Vortex lattices and crystalline geometries

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We consider $\mathrm{AdS}_2 \times R^2$ solutions supported by a magnetic field, such as those which arise in the near-horizon limit of magnetically charged AdS_4 Reissner-Nordstrom black branes. In the presence of an electrically charged scalar field, such magnetic solutions can be unstable to spontaneous formation of a vortex lattice. We solve the coupled partial differential equations that govern the charged scalar, gauge field, and metric degrees of freedom to lowest nontrivial order in an expansion around the critical point and discuss the corrections to the free energy and thermodynamic functions arising from the formation of the lattice. We describe how such solutions can also be interpreted, via S-duality, as characterizing infrared crystalline phases of conformal field theories doped by a chemical potential, but in zero magnetic field; the doped conformal field theories are dual to geometries that exhibit dynamical scaling and hyperscaling violation.

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I. INTRODUCTION

A topic of recent interest has been the holographic description of phases of quantum field theory with spatial anisotropy and/or inhomogeneity [1–15]. This is motivated in part by the crucial role that momentum relaxation due to inhomogeneities plays in transport phenomena in condensed matter systems, and in part by intrinsic interest in the rich physics of such phases.

Our goal in this work is twofold. On the one hand, as an extension of the ideas discussed in [16], we would like to illustrate the emergence of crystalline ground states ("solids") in conformal field theories doped by a chemical potential coupling to a globally conserved U(1) charge, but in zero magnetic field. In $2 + 1$ dimensions, monopole operators associated with the global U(1) symmetry [16–18] serve as order parameters for solid phases in doped conformal field theories (CFTs). Electric-magnetic duality allows one to find a dual description where the magnetic degrees of freedom are manifested in terms of electrically charged operators. In the bulk gravitational description, this allows us to view the formation of the solid by studying vortex lattice formation in the theory of a charged scalar moving in a background magnetic field. A crucial advantage of studying the solid phases of doped CFTs by using this dual charged scalar is that the Dirac quantization condition on the monopole charge translates into an exact commensurability relation between the area of the unit cell of the crystal and the density of doped charges [16,19].

On the other hand, an open problem in the study of holographic lattices has been to find, analytically, gravitationally back-reacted solutions for a crystalline lattice of dimension $d > 1$. This has largely been because of the relative difficulty of solving coupled systems of partial differential equations, instead of the ordinary differential equations which normally govern simple backgrounds in gauge/gravity duality. Here, we give an example of such a crystalline metric in $d = 2$. Our work builds on the earlier papers [3], which found an elegant solution for a vortex lattice in the probe approximation, and [11], where the backreaction of such a lattice on bulk gauge fields was studied in a different setting.

The organization of this paper is as follows. In Sec. II, we review the basic unperturbed $\mathrm{AdS}_2 \times R^2$ solution. In Sec. III, we incorporate a charged scalar field and describe the vortex lattice solution. In Sec. IV, we describe the basic physics visible in the perturbative vortex lattice solution. In Sec. V, we characterize how such a lattice could also emerge in the IR geometry of a gravitational solution which exhibits dynamical scaling with hyperscaling violation, as the S-dual of a doped CFT in zero magnetic field. Possible directions for future research are discussed in Sec. VI.

II. MAGNETIC $\mathrm{AdS}_2 \times R^2$ SOLUTIONS

Consider the theory with action

$$S = \int d^4x \sqrt{-g}\left(R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - 2\Lambda\right), \qquad (2.1)$$

where $\Lambda < 0$. It has $\mathrm{AdS}_2 \times R^2$ solutions with metric

$$ds^2 = L^2\left(-\frac{dt^2}{r^2} + \frac{dr^2}{r^2} + dx^2 + dy^2\right), \qquad (2.2)$$

where

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$$\frac{1}{L^2} = -2\Lambda. \tag{2.3}$$

The gauge field supporting the solution is

$$F_{xy} = Q_m dx \wedge dy, \tag{2.4}$$

with Q_m fixed in terms of the AdS radius by the equation

$$Q_m = \sqrt{2}L. \tag{2.5}$$

In particular, this means that for these solutions, fixing the magnetic field fixes also the cosmological constant and the AdS radius.

In addition to its intrinsic interest, this solution arises as the near-horizon geometry of extremal magnetically charged AdS/Reissner-Nordstrom black branes with AdS_4 asymptotics. In this context, the AdS_2 near-horizon region has played a crucial role in elucidating the non-Fermi–liquid behavior of probe fermions [20–22] scattering off the bath of locally critical excitations represented by the AdS_2 geometry [23,24].

III. THE VORTEX LATTICE

Our interest is not in the pure AdS_2 solution (2.2). We wish to include also an electrically charged scalar field, ψ, in the full action. In part, this is because a generic such theory could include such scalars; in part, it is motivated by the duality considerations to be described in Sec. V.

In any case, here, we will see that in some ranges of parameters, the charged scalar will qualitatively change the IR physics. The simplest case in which we can see this effect will be directly in the $AdS_2 \times R^2$ background of Sec. II. We will impose a hard wall cutoff at $r = r_0$ in the deep IR, along with suitable boundary conditions, to be described below. We can think of r_0 as a proxy for a "confinement scale" or a "temperature." Tuning the magnetic field relative to the "temperature" will trigger the scalar instability.

After including the ψ coupling to the gauge field, the action becomes

$$S = \int d^4x\sqrt{-g}\Big(R - 2\Lambda - \frac{1}{4}F^2 - |\nabla_\mu \psi|^2 - m^2|\psi|^2 - \lambda|\psi|^4\Big). \tag{3.1}$$

We have defined $\nabla_\mu = \partial_\mu + ieA_\mu$, where e is the electric charge of the scalar field. From this point on we will set $e = 1$. This action has a stress-energy tensor of the form

$$T_{\mu\nu} = -\frac{g_{\mu\nu}}{2}\mathcal{L}_{\rm mat} + \frac{1}{2}F_{\mu\lambda}F^{\lambda}_{\nu} + e^2A_\mu A_\nu|\psi|^2 + \frac{1}{2}[\partial_\mu\psi\partial_\nu\psi^* + ie\psi(A_\mu\partial_\nu + A_\nu\partial_\mu)\psi^* + \text{H.c.}], \tag{3.2}$$

where

$$\mathcal{L}_{\rm mat} = \frac{1}{4}F^2 + |\nabla_\mu\psi|^2 + m^2|\psi|^2 + \lambda|\psi|^4. \tag{3.3}$$

We may expand the magnitude of $|\nabla_\mu\psi|^2$ as

$$|\nabla_\mu\psi|^2 = |\partial_\mu\psi|^2 + iA_\mu(\psi\partial^\mu\psi^* - \psi^*\partial^\mu\psi) + A^2|\psi|^2. \tag{3.4}$$

At this point, we can calculate the Euler-Lagrange equation for ψ by differentiating with respect to ψ^*,

$$\partial_\mu(\sqrt{-g}\nabla^\mu\psi) = -\sqrt{-g}(iA^\mu\nabla_\mu\psi - m^2\psi - 2\lambda|\psi|^2\psi), \tag{3.5}$$

and the equation of motion for the gauge field,

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}F^{\mu\nu}) = i(\psi\partial^\nu\psi^* - \psi^*\partial^\nu\psi) + 2A^\nu|\psi|^2. \tag{3.6}$$

In addition to these equations of motion, we will also need to solve the Einstein equations,

$$R_{\mu\nu} - \frac{(R - 2\Lambda)}{2}g_{\mu\nu} = T_{\mu\nu}, \tag{3.7}$$

when we include backreaction of the ψ condensate on the gauge field and the metric.

