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Gauge invariant bound state equations for quark-antiquark systems in QCD

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Using gauge invariant quark Green's functions, defined with path-ordered gluon field phase factors along polygonal lines, and functional relations among them, two compatible bound state equations of the Dirac type are established for quark-antiquark systems, each relative to the quark or to the antiquark of the system. The kernels of the bound state equations are defined through a series of Wilson loop averages along closed polygonal contours and their functional derivatives on them. A sufficient criterion for spontaneous chiral symmetry breaking is derived, relating the Goldstone boson wave function in the zero total momentum limit with the scalar part of the gauge invariant quark two-point Green's function.

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I. INTRODUCTION

Gauge invariant quark Green's functions (GIQGF) [1,2], together with Wilson loops [3–9], represent the natural tools for the investigation of the properties of observable quantities in QCD [10–16]. Approaches using these ingredients meet, however, difficulties arising mainly from the nonlocal structure of the GIQGFs. For this reason, calculations of physical quantities like scattering amplitudes, bound state energies, form factors, have usually been carried out up to now with the more familiar formalism of ordinary, gauge variant, Green's functions using particular gauges. Nevertheless, a gauge invariant formalism would bring several advantages that are worth considering. First, one expects to find in the quantities under consideration an infrared safe behavior, free of artificial singularities and divergences. This is also true for the spectral functions underlying the gauge invariant Green's functions. Second, Wilson loops, when saturated for instance by minimal surfaces, allow for a systematic study of the confining properties of the theory. Third, the resolution of bound state problems provides the knowledge of gauge invariant bound state wave functions which are particularly useful for the calculation of matrix elements of operators involving path-ordered phase factors.

In this respect, to optimize the methods of investigation based on GIQGFs, an approach was undertaken by the present author with the aim of obtaining integrodifferential equations that the latter would satisfy, in a parallel way as for the Dyson-Schwinger equations in the case of ordinary Green's functions [17–20]. This was possible in the case of two-point GIQGFs (2PGIQGF) defined with a path-ordered gluon field phase factor along polygonal lines between the quark fields [21]. The 2PGIQGFs can then be classified according to the number of segments their phase factor line contains. Functional relations are then obtained among these 2PGIQFs, which, together with the equations of motion relative to the quark fields, lead to an integrodifferential equation satisfied by the 2PGIQGF defined with one straight line segment for the phase factor. The kernel of this equation involves, with increasing complexity, a series of Wilson loop averages along polygonal contours and their functional derivatives.

As a first step for the resolution of the above equation and the determination of the most important piece of the kernel, the case of two-dimensional QCD in the large-N_c limit [22] was considered. The equation could then be solved exactly and analytically, displaying the main features of the spectral properties of the quark fields [23]: The quarks contribute to the 2PGIQGF like physical particles with positive energies, respecting the causality property; the singularities of the GIQGF are located on the positive real axis of the momentum squared variable (timelike region) and are represented by an infinite series of branch cuts. Lehmann's positivity conditions of the spectral functions [24] are also satisfied. Although results obtained in two-dimensional theories cannot straightforwardly be transposed into four dimensions, they underline here the following two features: (i) the equation obtained for the 2PGIQGF is neither empty, nor unresolvable. With plausible assumptions about the properties of Wilson loops, it might also be analyzed in four dimensions. In particular, the only part of the kernel that survives in two dimensions is precisely the simplest Wilson loop, corresponding to a triangular contour. (ii) The resolution of the equation has provided new results, not known previously from more conventional approaches.

The aim of the present paper is to enlarge the scope of investigations of the GIQGFs by also including in it the four-point GIQGFs (4PGIQGF), which allows us to study the bound state problem of quark-antiquark systems with a gauge invariant formalism. This is done by using again the functional relationships between GIQGFs with different numbers of segments on their polygonal lines. One then ends up with two bound state equations of the Dirac type, each relative to the quark or to the antiquark of the system. The two equations are compatible among themselves due to the validity of the Bianchi identities satisfied by the gluon fields. The kernels of these equations involve, as in

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the 2PGIQGF case, series of Wilson loop averages along polygonal contours and their derivatives, as well as the 2PGIQGFs of each quark field. A functional relationship is established between these kernels and those of the 2PGIQGFs. It is assumed, for later investigations, that the bound state wave functions satisfy usual spectral properties, based on the positivity of the energies of the quark and gluon fields and on causality, leading here to a generalization of the Deser-Gilbert-Sudarshan representation [25].

The bound state equations thus obtained allow us to further investigate the question of spontaneous chiral symmetry breaking. It is shown that in the chiral limit (massless quarks), the bound state equations possess a massless solution with zero total momentum if the 2PGIQGF possesses a normalizable nonvanishing scalar part in this limit. This result is the analogue of the one established by Baker, Johnson and Lee [26] for the Bethe-Salpeter equation [27–29] and provides, prior to the resolution of the bound state equations, a sufficient criterion for chiral symmetry breaking in the case of bound states made of quarks and antiquarks with different flavors.

The organization of this paper follows. In Sec. II, properties of the 2PGIQGFs are summarized. In Sec. III, 4PGIQGFs are introduced and their properties are displayed. In Sec. IV, the bound state equations are established. Section V deals with the question of the spectral representation of the wave functions. In Sec. VI, a criterion for chiral symmetry breaking is derived. Summary and concluding remarks follow in Sec. VII. An Appendix is devoted to the normalization condition of the wave functions.

II. TWO-POINT GREEN'S FUNCTIONS

We summarize in this section the main results obtained for the 2PGIQGFs in Ref. [21]. We shall mainly be interested in path-ordered phase factors that are defined along polygonal lines in space (skew-polygonal) and made of junctions of straight line segments. Designated by $U(y, x)$, a phase factor along the straight line segment xy, with an orientation from x to y, a displacement of one end of the segment, while the other one is fixed, generates a displacement of all points of the segment with appropriate weight factors. We may characterize this as representing a rigid path displacement. Parametrizing linearly the segment with a parameter λ, $0 \leq \lambda \leq 1$, such that a point of the segment is represented as $z(\lambda)$, with $z(0) = x$ and $z(1) = y$, a displacement of one end point of the segment gives rise to two types of contribution, the first coming from the end point itself and the second coming from the inner points of the segment. One has for the rigid path derivatives the formulas

$$\frac{\partial U(y,x)}{\partial y^\alpha} = -igA_\alpha(y)U(y,x) + ig(y-x)^\beta \times \int_0^1 d\lambda\lambda U(y,z(\lambda))F_{\beta\alpha}(z(\lambda))U(z(\lambda),x), \tag{2.1}$$

$$\frac{\partial U(y,x)}{\partial x^\alpha} = +igU(y,x)A_\alpha(x) + ig(y-x)^\beta \times \int_0^1 d\lambda(1-\lambda)U(y,z(\lambda))F_{\beta\alpha}(z(\lambda))U(z(\lambda),x), \tag{2.2}$$

where A is the gluon potential, F its field strength, and g the coupling constant. In the above equations, the integrals represent the inner contributions of the segment. When dealing with gauge invariant quantities, the end point contributions are usually cancelled by similar contributions coming from neighboring segments or fields, and it is the inner contributions of the segments that remain. We adopt for them the following notations:

$$\frac{\bar\delta U(y,x)}{\bar\delta y^{\alpha+}} \equiv ig(y-x)^\beta \times \int_0^1 d\lambda\lambda U(y,z(\lambda))F_{\beta\alpha}(z(\lambda))U(z(\lambda),x), \tag{2.3}$$

$$\frac{\bar\delta U(y,x)}{\bar\delta x^{\alpha-}} \equiv ig(y-x)^\beta \times \int_0^1 d\lambda(1-\lambda)U(y,z(\lambda))F_{\beta\alpha}(z(\lambda))U(z(\lambda),x). \tag{2.4}$$

Taking into account the orientation on U, the superscript $+$ or $-$ of the derivative variable indicates the segment on which it is acting when we are in the presence of two joined segments. Thus if we have the expression $U(y,u)U(u,x)$, then the operator $\bar\delta/\bar\delta u^+$ will act on $U(u,x)$ only, through the end point u of the segment xu, while the operator $\bar\delta/\bar\delta u^-$ will act on $U(y,u)$.

The vacuum expectation value W of a Wilson loop (or, equivalently, the Wilson loop average) along a closed polygonal contour with n sides and n junction points $x_1, x_2, \ldots, x_n$ will be denoted W_n and will be represented as the exponential of a functional F_n [5,8],

$$W_n = W(x_n, x_{n-1}, \ldots, x_1) = e^{F_n(x_n,x_{n-1},\ldots,x_1)} = e^{F_n}, \tag{2.5}$$

the orientation of the contour going from x_1 to x_n through x_2, x_3, etc. Then, the notation $\bar\delta F_n/\bar\delta x_i^-$ means that the derivative acts on the internal part of the segment x_ix_{i+1} with x_{i+1} held fixed ($x_{n+1} = x_1$), while $\bar\delta F_n/\bar\delta x_i^+$ means that the derivative acts on the internal part of the segment $x_{i-1}x_i$ with x_{i-1} held fixed ($x_0 = x_n$).

