V}$ that describes this replacement is then

$$\tilde{V}_{ml,m'l'} = \delta_{lj}\delta_{l'j}(\delta_{m,k+1}\delta_{m',k} + \delta_{m,k}\delta_{m',k+1} + \delta_{m,k-1}\delta_{m',k} + \delta_{m,k}\delta_{m',k-1} + 2(\kappa-1)\delta_{m,k}\delta_{m',k}). \tag{125}$$

In contrast to a broken Neumann coordinate, which involves two lattice sites, the broken Dirichlet coordinate involves three lattice sites: $k-1$, k, and $k+1$. In matrix form, it is

$$\tilde{V} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2(\kappa-1) & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{126}$$

Writing the corresponding 3×3 block of the propagator as

$$\Delta = \begin{pmatrix} e & a & f \\ a & c & b \\ f & b & d \end{pmatrix}, \tag{127}$$

we find after a simple calculation

$$\det(1 + \tilde{V}\Delta) = (1 + a + b)^2 - c(e + d + 2f - 2(\kappa - 1)). \tag{128}$$

The matrix elements are related to the worldsheet propagator as follows:

[8]In the open-string nonplanar one-loop diagram, which contains singularities in the pomeron-channel invariant t due to closed-string states, this branch point in q causes the "unitarity violating" branch point (instead of a pole) in t that led Lovelace to anticipate the need for the critical dimension $D = 26$ [21]. Here we see that in addition to $D = 26$, we also need Neumann boundary conditions on all string coordinates to cancel the cut. As clarified in Ref. [22], the branch point is not really "unitarity violating," but rather simply a reflection of a continuous closed-string mass spectrum, or the holographic emergence of extra dimensions for the propagation of closed strings. In any case, the branch point in t also spells difficulty for the Goddard-Neveu-Scherk analytic continuation method [13] of regulating these divergences.

[9]Supersymmetry can potentially mitigate these difficulties through cancellation of the divergences due to tachyonic closed-string states. In such models, the UV divergence due to the dilaton is rendered finite provided that more than two coordinates are Dirichlet: $\int dq q^{-1}(-\ln q)^{-3/2}$ is convergent at $q \sim 0$ [23].

$$a = \Delta_{kj,(k-1)j} = \Delta_{(k-1)j,kj}, \qquad b = \Delta_{kj,(k+1)j} = \Delta_{(k+1)j,kj}, \tag{129}$$

$$c = \Delta_{kj,kj}, \qquad d = \Delta_{(k+1)j,(k+1)j}, \qquad e = \Delta_{(k-1)j,(k-1)j}, \tag{130}$$

$$f = \Delta_{(k+1)j,(k-1)j} = \Delta_{(k-1)j,(k+1)j}. \tag{131}$$

We will now proceed to evaluate the contribution of Dirichlet coordinates to the one-loop self-energy calculation with $K = 2$. As we did in the previous sections, we will again relabel $k \to M_1$ to better convey that it is referring to the position of the slit. We start by using the representation in Eq. (26) for the Dirichlet worldsheet propagator in order to obtain

$$c = \int_0^1 dx \frac{\sinh M_1\lambda \sinh (M-M_1)\lambda}{\sinh M\lambda \sinh \lambda}, \tag{132}$$

$$\begin{aligned} a &= \int_0^1 dx \frac{\sinh (M_1-1)\lambda \sinh (M-M_1)\lambda}{\sinh M\lambda \sinh \lambda} \\ &= \int_0^1 dx \frac{\sinh M_1\lambda \sinh (M-M_1)\lambda}{\sinh M\lambda} \coth \lambda \\ &\quad - \int_0^1 dx \frac{\cosh M_1\lambda \sinh (M-M_1)\lambda}{\sinh M\lambda}, \end{aligned} \tag{133}$$

$$\begin{aligned} b &= \int_0^1 dx \frac{\sinh M_1\lambda \sinh (M-M_1)\lambda}{\sinh M\lambda} \coth \lambda \\ &\quad - \int_0^1 dx \frac{\sinh M_1\lambda \cosh (M-M_1)\lambda}{\sinh M\lambda}, \end{aligned} \tag{134}$$

$$\begin{aligned} 1+a+b &= 2\int_0^1 dx \frac{\sinh M_1\lambda \sinh (M-M_1)\lambda}{\sinh M\lambda} \coth \lambda \\ &= 2\int_0^1 dx \frac{\sinh M_1\lambda \sinh (M-M_1)\lambda}{\sinh M\lambda} \\ &\quad \times \left(\frac{1}{\sinh \lambda} + \tanh \frac{\lambda}{2}\right) \\ &= 2c + 2\int_0^1 dx \frac{\sinh M_1\lambda \sinh (M-M_1)\lambda}{\sinh M\lambda} \tanh \frac{\lambda}{2}, \end{aligned} \tag{135}$$

$$\begin{aligned} d+e+2f &= 4\int_0^1 dx \frac{\sinh M_1\lambda \sinh (M-M_1)\lambda}{\sinh M\lambda} \\ &\quad \times \left(\frac{1}{\sinh \lambda} + \sinh \lambda\right) - 2\int_0^1 dx \cosh \lambda \\ &= 4c + 4\int_0^1 dx \frac{\sinh M_1\lambda \sinh (M-M_1)\lambda}{\sinh M\lambda} \sinh \lambda \\ &\quad - 2\int_0^1 dx \cosh \lambda. \end{aligned} \tag{136}$$

Inserting these into the formula for the determinant leads to

$$\begin{aligned} &\det (I + \tilde{V}\Delta) \\ &= c\int_0^1 dx \bigg(4\sinh^2(\lambda/2) - 8\frac{\sinh M_1\lambda \sinh (M-M_1)\lambda}{\sinh M\lambda} \\ &\quad \times \left[\frac{\sinh^3(\lambda/2)}{\cosh (\lambda/2)}\right] + 2\kappa\bigg) \\ &\quad + 4\bigg(\int_0^1 dx \frac{\sinh M_1\lambda \sinh (M-M_1)\lambda}{\sinh M\lambda} \tanh \frac{\lambda}{2}\bigg)^2. \end{aligned} \tag{137}$$

We will require the large-M limit of this determinant. We notice that the quantity squared in the last term is just the determinant we encountered for Neumann coordinates [Eq. (51)],

$$\begin{aligned} &2\int_0^1 dx \frac{\sinh M_1\lambda \sinh (M-M_1)\lambda}{\sinh M\lambda} \tanh \frac{\lambda}{2} \\ &= \mathrm{D}_{M_1} = \frac{1}{2} - I_{M_1} + \frac{f_2(x)}{2\pi M^2} + \frac{f_4(x)}{12\pi M^4} + \cdots, \end{aligned} \tag{138}$$

where we recall Eq. (60) and our definition $x = M_1/M$.

We also have some new integrals to analyze:

$$\int_0^1 dx \sinh^2 \frac{\lambda}{2} = \frac{1}{\pi} \int_0^{\lambda_0} d\lambda \frac{\cosh (\lambda/2)\sinh^2(\lambda/2)}{\sqrt{1-\sinh^2(\lambda/2)}} = \frac{1}{2}, \tag{139}$$

and for $M_1 \le M/2$ we use

$$\begin{aligned} &\frac{\sinh M_1\lambda \sinh (M-M_1)\lambda}{\sinh M\lambda} \\ &= \frac{1}{2}(1-e^{-2M_1\lambda}) - \frac{e^{-M\lambda}\sinh^2 M_1\lambda}{\sinh M\lambda}, \qquad M_1 \le \frac{M}{2} \end{aligned} \tag{140}$$

to decompose the second integral into two terms:

$$\begin{aligned} &\int_0^1 dx \frac{1}{2}(1-e^{-2M_1\lambda}) \frac{\sinh^3(\lambda/2)}{\cosh (\lambda/2)} \\ &= \frac{1}{2\pi} \int_0^{\lambda_0} d\lambda (1-e^{-2M_1\lambda}) \frac{\sinh^3(\lambda/2)}{\sqrt{1-\sinh^2(\lambda/2)}} \\ &= \frac{1}{2\pi} - J_{M_1}, \end{aligned} \tag{141}$$

$$J_{M_1} = \frac{1}{2\pi} \int_0^{\lambda_0} d\lambda e^{-2M_1\lambda} \frac{\sinh^3(\lambda/2)}{\sqrt{1-\sinh^2(\lambda/2)}}; \tag{142}$$

$$\begin{aligned} &\int_0^1 dx \frac{-e^{-M\lambda}\sinh^2 M_1\lambda}{\sinh M\lambda} \frac{\sinh^3(\lambda/2)}{\cosh (\lambda/2)} \\ &= \frac{1}{\pi} \int_0^{\lambda_0} d\lambda \frac{-e^{-M\lambda}\sinh^2 M_1\lambda}{\sinh M\lambda} \frac{\sinh^3(\lambda/2)}{\sqrt{1-\sinh^2(\lambda/2)}} \\ &= \frac{h_4(x)}{M^4} + \mathcal{O}(M^{-6}, e^{-M\lambda_0}), \end{aligned} \tag{143}$$

$$h_4(x) = -\frac{1}{8\pi}\int_0^\infty \lambda^3 d\lambda \frac{e^{-\lambda}\sinh^2 x\lambda}{\sinh\lambda}. \qquad (144)$$

