

First order actions for new massive dual gravities

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We present a first order formulation for the fourth order action of the new massive dual gravity in four dimensions, from which the dual equivalence with the massive Curtright theory is established. This proposal is easily generalized to arbitrary dimension. Also, we obtain the dual actions for massless and massive Curtright fields in D dimensions.

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I. INTRODUCTION

Higher order derivative theories usually contain ghost excitations, which make them nonunitary [1], although other quantum properties can be improved [2]. This is the case for gravities with curvature squared terms [3]. There are exceptions. For instance, some scalar fields coupled to gravitation admit a higher order action description ($R + R^2$ theory; see [4]). Also, the action for “Galileons” [5] contains higher derivatives; however, their field equations are nonlinear second order differential equations. This is analogous to the Lovelock gravity [6], whose second order field equations arise from an action that is higher order in curvature. In three dimensions, it is possible to have higher order gravity theories without loss of unitarity. The nondynamical Einstein-Hilbert action can be augmented with a third order derivative Lorentz-Chern-Simons term, the topologically massive gravity theory [7], which describes the local propagation of one, ghost-free and parity sensitive excitation: a massive graviton. Furthermore, the dubbed new massive gravity in three dimensions was formulated by Bergshoeff, Hohm, and Townsend [8] four years ago. It consists, again, in the nondynamical Einstein-Hilbert action supplemented with a specific curvature squared term, which leads to a fourth order field equation. At the linearized level, the new massive gravity is equivalent to the standard massive spin 2 Fierz-Pauli theory, which propagates 2 degrees of freedom with the same mass and opposite helicities ± 2 . In both cases, the Einstein-Hilbert action has a wrong sign, which is essential in order to have free ghost excitations. The new massive gravity has a discontinuity in its degrees of freedom when the massless limit is considered. This limit leads to the fourth order Schouten gravity, which describes a massless, conformal invariant, ghost-free local excitation [9]. Also, the general massive gravity [10], the most general gravity theory in three dimensions, propagates two massive gravitons of helicities ± 2 but with different masses. Its gauge invariant linearized action was shown to be dual equivalent [11] to the massive spin 2 Fierz-Pauli

with a term that breaks explicitly the local Lorentz symmetry [12].

Remarkably enough, it is the observation made in [13,14] that solving on shell algebraic and differential constraints, it is possible to obtain in three dimensions higher order field equations for higher spin. This procedure was extended to four dimensions, but considering the massive Curtright field [15], which is the dual field to the massive spin 2 Fierz-Pauli theory in four dimensions [16,17]. In consequence, a fourth order derivative action for the massive Curtright field is obtained. A necessary condition is that the field, which solves the constraints, belongs to the same Lorentz representation as the original field. This theory was called new massive dual gravity [18] and would be a unitary theory, by construction. The extension to arbitrary dimension has been achieved recently [19,20].

The aim of this work is to present a first order formulation for new massive dual gravities. The plan of this work is the following: in the next section, we introduce a first order formulation for a massless, mixed symmetry $\Phi_{mn,p}$ field, valid for arbitrary dimensions. From this action, we obtain the corresponding dual theory. In Sec. III, we perform the dimensional reduction of the action presented in Sec. II, keeping only first massive modes, to obtain the first order formulation for the massive $\Phi_{mn,p}$ field. In Sec. IV, we propose the first order action for the new massive dual gravity in four dimensions. Also, a second order action is established. From these actions the dual equivalence between the massive Curtright theory and the new massive dual gravity is easily achieved. These lower order derivative actions must be appropriate in order to carry out the quantization and check out the unitarity of the fourth order derivative new massive dual gravity. The generalization to any dimension D will be straightforward. Throughout this work we use η_{mn} mostly positive, and brackets indicate antisymmetrization without any normalization factor, e.g. $V_{[mnp]} = V_{mnp} + V_{npm} + V_{pnm}$.

II. THE CURTRIGHT FIELD

For general mixed symmetry fields $\Phi_{M_1 M_2 \dots M_i N}$, with second order actions, Zinoviev has achieved first order

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actions [21]. For our purpose, we will start with the following first order action for the massless field $\Phi^{MN,P} = -\Phi^{NM,P}$ in D dimensions:

$$I = \int d^D x \left(\frac{3}{4} Y_{MNP,Q} Y^{MNQ,P} - \frac{3}{4} \frac{1}{(D-3)} Y^{MN} Y_{MN} - \frac{1}{2} Y^{MNP,Q} F_{MNP,Q} \right), \quad (1)$$

where $Y^{MNP,Q}$ is an auxiliary field ($Y^{MN} = Y^{MNP,P}$) and $F_{MNP,Q}$ is the field strength of the field $\Phi^{MN,P}$

$$F_{MNP,Q} = \partial_M \Phi_{NP,Q} + \partial_N \Phi_{PM,Q} + \partial_P \Phi_{MN,Q}. \quad (2)$$