The 2PGIQGFs with phase factors along polygonal lines can be classified according to the number of segments they contain. The 2PGIQGF with a phase factor line with n sides and $n-1$ junction points $t_1, t_2, \ldots, t_{n-1}$ between the segments is defined as

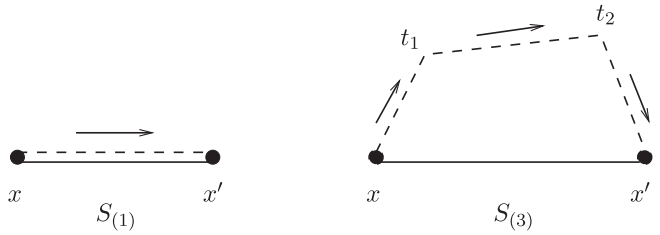

FIG. 1. Graphical representation of the 2PGIQGFs $S_{(1)}$ and $S_{(3)}$. The solid lines represent the quark field contractions, the dotted lines the phase factor along the polygonal lines, and the arrows the orientation on them.

$$S_{(n)}(x,x';t_{n-1},\ldots,t_1) = -\frac{1}{N_c}\langle\bar\psi(x')U(x',t_{n-1})U(t_{n-1},t_{n-2})\ldots U(t_1,x)\psi(x)\rangle, \tag{2.6}$$

the ψs being the quark fields, with mass term m, belonging to the defining fundamental representation of the color gauge group $SU(N_c)$ and the vacuum averaging being defined in the path integral formalism. (Spinor indices are not written and the color indices are implicitly summed.) The orientation of the path in $S_{(n)}(x,x';t_{n-1},\ldots,t_1)$ runs from x to x', passing by $t_1, t_2, \ldots, t_{n-1}$.

The simplest 2PGIQGF corresponds to the case where $n = 1$, for which the points x and x' are joined by a single straight line,

$$S_{(1)}(x,x') \equiv S(x,x') = -\frac{1}{N_c}\langle\bar\psi(x')U(x',x)\psi(x)\rangle. \tag{2.7}$$

(We shall generally omit the index 1 from that function.) A graphical representation of the 2PGIQGFs $S_{(1)}$ and $S_{(3)}$ is shown in Fig. 1.

For the internal parts of rigid path derivatives, we have the definitions

$$\frac{\bar\delta S_{(n)}(x,x';t_{n-1},\ldots,t_1)}{\bar\delta x^{\mu-}} = -\frac{1}{N_c}\Big\langle\bar\psi(x')U(x',t_{n-1})U(t_{n-1},t_{n-2})\ldots \times\frac{\bar\delta U(t_1,x)}{\bar\delta x^{\mu-}}\psi(x)\Big\rangle, \tag{2.8}$$

$$\frac{\bar\delta S_{(n)}(x,x';t_{n-1},\ldots,t_1)}{\bar\delta x'^{\nu+}} = -\frac{1}{N_c}\Big\langle\bar\psi(x')\frac{\bar\delta U(x',t_{n-1})}{\bar\delta x'^{\nu+}}U(t_{n-1},t_{n-2})\ldots \times U(t_1,x)\psi(x)\Big\rangle. \tag{2.9}$$

$S_{(1)}$ and $S_{(n)}$ satisfy the following equations of motion:

$$(i\gamma.\partial_{(x)} - m)S(x,x') = i\delta^4(x-x') + i\gamma^\mu\frac{\bar\delta S(x,x')}{\bar\delta x^{\mu-}}, \tag{2.10}$$

$$S(x,x')(-i\gamma.\overleftarrow{\partial}_{(x')} - m) = i\delta^4(x-x') - i\frac{\bar\delta S(x,x')}{\bar\delta x'^{\mu+}}\gamma^\mu, \tag{2.11}$$

$$(i\gamma.\partial_{(x)} - m)S_{(n)}(x,x';t_{n-1},\ldots,t_1) = i\delta^4(x-x')e^{F_n(x,t_{n-1},\ldots,t_1)} + i\gamma^\mu\frac{\bar\delta S_{(n)}(x,x';t_{n-1},\ldots,t_1)}{\bar\delta x^{\mu-}}, \tag{2.12}$$

$$S_{(n)}(x,x';t_{n-1},\ldots,t_1)(-i\gamma.\overleftarrow{\partial}_{(x')} - m) = i\delta^4(x-x')e^{F_n(x,t_{n-1},\ldots,t_1)} - i\frac{\bar\delta S_{(n)}(x,x';t_{n-1},\ldots,t_1)}{\bar\delta x'^{\mu+}}\gamma^\mu. \tag{2.13}$$

The $S_{(n)}$s also satisfy equations related to the junction points $t_1,\ldots,t_{n-1}$ on the polygonal line or more generally to local deformations of the paths. These involve the gluon field equations of motion and lead to equations related to the properties of phase factors and Wilson loops [4–7]. They should mainly be used for the determination of the expressions of the Wilson loop averages. In the present paper, the latter are assumed to be known, and therefore the corresponding equations will not be considered.

Multiplying the equations of motion (2.12) and (2.13) with $S(t_1,x)$ and $S(x',t_{n-1})$, respectively, and integrating with respect to x or x', one can establish functional relations between the various 2PGIQGFs. For $S_{(n)}$, one has

$$S_{(n)}(x,x';t_{n-1},\ldots,t_1) = S(x,x')e^{F_{n+1}(x',t_{n-1},\ldots,t_1,x)} + \Big(\frac{\bar\delta S(x,y_1)}{\bar\delta y_1^{\alpha_1+}} + S(x,y_1)\frac{\bar\delta}{\bar\delta y_1^{\alpha_1-}}\Big)\gamma^{\alpha_1}S_{(n+1)}(y_1,x';t_{n-1},\ldots,t_1,x)$$
$$= S(x,x')e^{F_{n+1}(x',t_{n-1},\ldots,t_1,x)} - S_{(n+1)}(x,z_1;x',t_{n-1},\ldots,t_1)\gamma^{\beta_1}\Big(\frac{\bar\delta S(z_1,x')}{\bar\delta z_1^{\beta_1-}} + \frac{\overleftarrow{\bar\delta}}{\bar\delta z_1^{\beta_1+}}S(z_1,x')\Big). \tag{2.14}$$

(Integrations on intermediate variables are implicit and will not be written throughout this paper. Here, y_1 and z_1 are integration variables.) By iterating these equations with respect to the higher $S_{(n)}$s of the right-hand sides and assuming that the last term rejected to infinity tends to zero, one ends up with a series expansion of any $S_{(n)}$ $(n>1)$ in terms of S and its derivative and derivatives of logarithms of Wilson loop averages. This shows that among the infinite set of 2PGIQGFs with polygonal lines, only the first one, with one single straight line, is a genuine dynamical independent quantity; all others are in principle calculable from it, provided one knows to evaluate the rigid path derivative of S and the Wilson loop averages.

The calculation of S proceeds from the equation of motion (2.10) [or (2.11)]. It is then necessary to devise a method for evaluating the rigid path derivative $\bar{\delta}S(x,x')/\bar{\delta}x^{\mu-}$. This is done by applying the rigid path derivative operator on both sides of Eq. (2.14) and repeating iteration operations. Specializing the result for S, one finds

$$\frac{\bar{\delta}S(x,x')}{\bar{\delta}x^{\mu-}} = \frac{\bar{\delta}F_2(x',x)}{\bar{\delta}x^{\mu-}}S(x,x') - \frac{\bar{\delta}^2F_3(x',x,y_1)}{\bar{\delta}x^{\mu-}\bar{\delta}y_1^{\alpha_1+}}S(x,y_1)\gamma^{\alpha_1}S_{(2)}(y_1,x';x) - \sum_{n=3}^{\infty}\left(\frac{\bar{\delta}S(x,y_1)}{\bar{\delta}y_1^{\alpha_1+}} + S(x,y_1)\frac{\bar{\delta}}{\bar{\delta}y_1^{\alpha_1-}}\right)\gamma^{\alpha_1}$$
$$\times\cdots\times\left(\frac{\bar{\delta}S(y_{n-3},y_{n-2})}{\bar{\delta}y_{n-2}^{\alpha_{n-2}+}} + S(y_{n-3},y_{n-2})\frac{\bar{\delta}}{\bar{\delta}y_{n-2}^{\alpha_{n-2}-}}\right)\gamma^{\alpha_{n-2}}\frac{\bar{\delta}^2F_{n+1}(x',x,y_1,\ldots,y_{n-1})}{\bar{\delta}x^{\mu-}\bar{\delta}y_{n-1}^{\alpha_{n-1}+}}S(y_{n-2},y_{n-1})$$
$$\times\gamma^{\alpha_{n-1}}S_{(n)}(y_{n-1},x';x,y_1,\ldots,y_{n-2}). \tag{2.15}$$

The right-hand side involves a series of terms in which the nth-order one contains $S_{(n)}$ and a Wilson loop with a polygonal contour with $(n+1)$ sides. One notices from the locations of the second-order derivatives acting on the Fs the absence of reducible-type contributions in the corresponding expressions; the latter are expected to be part of the definition of the $S_{(n)}$s when expressed in terms of free propagators. The calculation should be completed by bringing all derivative operators to the right; the final form shows that the nth-order term contains globally n derivatives acting on the logarithm of the corresponding Wilson loop average and/or on the Green's function S (at most at first order for the latter). Furthermore, each derivative acting on the Wilson loop operates on a different segment from the others; this prevents the appearance of singularities arising from derivatives acting on the same point. The Wilson loop contributions have the characteristics of being irreducible and are classified into the following three categories: connected, crossed, and nested [21]. The form (2.15) is the most convenient one for comparisons with other cases, such as those of the 4PGIQGFs.