In contrast to the above integrals, which are finite as $M \to \infty$, the integral defining c increases logarithmically with M. This feature is a direct consequence of Dirichlet boundary conditions. Remembering that the determinant enters the self-energy with a negative power, this means that Dirichlet conditions soften the leading UV divergence by powers of $(\ln M)^{-1}$. To investigate this phenomenon we look at

$$c = \int_0^{\lambda_0} d\lambda \frac{\coth(\lambda/2)}{\pi\sqrt{1-\sinh^2(\lambda/2)}} \frac{\sinh M_1\lambda \sinh(M-M_1)\lambda}{\sinh M\lambda} \equiv G_{M_1} + \hat{c}, \qquad (145)$$

$$G_{M_1} = \frac{1}{2\pi}\int_0^{\lambda_0} d\lambda \frac{(1-e^{-2M_1\lambda})\coth(\lambda/2)}{\sqrt{1-\sinh^2(\lambda/2)}}, \qquad (146)$$

$$\hat{c} = \frac{1}{\pi}\int_0^{\lambda_0} d\lambda \frac{\coth(\lambda/2)}{\sqrt{1-\sinh^2(\lambda/2)}}\left[\frac{e^{-M\lambda}\sinh^2 M_1\lambda}{\sinh M\lambda}\right]. \qquad (147)$$

The large-M behavior of $\hat{c}$ can be obtained by expanding

$$\frac{\coth(\lambda/2)}{\sqrt{1-\sinh^2(\lambda/2)}} = \frac{2}{\lambda} + \frac{5\lambda}{12} + \mathcal{O}(\lambda^3). \qquad (148)$$

Then, term by term we can extend the upper limit to ∞, with errors smaller than order $e^{-M\lambda_0}$ to get an asymptotic expansion

$$\hat{c} = g_0(x) + \frac{g_2(x)}{M^2} + \mathcal{O}(M^{-4}, e^{-M\lambda_0}), \qquad (149)$$

$$g_0(x) = \frac{2}{\pi}\int_0^\infty \frac{d\lambda}{\lambda}\left[\frac{e^{-\lambda}\sinh^2(x\lambda)}{\sinh\lambda}\right], \quad g_2(x) = \frac{5}{12\pi}\int_0^\infty \lambda d\lambda\left[\frac{e^{-\lambda}\sinh^2(x\lambda)}{\sinh\lambda}\right]. \qquad (150)$$

In this expansion, the coefficients depend on the ratio $x = M_1/M$, which is smaller than $1/2$: if this ratio is of order $1/M$, all of the terms are down by a further factor of $1/M^2$.

Next we study the difference $G_{M_1} = c - \hat{c}$, which depends only on M_1. Since M_1 has to be summed over the range $0 < M_1 < M/2$, we will need the large-M_1 behavior of this term, which we will find behaves like $\ln M_1$:

$$G_{M_1} = \frac{1}{2\pi}\int_0^{\lambda_0} d\lambda \frac{(1-e^{-2M_1\lambda})\coth(\lambda/2)}{\sqrt{1-\sinh^2(\lambda/2)}} = \frac{1}{2\pi}\int_0^{\lambda_0} d\lambda(1-e^{-2M_1\lambda})\left[\frac{\coth(\lambda/2)}{\sqrt{1-\sinh^2(\lambda/2)}} - \frac{2}{\lambda}\right] + \frac{1}{\pi}\int_0^{\lambda_0} d\lambda \frac{1-e^{-2M_1\lambda}}{\lambda}. \qquad (151)$$

The quantity in square brackets has good small-λ dependence, so the terms in the first integral can be evaluated separately:

$$\frac{1}{2\pi}\int_0^{\lambda_0} d\lambda\left[\frac{\coth(\lambda/2)}{\sqrt{1-\sinh^2(\lambda/2)}} - \frac{2}{\lambda}\right] = \frac{1}{\pi}\ln\frac{2}{\lambda_0} + \frac{\Gamma'(1)}{2\pi} - \frac{\Gamma'(1/2)}{2\pi\sqrt{\pi}}, \qquad (152)$$

$$\frac{1}{2\pi}\int_0^{\lambda_0} d\lambda e^{-2M_1\lambda}\left[\frac{\coth(\lambda/2)}{\sqrt{1-\sinh^2(\lambda/2)}} - \frac{2}{\lambda}\right] \sim \frac{5}{24\pi(2M_1)^2} + \mathcal{O}({M_1}^{-4}, e^{-2M_1\lambda_0}). \qquad (153)$$

Finally, the large-M_1 behavior of the last integral

$$\frac{1}{\pi}\int_0^{\lambda_0} d\lambda \frac{1-e^{-2M_1\lambda}}{\lambda} = -\frac{1}{\pi}\int_0^{\lambda_0} d\lambda(2M_1 e^{-2M_1\lambda})\ln\frac{\lambda}{\lambda_0} = -\frac{1}{\pi}\int_0^\infty d\lambda e^{-\lambda}\ln\frac{\lambda}{2M_1\lambda_0} + \mathcal{O}(e^{-2M_1\lambda_0}) = \frac{1}{\pi}\ln(2M_1\lambda_0) - \frac{\Gamma'(1)}{\pi} + \mathcal{O}(e^{-2M_1\lambda_0}). \qquad (154)$$

All together, then, we have for the large-M_1 behavior of $c - \hat{c}$

$$G_{M_1} = \frac{1}{\pi}\ln(4M_1) - \frac{\Gamma'(1) + \Gamma'(1/2)/\sqrt{\pi}}{2\pi} - \frac{5}{96\pi {M_1}^2} + \mathcal{O}({M_1}^{-4}, e^{-2M_1\lambda_0}) = \frac{1}{\pi}\ln(4M_1) - \frac{\psi(1) + \psi(1/2)}{2\pi} - \frac{5}{96\pi {M_1}^2} + \mathcal{O}({M_1}^{-4}, e^{-2M_1\lambda_0}), \qquad (155)$$

where $\psi(z) \equiv \Gamma'(z)/\Gamma(z)$ is the digamma function.

Finally, we quote the determinant for $K = 2$ due to a Dirichlet coordinate, keeping terms up to order M^{-2}:

$$\det(I + \tilde{V}\Delta) = \left(G_{M_1} + g_0(x) + \frac{g_2(x)}{M^2}\right) \times \left(2(\kappa+1) - \frac{4}{\pi} + 8J_{M_1}\right) + \left(\frac{1}{2} - I_{M_1}\right)^2 + \frac{(1-2I_{M_1})f_2(x)}{2\pi M^2} + \mathcal{O}(M^{-4}). \qquad (156)$$

Then the self-energy shift of the tachyon in the presence of a Dp-brane is given by

$$\delta P_K^- = 2g^2 \sum_{M_1=1}^{(M-1)/2} \det{}^{-(p-1)/2}(I + V\Delta) \times \det{}^{-(25-p)/2}(I + \tilde{V}\Delta) e^{-24B(K-1)}. \qquad (157)$$

The leading behavior for large M occurs from the region $1 \ll M_1 = \mathcal{O}(M)$. For $K = 2$ this gives

$$\delta P^-_{K=2} \sim 2g^2 \sum_{M_1 \gg 1}^{(M-1)/2} 2^{(p-1)/2} \left(\frac{2(\kappa+1) - 4/\pi}{\pi} \ln M_1\right)^{-(25-p)/2} e^{-24B}$$
$$\sim g^2 2^{p-13} \frac{M}{(\ln M)^{(25-p)/2}} \left(\frac{\pi^2}{(\kappa+1)\pi - 2}\right)^{(25-p)/2} e^{-24B}. \tag{158}$$

Subleading divergences of the forms $M/(\ln M)^{n+(25-p)/2}$ and $1/(\ln M)^{n+(25-p)/2}$ will also appear. In fact, each power of M can be expected to be multiplied by a power series in $(\ln M)^{-1}$. We leave the interpretation of these nonanalytic divergences to future work.

