

Dualities and bosonization of massless fermions in three-dimensional space-timeE. F. Moreno^{1,*} and F. A. Schaposnik^{2,†}¹*Department of Physics, Northeastern University Boston, Massachusetts 02115, USA*²*Departamento de Física, Universidad Nacional de La Plata, Instituto de Física La Plata C.C. 67, 1900 La Plata, Argentina*
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We study the bosonization of massless fermions in three-dimensional space-time. Using the path-integral approach as well as the operator formalism, we investigate new duality relations between fermionic and bosonic theories. In particular, we show that a theory of massless fermions is dual, within a quadratic approximation in the fields, to three different but equivalent bosonic theories: a nonlocal Maxwell–Chern–Simons–type theory, a nonlocal self-dual–type vector theory, and a local free massless bosonic theory. The equivalence is proven at the level of current correlation functions and current algebra analysis.

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I. INTRODUCTION

Bosonization of fermionic models in $d = 3$ space-time dimensions has been discussed in the past following different approaches. Based in an order-disorder duality, Marino [1] was able to completely express a massless $d = 3$ fermion field in terms of a bosonic vector field. In this way, fermionic current correlation functions can be reproduced in the framework of the bosonic theory described by a quadratic Lagrangian which contains a Chern–Simons and a nonlocal Maxwell-like term.

Concerning massive fermions, bosonization rules were established in Ref. [2] within the path-integral framework. The dual bosonic Lagrangian, calculated to leading order in $1/m$ (with m the fermion mass) was shown to correspond to a Chern–Simons Lagrangian coupled to an ordinary (local) Maxwell term. The same result was obtained calculating the one-loop quadratic part of the effective bosonic action, a method which is valid for arbitrary mass m [3] and reproduces in the $m \rightarrow 0$ limit the massless result. The current algebra of such a quadratic bosonic dual theory was proven to exactly reproduce the fermionic current algebra [4], and the extension to the non-Abelian case was developed in Ref. [5]. Both in the massless and in the massive cases, the Chern–Simons term, introduced in the pioneering works of Deser, Jackiw, and Templeton [6], plays a central role in all these results.

There has been recently a renewed interest in $d = 3$ bosonization after the report of new results on Chern–Simons theories coupled to scalars and fermion fields, in the large N limit [7–11] and their relation with Vassiliev higher spin theories [12,13]. In particular, Aharony *et al.* [10] found, exploiting the AdS/CFT correspondence, that the theory of N massless scalars coupled to a level k $U(N)$ Chern–Simons term is equivalent, for large N , to the

Legendre transform of a theory of k massless fermions coupled to a level N $U(k)$ Chern–Simons term. Moreover, it has been conjectured that the equivalence of the two theories could be valid also at finite N [10].

It is the purpose of this paper to discuss the fermion/boson connection in $d = 3$ for massless fields using the approach developed in Refs. [2–5]. In particular, we search for new duality relations that may provide a link with the results of Ref. [10]. Our work is organized as follows. We review in Sec. II the derivation of bosonization rules for massless fermions in $d = 3$ space-time dimensions using the path-integral approach [2–4]. We then derive in Sec. III new dualities between the massless fermion theory and different bosonic theories with actions related to the Maxwell–Chern–Simons theory [6], the self-dual theory [14], and a theory of decoupled scalars with a higher-derivative kinetic term. In Sec. IV we show the equivalence between one of the bosonic duals of massless fermions in $2 + 1$ dimensions and a massless spin zero bosonic model, thus establishing a fermion-boson duality which resembles that found in Ref. [10] particularized to the $U(1)$ case. Finally, we discuss our results in Sec. V.

II. BOSONIZATION OF MASSLESS FERMIONS IN THREE SPACE-TIME DIMENSIONS IN TERMS OF VECTOR FIELDS

We work within the path-integral approach focusing in the Abelian case where the bosonization rule for fermion currents takes the form, in $d = 3$ dimensional Euclidean space-time [1,2],

$$j_{\mu}(x) \equiv i \bar{\psi}(x) \gamma_{\mu} \psi(x) \rightarrow -\frac{1}{\sqrt{4\pi}} \epsilon_{\mu\nu\alpha} \partial_{\nu} A_{\alpha}(x),$$

$$\mu = 1, 2, 3. \quad (1)$$

The vector field A_{μ} is the bosonic field dual to the original fermion field. An analogous formula for general space-time dimension d was obtained in Ref. [4],

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$$j_\mu(x) \rightarrow C_d \epsilon_{\mu_1 \mu_2 \dots \mu_d} \partial_{\mu_2} \chi_{\mu_3 \dots \mu_d}(x), \quad (2)$$

with C_d a normalization constant and $\chi_{\mu_3 \dots \mu_d}(x)$ an antisymmetric $d - 2$ Kalb–Ramond field in d dimensional space-time [15]. Note that dualization of antisymmetric Kalb–Ramond fields in terms of derivatively coupled scalars is possible for all d [16,17]; this could open a route to $d = 3$ bosonization in terms of scalar bosons as in Ref. [10]. We shall see, however, that in this case the scalar field is trivially decoupled in the path-integral generating functional of current correlation functions, a result already discussed in Ref. [15].