This action differs from the Zinoviev action by a change of variables:

$$Y_{MNP,Q} = \Omega_{Q,MNP} - \eta_{[QM} \Omega_{NP]}. \quad (3)$$

We have invariance under the following gauge transformations:

$$\delta \Phi_{MN,P} = \partial_M z_{NP} - \partial_N z_{MP}, \quad (4)$$

being the gauge parameters z_{MN} an arbitrary general second order tensor. Besides, we have invariance under local ‘‘Lorentz’’ transformations

$$\delta \Phi_{MN,P} = \Lambda_{MNP} \quad (5)$$

and

$$\delta Y_{MNP,Q} = \frac{1}{3} \partial_Q \Lambda_{[MNP]} - \partial_R \eta_{Q[M} \Lambda_{NP]R}, \quad (6)$$

where the parameters Λ_{MNP} are completely antisymmetric. We decompose the $\Phi^{MN,P}$ field as

$$\Phi^{MN,P} = T^{MN,P} + C^{MN,P}, \quad (7)$$

where $T^{MN,P} = -T^{NM,P}$ is the Curtright field, which satisfies the cyclic identity $T_{[MN,P]} \equiv 0$ and C_{MNP} is completely antisymmetric. The Lorentz symmetry indicates that the C_{MNP} is a nondynamical field and the action will be written only in terms of the Curtright field. Let us see this. The equation of motion obtained after independent variations of the auxiliary field $Y^{MNP,Q}$ is

$$Y_{Q[MN,P]} - \frac{1}{D-3} \eta_{Q[M} Y_{NP]} = F_{MNP,Q}. \quad (8)$$

This equation can be solved for $Y_{MNP,Q}$ as

$$Y_{MNP,Q} = \frac{2}{3} F_{MNP,Q} + \frac{1}{3} F_{Q[MN,P]} - \eta_{Q[M} F_{NP]}. \quad (9)$$

Substituting back into (1) we obtain the second order action

$$I = \int d^D x \left[-\frac{1}{6} F_{MNP,Q} F^{MNP,Q} - \frac{1}{4} F_{MNP,Q} F^{MNQ,P} + \frac{3}{4} F^{MN} F_{MN} \right], \quad (10)$$

where the field strength is now expressed only in terms of the Curtright field $T_{MN,P}$. Equivalently, after using the property of cyclic identity satisfied by $T_{MN,P}$ and omitting divergences, this last action can be rewritten as originally proposed by Curtright:

$$I = \frac{3}{2} \int d^D x \left[-\frac{1}{6} F_{MNP,Q} F^{MNP,Q} + \frac{1}{2} F^{MN} F_{MN} \right]. \quad (11)$$

On the other hand, we can consider the field equation obtained making independent variations on $\Phi_{MN,P}$

$$\partial_Q Y^{QMN,P} = 0. \quad (12)$$

Locally this equation can be solved in terms of a $\Phi_{S_1 \dots S_{D-4}, Q}$ field as

$$Y_{PMN,Q} = \frac{1}{2} \epsilon^{PMNRS_1 \dots S_{D-4}} \partial_R \Phi_{S_1 \dots S_{D-4}, Q} \equiv \frac{1}{2(D-3)} \epsilon^{PMNS_1 \dots S_{D-3}} F_{S_1 \dots S_{D-3}, Q}. \quad (13)$$

We have defined $F_{S_1 \dots S_{D-3}, Q} \equiv \partial_{S_1} \Phi_{S_2 \dots S_{D-3}, Q} +$ cyclic permutation. In five dimensions, we have $Y_{PMN,Q} \sim \epsilon^{PMNRS} \partial_R \Phi_{S,Q}$, and plugging into (1), the linearized Einstein action is obtained, illustrating the well known duality relation between Curtright and the massless spin 2 fields in five dimensions [22]. In six dimensions the Curtright field is self-dual, and in general D dimensions, we have the following duality relationship:

$$T_{MN,P} \Leftrightarrow \Phi_{M_1 \dots M_{D-4}, N}. \quad (14)$$

The dual action is

$$I = \int d^D x \left[-\frac{(D-4)}{2(D-3)} F^{M_1 \dots M_{D-3}, N} F_{M_1 \dots M_{D-3}, N} - \frac{1}{2} F_{M_1 \dots M_{D-4}, N, P} F^{M_1 \dots M_{D-4}, P, N} + \frac{1}{2(D-3)} F_{M_1 \dots M_{D-4}, N, N} F^{M_1 \dots M_{D-4}, P, P} \right]. \quad (15)$$