B. Discretization of the continuum expressions for the self-energy of the superstring

Although we do not yet have a completely satisfactory GT lattice for the superstring, we can get a glimpse of the benefits of supersymmetry by simply discretizing the known continuum formulas for the gluon self-energy diagrams for the superstring with supersymmetry broken by compactification of an extra dimension.[10] The critical dimension is $D = 10$, so there will be eight transverse coordinates $\boldsymbol{x}$, $\boldsymbol{\mathcal{P}}$ and eight worldsheet fermions denoted $\boldsymbol{\Gamma}$ in the Ramond (R) sector and $\boldsymbol{H}$ in the Neveu-Schwarz (NS) sector. We shall need the correlators

$$\langle \mathcal{P}\mathcal{P} \rangle = \frac{1}{4\sin^2(\theta/2)} - \sum_{n=1}^{\infty} \frac{2nq^{2n}}{1-q^{2n}} \cos n\theta \tag{159}$$

$$= \frac{1}{4\sin^2(\theta/2)} - 2q^2 \cos\theta + \mathcal{O}(q^4), \tag{160}$$

$$\langle HH \rangle_+ = \frac{1}{2\sin(\theta/2)} - 2\sum_{r=1/2}^{\infty} \frac{q^{2r}}{1+q^{2r}} \sin r\theta \tag{161}$$

$$= \frac{1}{2\sin(\theta/2)} - 2q\left(1 - q + 4q^2\cos^2\frac{\theta}{2}\right)\sin\frac{\theta}{2} + \mathcal{O}(q^4), \tag{162}$$

$$\langle HH \rangle_- = \frac{1}{2\tan(\theta/2)} - 2\sum_{n=1}^{\infty} \frac{q^{2n}}{1+q^{2n}} \sin n\theta \tag{163}$$

$$= \frac{1}{2\tan(\theta/2)} - 2q^2 \sin\theta + \mathcal{O}(q^4), \tag{164}$$

$$\langle \Gamma\Gamma \rangle = \frac{1}{2\sin(\theta/2)} + 2\sum_{r=1/2}^{\infty} \frac{q^{2r}}{1-q^{2r}} \sin r\theta \tag{165}$$

$$= \frac{1}{2\sin(\theta/2)} + 2q\left(1 + q + 4q^2\cos^2\frac{\theta}{2}\right)\sin\frac{\theta}{2} + \mathcal{O}(q^4), \tag{166}$$

where we use the notation and conventions of Ref. [7]. We have expressed the correlators in terms of the moduli q, θ of the cylinder. To break supersymmetry, we compactify one of the transverse target space dimensions, imposing periodic boundary conditions on bosonic states and antiperiodic conditions on fermionic states. In the expressions of the one-loop self-energies, this simply means an insertion of the factor

$$\sum_{m=-\infty}^{\infty} q^{m^2R^2T_0/4\pi^2}, \qquad \text{bosonic loop,}$$
$$\sum_{m=-\infty}^{\infty} (-)^m q^{m^2R^2T_0/4\pi^2}, \qquad \text{fermionic loop.} \tag{167}$$

Both these factors approach unity in the decompactification limit $R \to \infty$. The absence of tachyonic divergences in the loop integrals requires $R^2T_0 \geq 4\pi^2$. When the gluon polarization is in the direction of a compactified coordinate, the $\langle \mathcal{P}\mathcal{P} \rangle$ correlator quoted above acquires an extra term:

$$\langle \mathcal{P}\mathcal{P} \rangle = -\left\langle \frac{m^2R^2T_0}{4\pi^2} \right\rangle + \frac{1}{4\sin^2(\theta/2)} - \sum_{n=1}^{\infty} \frac{2nq^{2n}}{1-q^{2n}} \cos n\theta, \tag{168}$$

where

$$\langle f(m) \rangle \equiv \frac{\sum_m f(m) q^{m^2R^2T_0/4\pi^2}}{\sum_m q^{m^2R^2T_0/4\pi^2}}, \qquad \text{bosonic loop,} \tag{169}$$

$$\langle f(m) \rangle \equiv \frac{\sum_m (-)^m f(m) q^{m^2R^2T_0/4\pi^2}}{\sum_m (-)^m q^{m^2R^2T_0/4\pi^2}}, \qquad \text{fermionic loop.} \tag{170}$$

[10] With unbroken supersymmetry, the diagram is identically zero.

By taking the $R \to 0$ limit, we get the extra term in this correlator when gluon polarizations are transverse to a D-brane: it is just $1/(2\ln q)$. It is noteworthy that the extra term from either compactification or the presence of D-branes is negative. Since generally $C_s < 0$,[11] this term therefore contributes positively to the self-energy. Thus, while the mass shift of the gluon (polarizations in uncompactified directions) is zero, the mass-squared shift of the massless scalar (polarization in compactified directions) is positive.

The self-energy of the gluon state is given as a sum of three terms:

$$\Delta P^- = \frac{C_s}{2P^+}(\Sigma_+ + \Sigma_- + \Sigma_F), \tag{171}$$

$$\Sigma_+ = \frac{1}{2}\int_0^1 \frac{dq}{q^2}\int_0^{2\pi} d\theta \sum_{m=-\infty}^{\infty} q^{m^2R^2T_0/4\pi^2}\frac{\prod_r(1+q^{2r})^8}{\prod_n(1-q^{2n})^8}\langle\mathcal{P}\mathcal{P}\rangle, \tag{172}$$

$$\Sigma_- = -8\int_0^1 \frac{dq}{q^2}\int_0^{2\pi} d\theta \sum_{m=-\infty}^{\infty} q^{m^2R^2T_0/4\pi^2} \times \frac{\prod_n(1+q^{2n})^8}{\prod_n(1-q^{2n})^8} q\langle\mathcal{P}\mathcal{P}\rangle, \tag{173}$$

$$\Sigma_F = -\frac{1}{2}\int_0^1 \frac{dq}{q^2}\int_0^{2\pi} d\theta \sum_{m=-\infty}^{\infty} (-)^m q^{m^2R^2T_0/4\pi^2} \times \frac{\prod_r(1-q^{2r})^8}{\prod_n(1-q^{2n})^8}\langle\mathcal{P}\mathcal{P}\rangle. \tag{174}$$

Terms involving the fermionic correlators $\langle HH\rangle_\pm^2$ and $\langle\Gamma\Gamma\rangle^2$ do not contribute to the on-shell two-gluon function because they are multiplied by kinematic factors like $k_i \cdot k_j \epsilon_k \cdot \epsilon_l$ or $k_i \cdot \epsilon_j k_k \cdot \epsilon_l$. But for the two-point function, $k_2 = -k_1$, $k_i^2 = 0$, and $k_i \cdot \epsilon_i = 0$, so all these factors vanish. The combination $\Sigma_+ + \Sigma_-$ projects out the odd G-parity states of the NS sector circulating the loop, while Σ_F represents the R-sector states circulating the loop. Because of Jacobi's abstruse identity,

$$\prod_r(1+q^{2r})^8 - \prod_r(1-q^{2r})^8 - 16q\prod_n(1+q^{2n})^8 = 0. \tag{175}$$

We have, for gluon polarization in uncompactified directions, the simplification

$$\Delta P^- = \frac{C_s}{2P^+}\int_0^1 \frac{dq}{q^2}\int_0^{2\pi} d\theta \sum_{m=\text{odd}} q^{m^2R^2T_0/4\pi^2} \times \frac{\prod_r(1-q^{2r})^8}{\prod_n(1-q^{2n})^8}\langle\mathcal{P}\mathcal{P}\rangle \tag{176}$$

$$= \frac{C_s}{2P^+}\int \frac{dq}{q}\int_0^{2\pi} d\theta \sum_{m=\text{odd}} q^{m^2R^2T_0/4\pi^2} \times \left(\frac{1-8q+36q^2}{4q\sin^2(\theta/2)} - 2q + 4q\sin^2\frac{\theta}{2} + \mathcal{O}(q^2)\right), \tag{177}$$

where in the second line we have explicitly displayed the first few terms in the q expansion. It is worth emphasizing that the right side vanishes in the decompactification limit $R \to \infty$, which restores supersymmetry. Also notice that, although the q integral is convergent at the lower end $q \sim 0$ when $R^2T_0 > 4\pi^2$, the θ integral is still divergent at its end points. It is this divergence that discretization will show can be absorbed in the constant boundary counterterm B.

Incidentally, for gluon polarization in the compactified direction, the open-string state corresponds to a massless scalar particle which gains a mass by virtue of the extra term shown in Eq. (168):

$$\Delta M^2_{\text{Scalar}} = 2P^+\Delta P^- = -\frac{C_sR^2T_0}{2\pi}\int_0^1 \frac{dq}{q^2}\frac{\prod_r(1-q^{2r})^8}{\prod_n(1-q^{2n})^8} \times \sum_{m=\text{odd}} m^2 q^{m^2R^2T_0/4\pi^2}, \tag{178}$$

which is positive since $C_s < 0$. The integral on the right is convergent at $q \sim 0$ provided $R^2T_0 > 4\pi^2$, and becomes arbitrarily large as $R^2T_0 \to 4\pi^2$. The convergence of the integral at $q \sim 1$ becomes transparent after the change of variables by the Jacobi imaginary transformation $q = e^{2\pi^2/\ln w}$, which maps $q = 1$ to $w = 0$.