It should be stressed that although the bosonization rules (1) and (2) are exact, the bosonic Lagrangian dual to the original free fermionic theory is not in general known in closed form since, as we shall see, this would require having an exact expression for the determinant of the Dirac operator in a gauge field background. As it is well-known, only in the $d = 2$ case the determinant can be calculated exactly. Concerning the $d = 3$ case, we recall that the parity-odd contribution to such a determinant—a Chern-Simons term—is known in closed form (see for example Ref. [18]), while the parity-even one can only be calculated using some approximation method, namely order by order in powers of the bosonic fields, using the ζ -function method in the case of massless fermions, or in powers of ∂/m for the case of fermions with mass m . Now, it is precisely the Chern-Simons term that induces, on the bosonic side, the current bosonization rule given in Eq. (1) and *a fortiori* the current algebra. Moreover, that is the reason why it reproduces, in the bosonic dual theory, the exact Schwinger term result for the equal-time current commutators of the original $d = 3$ fermionic theory calculated in Ref. [19].

Let us briefly describe the main steps leading to Eq. (1). The partition function for $d = 3$ free massless fermions in the presence of an external source s_μ for the fermion current is

$$Z_{\text{fer}}[s_\mu] = \int D\bar{\psi} D\psi \exp\left(\int d^3x \bar{\psi}(i\not{\partial} + \not{s})\psi\right). \quad (3)$$

Performing a local change (Jacobian free) of the fermionic variables, $\psi(x) \rightarrow g(x)\psi(x)$, $g(x) \in U(1)$ gives

$$\begin{aligned} Z_{\text{fer}}[s_\mu] &= \int D\bar{\psi} D\psi \exp\left(\int d^3x \bar{\psi}(i\not{\partial} + \not{s} + ig^{-1}\not{\partial}g)\psi\right) \\ &= \int Db_\mu D\bar{\psi} D\psi \exp\left(\int d^3x \bar{\psi}(i\not{\partial} + \not{s} + \not{b})\psi\right) \\ &\quad \times \delta[f_\mu[b]], \end{aligned} \quad (4)$$

where $f_\mu(b) = \epsilon_{\mu\alpha\beta} \partial_\alpha b_\beta$ and the δ function in the second line ensures that b is a flat connection, so that integration on b_μ reproduces the first line.

Integrating out fermions and after the trivial shift $b_\mu + s_\mu \rightarrow b_\mu$, the partition function becomes

$$Z_{\text{fer}}[s] = \int Db_\mu \det(i\not{\partial} + \not{b}) \delta[f_\mu[b] - f_\mu[s]]. \quad (5)$$

It is at this point that the bosonic field dual to the original fermion enters into play. Indeed, representing the delta function in Eq. (5) in the form

$$\begin{aligned} \delta[f_\mu[b] - f_\mu[s]] &= \int DA_\mu \exp\left(i \int d^3x (f_\mu[b] - f_\mu[s])A_\mu\right), \end{aligned} \quad (6)$$

the fermionic partition function can be finally written as a bosonic partition function $Z_{\text{bos}}[s]$,

$$Z_{\text{fer}}[s] = Z_{\text{bos}}[s], \quad (7)$$

where

$$Z_{\text{bos}}[s] \equiv \int DA_\mu \exp(-S_{\text{bos}}[A]) \exp\left(-i \int d^3x f_\mu[s]A_\mu\right) \quad (8)$$

with the bosonic action given by

$$\begin{aligned} \exp(-S_{\text{bos}}[A]) &= \int Db_\mu \det(i\not{\partial} + \not{b}) \\ &\quad \times \exp\left(i \int d^3x f_\mu[b]A_\mu\right). \end{aligned} \quad (9)$$

We see that the bosonic field A_μ , dual to the original fermion field, arises as the Lagrange multiplier enforcing the flatness condition for the auxiliary field b_μ in Eq. (5).