We will expand perturbatively in a small parameter ϵ around the solution $\psi = 0$ with background gauge field of the form

$$A_x = Q_c y, \qquad A_y = 0. \tag{3.8}$$

The scalar field in the competing vortex phase will itself be of order ϵ. For fixed r_0 and boundary conditions (to be discussed below), we will choose Q_c to be just at the onset for the transition to forming vortices. At this critical value of the magnetic field, the $\psi = 0$ solution will be degenerate with a vortex lattice solution. As we increase the magnetic field to slightly above its critical value, $\psi = 0$ will no longer be the preferred solution, and the vortex lattice will be preferred. As is familiar, the onset of the transition is signalled by the existence of a purely normalizable solution for ψ that respects the IR boundary conditions.

We can parametrize the backreaction of the scalar on the gauge sector through a perturbative expansion in the distance away from the critical field. The scalar will have the form

$$\psi(r, x, y) = \epsilon\psi_1(r, x, y) + \epsilon^3\psi_3(r, x, y) + \cdots \tag{3.9}$$

and the gauge field will have the form

$$A_x(r, x, y) = Qy + \epsilon^2 a_2^x(r, x, y) + \cdots, \tag{3.10}$$

$$A_y(r, x, y) = \epsilon^2 a_2^y(r, x, y) + \cdots, \tag{3.11}$$

with $A_t = A_r = 0$. When we consider lattice solutions which are periodic in x and y, the backreaction of ψ on the gauge field will require both A_x and A_y to be nonzero at $O(\epsilon^2)$, with both x and y dependence.

A similar statement holds for the metric at $O(\epsilon^2)$. Our metric ansatz, to $O(\epsilon^2)$, will be

$$ds^2 = L^2\left\{\frac{1}{r^2}((-1+\epsilon^2 a(r,x,y))dt^2 + (1+\epsilon^2 a(r,x,y))dr^2) + (1+\epsilon^2 b(r,x,y))(dx^2+dy^2)\right\}. \tag{3.12}$$

Because at zeroth order in epsilon the $AdS_2 \times R^2$ metric is exactly supported by the magnetic field (i.e. the gauge field is not a probe), we find it necessary to include metric backreaction once we backreact on the gauge field. This distinguishes our situation from that considered in e.g. [11].

The radial magnetic field will be

$$B_r = Q + \epsilon^2(\partial_y a_2^x(r,x,y) - \partial_x a_2^y(r,x,y)). \tag{3.13}$$

In general, when we backreact on the magnetic field, we may expect there to be a nonnormalizable piece at order ϵ^2, i.e. $A_x(r \to 0) = (Q + \delta Q \epsilon^2)y$. This shifts the naive critical value of the field at the transition. However, because the critical point is actually only dependent on the dimensionless combination Q/r_0^2, we can (and will) impose that there is no non-normalizable correction to the gauge field in our backreacted solutions. That is, we will set $\delta Q = 0$. The value of the critical point will still have an $O(\epsilon^2)$ shift; it will manifest itself as an $O(\epsilon^2)$ shift in the location of the hard wall, $r_0 \to r_0 + \delta r_0 \epsilon^2$. These two scenarios are equivalent; in both cases we should think of the backreaction of ψ on the metric and gauge field as inducing a shift in the dimensionless parameter which controls the critical point at $O(\epsilon^2)$.

A. Basic droplet

We now examine the solutions of the field equation for ψ, in the limit where we can neglect the backreaction of ψ on the gauge field and on the metric. (This will be at order ϵ.) Very similar equations have been examined in the literature on vortices in holographic superconductors [3,25,26]. The basic building block for the solutions we will study is the "droplet" solution of [25].

We will begin by setting $\lambda = 0$ in the potential for the scalar and proceed with the metric and the mass of the (dualized) monopole field unspecified. Both of these will affect the radial solution for the scalar, but we will see that the spatial part of ψ_1 decouples from the radial equation for all metrics we might consider, and so we can find the basic form of the droplet solution while leaving the metric general.

For metrics with components that only depend on r and for which $g^{xx} = g^{yy}$, we can solve this equation by separation of variables, assuming that

$$\psi_1 = \rho_0 \rho(r) g(y) e^{ikx}, \tag{3.14}$$

where ρ_0 is an overall constant. Inserting our choice of gauge (3.8) yields, after some algebraic manipulation,

$$\frac{1}{\rho_n(r)}\left(\frac{g^{rr}}{g^{xx}}\rho_n''(r) + \frac{1}{\sqrt{-g}g^{xx}}\frac{\partial}{\partial r}(\sqrt{-g}g^{rr})\rho_n'(r)\right) - \frac{m^2}{g^{xx}} = -\frac{1}{g_n(y)}(g_n''(y) - (Qy+k)^2 g_n(y)) = -\lambda_n, \tag{3.15}$$

where λ_n is the eigenvalue from the separation of variables. First we will consider the equation for $g(y)$, which will yield the basic droplet solution. This solution will only exist in the parameter ranges that admit a normalizable solution to the radial equation; we will discuss this in the next section. The equation for g becomes

$$g_n'' - (Qy+k)^2 g_n = \lambda_n g_n. \tag{3.16}$$

Now, redefining $Y = \sqrt{Q}(y + \frac{k}{Q})$, the g_n equation becomes

$$g_n''(Y) - \left(Y^2 + \frac{\lambda_n}{Q}\right) g_n(Y) = 0. \tag{3.17}$$

Solving, we get that

$$g_n(Y) = c_+ D_{\nu_+}(\sqrt{2}Y) + c_- D_{\nu_-}(i\sqrt{2}Y), \tag{3.18}$$

where $c_\pm$ are constants, $\nu_\pm = \frac{1}{2}(-1 \pm \frac{\lambda_n}{Q})$, and $D_\nu(x)$ is the parabolic cylinder function.