III. DIMENSIONAL REDUCTION

In this section we perform a dimensional reduction to the action (1) from D to $D-1$ dimensions in order to provide mass to the Curtright field. We will keep only the first massive modes in a similar way as was accomplished for spin 2 in [23]. For this goal, we make the following definitions:

$$Y_{mnp, q(x,y)} = \sqrt{\frac{\mu}{\pi}} Y_{mnp, q(x)} \cos \mu y, \quad (16)$$

$$Y_{mny, q(x,y)} = \sqrt{\frac{\mu}{\pi}} Y_{mn, p(x)} \sin \mu y, \quad (17)$$

$$Y_{mnp, y(x,y)} = \sqrt{\frac{\mu}{\pi}} X_{mnp(x)} \sin \mu y, \quad (18)$$

$$Y_{mny,y(x,y)} = \sqrt{\frac{\mu}{\pi}} Z_{mn(x)} \cos \mu y \quad (19)$$

and

$$\Phi_{mn,p(x,y)} = \sqrt{\frac{\mu}{\pi}} \Phi_{mn,p(x)} \cos \mu y, \quad (20)$$

$$\Phi_{my,n(x,y)} = \sqrt{\frac{\mu}{\pi}} \Phi_{mn(x)} \sin \mu y, \quad (21)$$

$$\Phi_{mn,y(x,y)} = \sqrt{\frac{\mu}{\pi}} B_{mn(x)} \sin \mu y, \quad (22)$$

$$\Phi_{my,y(x,y)} = \sqrt{\frac{\mu}{\pi}} a_{m(x)} \cos \mu y. \quad (23)$$

The dependence with the extra compact dimension is denoted by y , and low letters: m, n, p, \dots indicate coordinates in the reduced spacetime. The fields $X_{mnp}, Z_{mn}, \Phi_{mn}$, and B_{mn} are completely antisymmetric, a_m is a vector, and $Y_{mnp,q}$ and $Y_{mn,p}$ will play the role of auxiliary fields. Substituting these definitions and performing the integration on the compact coordinate, we obtain the dimensionally reduced action y . The result is

$$I = \int d^{D-1}x \left[\frac{3}{4} Y_{mnp,q} Y^{mnq,p} - \frac{3}{4(D-3)} Y_{mnp,p} Y^{mnq,q} - \frac{3}{2} Y^{mnp,q} \partial_m \Phi_{np,q} - \frac{3}{2(D-3)} Y^{mnp,p} Z_{mn} + \frac{3}{4} \frac{D-4}{D-3} Z^{mn} Z_{mn} \right. \\ \left. - \frac{3}{2} Z^{mn} (\partial_m a_n - \partial_n a_m + \mu B_{mn}) - \frac{3}{2} Y^{mn,p} (\partial_m \Phi_{np} - \partial_n \Phi_{mp} - \mu \Phi_{mn,p}) - \frac{1}{2} X^{mnp} (\partial_m B_{np} + \partial_n B_{pm} + \partial_p B_{mn}) \right] \\ + \frac{3}{2} Y^{mn,p} Y_{mp,n} - \frac{3}{2(D-3)} Y^{mp,p} Y_{mq,q} + \frac{3}{2} X^{mnp} Y_{mp,n}. \quad (24)$$

This reduced action is invariant under the following gauge transformations:

$$\delta \Phi_{mn,p(x)} = \partial_m z_{np(x)} - \partial_n z_{mp(x)}, \quad (25)$$

$$\delta B_{mn} = \partial_m z_{ny} - \partial_n z_{my}, \quad (26)$$

$$\delta \Phi_{mn} = \partial_m z_{yn} + \mu z_{mn} \quad (27)$$

and

$$\delta a_m = \partial_m z_{yy} - \mu z_{my}, \quad (28)$$

where we split up the gauge parameters as z_{MN} : $(z_{mn}, z_{my}, z_{ym}, z_{yy})$.