Equation (2.15) can be considered as the analogue of the self-energy Dyson-Schwinger equation in the case of the ordinary Green's function [17,18]. Defining the latter as $\tilde{S}(x,x') = \frac{1}{N_c}\langle\psi(x)\bar{\psi}(x')\rangle$, its equation of motion takes the form

$$(i\gamma.\partial_{(x)} - m)\tilde{S}(x,x') = i\delta^4(x-x') + \Sigma(x,y)\tilde{S}(y,x'), \tag{2.16}$$

where Σ defines the self-energy; it is a functional of the Green's function $\tilde{S}$ itself, together with other Green's functions of interest, like that of the gluon field or of the photon field in the case of QED. The expression of $\Sigma[\tilde{S}]$ in terms of $\tilde{S}$ and the other two-point Green's functions defines the Dyson-Schwinger equation.

In the present case, however, one meets a more complicated situation in two respects. First, Eq. (2.15) involves in its right-hand side the whole set of 2PGIQGFs defined along polygonal lines, although all of them are ultimately expressible in terms of S. This feature, on the other hand, is an indication that the set of 2PGIQGFs along polygonal lines is closed, since no other types of contour are needed to reach the final equation for S. Second, the integrals that are present are not of the convolution type; they overlap all terms that accompany them, which is due to the presence of the Wilson loops, whose contours pass by all points that are present. Taking into account these facts, one needs to introduce matrix-type self-energy operators, Σ_{mn}, where the first index refers to the initial Green's function that is considered $(S_{(m)})$ and the second one to the Green's function $S_{(n)}$ that appears in the right-hand side of the equation. With this definition, the equation of motion (2.10) can be schematically written in the form

$$(i\gamma.\partial_{(x)} - m)S(x,x') = i\delta^4(x-x') + \sum_{n=1}^{\infty}(\Sigma_{1n}[S] * S_{(n)}), \tag{2.17}$$

where the star operation represents the integrals involved in the term containing $S_{(n)}$ in the right-hand side of Eq. (2.15), and the functional expression of $\Sigma_{1n}[S]$ is deduced from that equation by identification.

The above results can also be applied with obvious transpositions to the equation of motion (2.11). The evaluation of $\bar{\delta}S(x,x')/\bar{\delta}x'^{\mu+}$ can be done in two ways. First, one might use the second expression of the functional relationship of Eq. (2.14). In this case the iterative expansion is done in the reverse order to that of Eq. (2.15). The corresponding equation takes then the schematic form

$$S(x,x')(-i\gamma.\overleftarrow{\partial}_{(x')}-m)=i\delta^4(x-x')+\sum_{n=1}^{\infty}(S_{(n)}*\tilde{\Sigma}_{1n}[S]), \tag{2.18}$$

where $\tilde{\Sigma}_{1n}[S]$ is deduced from $\Sigma_{1n}[S]$ by reversing the orders of appearance of various matrices and Green's functions and some of their arguments, changing the sign of terms containing an odd number of explicit γ matrices and replacing the rigid path derivatives $\bar{\delta}/\bar{\delta}x^{\mu-}$ and $\bar{\delta}/\bar{\delta}y_j^{\alpha_j+}$ with $\bar{\delta}/\bar{\delta}x'^{\mu+}$ and $\bar{\delta}/\bar{\delta}z_j^{\beta_j-}$, respectively. Second, one might use the same functional relations as for obtaining Eq. (2.15). In this case, the only modification is the replacement of the rigid path derivative $\bar{\delta}/\bar{\delta}x^{\mu-}$ by $\bar{\delta}/\bar{\delta}x'^{\mu+}$, while in Eq. (2.11) we have to take into account the global change of sign in front of $\bar{\delta}S(x,x')/\bar{\delta}x'^{\mu+}$ and the position of the matrix γ^μ on the utmost right. We shall write the resulting expression of the equation in the form

$$S(x,x')(-i\gamma.\overleftarrow{\partial}_{(x')}-m)=i\delta^4(x-x')+\sum_{n=1}^{\infty}(\hat{\Sigma}_{1n}[S]*S_{(n)}). \tag{2.19}$$

It is to be emphasized that once the operator $\Sigma_{1n}[S]$ of Eq. (2.17) [or its analogue of Eqs. (2.18) and (2.19)] has been evaluated by means of Eq. (2.15), then the Green's functions $S_{(1)},\ldots,S_{(n)}$, etc., have to be considered as ordinary complex functions of the variables $x, x', y_1,\ldots,y_n$, etc., satisfying translation invariance and well defined Lorentz transformation properties. The information contained in their phase factors along the rigid straight line segments and polygonal lines is now expressed by means of the corresponding Wilson loop averages and their rigid path derivatives, which, after evaluation, are themselves ordinary functions of their arguments x, x', etc. (the junction points of the segments). The straight line segments and the polygonal lines no longer introduce additional degrees of freedom, since their geometry is completely determined by the knowledge of the positions of the junction points of the segments. In particular, the operator $\partial/\partial x$ of the Dirac operator in Eq. (2.17) acts as an ordinary derivative operator on S. This is why Eq. (2.17) and its analogues have the status of integrodifferential equations.

In the rest of the paper we shall consider systems involving two quarks with different flavors and generally with different masses. To distinguish their individual Green's functions, we shall introduce an additional index for their notation. Thus, they will be denoted $S_{1,(n)}$ and $S_{2,(n)}$, respectively, corresponding to quark 1 and quark 2. The simplest Green's functions (with one straight line) S ($\equiv S_{(1)}$) will be denoted S_1 and S_2.

III. FOUR-POINT GREEN'S FUNCTIONS

We now consider two different quark fields, labeled with indices 1 and 2, respectively, with mass terms m_1 and m_2. Four-point GIQGFs are constructed by including gluon field phase factors between the quark and the antiquark fields. Considering a polygonal line made of n segments and another made of one segment, we define the 4PGIQGF $G_{(n)}$ as

$$\begin{aligned}&G_{(n)\alpha\beta,\beta'\alpha'}(x_1,x_2;x_2',x_1';t_{n-1},t_{n-2},\ldots,t_1)\\&\quad=-\frac{1}{N_c}\langle\bar{\psi}_{2\beta}(x_2)U(x_2,t_{n-1})\ldots U(t_1,x_1)\psi_{1\alpha}(x_1)\\&\qquad\times\bar{\psi}_{1\alpha'}(x_1')U(x_1',x_2')\psi_{2\beta'}(x_2')\rangle,\end{aligned} \tag{3.1}$$

where α, β, β', α' are the spinor indices of the quark fields.

The simplest such Green's function is $G_{(1)}$,

$$\begin{aligned}&G_{(1)}(x_1,x_2;x_2',x_1')\\&\quad=-\frac{1}{N_c}\langle\bar{\psi}_2(x_2)U(x_2,x_1)\psi_1(x_1)\bar{\psi}_1(x_1')U(x_1',x_2')\\&\qquad\times\psi_2(x_2')\rangle.\end{aligned} \tag{3.2}$$

A graphical representation of $G_{(1)}$ and $G_{(3)}$ is shown in Fig. 2.