The reader may recognize the differential equation for g_n, (3.16), as the same eigenvalue problem that arises in the study of the quantum mechanics of the simple harmonic oscillator. More properly, this is the case for appropriate choices of the separation constant. In these cases, we can write the (normalizable) solution for g_n in terms of the familiar Hermite polynomials,

$$g_n = e^{-Y^2/2} H_n(Y), \tag{3.19}$$

with eigenvalues $\lambda_n = 2Q(n + 1/2)$. The nth eigenvalue here characterizes the nth Landau level of the ψ particles. The "droplet" solutions with this shape were first discussed in the series of papers in [25], in a related but distinct context. The single droplet solution is when $n = 0$, which is just a Gaussian centered at $y = -k/Q$, $g(y) = e^{-Y^2/2}$. Note that g_n = constant is not a solution to the equations of motion.

B. Vortex lattice

Of more interest to us is a solution which preserves some discrete subgroup of the translation invariance of the original system. The basic droplet of Sec. III A breaks translations entirely. However, more symmetric solutions can be obtained by taking linear combinations of droplets, which still solve the (linearized) equations of motion neglecting back reaction.

A vortex lattice can be constructed as follows [3], using the zeroth Landau level solutions for the ψ field. The basic solution is

$$\psi_0(y;k) = e^{-\frac{y^2}{2}} = e^{-\frac{Q}{2}(y+\frac{k}{Q})^2}. \tag{3.20}$$

An appropriate superposition to give a lattice in the $x - y$ plane is

$$\Psi_{\rm lat}(x,y) = \frac{1}{L}\sum_{l=-\infty}^{\infty} c_l e^{ik_l x}\psi_0(y;k_l), \tag{3.21}$$

where

$$c_l \equiv e^{-i\pi\frac{v_2}{v_1^2}l^2}, \qquad k_l \equiv \frac{2\pi l}{v_1}\sqrt{Q} \tag{3.22}$$

for *arbitrary* v_1 and v_2.

One can write this in terms of the elliptic theta function θ_3,

$$\theta_3(v,\tau) \equiv \sum_{l=-\infty}^{\infty} q^{l^2} z^{2l}, \qquad q \equiv e^{i\pi\tau}, \qquad z \equiv e^{i\pi v}, \tag{3.23}$$

as

$$\begin{aligned}\psi_1(x,y,r) &= \rho_0\rho(r)\Psi_{\rm lat}(x,y),\\ \Psi_{\rm lat}(x,y) &\equiv e^{\frac{-Qy^2}{2}}\theta_3(v,\tau)\end{aligned} \tag{3.24}$$

with

$$v \equiv \frac{\sqrt{Q}(x+iy)}{v_1}, \qquad \tau \equiv \frac{2\pi i - v_2}{v_1^2}. \tag{3.25}$$

That the solution (3.21) represents a lattice is now evident from the basic properties of the elliptic theta function. For instance

$$\theta_3(v+1,\tau) = \theta_3(v,\tau) \tag{3.26}$$

and

$$\theta_3(v+\tau,\tau) = e^{-2\pi i(v+\tau/2)}\theta_3(v,\tau), \tag{3.27}$$

implying that $\Psi_{\rm lattice}$ returns to its value (up to a phase) upon translation by the lattice generators

$$\mathbf{a} = \frac{1}{\sqrt{Q}}v_1\partial_x, \qquad \mathbf{b} = \frac{1}{\sqrt{Q}}\left(\frac{2\pi}{v_1}\partial_y + \frac{v_2}{v_1}\partial_x\right). \tag{3.28}$$

These have been fixed such that the area of a unit cell is $2\pi/Q$, containing exactly one flux quantum. It is this quantization condition which translates, in the electromagnetic dual, to the commensurability condition between the area of the unit cell and the density of doped charges [16,19].

That $\Psi_{\rm lat}$ should be called a vortex lattice, despite the fact that it is composed of an array of the droplet solutions of [25], is evident from the fact that θ_3 vanishes on the lattice spanned by half-integral multiples of the lattice generators (giving rise to vortex cores) and has a phase rotation of 2π around each such zero.

Some common lattice shapes are obtained by choosing particular values of the parameters v_1, v_2. A rectangular lattice can be obtained by setting $v_2 = 0$. In this case all coefficients in equation (3.21) are equal, $c_l = c = 1$. The ratio of length to width of the rectangle is parametrized by v_1. For the special choice $v_1 = \sqrt{2\pi}$, the lattice is square. Another special choice is $v_2 = \frac{1}{2}v_1^2$; this yields a rhombic lattice. In this case $c_l = 1$ for $l \equiv 0 \bmod 2$ and $c_l = -i$ for $l \equiv 1 \bmod 2$. For the special case $v_1 = 2\sqrt{\pi}$, the rhombus is square (but now rotated 45° with respect to the x axis), and for $v_1 = \frac{2\sqrt{\pi}}{3^{\frac{1}{4}}}$ the lattice is composed of equilateral triangles (though the unit cell is still a rhombus).

At this point, nothing has fixed the "moduli" v_1, v_2 of the vortex lattice, nor the overall magnitude ρ_0 of ψ_1. In standard Landau-Ginzburg theories, apparently the triangular lattice is preferred. One could find preferred shapes in the approach here by including leading non-linearities in $|\psi|$ in the free energy, and minimizing the free energy density. It might be interesting to do this, while introducing parameters to vary that could lead to phase transitions in the preferred lattice shape.

C. The radial equation and boundary conditions

Now we consider the differential equation for $\rho(r)$. At order ϵ we are still in an $AdS_2 \times R^2$ background, which means that ψ should scale as a power law in r. Choosing the solution that vanishes at the boundary, we get

$$\rho(r) = r^\alpha, \tag{3.29}$$

where $\alpha = \frac{1}{2}(1+\sqrt{1+4(Q+m^2L^2)})$.

At the hard wall cutoff, $r = r_0$, we will need to impose a consistent set of boundary conditions. One way to do this is to consider a method very similar to the prescription of [27]. We will add a mirror image of the spacetime to the other side of the wall and glue them together at the IR boundary, $r = r_0$. Thus, we have two asymptotic UV boundaries (in our coordinates at $r = 0$ and $r = 2r_0$) and mirror solutions for the metric and the fields on either side of the wall. We will require the metric and fields to be continuous at the wall, but their derivatives will have a discontinuity. That is, we impose Israel junction conditions at the wall, including any localized energy-momentum sources present there. At the end of the day, we can quotient by the Z_2 symmetry to leave just one copy of the desired spacetime.