We can break these gauge symmetries choosing appropriately the z_{mn} and z_{my} gauge parameters to fix the gauges

$$\Phi_{mn} = 0 \quad \text{and} \quad a_m = 0. \quad (29)$$

The Φ_{mn} and a_m are Stueckelberg fields. Furthermore, the reduced action inherited Lorentz symmetries with parameters Λ_{MNP} : $(\Lambda_{mnp}, \Lambda_{mny} \equiv \Lambda_{mn})$, expressed in the following transformations:

$$\delta \Phi_{mn,p} = \Lambda_{mnp}, \quad (30)$$

$$\delta \Phi_{mn} = -\Lambda_{mn} = -\delta B_{mn}, \quad (31)$$

$$\delta a_m = 0, \quad (32)$$

$$\delta Y_{mnp,q} = \partial_q \Lambda_{mnp} - \partial_r \eta_{q[m} \Lambda_{n]p} - \mu \eta_{q[m} \Lambda_{n]p}, \quad (33)$$

$$\delta Y_{mn,p} = \partial_p \Lambda_{mn} + \eta_{p[m} \partial_q \Lambda_{n]q}, \quad (34)$$

$$\delta X_{mnp} = -\mu \Lambda_{mnp} \quad (35)$$

and

$$\delta Z_{mn} = -\partial_p \Lambda_{pmn}. \quad (36)$$

Similarly, we can break the Lorentz symmetry associated with the Λ_{mn} parameter by choosing

$$B_{mn} = 0. \quad (37)$$

At this stage, we can eliminate the Z_{mn} field since it appears as a quadratic multiplier. Its equation of motion leads to determining its value,

$$Z_{mn} = \frac{1}{D-4} Y_{mnp,p}. \quad (38)$$

With these gauges fixing and substituting this value of Z_{mn} , the reduced action boils down to

$$I = \int d^D x \left[\frac{3}{4} Y_{mnp,q} Y^{mnq,p} - \frac{3}{4(D-3)} Y_{mnp,p} Y^{mnq,q} \right. \\ \left. - \frac{3}{2} Y^{mnp,q} \partial_m \Phi_{np,q} + \frac{3}{2} X^{mnp} Y_{np,m} + \frac{3}{2} Y^{mn,p} Y_{mp,n} \right. \\ \left. - \frac{3}{2(D-2)} Y^{mp,p} Y_{mq,q} + \frac{3\mu}{2} Y^{mn,p} \Phi_{mn,p} \right]. \quad (39)$$

The local Lorentz symmetry associated with the parameter Λ_{mnp} can be used to fix gauge in two different ways. The first alternative consists in choosing $X^{mnp} = 0$, and the other possibility allows us to eliminate the antisymmetric part of $\Phi_{mn,p}$ (i.e. $\Phi_{[mn,p]} \equiv 0$). Both choices will lead to the same result: the first order formulation for the massive Curtright action. Fixing the gauge $X^{mnp} = 0$, the field equation for $Y^{mn,p}$ is

$$Y_{mp,n} - Y_{np,m} - \frac{1}{D-2}(\eta_{pn}Y_m - \eta_{pm}Y_n) = -\mu\Phi_{mn,p}, \quad (40)$$

which can be solved as

$$Y_{mn,p} = -\frac{\mu}{2}[\Phi_{mn,p} + \Phi_{mp,n} - \Phi_{np,m}] - \mu[\eta_{mp}\Phi_n - \eta_{np}\Phi_m], \quad (41)$$

and when this expression is plugged into (39), the result is the following nongauge invariant action:

$$I = \int d^Dx \left[\frac{3}{4}Y_{mn,p,q}Y^{mnq,p} - \frac{3}{4(D-3)}Y_{mn,p}Y^{mnq,q} - \frac{3}{2}Y^{mn,p,q}\partial_m\Phi_{np,q} - \frac{3\mu^2}{8}(\Phi^{mn,p}\Phi_{mp,n} + 2\Phi^{mn,p}\Phi_{mp,n} - 4\Phi^m\Phi_m) \right]. \quad (42)$$

The first line in this dimensional reduced action (42) is the first order action for the massless Curtright field. Now, if we decompose the $\Phi_{mn,p}$ in its irreducible parts: $\Phi_{mn,p} = T_{mn,p} + C_{mn,p}$, with $T_{[mn,p]} \equiv 0$ and C_{mnp} completely anti-symmetric, this part does not depend on C_{mnp} , while the coefficients of the massive term, the second line in action (42), are such that the constituents of $\Phi_{mn,p}$ are decoupled. Indeed

$$\Phi^{mn,p}\Phi_{mn,p} + 2\Phi^{mn,p}\Phi_{mp,n} - 4\Phi^m\Phi_m = 2T_{mn,p}T^{mn,p} - 4T_mT^m - C_{mnp}C^{mnp}. \quad (43)$$

In consequence, C_{mnp} is a nondynamical field, and then by eliminating $Y_{mn,p,q}$ through its equation of motion, we obtain the second order action for the massive Curtright field,

$$I = \frac{3}{2} \int d^Dx \left[-\frac{1}{6}F_{mn,p,q}F^{mnq,p} + \frac{1}{2}F^{mn}F_{mn} - \frac{\mu^2}{2}(T_{mn,p}T^{mn,p} - 2T_mT^m) \right]. \quad (44)$$

The number of degrees of freedom of this theory is given by $\frac{1}{3}(D-1)(D+1)(D-3)$, while in the massless case the Curtright field propagates $\frac{1}{3}D(D-2)(D-4)$ transverse modes. The massless Curtright field in $D+1$ dimensions has the same number of degrees of freedom as the massive Curtright field in D dimensions.