Now let us discretize the gluon self-energy [Eq. (176)] in the variables of the lightcone lattice and examine how the continuum limit is regained. Recalling Eqs. (34)–(36), we have

$$\Delta P^- \to \frac{64C_s}{a\pi T_0}\sum_K \frac{2+\mathcal{O}(K^4)}{K^3}\sum_{M_1=1}^{(M-1)/2}\sum_{k=-\infty}^{\infty} q^{1+(2k-1)^2R^2T_0/4\pi^2} \times\left[1 - 8q + 36q^2 + 4q^2\sin^4\frac{\pi M_1}{M} - 2q^2\sin^2\frac{\pi M_1}{M} + \mathcal{O}(q^3)\right], \tag{179}$$

where, to avoid ungainly expressions, we have deferred replacing q with its discretized version given by Eq. (35).

[11] From the lightcone viewpoint, the self-energy shift is a result from second-order perturbation theory which is necessarily negative by unitarity. Since the divergence in the integral has a positive coefficient, this implies that C_s must be negative. The negative divergent contribution to the shift is, of course, canceled against the boundary cosmological constant counterterm B.

Instead, we work out the large-M limit of the contribution of each power of q in what follows.

As we did in Sec. III A, we next study the large-M limit of the terms at fixed K. The summand involves terms of the form $q^p \sin^{2n}(\pi M_1/M)$, with $p \geq 1 + R^2 T_0/(4\pi^2) \geq 2$ and $n = 0, 1, 2$. For simplicity of discussion, let us choose $R^2 T_0 = 4\pi^2$, so the lowest power is q^2, and begin by examining the large-M limit of

$$\sum_{M_1=1}^{(M-1)/2} q^2 \sim \left(1 + \frac{\pi^2 K^2}{48M^2}\right) \sum_{M_1=1}^{(M-1)/2} \left(\frac{\pi K}{8M \sin \pi M_1/M}\right)^2 - 4 \sum_{M_1=1}^{(M-1)/2} \left(\frac{\pi K}{8M \sin \pi M_1/M}\right)^4$$
$$= \left(1 + \frac{\pi^2 K^2}{48M^2}\right)\left(\frac{\pi^2 K^2}{384} + \mathcal{O}(M^{-2})\right) - 4\zeta(4)\frac{K^4}{8^4} + \mathcal{O}(M^{-2})$$
$$= \frac{\zeta(2)K^2}{64} - \frac{\zeta(4)K^4}{1024} + \mathcal{O}(M^{-2}), \tag{180}$$

where the last two lines follow from the argument in footnote [2]. A similar analysis applies to the higher powers of q as well. As this q^2 example shows, the presence of additional factors of $\sin(\pi M_1/M)$ has the effect of suppressing the contribution by additional powers of M^{-1}. This means that in collecting the contributions to the constant boundary counterterm, one only needs to keep the first three terms in the square brackets:

$$\sum_{M_1=1}^{(M-1)/2} q^3 \sim \sum_{M_1=1}^{(M-1)/2} \left(\frac{\pi K}{8M \sin \pi M_1/M}\right)^3$$
$$\sim \frac{\pi^3 K^3}{8^3 M^2} \int_0^{1/2} dx \left(\frac{1}{\sin^3 \pi x} - \frac{1}{\pi^3 x^3} - \frac{1}{2\pi x}\right) + \frac{K^3}{8^3} \sum_{M_1=1}^{(M-1)/2} \frac{1}{M_1^3} + \frac{\pi^2 K^3}{8^3 M^2} \sum_{M_1=1}^{(M-1)/2} \frac{1}{2M_1}$$
$$= \frac{\zeta(3)K^3}{8^3} + \mathcal{O}(M^{-2}\ln M), \tag{181}$$

$$\sum_{M_1=1}^{(M-1)/2} q^4 \sim \sum_{M_1=1}^{(M-1)/2} \left(\frac{\pi K}{8M \sin \pi M_1/M}\right)^4 = \frac{\zeta(4)K^4}{8^4} + \mathcal{O}(M^{-2}). \tag{182}$$

Thus, the total contribution from these terms to the divergence is

$$\Delta P^-_{\rm div} = \frac{C_s}{a\pi T_0} \sum_K \frac{1}{K}[2\zeta(2) - 2\zeta(3)K + \zeta(4)K^2 + \mathcal{O}(K^4)]. \tag{183}$$

C. D-branes and the superstring

We now return to the effect of imposing Dirichlet conditions on some of the coordinates. As before, the self-energy integrands now acquire extra factors $(-\ln q)^{-(9-p)}$, which become

$$\left(-\ln \frac{\pi K}{8M \sin \pi M_1/M} + \frac{6 - 2\sin^2 \pi M_1/M}{3}\left(\frac{\pi K}{8M \sin \pi M_1/M}\right)^2 + \mathcal{O}(K^4)\right)^{-(9-p)}. \tag{184}$$

After expanding in powers of K, we see that in general we will encounter, in addition to more terms in higher powers of K, a series of terms with more negative powers of $\ln K$ of the form

$$\left(-\ln \frac{\pi K}{8M \sin \pi M_1/M}\right)^{-(9-p+n)}. \tag{185}$$

These extra factors modify the extraction of the divergent parts of the discretized self-energy expressions. It will suffice to illustrate this with the lowest contributing power of K:

$$\sum_{M_1=1}^{(M-1)/2} q^2 (-\ln q)^{-n} \rightarrow \sum_{M_1=1}^{(M-1)/2} \left(\frac{\pi K}{8M \sin \pi M_1/M}\right)^2 \left(-\ln \frac{\pi K}{8M \sin \pi M_1/M}\right)^{-n}$$
$$\sim \frac{\pi^2 K^2}{64M} \int_0^{1/2} dx \left(\frac{1}{\sin^2 \pi x}\left(-\ln \frac{\pi K}{8M \sin \pi x}\right)^{-n} - \frac{1}{\pi^2 x^2}\left(-\ln \frac{K}{8Mx}\right)^{-n}\right) + \sum_{M_1=1}^{\infty} \frac{K^2}{64M_1^2}\left(-\ln \frac{K}{8M_1}\right)^{-n}$$
$$- \sum_{M_1=(M+1)/2}^{\infty} \frac{K^2}{64M_1^2}\left(-\ln \frac{K}{8M_1}\right)^{-n}$$
$$= \sum_{M_1=1}^{\infty} \frac{K^2}{64M_1^2}\left(-\ln \frac{K}{8M_1}\right)^{-n} + \mathcal{O}(M^{-1}(\ln M)^{-n-1}). \tag{186}$$

So we see that the presence of D-branes for the superstring does not spoil the ability to absorb divergences in the boundary counterterm. Also notice that the negative powers of $\ln M$ suppress the M^{-1} correction term which, if nonzero, would signal a nonzero shift to the gluon mass.

ACKNOWLEDGMENTS

This research was supported in part by the Department of Energy under Grant No. DE-FG02-97ER-41029.

APPENDIX A: CLOSED STRINGS IN THE PRESENCE OF D-BRANES

In this appendix, we will briefly examine how the scattering of a closed-string tachyon off a D-brane can be described within the worldsheet-based approach, and compare with the string field theory analysis of the same process, which was carried out in Sec. 3 of Ref. [11].