In order to support the discontinuity at the IR wall and thus solve the equations of motion at the wall, there must be a source of stress-energy at $r = r_0$. Therefore we will add an action, $S_{\rm wall}$, localized to the wall and solve the equations of motion. One way that this is different from the situation discussed in [27] is that while those authors needed only to add a localized cosmological constant to the wall (as everything was only a function of the radial variable), we now have spatial (x, y) dependence, so our boundary action must also have spatial dependence, $S_{\rm wall} = S_{\rm wall}(x, y)$.

Which terms in the equations of motion will contribute to the boundary stress-energy? When integrating the equations of motion across the wall, the first derivative of any function of r will not contribute, whereas the second derivative will

$$\int_{r_0-\epsilon}^{r_0+\epsilon} drf'(r) = f(r_0+\epsilon) - f(r_0-\epsilon) = 0; \tag{3.30}$$

$$\int_{r_0-\epsilon}^{r_0+\epsilon} drf''(r) = f'(r_0+\epsilon) - f'(r_0-\epsilon) = -2f'(r_0-\epsilon). \tag{3.31}$$

Therefore, in order to solve for the action at the wall, we only need to consider the terms in the equations of motion which have second derivatives of functions of r. At zeroth order in ϵ the gauge field is independent of r, and the Einstein equations only depend on up to first derivatives of the metric functions. In this case, integrating the equations across the wall we find no contributions, and we find that we do not need an S_{wall} at zeroth order in ϵ.

At first order in ϵ, we need to consider integrating the ψ equation of motion across the wall. We will add the term

$$S^{\psi}_{\mathrm{wall}} = \int_{r=r_0} d^3x\sqrt{-h}\,\delta m_w^2|\psi|^2 \tag{3.32}$$

to the action, where $h_{\mu\nu}$ is the induced metric at $r = r_0$, and δm_w is a localized shift in the mass of ψ. The nonzero contributions to the equation of motion when integrated over the wall are

$$-\int_{r_0-\epsilon}^{r_0+\epsilon} dr\sqrt{-g}g^{rr}\psi'' = \int_{r_0-\epsilon}^{r_0+\epsilon} dr\sqrt{-h}\,\delta m_w^2\psi\,\delta(r-r_0), \tag{3.33}$$

which gives the result $\delta m_w^2 = 2\alpha/L$.

Note that after adding the wall-localized mass term (3.32), the strategy for finding the critical field at which a phase transition occurs is the following. For a fixed choice of the wall localized mass and the location of the wall r_0, there is a critical value of the B-field at which the purely normalizable solution for ψ obeys the boundary conditions. In this paper, we are always expanding about this critical field, with ϵ parametrizing the distance from criticality.

We note also that we will need to add additional terms to S_{wall} when we consider the equations of motion at $O(\epsilon^2)$ in the next section.

D. Higher order corrections to the gauge field and metric

The relevant equations of motion are the Einstein equations and the Euler-Lagrange equations for ψ, A_μ, Eqs. (3.5) and (3.6).

In an $AdS_2 \times R^2$ background, all the unknown functions scale as power laws in r. At $O(\epsilon)$ there is only the ψ equation of motion. From Sec. III we know there exists a lattice solution of the form

$$\psi_1(r,x,y) = \rho_0 r^{\alpha}\sum_{l=-\infty}^{\infty} e^{\frac{2\pi i l\sqrt{Q}x}{v_1}} e^{-\frac{Q}{2}(y+\frac{2\pi i l}{v_1\sqrt{Q}})^2}, \tag{3.34}$$

where ρ_0 is the magnitude of ψ_1 and the scaling exponent is

$$\alpha = \frac{1}{2}\left(1+\sqrt{1+4(Q+m^2L^2)}\right). \tag{3.35}$$

ψ_1 acts as a source in the gauge field equation of motion and Einstein equations at $O(\epsilon^2)$. Therefore we can extract the r scaling in the $O(\epsilon^2)$ corrections and solve the equations of motion for the spatial dependence. We write

$$f_i(r,x,y) = \rho_0^2 r^{2\alpha} f_i(x,y), \tag{3.36}$$

where $f_i = a, b, a_2^x, a_2^y$. By assuming a normalizable radial dependence of this form for each field, we are implicitly setting one integration constant to zero per function. Our choice of solution for each of these fields will also fix the form of the localized stress energy we will need to add at the wall in order to have a consistent solution.

Now we examine the differential equations at $O(\epsilon^2)$. The equation of motion for A_r gives us the constraint

$$\partial_x a_2^x + \partial_y a_2^y = 0. \tag{3.37}$$

Besides this, we have two additional gauge field equations of motion (one each for x, y) and 5 nontrivial Einstein equations (for G_{tt}, G_{rr}, $G_{xx} = G_{yy}$, G_{rx}, G_{ry}) at $O(\epsilon^2)$. These are seven equations and four unknown functions. Luckily, three of them are redundant, and we can find a consistent solution once we have chosen the form of the source, ψ_1. The Einstein equations are

$$\begin{aligned}
2(\partial_y a_2^x - \partial_x a_2^y) + Q(\partial_x^2+\partial_y^2)(a+b) + 2Q(4\alpha^2-1)b &= S_1(\psi_1)\\
2(\partial_x a_2^y - \partial_y a_2^x) + Q(\partial_x^2+\partial_y^2)(a-b) + 2Q(2\alpha+1)b &= S_2(\psi_1)\\
2\alpha a_2^y + Q(\alpha-1)\partial_x a - Q\alpha\partial_x b &= S_3(\psi_1)\\
2\alpha a_2^x - Q(\alpha-1)\partial_y a + Q\alpha\partial_y b &= S_4(\psi_1)\\
2(\partial_x a_2^y - \partial_y a_2^x) - 2Q(\alpha-1)(2\alpha-1)a + 2Q(2\alpha^2-\alpha+1)b &= S_5(\psi_1)
\end{aligned} \tag{3.38}$$

and the gauge field equations are

$$(\partial_x^2 + \partial_y^2 + 2\alpha(2\alpha-1))a_2^x - Q\partial_y b = S_6(\psi_1) \qquad (\partial_x^2 + \partial_y^2 + 2\alpha(2\alpha-1))a_2^y + Q\partial_x b = S_7(\psi_1), \tag{3.39}$$