Equation (8) shows that the external source s_μ has factored out from the auxiliary field b_μ integration so that already at this stage one can obtain the exact bosonization rule (1) by differentiating $Z_{\text{bos}}[s]$:

$$\langle j_\mu(x) \rangle = \frac{1}{Z} \frac{\delta Z}{\delta s^\mu(x)} \Big|_{s=0} = \left\langle -\frac{1}{\sqrt{4\pi}} \epsilon_{\mu\nu\alpha} \partial_\nu A_\alpha(x) \right\rangle. \quad (10)$$

Of course one has still to determine the bosonic action $S_{\text{bos}}[A]$ so as to be able to effectively compute the vacuum expectation value in the rhs in Eq. (10).

As already pointed out above, the derivation we have presented can be easily extended to any number of space-time dimensions, in which case instead of the vector field A_μ , the field dual to the d -dimensional fermion will be an antisymmetric Kalb–Ramond field $\chi_{\mu_3 \dots \mu_d}$ with the bosonization rule for the fermion current given in Eq. (2). Again, the dual bosonic field $\chi_{\mu_3 \dots \mu_d}$ arises as a Lagrange multiplier with its rank determined from the form of the flatness conditions of the auxiliary field strength $f_{\mu\nu}[b]$ in different space-time dimensions.

It is interesting to note that in the $d = 2$ case with an appropriate choice of constant C_2 , the bosonization rule Eq. (2) takes well-honored form [20],

$$j_\mu(x) \rightarrow -\frac{1}{\sqrt{\pi}} \epsilon_{\mu\nu} \partial_\nu \phi(x). \quad (11)$$

The rhs of this formula is the topological current in $d = 2$ space-time dimensions constructed from a bosonic scalar ϕ . Analogously, the rhs in Eq. (1) is nothing but the topological current in $d = 3$ dimensions for the vector field A_μ .

As explained above, the parity-even part of the fermion determinant appearing in the definition of $S_{\text{bos}}[A]$, Eq. (9), cannot be computed in closed form for $d > 2$. One can perform a quadratic approximation as developed in Ref. [3], or, following Ref. [1], one can use an order-disorder operator approach.

The quadratic approximation, and in general the field expansion that we propose, can be interpreted in an effective field theory approach *à la* Weinberg (see for example Refs. [21,22]) and corresponds to a low energy expansion of the Lagrangian. This is evident in massive theories where the decoupling theorem ensures that higher order terms at energy E are suppressed by powers of E/mass . For a massless theory the problem is slightly subtler: there is in principle no natural mass to use in the expansion, and also the low energy terms can be nonlocal. Nevertheless, the effective field theory program can be implemented also in this case (see Ref. [23] for a discussion). In general one uses the renormalization mass scale μ (needed by dimensional arguments) as the expansion parameter. In our case we introduce such a mass scale through a constant e with units of $\text{mass}^{1/2}$. This parameter is needed by dimensional grounds (the field b_μ has units of mass, so we need a dimensionful parameter), and, moreover, this constant can be related to a coupling constant in interacting theories. Indeed, it appears as the dimensionful three-dimensional electric charge when the fields interact with an electromagnetic field or as the coupling constant multiplying the auxiliary vector field one introduces to handle the quartic self-interaction term in Thirring-like models. Thus, for the massless free fermion theory, the natural low energy expansion in our approach will be one in powers of E/e^2 , and hence the quadratic field expansion corresponds to the leading order approximation in the low energy expansion. The physical results of our bosonization rules will be valid in this regime and could then be directly applicable to study interacting theories.

Inserting the results of Ref. [3] for the fermion determinant $\det(\not{\partial} + \not{b})$ in (9) and integrating over b_μ , one then gets

$$\begin{aligned} S_{\text{bos}}[A] &= \int d^3x \left(\frac{1}{2\pi} F_{\mu\nu} T^{-1} F_{\mu\nu} - \frac{i}{\pi} \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \right) \\ &\equiv S_{\text{TMCS}}[A]. \end{aligned} \quad (12)$$

We again stress that the Chern-Simons parity-violating term does not pick up higher order corrections. Concerning the Maxwell-like term, which does receive

corrections, we denote with T^{-1} the Green function of the square root of the (minus) Euclidean d’Alambertian (i.e., the $d = 3$ Laplacian),

$$T \equiv (-\square)^{1/2}, \quad (T * T^{-1})_{x,y} = \delta^{(3)}(x - y). \quad (13)$$

Since we are working in Euclidean space, one has

$$\square = \partial_0^2 + \nabla^2, \quad \nabla^2 = \partial_{x_1}^2 + \partial_{x_2}^2 \quad (14)$$

$$-\square \exp(ikx) = |k|^2 \exp(ikx) \quad (15)$$

and using the natural definition of the square root of the Laplacian; it can be proved that

$$T \exp(ikx) = |k| \exp(ikx) \quad (16)$$

and

$$T^{-1}(x, 0) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{|k|} \exp(ikx). \quad (17)$$

A rigorous definition and properties of the square root of Laplacians can be found in Refs. [24,25].