If we have chosen the gauge $C_{mnp} = 0$ ($\Phi_{mn,p} = T_{mn,p}$), then X^{mnp} is a multiplier, which tells us that $Y_{[mn,p]} = 0$. After eliminating $Y_{mn,p}$ through the use of its equation of motion, we again reach the action for the massive Curtright field. From now on, we will refer only to the Curtright $T_{mn,p}$.

From the first order action (42), we can obtain the dual theory for the massive Curtright field if we eliminate the

$\Phi^{mn,p}$ field through its equation of motion. In fact, we obtain

$$\Phi^{mn,p} = -\frac{1}{\mu^2}\partial_q(Y^{qpm,n} - Y^{qpn,m}) + \frac{1}{\mu^2(D-2)}\partial_q(\eta^{pm}Y^{qn} - \eta^{pn}Y^{qm}), \quad (45)$$

and substituting into (42), the following action for $Y^{mnp,q}$ is obtained:

$$\frac{2}{3}\mu^2I = \int d^Dx \left[\partial_q Y^{qmn,p} \partial^r Y_{rmp,n} - \frac{1}{D-2} \partial_q Y^{qm} \partial_r Y^{rn} + \frac{\mu^2}{2} \left(Y^{mnp,q} Y_{mnq,p} - \frac{1}{D-3} Y^{mn} Y_{mn} \right) \right]. \quad (46)$$

Now, if we decompose $Y^{mnp,q}$ in its trace and traceless parts

$$Y^{mnp,q} = W^{mnp,q} + \frac{1}{D-2}(\eta^{qm}Y^{np} + \eta^{qn}Y^{pm} + \eta^{qp}Y^{mn}), \quad (47)$$

where $W^{mnp,q}$ is the traceless part of $Y^{mnp,q}$ ($W_p^{mnp} = 0$), the action is written as

$$\frac{2}{3}\mu^2I = \int d^Dx \left[\partial_q W^{qmn,p} \partial^r W_{rmp,n} + \frac{\mu^2}{2} \left(W^{mnp,q} W_{mnq,p} - \frac{1}{(D-2)(D-3)} Y^{mn} Y_{mn} \right) \right], \quad (48)$$

which clearly shows that the trace of $Y^{mnp,q}$ is a non-dynamical variable. Next, we introduce the dual of $W^{mnp,q}$,

$$W^{mnp,q} = \frac{1}{(D-3)} \epsilon^{mnp r_1 \dots r_{D-3}} T_{r_1 \dots r_{D-3} q}. \quad (49)$$

Since $W^{mnp,q}$ is traceless, then $T_{r_1 \dots r_{D-3}, q}$ satisfy the cyclic identity: $T_{[r_1 \dots r_{D-3}, q]} \equiv 0$ and introducing the field strength $F_{r_1 r_2 \dots r_{D-2}, q} \equiv \partial_{r_1} T_{r_2 \dots r_{D-3}, q} +$ cyclic permutations, the dual action for the massive Curtright field is

$$I = \int d^Dx \left[-\frac{1}{(D-2)} F^{r_1 \dots r_{D-3} p, q} F_{r_1 \dots r_{D-3} p, q} - F_{r_1 \dots r_{D-3} p, p} F^{r_1 \dots r_{D-4} q, q} - \mu^2 (T_{r_1 \dots r_{D-3}, p} T^{r_1 \dots r_{D-3}, p} - (D-3) T_{r_1 \dots r_{D-4} p, p} T^{r_1 \dots r_{D-4} p, p}) \right]. \quad (50)$$

Thus, we have established the duality relationship between the massive Curtright field ($T_{mn,p}$) and a massive mixed symmetry field $T_{m_1 \dots m_{D-3}, p}$.