To this end, we will first have to generalize the introductory remarks of Sec. VII, and derive a determinant formula for the path integral where instead of one site obeying Dirichlet boundary conditions, we now have $K-1$ consecutive sites in the temporal direction, and in the same spatial position k. This requires taking the direct product of the matrix in Eq. (126) with a diagonal matrix with entries 1 for the sites in question, and 0 otherwise. Then $\det(I+\tilde{V}\Delta)$ may be written in the block form

$$\det(I+\tilde{V}\Delta)=\begin{vmatrix} I+\mathbf{\Delta}_{-10} & \mathbf{\Delta}_{00} & \mathbf{\Delta}_{01} \\ \mathbf{\Delta}_{-1,-1}+2(\kappa-1)\mathbf{\Delta}_{-10}+\mathbf{\Delta}_{-11} & I+\mathbf{\Delta}_{-10}+2(\kappa-1)\mathbf{\Delta}_{00}+\mathbf{\Delta}_{01} & \mathbf{\Delta}_{-11}+2(\kappa-1)\mathbf{\Delta}_{01}+\mathbf{\Delta}_{11} \\ \mathbf{\Delta}_{-10} & \mathbf{\Delta}_{00} & I+\mathbf{\Delta}_{01} \end{vmatrix},$$

where each of the blocks corresponds again to spatial positions $k-1$, k, and $k+1$, and $\mathbf{\Delta}_{lm}$ is the $(K-1)$-dimensional matrix with elements $\Delta_{i(k+l),j(k+m)}$, namely with only the temporal indices i, j varying. By elementary row and column operations, we can reduce the size of the matrix and bring it to the form

$$\begin{aligned}\det(I+\tilde{V}\Delta)&=\begin{vmatrix} I+\mathbf{\Delta}_{-10}+\mathbf{\Delta}_{01} & -2(\kappa-1)I+\mathbf{\Delta}_{-1,-1}+\mathbf{\Delta}_{11}+2\mathbf{\Delta}_{-11} \\ \mathbf{\Delta}_{00} & I+\mathbf{\Delta}_{-10}+\mathbf{\Delta}_{01} \end{vmatrix}\\ &=\begin{vmatrix} I+\mathbf{\Delta}_{-10}-2\mathbf{\Delta}_{00}+\mathbf{\Delta}_{01} & -2(\kappa+1)I+\mathbf{\Delta}_{-1,-1}-4\mathbf{\Delta}_{-10}+2\mathbf{\Delta}_{-11}+4\mathbf{\Delta}_{00}-4\mathbf{\Delta}_{01}+\mathbf{\Delta}_{11} \\ \mathbf{\Delta}_{00} & I+\mathbf{\Delta}_{-10}-2\mathbf{\Delta}_{00}+\mathbf{\Delta}_{01} \end{vmatrix}. \end{aligned} \tag{A1}$$

The latter equation may be used for the study of the open- or closed-string case, by substituting the corresponding expression for the propagator. In what follows, we will focus on the closed string, where the propagator is a function of $|l-m|$ alone, and more concretely is given by Eq. (29). Clearly, the zero-mode piece of the propagator will dominate as $N\to\infty$, and following the same logic as in the string field theory approach [11], the quantity of interest will be precisely the coefficient of the zero mode in $\det(I+\tilde{V}\Delta)$,

$$\mathcal{M}_K=\lim_{N\to\infty}\frac{4M\det(I+\tilde{V}\Delta)}{N+1}. \tag{A2}$$

The expression of the second line of Eq. (A1) is advantageous for obtaining $\mathcal{M}_K$, as the $\mathcal{O}(N)$ term is contained only in the lower left block. In fact, since this term will be the same for all elements in the block, we can perform further row and column operations in order to remove it from all but one element. This implies that $\mathcal{M}_K$ will simply equal the minor of the latter element, or in other words it will be given by a determinant of dimension $2(K-1)-1$.

For specific K, here it is also possible to obtain an asymptotic expansion in M for $\mathcal{M}_K$ with the help of the Euler-Maclaurin formula. For example, setting $\kappa=1$ for simplicity, we find

$$\mathcal{M}_2=4\left(1-\frac{1}{\pi}\right)+\frac{\pi^3}{15}\frac{1}{M^4}+\mathcal{O}(M^{-6})\simeq 2.727+\frac{2.067}{M^4}, \tag{A3}$$

$$\begin{aligned}\mathcal{M}_3&=\frac{(5\pi-8)(3\pi^2+16)}{2\pi^3}+\frac{\frac{5\pi}{2}-4}{M^2}+\mathcal{O}(M^{-4})\\ &\simeq 5.669+\frac{3.854}{M^2},\end{aligned} \tag{A4}$$

$$\begin{aligned}\mathcal{M}_4&=-\frac{32(1024-296\pi-75\pi^2+12\pi^3)}{9\pi^4}\\ &\quad-\frac{16(2048-720\pi-129\pi^2+36\pi^3)}{27\pi^2M^2}+\mathcal{O}(M^{-4})\\ &\simeq 10.003+\frac{22.270}{M^2}.\end{aligned} \tag{A5}$$

It is worth noting that, particularly for $K=2$, the coefficient of the $\mathcal{O}(M^{-2})$ term is zero. We may compare the results of our current approach to D-branes with the string-field-theory-based approach of Ref. [11] by noting that the quantity we defined in equation Eq. (87) of the latter paper equals, in our current notations,

$$r_K=\sqrt{\eta^{-K+1}\frac{1-\eta^{2K}}{1-\eta^2}\frac{\det(h_{lp})}{\mathcal{M}_K}}, \tag{A6}$$

where $\eta = 1 + \kappa - \sqrt{\kappa(2+\kappa)}$ and $\det(h_{lp})$ is given by Eq. (33) of Ref. [12]. Indeed, we have verified that the two approaches yield the same values for r_K for a wide range of M and K, and that the fits we obtained in Ref. [11] for fixed K and varying $M \gg K$ are in good agreement with the asymptotic expressions we can now obtain analytically. As an illustration, with the current methods, we find for $\kappa = 1$

$$r_2 = \frac{\sqrt{\pi}}{\sqrt{2(\pi-1)}}\left(1 + \frac{\pi}{6M^2}\right) + \mathcal{O}(M^{-4}) \simeq 0.856429 + \frac{0.448425}{M^2} + \mathcal{O}(M^{-4}), \qquad \text{(A7)}$$

which compares very well with the fit on the left-hand side of Fig. 14 in the latter reference.

APPENDIX B: OPEN-STRING SELF-ENERGY: STRING FIELD VIEWPOINT

In this appendix, we give the alternative expression for the self-energy as a concatenation of open-string propagators, along the lines of Ref. [11]. For the open-string self-energy, depicted in Fig. 1, we have a total of $N + K - 1$ missing links, N for the open-string ends and $K - 1$ for the extra two ends of the two intermediate strings. Let the external string have M sites and the two intermediate strings have M_1, $M_2 = M - M_1$ sites, respectively. At each time j, there will be M coordinates x_i^j, $i = 1, \ldots, M$. If at time j there are two open strings, we shall identify x_i^j, $i = 1, \ldots, M_1$, with the first and x_i^j, $i = M_1 + 1, \ldots, M$ with the second. The summand of the self-energy diagram will then depend on M_1, J, K, so we write

$$\begin{aligned}\langle N+1, \{x^f\}|0,\{x^i\}\rangle^{\rm open}_{M_1,K,J} &= \int dx_i^K dx_i^L \langle L,\{x^f\}|0,\{x^L\}\rangle^{\rm open}_M \langle K,\{x^L_<\}|0,\{x^K_<\}\rangle^{\rm open}_{M_1}\langle K,\{x^L_>\}|0,\{x^K_>\}\rangle^{\rm open}_{M_2}\langle J,\{x^K\}|0,\{x^i\}\rangle^{\rm open}_M \\ &\quad\times e^{-T_0[(x^L_{M_1+1}-x^L_{M_1})^2+(x^K_{M_1+1}-x^K_{M_1})^2]/4} \\ &= \mathcal{D}^{\rm open}_M(J)\mathcal{D}^{\rm open}_{M_1}(K)\mathcal{D}^{\rm open}_{M_2}(K)\mathcal{D}^{\rm open}_M(L)\int dx_i^K dx_i^L e^{iW+(N+K-1)B_0-T_0[(x^L_{M_1+1}-x^L_{M_1})^2+(x^K_{M_1+1}-x^K_{M_1})^2]/4},\end{aligned} \qquad \text{(B1)}$$

where we have introduced the notation $x_<$ for x_i, $i = 1, \ldots, M_1$ and $x_>$ for x_i, $i = M_1 + 1, \ldots, M$.

We will again want to change integration variables to normal modes of either the single external string or the two intermediate strings as follows:

$$x_i = \frac{1}{\sqrt{M}}q_0 + \sqrt{\frac{2}{M}}\sum_{m=1}^{M-1} q_m \cos\frac{m\pi}{M}\left(i - \frac{1}{2}\right) \qquad \text{(B2)}$$

$$= \begin{cases} \frac{1}{\sqrt{M_1}}q_0^{(1)} + \sqrt{\frac{2}{M_1}}\sum_{m=1}^{M_1-1} q_m^{(1)}\cos\frac{m\pi}{M_1}\left(i-\frac{1}{2}\right) & i = 1,\ldots,M_1, \\ \frac{1}{\sqrt{M_2}}q_0^{(2)} + \sqrt{\frac{2}{M_2}}\sum_{m=1}^{M_2-1} q_m^{(2)}\cos\frac{m\pi}{M_2}\left(i - M_1 - \frac{1}{2}\right) & i = M_1+1,\ldots,M. \end{cases} \qquad \text{(B3)}$$