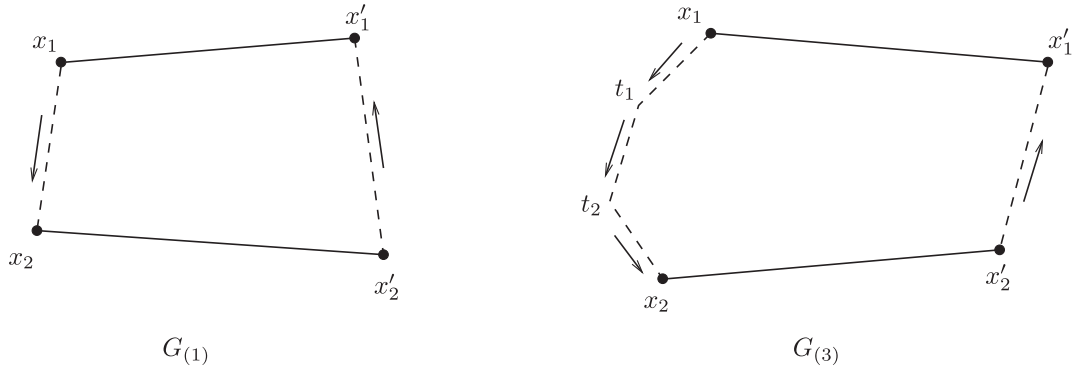

FIG. 2. Graphical representation of the 4PGIQGFs $G_{(1)}$ and $G_{(3)}$. Same conventions as in Fig. 1.

In the following, for the study of the bound state problem, the points x_1' and x_2' will be sent to $-\infty$ in time; they will then disappear by factorization from the bound state equations, which is why the classification of the 4PGIQGFs hinges here only on the line between the points x_1 and x_2.

The 4PGIQGFs satisfy the following equations of motion:

$$(i\gamma.\partial_1 - m_1)G_{(n)}(x_1, x_2; x_2', x_1'; t_{n-1}, \ldots, t_1) = i\delta^4(x_1 - x_1')S_{2,(n+1)}(x_2', x_2; t_{n-1}, \ldots, t_1, x_1) + i\gamma^\mu \frac{\bar\delta}{\bar\delta x_1^{\mu-}} G_{(n)}(x_1, x_2; x_2', x_1'; t_{n-1}, \ldots, t_1)|_{x_1 t_1}, \tag{3.3}$$

$$G_{(n)}(x_1, x_2; x_2', x_1'; t_{n-1}, \ldots, t_1)(-i\gamma.\overleftarrow{\partial}_2 - m_2) = i\delta^4(x_2 - x_2')S_{1,(n+1)}(x_1, x_1'; x_2, t_{n-1}, \ldots, t_1) - i\frac{\bar\delta}{\bar\delta x_2^{\nu+}} G_{(n)}(x_1, x_2; x_2', x_1'; t_{n-1}, \ldots, t_1)|_{t_{n-1} x_2}\gamma^\nu. \tag{3.4}$$

($\partial_1 = \partial/\partial x_1$ and $\partial_2 = \partial/\partial x_2$.) The γ matrices that act on G from the left, act on its first spinor index (α), while those acting from the right act on its second spinor index (β), as defined in Eq. (3.1).

Multiplying Eqs. (3.3) and (3.4) with $S_1(t_1, x_1)$ and $S_2(x_2, t_{n-1})$, respectively, and integrating, one obtains the following functional relations between different 4PGIQGFs:

$$\begin{aligned} G_{(n)}(x_1, x_2; x_2', x_1'; t_{n-1}, \ldots, t_1) &= S_1(x_1, x_1')S_{2,(n+2)}(x_2', x_2; t_{n-1}, \ldots, t_1, x_1, x_1') \\ &\quad + \left(\frac{\bar\delta S_1(x_1, y_1)}{\bar\delta y_1^{\alpha_1+}} + S_1(x_1, y_1)\frac{\bar\delta}{\bar\delta y_1^{\alpha_1-}}\right)\gamma^{\alpha_1} G_{(n+1)}(y_1, x_2; x_2', x_1'; t_{n-1}, \ldots, t_1, x_1) \\ &= S_{1,(n+2)}(x_1, x_1'; x_2', x_2, t_{n-1}, \ldots, t_1)S_2(x_2', x_2) \\ &\quad - G_{(n+1)}(x_1, z_1; x_2', x_1'; x_2, t_{n-1}, \ldots, t_1)\gamma^{\beta_1}\left(\frac{\bar\delta S_2(z_1, x_2)}{\bar\delta z_1^{\beta_1-}} + \frac{\overleftarrow{\bar\delta}}{\bar\delta z_1^{\beta_1+}} S_2(z_1, x_2)\right). \end{aligned} \tag{3.5}$$

IV. BOUND STATE EQUATIONS

In order to obtain bound state equations, one considers in the 4PGIQGFs the limit of large timelike separations between the set of points $(x_1, t_1, \ldots, t_{n-1}, x_2)$ and the set (x_2', x_1'). In this limit, the Green's functions can be saturated by a complete set of hadronic states, among which are single mesons representing bound states of quarks and antiquarks. To simplify the analysis, one may either consider the large-N_c limit, in which case only single poles survive [22,30], or simply neglect inelasticity effects to ensure the stability of the bound state. By appropriate projection operations [28] or limiting procedures to the pole position in the total momentum space, it is possible to select one particular bound state among the whole set of intermediate states.

We shall refer to the selected bound state with its total four-momentum P only, discarding other quantum numbers, which will not play any role in the following. (We assume that the bound state is nondegenerate.) The wave functions are classified according to the number of straight line segments existing on the polygonal lines of the phase factors ($n, j = 1, 2, \ldots, \infty$),

$$\begin{aligned} &\Phi_{(n)\alpha\beta}(x_1, x_2; t_{n-1}, \ldots, t_1) \\ &\quad = -\frac{1}{\sqrt{N_c}}\langle 0|T(\bar\psi_{2\beta}(x_2)U(x_2, t_{n-1})\ldots \\ &\qquad \times U(t_1, x_1)\psi_{1\alpha}(x_1))|P\rangle, \end{aligned} \tag{4.1}$$

$$\begin{aligned} &\bar\Phi_{(j)\beta'\alpha'}(x_2', x_1'; t_{j-1}, \ldots, t_1) \\ &\quad = \frac{1}{\sqrt{N_c}}\langle P|T(\bar\psi_{1\alpha'}(x_1')U(x_1', t_{j-1})\ldots U(t_1, x_2') \\ &\qquad \times \psi_{2\beta'}(x_2'))|0\rangle. \end{aligned} \tag{4.2}$$

They become in the simplest cases $n = 1$ and $j = 1$,

$$\Phi_{(1)}(x_1, x_2) = -\frac{1}{\sqrt{N_c}}\langle 0|T(\bar\psi_2(x_2)U(x_2, x_1)\psi_1(x_1))|P\rangle, \tag{4.3}$$

$$\bar\Phi_{(1)}(x_2', x_1') = \frac{1}{\sqrt{N_c}}\langle P|T(\bar\psi_1(x_1')U(x_1', x_2')\psi_2(x_2'))|0\rangle. \tag{4.4}$$

(The dependence of the wave functions on the total four-momentum P of the bound state is omitted from their

arguments for notational simplification.) We note that the above wave functions for all ns describe the same bound state, but differ in their expressions due to differences in their contents with respect to the phase factor lines.

Taking large timelike separations between initial and final coordinates as described above, or going in total momentum space to the pole position, and selecting one bound state, the equations of motion (3.3) and (3.4) of the 4PGIQGFs are transformed into wave equations (their inhomogeneous parts not contributing to the bound states poles),

$$(i\gamma.\partial_1 - m_1)\Phi_{(n)}(x_1, x_2; t_{n-1}, \ldots, t_1) = +i\gamma^\mu \frac{\bar\delta}{\bar\delta x_1^{\mu-}} \Phi_{(n)}(x_1, x_2; t_{n-1}, \ldots, t_1)|_{x_1 t_1}, \quad (4.5)$$

$$\Phi_{(n)}(x_1, x_2; t_{n-1}, \ldots, t_1)(-i\gamma.\overleftarrow{\partial}_2 - m_2) = -i \frac{\bar\delta}{\bar\delta x_2^{\nu+}} \Phi_{(n)}(x_1, x_2; t_{n-1}, \ldots, t_1)|_{t_{n-1} x_2} \gamma^\nu, \quad (4.6)$$

which become for the case $n = 1$

$$(i\gamma.\partial_1 - m_1)\Phi_{(1)}(x_1, x_2) = +i\gamma^\mu \frac{\bar\delta}{\bar\delta x_1^{\mu-}} \Phi_{(1)}(x_1, x_2), \quad (4.7)$$

$$\Phi_{(1)}(x_1, x_2)(-i\gamma.\overleftarrow{\partial}_2 - m_2) = -i \frac{\bar\delta}{\bar\delta x_2^{\nu+}} \Phi_{(1)}(x_1, x_2)\gamma^\nu. \quad (4.8)$$

A wave function $\Phi_{(n)}$ thus satisfies two independent Dirac-type equations. They should, however, be compatible among themselves in order not to give rise to a vanishing solution. The compatibility condition is obtained by making the two Dirac operators act on the wave function in different orders and subtracting the corresponding results from each other. Since the two Dirac operators commute among themselves, the result should be zero. One finds

$$\left(\frac{\bar\delta}{\bar\delta x_2^{\nu+}} \frac{\bar\delta}{\bar\delta x_1^{\mu-}} - \frac{\bar\delta}{\bar\delta x_1^{\mu-}} \frac{\bar\delta}{\bar\delta x_2^{\nu+}} \right) \Phi_{(n)} = 0. \quad (4.9)$$

For $n \geq 2$, the commutativity of the two rigid path derivatives results from the fact that they operate on different segments in an uncorrelated way. For $n = 1$, since they operate on the same segment, they may act on the same point, giving rise to additional singularities. It can, however, be shown that because of the Bianchi identities satisfied by the gluon fields, even in this case the two operators commute [16]. The two wave equations (4.5) and (4.6), or (4.7) and (4.8), are therefore compatible among themselves.