IV. THE FIRST ORDER ACTION

In four dimensions the massless Curtright does not propagate any local excitations like the massless spin 2

in three dimensions. Generally, a massless $\Phi_{m_1, m_2, m_{D-2}, n}$ field in D dimensions does not have any degrees of freedom. In four dimensions, Zinoviev has written down a first order action for the nondynamical Curtright field, akin to the well-known first order action of linearized Einstein action in three dimensions. This action is expressed in terms of the $T_{mn,p}$ field and an auxiliary field h_{mn} neither symmetric nor antisymmetric. This action is written out as

$$I_C = \int d^4x \left[-\frac{1}{4}(h_{mn}h^{nm} - h^2) + \frac{1}{2}\epsilon^{mnpq}h_{mr}\partial_n T_{pq,r} \right]. \quad (51)$$

In fact, the field equation obtained by varying the auxiliary field h^{nm} tells us

$$h^{nm} = \epsilon^{mpqr}\partial_p T_{qr,n}. \quad (52)$$

Note that $h = 0$ on shell, but we will keep the h^2 term in the action because it will be necessary to have the dual off-shell equivalence between massive spin 2 and Curtright fields in four dimensions. When this value of h_{nm} is substituted into I_C , the second order Curtright action is obtained

$$I_C = \frac{1}{2} \int d^4x T_{mn,p} G^{mn,p}, \quad (53)$$

where

$$G^{mn,p} \equiv \frac{1}{2}\epsilon^{mnsr}\epsilon^{pquv}\partial_r\partial_q T_{uv,s} \quad (54)$$

is the ‘‘generalized Einstein tensor’’ for the Curtright field introduced in Ref. [18].

Now, we are ready to propose the first order action in four dimensions for the fourth order ‘‘new massive gravity.’’ This action involves three independent variables: the Curtright field $T_{mn,p}$, a h_{mn} field and an auxiliary field $Y_{mn,p}$. Both $T_{mn,p}$ and $Y_{mn,p}$ satisfy cyclic identities. The action is written as

$$I = \int d^4x \left[-\frac{1}{4}(h_{mn}h^{nm} - h^2) + \frac{1}{2}\epsilon^{mnpq}h_{mr}\partial_n T_{pq,r} + Y^{mn,p}Y_{mp,n} - \frac{1}{2}Y^m Y_m + \mu Y^{mn,p}T_{mn,p} \right]. \quad (55)$$

Making independent variations on the fields, we obtain

$$\frac{\delta I}{\delta h_{nm}} = 0 \Rightarrow h^{mn} = \epsilon^{npqr}\partial_p T_{qr,m}, \quad (56)$$

$$\begin{aligned} \frac{\delta I}{\delta Y_{mn,p}} &= 0 \\ \Rightarrow Y_{mn,p} &= -\frac{\mu}{2}[T_{mn,p} + T_{mp,n} - T_{np,m}] \\ &\quad - \mu[\eta_{mp}T_n - \eta_{np}T_m] \end{aligned} \quad (57)$$

and

$$\frac{\delta I}{\delta T_{mn,p}} = 0 \Rightarrow Y^{mn,p} = -\frac{1}{2\mu}\epsilon^{mnpq}\partial_q h_{rp}. \quad (58)$$

Equations (56) and (58) are first order in derivatives. As usual, the auxiliary fields (h_{mn} and $Y_{mn,p}$) can be substituted, using Eqs. (56) and (57), in order to reach the second order action (44) for the massive Curtright field $T_{mn,p}$.

Alternatively, we can express the $Y_{mn,p}$ field [Eq. (58)] in terms of the second derivative of the Curtright field $T_{mn,p}$ using Eq. (56),

$$Y_{mn,p} = -\frac{1}{\mu}G_{mn,p[T]}. \quad (59)$$

Since $Y_{mn,p}$ satisfies a cyclic identity, we can write

$$Y^{mn,p}Y_{mp,n} - \frac{1}{2}Y^m Y_m = \frac{1}{2}Y^{mn,p}Y_{mn,p} - \frac{1}{2}Y^m Y_m. \quad (60)$$

Plugging into the first order action (55), the values of h_{mn} and $Y_{mn,p}$ given by Eqs. (56) and (59), we obtain the fourth order action of the new massive gravity in four dimensions,

$$I = \frac{1}{2} \int d^4x \left[-T_{mn,p}G^{mn,p} + \frac{1}{\mu^2}G^{mn,p}S_{mn,p} \right], \quad (61)$$

where

$$S_{mn,p} \equiv G_{mn,p} - \frac{1}{2}(\eta_{np}G_m - \eta_{mp}G_n) \quad (62)$$

is a generalized ‘‘Schouten’’ tensor. Note the wrong sign of the kinematical Curtright term; it has been flipped over. In this way we have implemented a first order parent action that establishes the equivalence between the second order massive Curtright theory and the higher order action of [18].