where the ψ-dependent source terms are given by

$$\begin{aligned}
S_1(\psi_1) &= -\frac{Q}{2}(2\alpha^2 + Q^2m^2 + 2Q^2y^2)|\psi_1|^2 + iQ^2y(\psi_1^*\partial_x\psi_1 - \psi_1\partial_x\psi_1^*) - Q(|\partial_x\psi_1|^2 + |\partial_y\psi_1|^2)\\
S_2(\psi_1) &= \frac{Q}{2}(-2\alpha^2 + Q^2m^2 + 2Q^2y^2)|\psi_1|^2 - iQ^2y(\psi_1^*\partial_x\psi_1 - \psi_1\partial_x\psi_1^*) + Q(|\partial_x\psi_1|^2 + |\partial_y\psi_1|^2)\\
S_3(\psi_1) &= \frac{Q\alpha}{2}\partial_x|\psi_1|^2 \qquad S_4(\psi_1) = -\frac{Q\alpha}{2}\partial_y|\psi_1|^2\\
S_5(\psi_1) &= -\frac{Q}{2}(2\alpha^2 + Q^2m^2 + 2Q^2y^2)|\psi_1|^2 + iQ^2y(\psi_1^*\partial_x\psi_1 - \psi_1\partial_x\psi_1^*) - Q(|\partial_x\psi_1|^2 - |\partial_y\psi_1|^2)\\
S_6(\psi_1) &= -\frac{i}{2}Q^2(\psi_1^*\partial_x\psi_1 - \psi_1\partial_x\psi_1^* + 2iyQ|\psi_1|^2) \qquad S_7(\psi_1) = \frac{i}{2}Q^2(\psi_1\partial_y\psi_1^* - \psi_1^*\partial_y\psi_1)
\end{aligned} \tag{3.40}$$

and a, b, a_2^x, a_2^y, ψ_1 are now only functions of x, y as we have omitted the power law r dependence.

We know that the vortex lattice solution is periodic in x, y with periodicity $\frac{v_1}{\sqrt{Q}}$ in the x direction and $\frac{2\pi}{v_1\sqrt{Q}}$ in the y direction (this is only for the rectangular lattice); therefore, we can expand each of these functions as a double Fourier series in x, y as

$$f_i(x,y) = \sum_{k,l} v_1 e^{\frac{2\pi ik\sqrt{Q}x}{v_1}} e^{ilv_1\sqrt{Q}y} e^{-\frac{k^2\pi^2}{v_1^2} - i\pi kl - \frac{l^2v_1^2}{4}} \tilde{f}_i(k,l), \tag{3.41}$$

where $f_i = a, b, a_2^x, a_2^y$, and we have pulled out the exponential function of m, n which will be present in all of the source terms. Notice that the periodicity implies that each unit cell has a net flux density of $\frac{2\pi}{Q}$. It remains to Fourier transform the source terms in the equations of motion in order to bring them into the form of Eq. (3.41) and then solve algebraic equations for the polynomial coefficients $\tilde{f}_i(k,l)$. In order to do this, we will use properties of exponentials and the Fourier transform to write an infinite sum of Gaussians as an infinite sum of exponentials,

$$\sum_k e^{-\frac{1}{2}(y+\frac{2\pi k}{v_1})^2} e^{-\frac{1}{2}(y+\frac{2\pi(k+l)}{v_1})^2} = \sum_k e^{iv_1ky}\frac{v_1}{2\sqrt{\pi}} e^{-\frac{v_1^2k^2}{4} - i\pi kl - \frac{l^2\pi^2}{v_1^2}}. \tag{3.42}$$

First we will do this for $Q = 1$. In this case we also have $L = 1/\sqrt{2}$, $\Lambda = -1$, and $m^2 = 2(\alpha^2 - \alpha - 1)$. Plugging in our ansatz of Eq. (3.41), we get the following algebraic equations for the $\tilde{f}_i(k,l)$:

$$\begin{aligned}
2i\left(lv_1\tilde{a}_2^x - \frac{2\pi k}{v_1}\tilde{a}_2^y\right) + 2(4\alpha^2-1)\tilde{b} - \left[\left(\frac{2\pi k}{v_1}\right)^2 + (lv_1)^2\right](\tilde{a}+\tilde{b}) &= \frac{\alpha(1-2\alpha)}{2\sqrt{\pi}} + \frac{\pi^{3/2}k^2}{v_1^2} + \frac{l^2v_1^2}{4\sqrt{\pi}}\\
2i\left(\frac{2\pi k}{v_1}\tilde{a}_2^y - lv_1\tilde{a}_2^x\right) + 2(2\alpha+1)\tilde{b} - \left[\left(\frac{2\pi k}{v_1}\right)^2 + (lv_1)^2\right](\tilde{a}-\tilde{b}) &= -\frac{\alpha}{2\sqrt{\pi}} - \frac{\pi^{3/2}k^2}{v_1^2} - \frac{l^2v_1^2}{4\sqrt{\pi}}\\
2\alpha\tilde{a}_2^y + \frac{2\pi ik}{v_1}(\alpha(\tilde{a}-\tilde{b}) - \tilde{a}) &= \frac{i\alpha\sqrt{\pi}k}{2v_1}\\
2\alpha\tilde{a}_2^x + ilv_1(\tilde{a} - \alpha(\tilde{a}-\tilde{b})) &= -\frac{il\alpha v_1}{4\sqrt{\pi}}\\
2i\left(\frac{2\pi k}{v_1}\tilde{a}_2^y - lv_1\tilde{a}_2^x\right) - 2(\alpha-1)(2\alpha-1)\tilde{a} + 2(2\alpha^2-\alpha+1)\tilde{b} &= \frac{1+\alpha-2\alpha^2}{2\sqrt{\pi}}\\
\left[2\alpha(2\alpha-1) - \left(\frac{2\pi k}{v_1}\right)^2 - (lv_1)^2\right]\tilde{a}_2^x - ilv_1\tilde{b} &= -\frac{ilv_1}{4\sqrt{\pi}}\\
\left[2\alpha(2\alpha-1) - \left(\frac{2\pi k}{v_1}\right)^2 - (lv_1)^2\right]\tilde{a}_2^y + \frac{2\pi ik}{v_1}\tilde{b} &= \frac{ik\sqrt{\pi}}{2v_1}.
\end{aligned} \tag{3.43}$$