In order to evaluate the interaction part of the wave equations, it is necessary to express the rigid path derivatives in terms of calculable kernels. We follow here a method similar to that used for the 2PGIQGFs (Sec. II). By selecting in Eqs. (3.5) the bound state contribution, one obtains the following two equivalent equations, expressing a wave function $\Phi_{(n)}$ in terms of $\Phi_{(n+1)}$ and the 2PGIQGFs S_1 or S_2:

$$\Phi_{(n)}(x_1, x_2; t_{n-1}, \ldots, t_1) = +\left(\frac{\bar\delta S_1(x_1, y_1)}{\bar\delta y_1^{\alpha_1+}} + S_1(x_1, y_1) \frac{\bar\delta}{\bar\delta y_1^{\alpha_1-}} \right) \gamma^{\alpha_1} \Phi_{(n+1)}(y_1, x_2; t_{n-1}, \ldots, t_1, x_1)$$
$$= -\Phi_{(n+1)}(x_1, z_1; x_2, t_{n-1}, \ldots, t_1)\gamma^{\beta_1} \left(\frac{\bar\delta S_2(z_1, x_2)}{\bar\delta z_1^{\beta_1-}} + \frac{\overleftarrow{\bar\delta}}{\bar\delta z_1^{\beta_1+}} S_2(z_1, x_2) \right). \quad (4.10)$$

However, contrary to the 2PGIQGF case [Eqs. (2.14)], they do not contain the lowest-index wave function $\Phi_{(1)}$, which could generate an iterative series and allow for the calculation of the $\Phi_{(n)}$s in terms of $\Phi_{(1)}$. The difficulty can be overcome by adding to the above equations identities involving $\Phi_{(1)}$. Considering the equations of motion (2.12) and (2.13), multiplying them with $\Phi_{(1)}(t_1, x_1)$ and $\Phi_{(1)}(x_2, t_{n-1})$, respectively, and integrating, one obtains the two equations,

$$\Phi_{(1)}(x_1, x_2) e^{F_{n+1}(x_2, t_{n-1}, \ldots, t_1, x_1)} + \left(\frac{\bar\delta \Phi_{(1)}(x_1, y_1)}{\bar\delta y_1^{\alpha_1+}} + \Phi_{(1)}(x_1, y_1) \frac{\bar\delta}{\bar\delta y_1^{\alpha_1-}} \right) \gamma^{\alpha_1} S_{2,(n+1)}(y_1, x_2; t_{n-1}, \ldots, t_1, x_1) = 0, \quad (4.11)$$

$$\Phi_{(1)}(x_1, x_2) e^{F_{n+1}(x_2, t_{n-1}, \ldots, t_1, x_1)} - S_{1,(n+1)}(x_1, z_1; x_2, t_{n-1}, \ldots, t_1)\gamma^{\beta_1} \left(\frac{\bar\delta \Phi_{(1)}(z_1, x_2)}{\bar\delta z_1^{\beta_1-}} + \frac{\overleftarrow{\bar\delta}}{\bar\delta z_1^{\beta_1+}} \Phi_{(1)}(z_1, x_2) \right) = 0. \quad (4.12)$$

These can now be added, respectively, to the two expressions of $\Phi_{(n)}$ in Eqs. (4.10), yielding the following new functional relations:

$$\Phi_{(n)}(x_1,x_2;t_{n-1},\ldots,t_1)=\Phi_{(1)}(x_1,x_2)e^{F_{n+1}(x_2,t_{n-1},\ldots,t_1,x_1)}+\left(\frac{\bar\delta\Phi_{(1)}(x_1,y_1)}{\bar\delta y_1^{\alpha_1+}}+\Phi_{(1)}(x_1,y_1)\frac{\bar\delta}{\bar\delta y_1^{\alpha_1-}}\right)\gamma^{\alpha_1}S_{2,(n+1)}$$
$$\times(y_1,x_2;t_{n-1},\ldots,t_1,x_1)+\left(\frac{\bar\delta S_1(x_1,y_1)}{\bar\delta y_1^{\alpha_1+}}+S_1(x_1,y_1)\frac{\bar\delta}{\bar\delta y_1^{\alpha_1-}}\right)\gamma^{\alpha_1}\Phi_{(n+1)}(y_1,x_2;t_{n-1},\ldots,t_1,x_1),\tag{4.13}$$

$$\Phi_{(n)}(x_1,x_2;t_{n-1},\ldots,t_1)=\Phi_{(1)}(x_1,x_2)e^{F_{n+1}(x_2,t_{n-1},\ldots,t_1,x_1)}-S_{1,(n+1)}(x_1,z_1;x_2,t_{n-1},\ldots,t_1)\gamma^{\beta_1}$$
$$\times\left(\frac{\bar\delta\Phi_{(1)}(z_1,x_2)}{\bar\delta z_1^{\beta_1-}}+\frac{\overleftarrow{\bar\delta}}{\bar\delta z_1^{\beta_1+}}\Phi_{(1)}(z_1,x_2)\right)-\Phi_{(n+1)}(x_1,z_1;x_2,t_{n-1},\ldots,t_1)\gamma^{\beta_1}$$
$$\times\left(\frac{\bar\delta S_2(z_1,x_2)}{\bar\delta z_1^{\beta_1-}}+\frac{\overleftarrow{\bar\delta}}{\bar\delta z_1^{\beta_1+}}S_2(z_1,x_2)\right).\tag{4.14}$$

They can be used to express, through iterative calculations, $\Phi_{(n)}$ in terms of $\Phi_{(1)}$, S_1, S_2 and Wilson loop averages. They parallel relations (2.14) of the 2PGIQGFs, but with the additional complication that the iteration should be carried out simultaneously in $\Phi_{(n+1)}$ and $S_{(n+1)}$.

The final step consists in using expressions (4.13) and (4.14) to bring the right-hand sides of the wave equations (4.5), (4.6), (4.7), and (4.8) into a form where the wave functions appear as acted on by kernels made of Wilson loop averages and their derivatives, as well as 2PGIQGFs. The procedure follows similar lines as those adopted for the 2PGIQGFs [21]. Considering in particular the wave equation (4.7), one finds