The missing link terms in the exponent involve

$$x_{M_1+1} - x_{M_1} = -2\sqrt{\frac{2}{M}}\sum_{m=1}^{M-1} q_m \sin\frac{mM_1\pi}{M}\sin\frac{m\pi}{2M}, \qquad \text{(B4)}$$

$$(x_{M_1+1} - x_{M_1})^2 = \frac{8}{M}\sum_{m',m''=1}^{M-1} q_{m'}q_{m''}\sin\frac{m'M_1\pi}{M}\sin\frac{m''M_1\pi}{M} \times \sin\frac{m'\pi}{2M}\sin\frac{m''\pi}{2M}. \qquad \text{(B5)}$$

It is straightforward to relate $q_m^{(1)}$, $q_m^{(2)}$ to q_m:

$$q_0^{(1)} = \sqrt{\frac{M_1}{M}}q_0 + \sqrt{\frac{2}{MM_1}}\sum_{m'=1}^{M-1} q_{m'}U^{(1)}_{m'0}, \qquad q_m^{(1)} = \frac{2}{\sqrt{MM_1}}\sum_{m'=1}^{M-1} q_{m'}U^{(1)}_{m'm}, \qquad \text{(B6)}$$

$$q_0^{(2)} = \sqrt{\frac{M_2}{M}}q_0 + \sqrt{\frac{2}{MM_2}}\sum_{m'=1}^{M-1} q_{m'}U^{(2)}_{m'0}, \qquad q_m^{(2)} = \frac{2}{\sqrt{MM_2}}\sum_{m'=1}^{M-1} q_{m'}U^{(2)}_{m'm}, \qquad \text{(B7)}$$

and we note the identity $q_0^{(1)}\sqrt{M_1} + q_0^{(2)}\sqrt{M_2} = q_0\sqrt{M}$, as expected from the fact that $q_0/\sqrt{M}$ is the center of momentum of the open string. The matrices $U^{(1)}$, $U^{(2)}$ are listed in Appendix E.

1. Correction to the open-string ground energy

For the ground state it suffices to set $x^i = x^f = 0$, so that the expression for iW simplifies somewhat:

$$iW \to -\frac{T_0}{2}\Bigg[\frac{(q_0^L)^2}{L} + \frac{(q_0^K)^2}{J} + \sum_{m=1}^{M-1} \sinh\lambda_m^o((q_m^L)^2 \coth L\lambda_m^o + (q_m^K)^2\coth J\lambda_m^o) + \frac{(q_0^{L,1}-q_0^{K,1})^2}{K}$$
$$+ \sum_{m=1}^{M_1-1}\sinh\lambda_m^{o,1}\left([(q_m^{L,1})^2+(q_m^{K,1})^2]\coth K\lambda_m^{o,1} - 2\frac{q_m^{K,1}q_m^{L,1}}{\sinh K\lambda_m^{o,1}}\right) + \frac{(q_0^{L,2}-q_0^{K,2})^2}{K}$$
$$+ \sum_{m=1}^{M_2-1}\sinh\lambda_m^{o,2}\left([(q_m^{L,2})^2+(q_m^{K,2})^2]\coth K\lambda_m^{o,2} - 2\frac{q_m^{K,2}q_m^{L,2}}{\sinh K\lambda_m^{o,2}}\right)\Bigg], \quad \text{(B8)}$$

where $\lambda_m^{o,1}$, $\lambda_m^{o,2}$ are obtained from λ_m^o through the substitutions $M \to M_1, M_2$, respectively.

Finally, we eliminate $q_m^{(1,2)}$ in favor of q_m. Because $U^{(2)}_{m'0} = -U^{(1)}_{m'0}$, we find that the zero modes combine nicely:

$$(q_0^{L,1}-q_0^{K,1})^2 + (q_0^{L,2}-q_0^{K,2})^2$$
$$= (q_0^K - q_0^L)^2 + \frac{2}{M_1M_2}\sum_{m',m''=1}^{M-1}(q_{m'}^K - q_{m'}^L)$$
$$\times (q_{m''}^K - q_{m''}^L)U^{(1)}_{m'0}U^{(1)}_{m''0}. \quad \text{(B9)}$$

From this we see that the zero modes enter the exponent in the combination

$$iW_0 = -\frac{T_0}{2}\left[\frac{(q_0^L)^2}{L} + \frac{(q_0^K)^2}{J} + \frac{(q_0^K-q_0^L)^2}{K}\right]. \quad \text{(B10)}$$

So, integrating them out simply implements closure on the zero modes:

$$\int dq_0^K dq_0^L e^{iW_0} = \frac{2\pi}{T_0}\sqrt{\frac{JKL}{J+K+L}} = \frac{2\pi}{T_0}\sqrt{\frac{JKL}{N+1}}. \quad \text{(B11)}$$

The contribution of the nonzero modes to the exponent can be expressed using the following matrix definitions:

$$A^{(1)}_{m'm''} \equiv \frac{4}{MM_1}\sum_{m=1}^{M_1-1}U^{(1)}_{m'm}U^{(1)}_{m''m}\sinh\lambda_m^{o,1}\coth K\lambda_m^{o,1}, \quad \text{(B12)}$$

$$B^{(1)}_{m'm''} \equiv -\frac{4}{MM_1}\sum_{m=1}^{M_1-1}U^{(1)}_{m'm}U^{(1)}_{m''m}\frac{\sinh\lambda_m^{o,1}}{\sinh K\lambda_m^{o,1}}, \quad \text{(B13)}$$

$$A^{(2)}_{m'm''} \equiv \frac{4}{MM_2}\sum_{m=1}^{M_2-1}U^{(2)}_{m'm}U^{(2)}_{m''m}\sinh\lambda_m^{o,2}\coth K\lambda_m^{o,2}, \quad \text{(B14)}$$

$$B^{(2)}_{m'm''} \equiv -\frac{4}{MM_2}\sum_{m=1}^{M_2-1}U^{(2)}_{m'm}U^{(2)}_{m''m}\frac{\sinh\lambda_m^{o,2}}{\sinh K\lambda_m^{o,2}}. \quad \text{(B15)}$$

Taking the limit $L, J \to \infty$, we define the nonzero-mode contribution to iW plus the missing link terms as

$$iW' = -\frac{T_0}{2}\Bigg[\sum_{m=1}^{M-1}\sinh\lambda_m^o((q_m^L)^2+(q_m^K)^2) + \frac{2}{KM_1M_2}\sum_{m',m''=1}^{M-1}(q_{m'}^K-q_{m'}^L)(q_{m''}^K-q_{m''}^L)U^{(1)}_{m'0}U^{(1)}_{m''0}$$
$$+ \sum_{m',m''=1}^{M-1}(q_{m'}^Kq_{m''}^K + q_{m'}^Lq_{m''}^L)(A^{(1)}+A^{(2)})_{m'm''} + 2\sum_{m',m''=1}^{M-1}q_{m'}^Kq_{m''}^L(B^{(1)}+B^{(2)})_{m'm''}$$
$$+ \frac{4}{M}\sum_{m',m''=1}^{M-1}(q_{m'}^Kq_{m''}^K + q_{m'}^Lq_{m''}^L)\sin\frac{m'M_1\pi}{M}\sin\frac{m''M_1\pi}{M}\sin\frac{m'\pi}{2M}\sin\frac{m''\pi}{2M}\Bigg] \quad \text{(B16)}$$

$$\equiv -\frac{T_0}{2}\left[\sum_{m',m''=1}^{M-1}(q_{m'}^Kq_{m''}^K + q_{m'}^Lq_{m''}^L)A_{m'm''} + 2q_{m'}^Kq_{m''}^LB_{m'm''}\right], \quad \text{(B17)}$$

where we have defined

$$A_{m'm''} \equiv \delta_{m'm''}\sinh\lambda_{m'}^o + (A^{(1)}+A^{(2)})_{m'm''}$$
$$+ \frac{2}{KM_1M_2}U^{(1)}_{m'0}U^{(1)}_{m''0} + \frac{4}{M}\sin\frac{m'M_1\pi}{M}$$
$$\times \sin\frac{m''M_1\pi}{M}\sin\frac{m'\pi}{2M}\sin\frac{m''\pi}{2M}, \quad \text{(B18)}$$

$$B_{m'm''} \equiv (B^{(1)}+B^{(2)})_{m'm''} - \frac{2}{KM_1M_2}U^{(1)}_{m'0}U^{(1)}_{m''0}. \quad \text{(B19)}$$