From this we can see that we expect $\tilde{a}$ and $\tilde{b}$ to be real and $\tilde{a}_2^x$ and $\tilde{a}_2^y$ to be purely imaginary. For $k = l = 0$ we get the solution

$$\tilde{a} = \frac{\alpha(2\alpha^2 + \alpha - 4) - 1}{4(\alpha - 1)(4\alpha^2 - 1)\sqrt{\pi}}, \qquad \tilde{b} = -\frac{\alpha}{4(2\alpha + 1)\sqrt{\pi}}, \qquad \tilde{a}_2^x = \tilde{a}_2^y = 0, \tag{3.44}$$

and in all other cases the solutions are

$$\begin{aligned}
\tilde{a}(k, l) &= -\frac{\alpha v_1^2((\alpha + 1)4k^2\pi^2 + v_1^2(2 + (\alpha + 1)l^2 v_1^2 - 2\alpha(2\alpha^2 + \alpha - 4)))}{D} \\
\tilde{b}(k, l) &= -\frac{16k^4\pi^4 + (8k^2\pi^2 v_1^2 + 2l^2 v_1^6)(1 + l^2 v_1^2 + \alpha(2 - 3\alpha)) + v_1^4(-l^4 v_1^4 + 4\alpha^2(2\alpha^2 - 3\alpha + 1))}{2D} \\
\tilde{a}_2^x(k, l) &= \frac{ilv_1^3(4k^2\pi^2 + v_1^2(1 + l^2 v_1^2 + (2 - 3\alpha)\alpha))}{D} \\
\tilde{a}_2^y(k, l) &= -\frac{2\pi ikv_1(4k^2\pi^2 + v_1^2(1 + l^2 v_1^2 + (2 - 3\alpha)\alpha))}{D},
\end{aligned} \tag{3.45}$$

where

$$D = 2\sqrt{\pi}(16k^4\pi^4 + 8k^2\pi^2 v_1^2(l^2 v_1^2 + 2\alpha(1 - 2\alpha)) + v_1^4(l^4 v_1^4 + 4l^2 v_1^2\alpha(1 - 2\alpha) + 4\alpha(\alpha - 1)(4\alpha^2 - 1))). \tag{3.46}$$

Note that the Eqs. (3.43) are not solvable for $\alpha = \pm\frac{1}{2}$, 1. In the case we have chosen, where $Q = 1$, $\alpha = \frac{1}{2}(1 + \sqrt{5 + 2m^2})$. Thus, we can only solve these equations for some values of m.

E. $O(\epsilon^2)$ stress energy at the wall

At $O(\epsilon^2)$ we need to consider the gauge field equations of motion and the Einstein equations integrated across the wall. We will now consider the following action at the wall:

$$S_{\text{wall}} = \int_{r=r_0} d^3x\sqrt{-h}\{\delta m_w^2|\psi|^2 + \epsilon^2 A_\mu J_w^\mu + \epsilon^2 (T_w)_\mu{}^\mu\}, \tag{3.47}$$

where we have added a current J_w^μ that couples to the gauge field, as well as a source of stress-energy $(T_w)_{\mu\nu}$ localized at the wall. This is the most general form of action we can add to the wall and should easily lead to a solution. Because we don't want the boundary current or stress tensor to enter into the equations of motion at zeroth order, we have assumed that each term enters the action at $O(\epsilon^2)$.

First we will consider integrating the gauge field equations of motion across the wall. The relevant equations are those for A_x, A_y. The equations we must solve are

$$\int_{r_0-\epsilon}^{r_0+\epsilon} dr\sqrt{-g}g^{rr}g^{xx}(a_2^{r,x,y})'' = \int_{r_0-\epsilon}^{r_0+\epsilon} dr\sqrt{-h}h^{xx}(J_w)_{x,y}\delta(r - r_0), \tag{3.48}$$

which have the solutions

$$(J_w)_{x,y} = -\frac{4\alpha}{L}a_2^{x,y}(r_0, x, y). \tag{3.49}$$

Finally, we must consider the Einstein equations. There are three equations that include second derivatives of the fields, G_{tt}, G_{xx}, and G_{yy}. The total stress-energy from the action at the wall takes the form

$$-\frac{\sqrt{-h}h_{\mu\nu}}{2}\mathcal{L}_{\text{wall}} + \epsilon^2\sqrt{-h}(A_\mu(J_w)_\nu + (T_w)_{\mu\nu}), \tag{3.50}$$

where $\mathcal{L}_w$ is the integrand of S_{wall}. After integrating the Einstein equations, we get the following set of equations for T_w:

$$\begin{aligned}
\frac{\alpha}{r_0^3}b(r_0, x, y) &= \frac{L^5}{2r_0^3}\left(\delta m_w^2|\psi_1(r_0, x, y)|^2 + \frac{Qy}{L^2}(J_w)_x + \frac{r_0^2}{L^2}(T_w)_{tt} + \frac{1}{L^2}(T_w)_{xx} + \frac{1}{L^2}(T_w)_{yy}\right) \\
\frac{\alpha}{2r_0}(a(r_0, x, y) - b(r_0, x, y)) &= -\frac{L^5}{2r_0}\left(\delta m_w^2|\psi_1|^2 - \frac{Qy}{L^2}(J_w)_x - \frac{r_0^2}{L^2}(T_w)_{tt} - \frac{1}{L^2}(T_w)_{xx} + \frac{1}{L^2}(T_w)_{yy}\right) \\
\frac{\alpha}{2r_0}(a(r_0, x, y) - b(r_0, x, y)) &= -\frac{L^5}{2r_0}\left(\delta m_w^2|\psi_1|^2 + \frac{Qy}{L^2}(J_w)_x - \frac{r_0^2}{L^2}(T_w)_{tt} + \frac{1}{L^2}(T_w)_{xx} - \frac{1}{L^2}(T_w)_{yy}\right),
\end{aligned}$$

which have the solution

$$(T_w)_{tt} = \frac{\alpha}{r_0^2 L^3}(a(r_0,x,y) - b(r_0,x,y)) + \frac{2\alpha L}{r_0^2}|\psi_1(r_0,x,y)|.^2$$
$$(T_w)_{xx} = (T_w)_{yy} + \frac{4\alpha Q y}{L} a_2^x(r_0,x,y)$$
$$(T_w)_{yy} = -2\alpha L|\psi_1(r_0,x,y)|^2 + \frac{\alpha}{2L^3}(3b(r_0,x,y) - a(r_0,x,y)). \quad (3.51)$$

We note that, as with the original Randall-Sundrum matching [27], the wall-localized stress energy violates the null energy condition. This is not a significant concern here (as it was not there); warped solutions microscopically realizing Randall-Sundrum-like warping have been found in the full string theory, and we expect similar solutions could be found in this more involved case. It does mean that the wall should not be considered as a ''brane,'' which has Goldstone modes that allow it to fluctuate in the transverse dimensions.

F. Pictures of the modulated phase

We conclude this section with representative plots of the scalar supporting the vortex lattice $\psi_1(x,y)$ (Fig. 1), the modulation of flux density in the crystal (Fig. 2), and a representative crystalline metric function (plotted as a function of (x, y) in Fig. 3 and (r, y) in Fig. 4). All plots are for values of the parameters given by $Q = 1$, $\alpha = \frac{1}{2} + \frac{\sqrt{3}}{2}$, $v_1 = \sqrt{2\pi}$ (a square lattice). The functions have been approximated keeping 121 terms in the Fourier series (i.e., with k, l running from -5 to 5 in the formulas above).