Alternatively, we can arrive to the fourth order action from a second order action. After integrating out the h_{mn} auxiliary field, we introduce a new $v_{mn,p}$ auxiliary field, in order to rewrite the second order action for the massive Curtright field in the following way:

$$\begin{aligned} I &= \int d^4x \left[-\frac{1}{2}v_{mn,p}\epsilon^{mnsr}\epsilon^{pquv}\partial_r\partial_q v_{uv,s} \right. \\ &\quad + T_{mn,p}\epsilon^{mnsr}\epsilon^{pquv}\partial_r\partial_q v_{uv,s} + Y^{mn,p}Y_{mp,n} - \frac{1}{2}Y^m Y_m \\ &\quad \left. + \mu Y^{mn,p}T_{mn,p} \right]. \end{aligned} \quad (63)$$

Now, the $T_{mn,p}$ field is a linear multiplier, whose constraint is solved by $Y_{mn,p} = -\frac{1}{\mu}G_{mn,p[v]}$, which lead to the fourth order action for $v_{mn,p}$.

Moreover, we can establish the dual equivalence between massive Curtright field and the massive spin 2 field, in four dimensions, from our parent first order action. Indeed, using only (58), we obtain

$$I = \int d^4x \left[-\frac{1}{16} \left(C_{mn,p} C^{mn,p} - \frac{1}{16} C_{mn,p} C^{mp,n} + \frac{1}{4} C_m C^m \right) - \frac{1}{4} (h_{mn} h^{nm} - h^2) \right], \quad (64)$$

where $C_{mn,p} \equiv \partial_m h_{np} - \partial_n h_{mp}$ and $C_m = C_{mn,n}$. This is just the massive second order spin 2 Fierz-Pauli action.

The generalization to D dimensions is

$$I = \int d^Dx \left[-\frac{1}{4} (h_{mn} h^{nm} - h^2) + \frac{1}{2} \epsilon^{m_1 \dots m_{D-2}} \omega_{m_r} \partial_n T_{s_1 \dots s_{D-2}, r} + Y^{m_1 \dots m_{D-3}, p} Y_{m_1 \dots m_{D-3}, p} - \frac{1}{D-2} Y^{m_1 \dots m_{D-3}} Y_{m_1 \dots m_{D-3}} + \mu Y^{m_1 \dots m_{D-2}, p} T_{m_1 \dots m_{D-2}, p} \right], \quad (65)$$

where $T_{m_1, m_2, \dots, m_{D-2}, n}$ is the dual massive field of the massive h_{mn} field [17]. $Y_{m_1, m_2, \dots, m_{D-2}, n}$ is an auxiliary field ($Y_{m_1, m_2, \dots, m_{D-3}} = Y_{m_1, m_2, \dots, m_{D-3}, n}$). Both $T_{m_1, m_2, m_{D-2}, n}$ and $Y_{m_1, m_2, m_{D-2}, n}$ satisfy cyclic identities. Now, we have the following equations:

$$h^{mn} = \epsilon^{n p r_1 \dots r_{D-2}} \partial_p T_{r_1 \dots r_{D-2}, m} \quad (66)$$

and

$$Y^{m_1 \dots m_{D-2}, n} = -\frac{1}{\mu} G^{m_1 \dots m_{D-2}, n}, \quad (67)$$

where

$$G^{m_1 \dots m_{D-2}, n} = \frac{1}{2} \epsilon^{m_1 \dots m_{D-2} q p} \epsilon^{n s r_1 \dots r_{D-2}} \partial_q \partial_s T_{r_1 \dots r_{D-2}, p} \quad (68)$$

is the generalized Einstein tensor. Plugging (66) and (67) into (65), the fourth order action [19,20] is obtained,

$$I = \int d^Dx \left[\frac{1}{2} T_{m_1 \dots m_{D-2}, n} G^{m_1 \dots m_{D-2}, n} + \frac{1}{\mu^2} G^{m_1 \dots m_{D-2}, n} S_{m_1 \dots m_{D-2}, n} \right], \quad (69)$$

where

$$S_{m_1 \dots m_{D-2}, n} = G_{m_1 \dots m_{D-2}, n} - \frac{1}{D-2} \eta_{n[m_{D-2}} G_{m \dots m_{D-3}], p, p} \quad (70)$$

is the generalized Schouten tensor. Also, the first order action (65) permits one to show the dual equivalence

$$h_{mn} \Leftrightarrow T_{m_1 \dots m_{D-2}, n} \quad (71)$$

which was established in [17] for the massive linearized gravitation in arbitrary dimensions.