$$\frac{\bar\delta\Phi_{(1)}(x_1,x_2)}{\bar\delta x_1^{\mu-}}=\frac{\bar\delta F_2(x_2,x_1)}{\bar\delta x_1^{\mu-}}\Phi_{(1)}(x_1,x_2)-\frac{\bar\delta^2F_3(x_2,x_1,y_1)}{\bar\delta x_1^{\mu-}\bar\delta y_1^{\alpha_1+}}[\Phi_{(1)}(x_1,y_1)\gamma^{\alpha_1}S_{2,(2)}(y_1,x_2;x_1)+S_1(x_1,y_1)\gamma^{\alpha_1}\Phi_{(2)}(y_1,x_2;x_1)]$$
$$-\left\{\left(\frac{\bar\delta\Phi_{(1)}(x_1,y_1)}{\bar\delta y_1^{\alpha_1+}}+\Phi_{(1)}(x_1,y_1)\frac{\bar\delta}{\bar\delta y_1^{\alpha_1-}}\right)\gamma^{\alpha_1}\frac{\bar\delta^2F_4(x_2,x_1,y_1,y_2)}{\bar\delta x_1^{\mu-}\bar\delta y_2^{\alpha_2+}}S_2(y_1,y_2)\gamma^{\alpha_2}S_{2,(3)}(y_2,x_2;x_1,y_1)\right.$$
$$+\left(\frac{\bar\delta S_1(x_1,y_1)}{\bar\delta y_1^{\alpha_1+}}+S_1(x_1,y_1)\frac{\bar\delta}{\bar\delta y_1^{\alpha_1-}}\right)\gamma^{\alpha_1}\frac{\bar\delta^2F_4(x_2,x_1,y_1,y_2)}{\bar\delta x_1^{\mu-}\bar\delta y_2^{\alpha_2+}}[\Phi_{(1)}(y_1,y_2)\gamma^{\alpha_2}S_{2,(3)}(y_2,x_2;x_1,y_1)$$
$$\left.+S_1(y_1,y_2)\gamma^{\alpha_2}\Phi_{(3)}(y_2,x_2;x_1,y_1)]\right\}-\sum_{j=4}^{\infty}\sum_{r=1}^{j-2}\left(\frac{\bar\delta S_1(x_1,y_1)}{\bar\delta y_1^{\alpha_1+}}+S_1(x_1,y_1)\frac{\bar\delta}{\bar\delta y_1^{\alpha_1-}}\right)\gamma^{\alpha_1}$$
$$\times\cdots\times\left(\frac{\bar\delta S_1(y_{r-2},y_{r-1})}{\bar\delta y_{r-1}^{\alpha_{r-1}+}}+S_1(y_{r-2},y_{r-1})\frac{\bar\delta}{\bar\delta y_{r-1}^{\alpha_{r-1}-}}\right)\gamma^{\alpha_{r-1}}\left(\frac{\bar\delta\Phi_{(1)}(y_{r-1},y_r)}{\bar\delta y_r^{\alpha_r+}}+\Phi_{(1)}(y_{r-1},y_r)\frac{\bar\delta}{\bar\delta y_r^{\alpha_r-}}\right)$$
$$\times\gamma^{\alpha_r}\left(\frac{\bar\delta S_2(y_r,y_{r+1})}{\bar\delta y_{r+1}^{\alpha_{r+1}+}}+S_2(y_r,y_{r+1})\frac{\bar\delta}{\bar\delta y_{r+1}^{\alpha_{r+1}-}}\right)\gamma^{\alpha_{r+1}}\cdots\left(\frac{\bar\delta S_2(y_{j-3},y_{j-2})}{\bar\delta y_{j-2}^{\alpha_{j-2}+}}+S_2(y_{j-3},y_{j-2})\frac{\bar\delta}{\bar\delta y_{j-2}^{\alpha_{j-2}-}}\right)$$
$$\times\gamma^{\alpha_{j-2}}\frac{\bar\delta^2F_{j+1}(x_2,x_1,y_1,\ldots,y_{j-1})}{\bar\delta x_1^{\mu-}\bar\delta y_{j-1}^{\alpha_{j-1}+}}S_2(y_{j-2},y_{j-1})\gamma^{\alpha_{j-1}}S_{2,(j)}(y_{j-1},x_2;x_1,y_1,\ldots,y_{j-2})$$
$$-\sum_{j=4}^{\infty}\left(\frac{\bar\delta S_1(x_1,y_1)}{\bar\delta y_1^{\alpha_1+}}+S_1(x_1,y_1)\frac{\bar\delta}{\bar\delta y_1^{\alpha_1-}}\right)\gamma^{\alpha_1}\times\cdots\times\left(\frac{\bar\delta S_1(y_{j-3},y_{j-2})}{\bar\delta y_{j-2}^{\alpha_{j-2}+}}+S_1(y_{j-3},y_{j-2})\frac{\bar\delta}{\bar\delta y_{j-2}^{\alpha_{j-2}-}}\right)$$
$$\times\gamma^{\alpha_{j-2}}\frac{\bar\delta^2F_{j+1}(x_2,x_1,y_1,\ldots,y_{j-1})}{\bar\delta x_1^{\mu-}\bar\delta y_{j-1}^{\alpha_{j-1}+}}[\Phi_{(1)}(y_{j-2},y_{j-1})\gamma^{\alpha_{j-1}}S_{2,(j)}(y_{j-1},x_2;x_1,y_1,\ldots,y_{j-2})$$
$$+S_1(y_{j-2},y_{j-1})\gamma^{\alpha_{j-1}}\Phi_{(j)}(y_{j-1},x_2;x_1,y_1,\ldots,y_{j-2})].\tag{4.15}$$

In the double sum with respect to r and j, y_0 means x_1. When $r = 1$, factors with S_1 and derivatives do not exist on the left of $\Phi_{(1)}$; when $r = j - 2$, factors with S_2 and derivatives do not exist on the right of $\Phi_{(1)}$, except the one preceding $S_{2(j)}$.

The kernels appearing in the right-hand side of Eq. (4.15) can be expressed through functional relationships in terms of those of the 2PGIQGFs. Going back to Eq. (2.17) and to the definition of the operator $\Sigma_{1n}[S]$ there, to which we assign now a new index, 1 or 2, according to its content of quark 1 or quark 2, the wave equation (4.7) can be rewritten in the form

$$(i\gamma.\partial_1 - m_1)\Phi_{(1)}(x_1, x_2) = \sum_{n=1}^{\infty}(\Sigma_{1,1n}[S_1] * \Phi_{(n)}) + \sum_{n=2}^{\infty}\left(\left(\frac{\delta\Sigma_{1,1n}[S_1]}{\delta S_1} * *\Phi_{(1)}\right) * S_{2,(n)}\right), \quad (4.16)$$

where the double-star operator means that after functionally taking the derivative of $\Sigma_{1,1n}[S_1]$ with respect to S_1, the latter is replaced in the same place by $\Phi_{(1)}$ and all S_1**s** on the right of $\Phi_{(1)}$ are replaced by S_2. A similar expression can also be derived for the right-hand side of Eq. (4.8), using either of the forms (2.18) or (2.19).

Equation (4.15) parallels Eq. (2.15) of the 2PGIQGF. High-index wave functions and 2PGIQGFs should in principle be eliminated in terms of the lowest-index ones using the functional relations (4.13), (4.14), and (2.14). Derivative terms of $\Phi_{(1)}$ that appear in the right-hand side of Eq. (4.15) in high-order terms could be eliminated using the first terms of the expansions through an iterative procedure. Expansion (4.15) should be completed by bringing all derivative terms to the right; the result is very similar to what is obtained in the 2PGIQGF case. Connected parts of the Wilson loop averages of high-order terms involve an increasing number of derivatives, each on a different segment of the corresponding polygonal contours [21].

In perturbation theory, each derivative of a Wilson loop introduces one multiplicative power of the coupling constant. Therefore, for the perturbative regime, terms with the smallest number of derivatives would be the dominant ones. For large distances, Wilson loop averages are expected to be dominated by minimal surfaces [3,5,10]. Here also, the dominant terms are those with the smallest number of derivatives. This suggests that the expansion (4.15) has, at least formally, a perturbative structure, the decrease in magnitude of the terms being estimated by the global number of the derivative operators. The first term of the above expansions involves a Wilson loop with one derivative, which, however, vanishes for symmetry reasons. The first leading term of the expansion is then the two-derivative term, involving a Wilson loop along a triangular contour. In two-dimensional QCD in the large-N_c limit, this term is actually the only one that survives in the above expansions and brings an indirect confirmation to the previous analysis.

Our remark [in the paragraph following Eq. (2.19)] concerning the interpretation of Eq. (2.17) and its analogues also applies to Eq. (4.16) and its analogues. Once the rigid path derivative of the wave functions has been evaluated by means of Eq. (4.15) or similar ones, then the wave functions have to be considered as ordinary complex functions of their arguments $x_1, x_2, t_1, \ldots, t_n, P$, etc. The wave equations of the type of (4.16) become integrodifferential equations.

For completeness, the normalization condition of the wave functions is presented in the appendix.

V. SPECTRAL REPRESENTATION

A two-particle wave function of colorless fields is a vertex function and satisfies, on the basis of the properties of causality and positivity of physical particle energies, an integral representation known as the Deser-Gilbert-Sudarshan (DGS) representation [25]. One might try to apply the DGS analysis to the presently defined wave functions (4.1) and (4.3). However, an immediate difficulty arises from the fact that intermediate states needed for the determination of their singularities are necessarily colored objects. Intermediate states, placed inside the operators that are present in the matrix elements (4.1) or (4.3), should have the quantum numbers of operators made of one quark (belonging to the defining fundamental representation of the color gauge group) and a certain number of gluons (belonging to the adjoint representation); these combinations cannot produce color singlet operators. Therefore, hadronic intermediate states, which are expected to form the only physical states of the theory, do not contribute to the formation of the singularities of the wave functions. One then is tempted to conclude that the wave functions are free of singularities and are entire functions. However, the equations satisfied by the GIQGFs do display momentum-space singularities generated by the free quark propagators present in them. The same difficulty also appeared in the 2PGIQGF case.