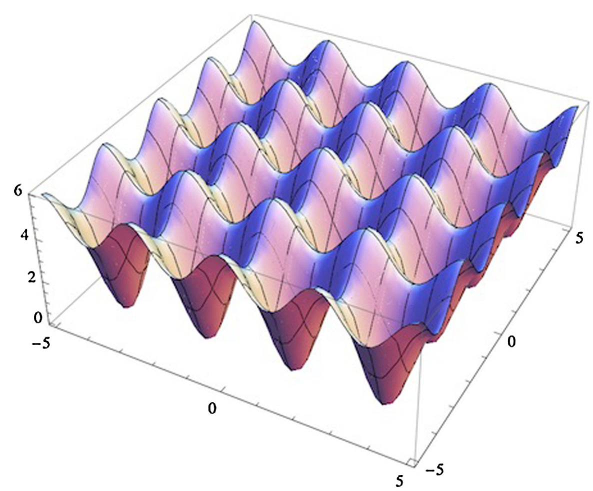

FIG. 1 (color online). The scalar vortex lattice configuration $\psi_1(x, y)$.

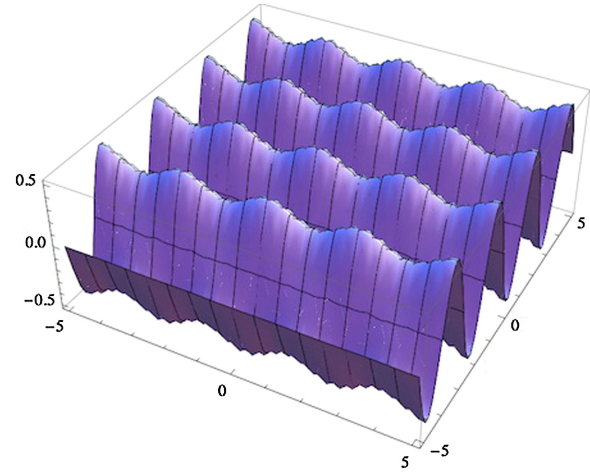

FIG. 2 (color online). The $\mathcal{O}(\epsilon^2)$ correction to the A_x gauge field, which controls the modulation of the flux density in the lattice.

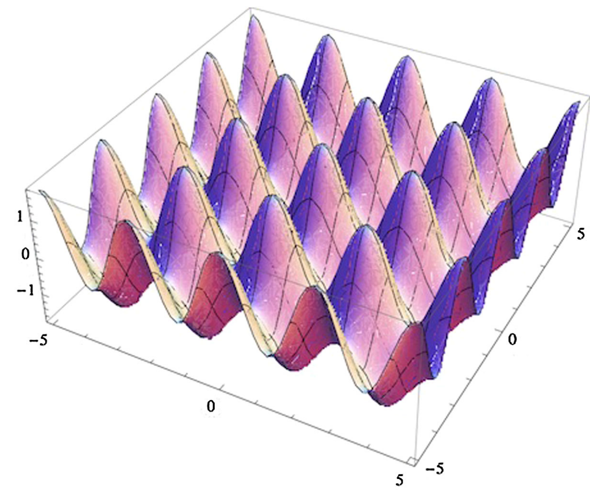

FIG. 3 (color online). The metric function $b(x, y)$ generated by the backreaction of a square vortex lattice.

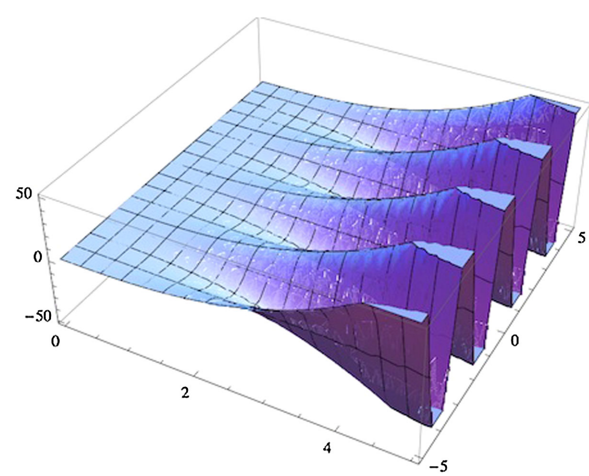

FIG. 4 (color online). The metric function in front of $dx^2 + dy^2$, now plotted as a function of y and r. We have chosen $x = 2$ for this plot.

IV. COMMENTS ON PHYSICS OF THE LATTICE MODEL

From the form of the deformed metric in Sec. III, we can infer some basic facts about the physics of the lattice solution. The IR wall geometry we have implemented is a bottom-up implementation of IR confinement [28]. In physical observables, powers of the IR radial cutoff r_0 can be replaced by powers of $1/\Lambda$, with Λ the scale of confinement. However, it is common in such solutions that also at finite temperature, one could (after the transition from confinement to deconfinement represented by a horizon at some $r < r_0$) replace powers of r_0 by $1/T$. Using this correspondence, we can infer the leading corrections to thermodynamic functions.

The free energy density $\mathcal{F}$ will receive a correction at $\mathcal{O}(\epsilon^2)$. It will have the general form

$$\mathcal{F} \sim T(1 + \epsilon^2 T^{-2\alpha} + \cdots), \tag{4.1}$$

where the leading term comes from the AdS_2 geometry (and gives rise to the notorious extensive ground-state entropy), and the subleading term is due to the physics of the vortex lattice. One can see that the $\mathcal{O}(\epsilon^2)$ corrections will scale like $\Lambda^{-2\alpha}$ in the confining geometry quite explicitly, both from the form of the wall action (3.47) and from the ϵ expansion of the contributions to the bulk action.

The schematic formula (4.1) makes it clear that for a given value of ϵ and α, there is an IR scale beneath which one should not trust perturbation theory. To avoid this region, one must keep

$$T > \epsilon^{\frac{1}{\alpha}}. \tag{4.2}$$

As α increases, the regime of trustworthiness of the linearized solution shrinks; this is in keeping with the simple intuition that the perturbation expansion in powers of ϵr^{α} will break down at smaller values of r for larger α.

Free classical defects would contribute a correction to the free energy density proportional to T (and, of course, inversely proportional to the lattice spacing). The exponent α therefore parametrizes an anomalous scaling of the free energy per vortex, characteristic of the strongly coupled field theory.

What happens beyond the regime where perturbation theory around the transition is valid? One natural speculation is that as one proceeds to the deep IR, the different lattice sites "decouple" in a manner similar to that seen in AdS_2 fragmentation [29]. It is possible that this would proceed via another phase transition (at a temperature/energy scale lower than the transition to the vortex lattice state) to a "fragmented" state. Such a fate was proposed in [24] for the D-brane lattice models of [2], where it was speculated that this might also characterize the physics of generic AdS_2 horizons. The growing localization of the dominant contribution to the low-temperature entropy on distinct lattice sites in the gravity solution provides support for this idea, in perturbation theory.