Finally, we consider the case for three dimensions. We must expect to recover the fourth order new massive gravity [8]. The first order action to consider is

$$I = \int d^3x \left[-\frac{1}{2} (\omega_{mn} \omega^{nm} - \omega^2) + \epsilon^{mnpq} \omega_{m_r} \partial_n e_{pq} + \frac{1}{2} Y^{mn} Y_{nm} - \frac{1}{4} Y^2 + \mu Y^{mn} e_{mn} \right]. \quad (72)$$

We have denoted the auxiliary fields by ω_{mn} (for convenience) and Y_{nm} , which are general second order tensor. The coefficients have been slightly changed in concordance with the result of Ref. [23]. The following equations of motion are obtained:

$$\frac{\delta I}{\delta \omega_{mn}} = 0 \Rightarrow \omega_{mn} - \eta_{nm} \omega + \epsilon_n^{pq} \partial_p e_{qn} = 0, \quad (73)$$

$$\frac{\delta I}{\delta Y_{mn}} = 0 \Rightarrow Y_{mn} - \frac{1}{2} \eta_{mn} Y + \mu e = 0 \quad (74)$$

and

$$\frac{\delta I}{\delta e_{mn}} = 0 \Rightarrow Y_{mn, p} = -\frac{1}{\mu} \epsilon_m^{pq} \partial_p \omega_{qn}. \quad (75)$$

Equation (73) is solved as

$$\omega_{mn} = \epsilon_n^{pq} \partial_p e_{qm} - \frac{1}{2} \eta_{mn} \epsilon^{pqr} \partial_p e_{qr} \equiv W_{mn[e]}, \quad (76)$$

while Eq. (74) determines Y_{mn} ,

$$Y_{mn} = -\mu (e_{mn} - \eta_{mn} e). \quad (77)$$

Inserting Eqs. (76) and (77) into Eq. (72), the following action is obtained:

$$I = \int d^3x \left[\frac{1}{2} e_{mq} \epsilon^{mnp} \partial_n W_{pq[e]} - \frac{\mu^2}{2} (e_{mn} e^{nm} - e^2) \right], \quad (78)$$

and after introducing the symmetric and antisymmetric parts of e_{mn} ($e_{mn} = h_{mn} + \epsilon_{mnp} v^q$), we recognize (78) as the action for the massive spin 2. The first term is the linearized Einstein action ($e_{mq} \epsilon^{mnp} \partial_n W_{pq[e]} = h_{mn} G^{mn}$, $G^{mn} = R_{mn} - \frac{1}{2} \eta_{mn} R$ is the linearized Einstein tensor) expressed only in terms of the symmetric part h_{mn} of e_{mn} . The antisymmetric component v^q appears decoupled in the massive term. Alternatively, as in the formulation for the Curtright field, we can express Y_{mn} as

$$Y_{mn} = -\frac{1}{\mu} \epsilon_m^{pq} \partial_p W_{qn} = -\frac{1}{\mu} G_{mn[h]}, \quad (79)$$

and substituting (76) and (79) into the action, the new massive gravity is reached,

$$I_{BHT} = \frac{1}{2} \int d^3x \left[-h_{mn} G^{mn} + \frac{1}{\mu^2} \left(R_{mn} R^{mn} - \frac{3}{8} R^2 \right) \right]. \quad (80)$$

Thus, we have implemented a first order formulation for the fourth order new massive gravity in three dimensions. On the other hand, the action expressed only in terms of

ω_{mn} is the same form as for the e_{mn} . In three dimensions, the massive spin 2 field is self-dual.

V. CONCLUSIONS

In this paper we have established a first order action for the fourth order action of the new massive gravity in four dimensions. To achieve this goal, two auxiliary fields were introduced, besides the Curtright field. One of these auxiliary fields comes from the dimensional reduction of the first order action for the Curtright field, while the other auxiliary field appears when we consider a first order kinematical term for a general mixed symmetry $T_{s_1 \dots s_{D-2}, r}$ [16],

$$I = \int d^D x \left[-\frac{1}{4} (h_{mn} h^{nm} - h^2) + \frac{1}{2} \epsilon^{mns_1 \dots s_{D-2}} \omega_{mr} \partial_n T_{s_1 \dots s_{D-2}, r} \right]. \quad (81)$$

This action is “topological” in the sense that it does not propagate any local degrees of freedom. Likewise,

a second order action was achieved. The extension to arbitrary dimensions was straightforward formulated, including $D = 3$ new massive gravity. Furthermore, we have established the dual actions for massless and massive Curtright field in any dimensions,

$$\begin{aligned} T_{mn,p} &\Leftrightarrow T_{m_1 \dots m_{D-4}, p} (\mu = 0) \\ T_{mn,p} &\Leftrightarrow T_{m_1 \dots m_{D-3}, p} (\mu \neq 0). \end{aligned} \quad (82)$$

With these lower derivative actions, the canonical analysis must be straightforward with the goal of confirming the absence of ghost. This issue is the proposal of a future work.

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