The Gaussian integral in the two-open-string function then becomes

$$\int dx_i^K dx_i^L e^{iW'} \to \left(\frac{2\pi}{T_0}\right)^M\sqrt{\frac{JKL}{N+1}}\det{}^{-1/2}\begin{pmatrix} A & B \\ B & A\end{pmatrix}, \quad \text{(B20)}$$

and the prefactors in the expression for the two-open-string function become in the limit $J, L \to \infty$

$$\mathcal{D}_M^{\rm open}(J)\mathcal{D}_{M_1}^{\rm open}(K)\mathcal{D}_{M_2}^{\rm open}(K)\mathcal{D}_M^{\rm open}(L) \to \frac{e^{-(J+L)\sum_m \lambda_m^o/2}}{K\sqrt{JL}}\left(\frac{T_0}{2\pi}\right)^{3M/2}\prod_{m=1}^{M-1}[2\sinh\lambda_m^o] \times \prod_{m=1}^{M_1-1}\left[\frac{\sinh K\lambda_m^{o,1}}{\sinh\lambda_m^{o,1}}\right]^{-1/2}\prod_{m=1}^{M_2-1}\left[\frac{\sinh K\lambda_m^{o,2}}{\sinh\lambda_m^{o,2}}\right]^{-1/2}. \tag{B21}$$

Combining these results, dividing by $\mathcal{D}^{\rm open}(N+1)e^{NB_0}$, and summing over M_1, K leads to our expression for the ground-state energy shift:

$$-a\Delta P^- = \sum_{K=1}^{\infty}\sum_{M_1=1}^{M-1}\left[\frac{e^{K\sum_m \lambda_m^o/2+(K-1)B_0}}{\sqrt{K}}\prod_{m=1}^{M-1}[2\sinh\lambda_m^o]^{1/2}\right]^{D-2}\left[\prod_{m=1}^{M_1-1}\left[\frac{\sinh K\lambda_m^{o,1}}{\sinh\lambda_m^{o,1}}\right]\prod_{m=1}^{M_2-1}\left[\frac{\sinh K\lambda_m^{o,2}}{\sinh\lambda_m^{o,2}}\right]\det\begin{pmatrix}A & B\\ B & A\end{pmatrix}\right]^{-(D-2)/2}. \tag{B22}$$

Doing the three products $\prod_{m=1}^{M-1}[2\sinh\lambda_m^o]$ explicitly yields

$$-a\Delta P^- = \sum_{K=1}^{\infty}\sum_{M_1=1}^{M-1}\left[\frac{e^{K\sum_m \lambda_m^o/2+(K-1)B_0}}{\sqrt{K}}\right]^{D-2}\prod_{m=1}^{M-1}[2\sinh\lambda_m^o]^{D-2}\left[\frac{M}{M_1M_2}\frac{\sinh 2\sinh^{-1}1\,\sinh 2M\sinh^{-1}1}{\sinh 2M_1\sinh^{-1}1\,\sinh 2M_2\sinh^{-1}1}\right]^{-(D-2)/4} \times\left[\prod_{m=1}^{M_1-1}[2\sinh K\lambda_m^{o,1}]\prod_{m=1}^{M_2-1}[2\sinh K\lambda_m^{o,2}]\det\begin{pmatrix}A & B\\ B & A\end{pmatrix}\right]^{-(D-2)/2}. \tag{B23}$$

2. Correction to the open-string gluon energy

Examination of the open-string propagator shows that the gluon state, the lightest spin-1 state with energy λ_1^o above the ground state, contributes via the first-order term in the expansion of

$$\exp\left[T_0\frac{\boldsymbol{q}_{1,f}^o\cdot\boldsymbol{q}_{1,f\iota}^o\sinh\lambda_1^o}{\sinh(N+1)\lambda_1^o}\right] \sim 1 + T_0\boldsymbol{q}_{1,f}^o\cdot\boldsymbol{q}_{1,i}^o 2\sinh\lambda_1^o e^{-(N+1)\lambda_1^o}. \tag{B24}$$

So to extract the one-loop correction, we isolate this term from the two external line propagators in the expression for the one-loop correction to the two-point function. It is then safe to take the $J, L \to \infty$ limit of what multiplies these factors. Then, in parallel with our extraction of the correction to the graviton self-energy, we find

$$-a\Delta P_{\rm gluon}^-\delta_{kl} = 2T_0\sinh\lambda_1^o\sum_{K=1}^{\infty}\sum_{M_1=1}^{M-1}e^{K\lambda_1^o}\langle q_{1,L}^{ok}q_{1,K}^{ol}\rangle\left[\frac{e^{K\sum_m \lambda_m^o/2+(K-1)B_0}}{\sqrt{K}}\prod_{m=1}^{M-1}[2\sinh\lambda_m^o]^{1/2}\right]^{D-2}\left[\prod_{m=1}^{M_1-1}\left[\frac{\sinh K\lambda_m^{o,1}}{\sinh\lambda_m^{o,1}}\right]^{-1/2} \times\prod_{m=1}^{M_2-1}\left[\frac{\sinh K\lambda_m^{o,2}}{\sinh\lambda_m^{o,2}}\right]^{-1/2}\det{}^{-1/2}\begin{pmatrix}A & B\\ B & A\end{pmatrix}\right]^{D-2}, \tag{B25}$$

where the correlator is given by

$$\langle q_{L,1}^{ok}q_{K,1}^{ol}\rangle = \frac{\int dq_{L,m}^o dq_{K,m}^o q_{L,1}^{ok}q_{K,1}^{ol}e^{iW'}}{\int dq_{L,m}^o dq_{K,m}^o e^{iW'}}. \tag{B26}$$

Again, with the notation

$$\begin{pmatrix}A & B\\ B & A\end{pmatrix}^{-1} = \begin{pmatrix}A' & B'\\ B' & A'\end{pmatrix}, \tag{B27}$$

it follows that

$$\langle q^{ok}_{L,1} q^{ol}_{K,1} \rangle = \delta_{kl} \frac{B'_{11}}{T_0}. \tag{B28}$$

APPENDIX C: NORMAL MODES

A string with $P^+ = MaT_0$ is described at a fixed time by M coordinates x_i or y_i, $i = 1, \ldots M$. In this article, we require several normal mode decompositions depending on the boundary conditions.

Neumann Open String:

$$x_i = \frac{1}{\sqrt{M}} q_0 + \sqrt{\frac{2}{M}} \sum_{m=1}^{M-1} q_{om} \cos \frac{m\pi(i-1/2)}{M}, \tag{C1}$$

$$q_0 = \sqrt{\frac{1}{M}} \sum_{i=1}^{M} x_i, \qquad q_{om} = \sqrt{\frac{2}{M}} \sum_i x_i \cos \frac{m\pi(i-1/2)}{M}. \tag{C2}$$

Closed String (Neumann):

M odd:

$$x_i = \frac{1}{\sqrt{M}} q_0 + \sqrt{\frac{2}{M}} \sum_{m=1}^{(M-1)/2} \left[q_{cm} \cos \frac{2m\pi(i-1/2)}{M} + q_{sm} \sin \frac{2m\pi(i-1/2)}{M} \right]. \tag{C3}$$

M even:

$$x_i = \frac{1}{\sqrt{M}} (q_0 + q_{sM/2}(-)^i) + \sqrt{\frac{2}{M}} \sum_{m=1}^{M/2-1} \left[q_{cm} \cos \frac{2m\pi(i-1/2)}{M} + q_{sm} \sin \frac{2m\pi(i-1/2)}{M} \right], \tag{C4}$$

$$q_{cm} = \sqrt{\frac{2}{M}} \sum_i x_i \cos \frac{2m\pi(i-1/2)}{M}, \qquad q_{sm} = \sqrt{\frac{2}{M}} \sum_i x_i \sin \frac{2m\pi(i-1/2)}{M}, \tag{C5}$$

$$q_{sM/2} = \sqrt{\frac{1}{M}} \sum_{i=1}^{M} (-)^i x_i, \quad \text{for } M \text{ even}, \qquad q_0 = \sqrt{\frac{1}{M}} \sum_{i=1}^{M} x_i. \tag{C6}$$

Dirichlet Open String:

$$y_k = \sqrt{\frac{2}{M}} \sum_{m=1}^{M-1} q_{Dm} \sin \frac{m\pi k}{M} \quad \text{for } k = 1, \ldots, M-1, \qquad y_M = q_{DM}, \tag{C7}$$

$$q_{Dm} = \sqrt{\frac{2}{M}} \sum_{k=1}^{M-1} y_k \sin \frac{m\pi k}{M}, \qquad 0 < m < M, \qquad q_{DM} = y_M. \tag{C8}$$

Closed String (Dirichlet):

M odd:

$$y_i = \frac{1}{\sqrt{M}} q_0 + \sqrt{\frac{2}{M}} \sum_{m=1}^{(M-1)/2} \left[q_{cm} \cos \frac{2m\pi i}{M} + q_{sm} \sin \frac{2m\pi i}{M} \right]. \tag{C9}$$