Finally, it is worth emphasizing that the lattices discussed here are quite distinct from those obtained in related literature by considering a periodic spatial variation in the chemical potential $\mu(x)$ [10]. The key difference lies in the nature of the IR behavior. In systems with a finite charge density, spatial modulations of μ can and will be cancelled by the background charge carriers—they will be screened. The hard lattices of the sort discussed here, in contrast, cannot be screened (physically, one cannot screen a magnetic field), and their effects should be expected to persist to the deep IR. In the S-dual perspective, such a feature is natural for the analog of "Wigner crystallization" of charged carries that are added to a conformal field theory.

V. CONNECTING WITH MORE GENERAL GRAVITY SOLUTIONS

Here, we describe how the lattice solutions we found in Sec. III should also arise in "IR completions" of metrics with rather general dynamical critical exponent z and hyperscaling violation parameter θ [30–33]. The basic point will be that, as in [34,35], the AdS_2 can arise in the deep IR, where corrections to the action supporting such solutions can become important.

The bigger physics picture follows. As discussed in [16], one can expect expectation values of monopole operators to serve as order parameters for translation-breaking phases in doped critical field theories. By S-duality in the four-dimensional bulk, one can map the magnetically charged field, dual to the monopole operator, to an electrically charged field. The doping maps to a background magnetic field. Then, the lattices found in Sec. III give concrete examples of the solids described in [16], in strongly coupled quantum field theory. The considerations of this section show that this can happen in models with rather general z and θ.

A. Basic EMD theory and magnetic solutions

We start with the bulk gravity theory represented by an Einstein-Maxwell-Dilaton action,

$$S = \int d^4x\sqrt{-g}(R - 2(\partial\phi)^2 - f(\phi)F_{\mu\nu}F^{\mu\nu} - V(\phi)), \tag{5.1}$$

where the gauge-coupling function is of the form

$$f(\phi) = e^{2\alpha\phi} \tag{5.2}$$

and the scalar potential takes the form

$$V(\phi) = \frac{1}{L^2}e^{-\eta\phi}. \tag{5.3}$$

This theory supports solutions of the form

$$ds^2 = L^2\left(-\bar{a}(r)^2 dt^2 + \frac{dr^2}{\bar{a}(r)^2} + \bar{b}(r)^2(dx^2 + dy^2)\right) \quad (5.4)$$

with scalar profile

$$\phi(r) = K \log(r). \quad (5.5)$$

In the simplest solutions, $\bar{a}$ and $\bar{b}$ take power-law scaling forms, and one finds

$$\bar{a}(r) = C_a r^{1-\gamma}, \qquad \bar{b}(r) = C_b r^{\beta} \quad (5.6)$$

in the vicinity of the horizon (with differences arising as one goes towards the UV, if one wishes to find asymptotically AdS solutions, as in e.g. [30]; note that we use the convention that $r \to \infty$ is the UV and $r \to 0$ is the IR in this section, in contrast with earlier sections). The two exponents γ and β, fixed by the parameters α and η in the action, capture the scaling properties of the IR fixed point induced by doping the CFT. The physically relevant dynamical critical exponent z and the hyperscaling violation exponent θ are determined in terms of γ and β by the equations

$$\beta = \frac{\theta - 2}{2(\theta - z)}, \qquad \gamma = \frac{\theta}{2(\theta - z)}. \quad (5.7)$$

General values of these exponents were first obtained in dilatonic systems in [31].

In many cases that have been studied, these solutions can in fact be supported in two different ways. If one studies an electrically charged black brane, Gauss's law yields the solution

$$F = \frac{Q_e}{f(\phi)\bar{b}(r)^2} dt \wedge dr. \quad (5.8)$$

One then finds extremal solutions where $K < 0$ and $\phi \to \infty$ near the horizon $r \sim 0$. This means that the coupling is vanishing. As discussed in [30], in the very near-horizon regime, the solution is then unreliable; in a full UV complete theory like string theory, higher derivative corrections will usually become important, because new light states appear as $g = e^{\alpha\phi} \to 0$. This difficulty can be avoided by turning on a small temperature, since this cuts off the running of the dilaton; and the near-horizon solutions for finite T are simple to write down as well.

However, in a four-dimensional bulk, one can also use bulk electric-magnetic duality to find a representation of the solution in terms of a magnetically charged black brane, i.e. a field theory immersed in a background magnetic field. This allows us to make contact with our discussion in Secs. II and III and with the picture of [16] as follows:

(i) Suppose one is interested in studying the physics of monopole operators to diagnose the phase structure of the "electric" model. One could introduce monopoles into the theory (5.1) and compute their correlators using semi-classical techniques in a multisoliton background. However, it is easier to realize that by electric/magnetic duality, one can represent the monopoles as quanta of fundamental electrically charged fields in a dual theory, where the electric background (5.8) is dualized to a background magnetic field.

(ii) As mentioned above, the running dilaton indicates an "IR incompleteness" of the solution—as the dilaton runs to extreme values, new corrections typically become important and deform the solution. For magnetically charged black branes in these dilatonic system, one possible result of the corrections is the emergence, in the deep IR, of an AdS_2 geometry. This was discussed for Lifshitz scaling metrics in [34] and for general θ and z in [35].

The end result is that in critical theories dual to dilatonic systems with fairly generic z and θ, if we are concerned mostly with the physics at very low energies, we can study monopole operators by considering the dynamics of electrically charged scalars in an AdS_2 throat supported by magnetic flux. This provides a rather general setting where our analysis in Sec. III could be relevant.

VI. DISCUSSION

There are many interesting directions for future exploration of the analytical vortex lattice solutions described here. We briefly mention some of these now.

(i) It should be possible to find analogous perturbative crystalline geometries emerging directly out of solutions with various values of z and θ, without invoking the transition to an AdS_2 spacetime [36].

(ii) It would be natural to explore replacing the IR Israel thin wall considered here, with a black brane horizon.

(iii) One would like to compute simple correlation functions in these backgrounds. For instance, quasiuniversal features have been seen in the transport properties of simple holographic lattice models in [10]. Their analogues in this system are worth exploring [36].

(iv) Most ambitiously, it would be nice to find the full nonlinear solution to the coupled set of partial differential equations that characterize the system. This would most likely rely on powerful numerical techniques. This program should yield new insights on the "fragmentation" phenomenon, and the eventual emergence of a solid in a "confined" phase of the boundary gauge theory.

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