To explain the presence of singularities in the Green's functions it is necessary to admit that completeness sums may be considered with colored quark and gluon states, irrespective of the fact that the latter may not be observable as asymptotic free states. (A similar conclusion was also drawn in Ref. [31] concerning ordinary propagators.) It is the solutions of the corresponding equations which should ultimately determine their precise properties. We further assume the usual spectral and causality properties of quantum field theory. The presence of gluon fields, treated here covariantly, might introduce in addition negative norm states (but still carrying positive energies) in the completeness sum, whose main effect could be the change of sign within certain intervals of the concerned spectral function

[19,20,31]. Therefore, no positivity conditions should be imposed in advance on the spectral functions (this concerns mainly the 2PGIQGFs).

More generally, the fact that the intermediate states are colored objects puts constraints on their specific properties: the colored states, when placed inside a gauge invariant operator, separate the latter into two gauge covariant operators; their contribution should reproduce at the end gauge invariant quantities. The study of the mechanism of that operation deserves attention in future investigations.

The above general hypotheses were applied to the 2PGIQGF case and led to a generalized Källén-Lehmann representation [21,24,32]. The analytic resolution of the 2PGIQGF equation in two-dimensional QCD in the large-N_c limit has confirmed the previous hypotheses: the quarks and gluons do contribute to the spectral functions with positive energies, with the difference that their singularities are no longer represented by simple poles, but by an infinite number of branch points with a stronger power than poles [23]. Furthermore, Lehmann's positivity conditions (or inequalities) [24] remain satisfied.

It is therefore reasonable to continue to apply the same approach to the study of the spectral representation of the wave functions. Considering the wave function $\Phi_{(1)}$ [Eq. (4.3)], one can repeat the same analysis as in Ref. [21] for the 2PGIQGF. The presence of the gluon field phase factor introduces an infinite series of additional powers of the denominator of the dispersive integral. Defining total and relative coordinates as $X = \frac{1}{2}(x_1 + x_2)$ and $x = x_1 - x_2$, one can factorize the plane wave part of the wave function and introduce the internal wave function as

$$\Phi_{(1)}(P, x_1, x_2) = e^{-iP.X}\phi_{(1)}(P, x). \tag{5.1}$$

$\phi_{(1)}(P, x)$ satisfies a generalized form of the DGS representation, which we write here, for simplicity, for the total spin-0 case, ignoring the spinorial content,

$$\phi_{(1)}(P, x) = \sum_{n=1}^{\infty} i \int \frac{d^4k}{(2\pi)^4} e^{-ik.x} \int_{-1/2}^{1/2} \frac{d\beta}{2\pi} \times \int_0^{\infty} ds' \frac{H_{(1),n}(s', \beta) e^{-i\beta P.x}}{(k^2 - s' + i\varepsilon)^n}. \tag{5.2}$$

The lower bound of the spectral variable s' may actually depend on the quark masses, on P^2 and on β. The above general form might still be simplified through integrations by parts and recombinations. In particular, the sum might lead to a global fractional power. In two-dimensional QCD, the resulting power of the denominator for the 2PGIQGF case was found 3/2 [23].

Representation (5.2), or a simpler version of it, could be used for the search for solutions of the wave equations. According to the solutions that are found, its detailed properties could be better specified. Nevertheless, we do not dispose of much theoretical freedom for qualitative changes of representation (5.2) without altering some basic property of quantum field theory. That question might still be reconsidered in the light of the confinement mechanism that would be found in four-dimensional QCD.

VI. CHIRAL SYMMETRY BREAKING

In the limit of $N_f \times N_c$ massless quarks, it is expected that the chiral $SU(N_f) \times SU(N_f)$ symmetry group of QCD undergoes a spontaneous breakdown to its diagonal subgroup $SU(N_f)_V$ [33–36]. On the other hand, the Nambu–Jona-Lasinio model [37] underlines the intimate relationship that exists between chiral symmetry breaking and the dynamical mass generation of fermions. Baker, Johnson and Lee considered the case of QED and showed that a similar relationship also exists there: the generation of a nonzero mass term in the fermion self-energy in the massless limit entails the existence of a zero-mass pseudoscalar bound state solution of the Bethe-Salpeter equation [26]. They argued, however, that such a solution might not necessarily correspond to an observable boson. The latter phenomenon is actually a consequence of the axial anomaly problem in Abelian sectors [38,39] and should be avoided by considering nonsinglet sectors of a non-Abelian chiral group.

In this section we aim at showing that the relationship between dynamical mass generation of fermions and chiral symmetry breaking also exists in QCD. That question is not new and in the past many works, based on approaches using the axial-vector Ward-Takahashi identities, Dyson-Schwinger equations and instantaneous approximations of the Bethe-Salpeter kernel with confining interactions, have shown the possible validity of such a mechanism [19,20,40–47]. Our proof below is more formal and independent of any approximation of the interaction kernels.

To this end, we consider the case of massless quarks 1 and 2; in this limit the labeling of the 2PGIQGFs with the quark indices 1 and 2 becomes irrelevant and we may discard them from our notations. The 2PGIQGF $S_{(1)}$ is decomposed into vector and scalar parts [21]. The part that is most sensitive to chiral symmetry breaking is the scalar one. We isolate it by taking the anticommutator of $S_{(1)}$ with the γ_5 matrix. It satisfies the equations

$$i\gamma.\partial_1[\gamma_5, S_{(1)}(x_1, x_2)]_+ = i\gamma^\mu \frac{\bar{\delta}}{\bar{\delta}x_1^{\mu-}}[\gamma_5, S_{(1)}(x_1, x_2)]_+, \tag{6.1}$$

$$[\gamma_5, S_{(1)}(x_1, x_2)]_+(-i\gamma.\overleftarrow{\partial}_2) = -i\frac{\bar{\delta}}{\bar{\delta}x_2^{\mu+}}[\gamma_5, S_{(1)}(x_1, x_2)]_+\gamma^\mu, \tag{6.2}$$

where $[,]_+$ represents the anticommutator.

The right-hand sides of Eqs. (6.1) and (6.2) are obtained from Eq. (2.15) and its adjoint with respect to x_2 and more

schematically from Eqs. (2.17), (2.18), and (2.19). We have for the rigid path derivative with x_1,

$$\begin{aligned}&\frac{\bar{\delta}}{\bar{\delta}x_1^{\mu-}}[\gamma_5, S_{(1)}(x_1,x_2)]_+\\&\quad=\frac{\bar{\delta}F_2(x_2,x_1)}{\bar{\delta}x_1^{\mu-}}[\gamma_5, S_{(1)}(x_1,x_2)]_+\\&\qquad-\frac{\bar{\delta}^2F_3(x_2,x_1,y_1)}{\bar{\delta}x_1^{\mu-}\bar{\delta}y_1^{\alpha_1+}}\{[\gamma_5, S_{(1)}(x_1,y_1)]_+\gamma^{\alpha_1}S_{2,(2)}(y_1,x_2;x_1)\\&\qquad+S_1(x_1,y_1)\gamma^{\alpha_1}[\gamma_5, S_{(2)}(y_1,x_2;x_1)]_+\}+\ldots,\end{aligned}\tag{6.3}$$

where the dots correspond to similar terms coming from the contribution of the sum contained in Eq. (2.15). In schematic form, the equation relative to x_1 is

$$\begin{aligned}&i\gamma.\partial_1[\gamma_5, S_{(1)}]_+\\&\quad=\sum_{n=1}^{\infty}(\Sigma_{1n}[S_{(1)}]*[\gamma_5, S_{(n)}]_+)\\&\qquad+\sum_{n=2}^{\infty}\left(\left(\frac{\delta\Sigma_{1n}[S_{(1)}]}{\delta S_{(1)}}**[\gamma_5, S_{(1)}]_+\right)*S_{(n)}\right),\end{aligned}\tag{6.4}$$

where Σ_{1n} and the star operation were introduced in Eq. (2.17), while the definition of the double-star operator is the same as in Eq. (4.16), with the only difference that the quark indices, 1 or 2, are now removed from the Green's functions.

Comparing the above equation with Eq. (4.16) (in the limits $m_1 = m_2 = 0$ and removing the quark indices 1 or 2), one immediately concludes that $[\gamma_5, S_{(1)}(x_1,x_2)]_+$ satisfies the same equation as $\Phi_{(1)}(x_1,x_2)$, provided the following correspondences are also done: $[\gamma_5, S_{(n)}]_+ \to \Phi_{(n)}$, $n = 1, 2, \ldots, \infty$, the general case of n being itself established from the equations of $S_{(n)}$. A similar conclusion also holds for the adjoint equations corresponding to x_2. On the other hand, the 2PGIQGFs do not depend on the bound state total momentum vector P. Therefore, the above correspondence is possible only if the Φs are independent of P; this is possible if $P = 0$ in the corresponding bound state, which in turn implies $P^2 = 0$.

We thus arrive at the conclusion that if the 2PGIQGF $S_{(1)}$ has, in the massless quark limit, a nontrivial normalizable scalar part, then the latter, multiplied by the γ_5 matrix, is a solution of the bound state equation with zero mass, in the limit of zero total momentum. This is an indication of the existence of a pseudoscalar Goldstone boson in the bound state spectrum. The complete expression of the corresponding wave function for nonzero P should be searched for from the bound state equations themselves; that part is not given by the 2PGIQGFs.