M even:

$$y_i = \frac{1}{\sqrt{M}}(q_0 + q_{cM/2}(-)^i) + \sqrt{\frac{2}{M}}\sum_{m=1}^{M/2-1}\left[q_{cm}\cos\frac{2m\pi i}{M} + q_{sm}\sin\frac{2m\pi i}{M}\right], \tag{C10}$$

$$q_{cm} = \sqrt{\frac{2}{M}}\sum_i y_i\cos\frac{2m\pi i}{M}, \qquad q_{sm} = \sqrt{\frac{2}{M}}\sum_i y_i\sin\frac{2m\pi i}{M}, \tag{C11}$$

$$q_0 = \sqrt{\frac{1}{M}}\sum_{i=1}^{M} y_i, \qquad q_{cM/2} = \sqrt{\frac{1}{M}}\sum_{i=1}^{M}(-)^i y_i, \quad \text{(for } M \text{ even)}. \tag{C12}$$

APPENDIX D: PROPAGATORS

1. Neumann open-string propagator

$$\langle N+1, x^f|0, x^i\rangle^{\text{open}} = \mathcal{D}^{\text{open}}(N+1)e^{iW_{\text{open}}}, \tag{D1}$$

where

$$\mathcal{D}^{\text{open}}(N+1) = \frac{1}{\sqrt{N+1}}\left(\frac{T_0}{2\pi}\right)^{M/2}\prod_{m=1}^{M-1}\left[\frac{\sinh(N+1)\lambda_m^o}{\sinh\lambda_m^o}\right]^{-1/2}, \tag{D2}$$

$$iW_{\text{open}} = -\frac{T_0}{2}\left[\frac{(q_{0,f}-q_{0,i})^2}{N+1} + \sum_{m=1}^{M-1}\sinh\lambda_m^o\left((q_{m,i}^2 + q_{m,f}^2)\coth(N+1)\lambda_m^o - 2\frac{q_{m,i}q_{m,f}}{\sinh(N+1)\lambda_m^o}\right)\right], \tag{D3}$$

$$\lambda_0^o = 0, \qquad \lambda_m^o = 2\sinh^{-1}\sin\frac{m\pi}{2M}, \qquad m = 1, \ldots, M-1, \tag{D4}$$

Where the q_m's are the normal mode coordinates for the x's. The right side is the result of doing the integrations over all the x_i^j with $i = 1, \ldots, M$ and $j = 1, \ldots N$. The propagator spans $N+1$ time steps, and this result corresponds to assigning half the potential energy $T_0\sum_{i=1}^{M-1}(x_{i+1}^j - x_i^j)^2/2$ to time $j = 0$ and half to $j = N+1$.

2. Dirichlet open-string propagator

The Dirichlet open-string propagator over a time of $K = N+1$ steps is evaluated to be

$$\langle q^f, N+1|q^i, 0\rangle^D = \mathcal{D}^D(N+1)e^{iW^D}, \tag{D5}$$

where

$$\mathcal{D}^D(N+1) = \left(\frac{T_0}{2\pi}\right)^{M/2}\prod_{m=1}^{M}\left[\frac{\sinh(N+1)\lambda_m^D}{\sinh\lambda_m^D}\right]^{-1/2}, \tag{D6}$$

$$iW^D = -\frac{T_0}{2}\sum_{m=1}^{M}\left((q_{Dm}^{f2} + q_{Dm}^{i2})\sinh\lambda_m^D\coth K\lambda_m^D - 2q_{Dm}^f q_{Dm}^i\frac{\sinh\lambda_m^D}{\sinh K\lambda_m^D}\right), \tag{D7}$$

$$\lambda_m^D = \lambda_m^o, \qquad m = 1, \ldots, M-1, \qquad \lambda_M^D = 2\sinh^{-1}\sqrt{\frac{\kappa}{2}}. \tag{D8}$$

We recall that the above expressions give the result of integrating over all the variables y_i^j for $j = 1, \ldots, N$, with half the potential energy assigned to $j = 0, N+1$, which is consistent with the closure requirement.

APPENDIX E: OVERLAP FORMULAS

Open-2 Open, Neumann:

$$q_0^{(1)} = \sqrt{\frac{M_1}{M}} q_0 + \sqrt{\frac{2}{MM_1}} \sum_{m'=1}^{M-1} q_{m'} U_{m'0}^{(1)}, \qquad q_m^{(1)} = \frac{2}{\sqrt{MM_1}} \sum_{m'=1}^{M-1} q_{m'} U_{m'm}^{(1)}, \tag{E1}$$

$$q_0^{(2)} = \sqrt{\frac{M_2}{M}} q_0 + \sqrt{\frac{2}{MM_2}} \sum_{m'=1}^{M-1} q_{m'} U_{m'0}^{(2)}, \qquad q_m^{(2)} = \frac{2}{\sqrt{MM_2}} \sum_{m'=1}^{M-1} q_{m'} U_{m'm}^{(2)}, \tag{E2}$$

$$U_{m'm}^{(1)} = \sum_{i=1}^{M_1} \cos\frac{m'\pi}{M}\left(i - \frac{1}{2}\right) \cos\frac{m\pi}{M_1}\left(i - \frac{1}{2}\right) = \frac{(-)^m}{2} \frac{\sin(m'\pi M_1/M)\sin(m'\pi/2M)\cos(m\pi/2M_1)}{\sin^2(m'\pi/2M) - \sin^2(m\pi/2M_1)}, \tag{E3}$$

$$U_{m'm}^{(2)} = \sum_{i=1+M_1}^{M} \cos\frac{m'\pi}{M}\left(i - \frac{1}{2}\right) \cos\frac{m\pi}{M_2}\left(i - M_1 - \frac{1}{2}\right) = -\frac{1}{2} \frac{\sin(m'\pi M_1/M)\sin(m'\pi/2M)\cos(m\pi/2M_2)}{\sin^2(m'\pi/2M) - \sin^2(m\pi/2M_2)}, \tag{E4}$$

and we note the identity $q_0^{(1)}\sqrt{M_1} + q_0^{(2)}\sqrt{M_2} = q_0\sqrt{M}$, as expected from the fact that $q_0/\sqrt{M}$ is the center of momentum of the open string.

We can also express the q's in terms of the $q^{(1)}$, $q^{(2)}$'s:

$$q_0 = q_0^{(1)}\sqrt{\frac{M_1}{M}} + q_0^{(2)}\sqrt{\frac{M_2}{M}}, \tag{E5}$$

$$q_{m'} = \sqrt{\frac{2}{MM_1}}\left(q_0^{(1)} U_{m'0}^{(1)} + \sqrt{2}\sum_{m=1}^{M_1-1} q_m^{(1)} U_{m'm}^{(1)}\right) + \sqrt{\frac{2}{MM_2}}\left(q_0^{(2)} U_{m'0}^{(2)} + \sqrt{2}\sum_{m=1}^{M_2-1} q_m^{(2)} U_{m'm}^{(2)}\right). \tag{E6}$$

Open-2 Open, Dirichlet:

$$q_{DM_1}^{(1)} = \sqrt{\frac{2}{M}} \sum_{m'=1}^{M-1} q_{Dm'} \sin\frac{m'\pi M_1}{M}, \qquad q_{Dm}^{(1)} = \frac{2}{\sqrt{MM_1}} \sum_{m'=1}^{M-1} q_{Dm'} U_{m'm}^{D(1)}, \tag{E7}$$

$$q_{DM_2}^{(2)} = y_M = q_{DM}, \qquad q_{Dm}^{(2)} = \frac{2}{\sqrt{MM_2}} \sum_{m'=1}^{M-1} q_{Dm'} U_{m'm}^{D(2)}, \tag{E8}$$

$$\begin{aligned} U_{m'm}^{D(1)} &= \sum_{k=1}^{M_1-1} \sin\frac{m'\pi k}{M} \sin\frac{m\pi k}{M_1} = \frac{(-)^m}{4} \frac{\sin(m\pi/M_1)\sin(m'\pi M_1/M)}{\sin^2(m'\pi/2M) - \sin^2(m\pi/2M_1)}, \\ U_{m'm}^{D(2)} &= \sum_{k=1+M_1}^{M-1} \sin\frac{m'\pi k}{M} \sin\frac{m\pi(k-M_1)}{M_1} = \frac{(-)^{m'}}{4} \frac{\sin(m\pi/(M-M_1))\sin(m'\pi(M-M_1)/M)}{\sin^2(m'\pi/2M) - \sin^2(m\pi/2(M-M_1))}. \end{aligned} \tag{E9}$$

3 Zero-momentum Tachyon Vertex:

$$V_3 = \frac{1}{\sqrt{|P_1^+ P_2^+ P_3^+|}} \left|\frac{P_1^+}{P_3^+}\right|^{(P_1^{+2}+P_2^{+2}+P_1^+P_2^+)/P_2^+P_3^+} \left|\frac{P_2^+}{P_3^+}\right|^{(P_1^{+2}+P_2^{+2}+P_1^+P_2^+)/P_1^+P_3^+}, \tag{E10}$$

$$P_3^+ = -P_1^+ - P_2^+. \tag{E11}$$

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