The previous exact relationship between the scalar part of the 2PGIQGF $S_{(1)}$ and the wave function with zero total momentum could also be stated in any truncation scheme adopted as an approximation for the resolution of the corresponding equations. Equation (6.4), when written explicitly, has a structure similar to that of Eq. (4.15). Therefore, any truncation scheme in one of the equations has its equivalent truncation scheme in the other equation. This is easily checked with Eq. (6.3), where dropping the dots would amount to truncating the series beyond two derivative terms in the kernels. One checks in Eq. (4.15) that the same approximation, keeping in the right-hand side the first three terms, reproduces the structure of Eq. (6.3). This property remains true order by order of the expansion based on the number of derivatives. It allows, on practical grounds, the use of approximations that remain compatible with chiral symmetry.

It is worth noting that the scalar part of the 2PGIQGF $S_{(1)}$ does not necessarily correspond to a conventional mass term which would give rise to a pole structure in the quark Green's function. Rather, the property that quarks are confined suggests that it would possess a more complicated structure. In two-dimensional QCD, the resolution of the problem led to the appearance of an infinite set of branch points, at dynamically generated mass values $M_1, M_2, \ldots, M_n, \ldots$, with stronger singularities than simple poles [23].

The above criterion for chiral symmetry breaking is only a sufficient one. This is a consequence of the fact that the equations satisfied by Φ are linear in Φ (given the expressions of the Ss), while the equations satisfied by $[\gamma_5, S]_+$ are nonlinear, since the latter are also contained in the Ss present in the equations.

VII. SUMMARY AND CONCLUDING REMARKS

Phase factors along polygonal lines allow a simple classification of gauge invariant quark Green's functions according to the number of segments they contain on the lines and permit a systematic investigation of their properties through their equations of motion. The latter can be reexpressed as integrodifferential equations in which the kernels are essentially represented by Wilson loop averages along polygonal contours with an increasing number of sides and derivatives.

This approach was applied to the cases of two-point and four-point Green's functions in QCD, leading in the latter case to bound state equations for quark-antiquark systems. The functional relationships between the kernels of the bound state equations and those of the two-point functions were displayed. A sufficient criterion for spontaneous chiral symmetry breaking was derived, relating the Goldstone boson wave function in the zero total momentum limit with the scalar part of the two-point Green's function.

The idea of relating the perturbative degree of a kernel with the number of sides and derivatives of the Wilson loop contours offers promising perspectives for practical applications of the equations obtained thus far. In this case, the dominant terms of the interaction kernels would come from the simplest contours and the least number of derivatives. That feature is also manifest in two-dimensional

QCD in the large-N_c limit. An analysis of the structure of the present equations in two dimensions would provide us with a simplified framework for the understanding of various mechanisms at work and this in turn might serve as helpful guidance for future resolutions of the relevant problems in four dimensions.

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APPENDIX: NORMALIZATION CONDITION

The normalization condition of the wave functions is usually obtained by acting on the four-point Green's function with its inverse. In the present case, the inhomogeneous parts of the equations of motion (3.3) and (3.4) being two-point Green's functions, one needs to use, for $G_{(1)}$, simultaneously both equations. One obtains

$$\left(i\gamma.\partial_1 - m_1 - i\gamma^\mu \frac{\bar\delta}{\bar\delta x_1^{\mu -}}\right) G_{(1)}(x_1, x_2; x_2', x_1') \times \left(-i\gamma.\overleftarrow{\partial}_2 - m_2 - i\gamma^\nu \frac{\overleftarrow{\bar\delta}}{\bar\delta x_2^{\nu +}}\right) = i^2\delta^4(x_1 - x_1')\delta^4(x_2 - x_2'). \tag{A1}$$

$[F_2(x_2, x_1) = 1.]$

After calculating the effects of the rigid path derivatives, one passes to total momentum space P by taking the Fourier transform with respect to the total coordinate difference $X - X' = \frac{1}{2}(x_1 + x_2) - \frac{1}{2}(x_1' + x_2')$, while remaining in the relative coordinate spaces $x = (x_1 - x_2)$ and $x' = (x_1' - x_2')$. Designating by $L(P, x)$ the operator acting in the left-hand side of Eq. (A1), the latter can be written in condensed form as

$$L(P, x) * G_{(1)}(P, x, x') = i^2\delta^4(x - x'), \tag{A2}$$

where the star notation represents here the complete series of terms obtained by the action of $L(P, x)$ on $G_{(1)}$ as in Eqs. (2.15) and (4.15), which involves linearly the set of Green's functions $G_{(n)}$ $(n = 1, 2, \dots)$.

We specify with the label k one of the bound states of the system and isolate its contribution and that of its antiparticle in $G_{(1)}$ from the rest,

$$G_{(1)}(P, x, x') = \frac{i\phi_{(1)k}(P_k, x)\bar\phi_{(1)k}(P_k, x')}{P^2 - P_k^2 + i\varepsilon} + iR_{(1)k}(P, x, x'), \tag{A3}$$

where the ϕs represent the internal part of the wave functions, as defined in Eq. (5.1). [The chronological product being evaluated with the time components of the xs, we have also $\mathbf{P_k} = \mathbf{P}$.] Decomposition (A3) is replaced in Eq. (A2) and then both sides of the latter are multiplied with $\bar\phi_{(1)k}(x)$ and integrated with respect to x, with the trace on the spinor indices taken. Taking into account the fact that the operator $L(P_k)$ annihilates $\phi_{(1)k}$, one obtains

$$\bar\phi_{(1)k}(x)\left(\frac{L(P, x) - L(P_k, x)}{P^2 - P_k^2 + i\varepsilon}\right) * \phi_{(1)k}(x)\bar\phi_{(1)k}(x') + \bar\phi_{(1)k}(x)L(P, x) * R_{(1)k}(P, x, x') = i\bar\phi_{(1)k}(x'). \tag{A4}$$

At this stage, when working with conventional Green's functions, one takes the limit $P^2 \to P_k^2$. The convolutive nature of $L(P)$ then allows it also to act on the left and the remainder contribution disappears from the equation. In the present case, the operator $L(P)$ acts specifically on the right and without further information it is not entitled to be converted to the left.

To go further, we consider, as in Eq. (A1), equations of motion of $G_{(1)}$ concerning its two right arguments and define correspondingly an operator $\overleftarrow{L}(P)$ acting from the right,

$$G_{(1)}(P, x'', x) * \overleftarrow{L}(P, x) = i^2\delta^4(x'' - x). \tag{A5}$$

From Eqs. (A2) and (A5) one deduces

$$\begin{aligned} &G_{(1)}(P, x'', x)L(P, x) * G_{(1)}(P, x, x') \\ &\quad = G_{(1)}(P, x'', x) * \overleftarrow{L}(P, x)G_{(1)}(P, x, x') \\ &\quad = i^2 G_{(1)}(P, x'', x'), \end{aligned} \tag{A6}$$

which entails a weak form of conversion of the operator $L(P)$ from right to left. Since this result is true for any P, x'' and x', one might adopt the assumption that it remains true also for parts of $G_{(1)}$. Adopting the latter assumption, and taking the limit $P^2 \to P_k^2$ in Eq. (A4), one obtains the two relations,

$$\frac{1}{i}\int d^4x\bar\phi_{(1)k}(x)\frac{\partial L(P, x)}{\partial P^2}\bigg|_{P^2 = P_k^2} * \phi_{(1)k}(x) = 1, \tag{A7}$$

$$\int d^4x\bar\phi_{(1)k}(x)L(P_k, x) * R_{(1)k}(P_k, x, x') = 0, \tag{A8}$$

which display the normalization and orthogonality conditions. In Eq. (A7), the derivative of $L(P, x)$ with respect to P^2 is understood in the sense that one first evaluates the effect of $L(P)$ on $\phi_{(1)k}$ and then takes in the resulting kernels the corresponding derivative.

Isolating in $R_{(1)k}$ the contribution of another bound state, specified by a label m, and assuming its independence from the rest, one obtains the more precise orthogonality relation,

$$\int d^4x\bar\phi_{(1)k}(x)\left(\frac{L(P_k, x) - L(P_m, x)}{P_k^2 - P_m^2}\right) * \phi_{(1)m}(x) = 0, \quad m \neq k. \tag{A9}$$

We notice that the free part of the normalization kernel in Eq. (A7), coming from the two Dirac operators, is the same as for the Bethe-Salpeter wave functions [29]. In the nonrelativistic limit, decomposing ϕ into 2×2 components, those which are dominant satisfy the properties $\gamma_0 \phi = -\phi \gamma_0 = \phi$.

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