Were fermions massive, one would have a Maxwell term $(1/(|m|)F_{\mu\nu}^2$ instead of the first term in Eq. (12). In this respect, since in the massless fermionic case there is no dimensionful parameter, it is natural to find in the second order term of the even-parity contribution to the fermion determinant a T^{-1} factor leading to the correct dimensionless Maxwell-like term.

Note that while the fermionic action is invariant under global $U(1)$ transformations, the bosonic action exhibits a local $U(1)$ gauge invariance, provided A_μ is endowed with appropriate boundary conditions. This invariance is a consequence of the natural choice of the gauge-invariant procedure adopted in the path-integral bosonization approach that we employed and also arises within the order-disorder operator algebra approach of Ref. [1]. As it will be discussed below, there is a regularization ambiguity parameter α that for the moment we have chosen as $\alpha = i$ in the previous formulas. It should be stressed that, in spite of being nonlocal, action S_{TMCS} is perfectly manageable. Indeed, the choice of a retarded prescription shows that the nonlocality preserves causality, and a canonical quantization of the theory can be formulated [26].

Returning to the bosonization rule (1), one can easily see that it reproduces the correct commutation relations, both for the fermionic and the bosonic dual theories. Indeed, the well-known current-current commutators for fermions giving a Schwinger term, which in $d = 3$ takes the form [19]

$$[j_0(\vec{x}, t), j_i(\vec{y}, t)] = -\frac{1}{8\pi} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \partial_i \delta^{(2)}(\vec{x} - \vec{y}), \quad (18)$$

can be obtained from the bosonic theory if, according to Eq. (1), one identifies

$$j_0 \rightarrow \frac{1}{\sqrt{4\pi}} B, \quad j_i \rightarrow \frac{1}{\sqrt{4\pi}} \epsilon_{ij} E_j \quad (19)$$

and uses the bosonic action S_{TMCS} to compute current commutators using the Bjorken–Johnson–Low approach. We have written $E_i = F_{i0}$ the electric field and $B = \epsilon_{ij}\partial_i A_j$ ($i, j = 1, 2$) the magnetic field. Interestingly enough, from this point of view, the origin of the Schwinger term in the free fermion theory can be traced back to the appearance of a Chern-Simons term in the dual bosonic action (see Ref. [27] for a recent discussion on this issue).

III. DUALITIES AMONG BOSONIC ACTIONS

We shall now see that the action S_{TCSM} is dual to a self-dual bosonic action. This was proven for massive fermions in Ref. [2]; here we extend the result to the massless case. To this end, let us start by considering the Maxwell-like term in the bosonic action (12),

$$\frac{1}{2\pi} \int d^3x F_{\mu\nu}[A] T^{-1} F_{\mu\nu}[A] = \frac{1}{2} \int d^3x J_\mu[A] T^{-1} J_\mu[A], \quad (20)$$

where

$$J_\mu[A] = \sqrt{\frac{2}{\pi}} \epsilon_{\mu\nu\alpha} \partial_\alpha A_\beta. \quad (21)$$

Using the identity

$$\begin{aligned} & \exp\left(-\int d^3x J_\mu[A] T^{-1} J_\mu[A]\right) \\ &= \int Da_\mu \exp\left(-\frac{1}{2} a_\mu T a_\mu + a_\mu J_\mu[A]\right), \end{aligned} \quad (22)$$

$Z_{\text{bos}}[s]$ can be written as

$$\begin{aligned} Z_{\text{bos}}[s] &= \int DA_\mu Da_\mu \exp\left(+\frac{1}{2} a_\mu T a_\mu - a_\mu J_\mu[A]\right. \\ &\quad \left.+ \frac{i}{\pi} \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \frac{1}{\sqrt{4\pi}} \epsilon_{\mu\nu\lambda} s_\mu \partial_\nu A_\lambda\right). \end{aligned} \quad (23)$$

If we now proceed to integrate over A_μ , we get

$$\begin{aligned} & \int DA \exp\left(-\int d^3x \left(\frac{i}{\pi} \epsilon_{\mu\nu\alpha} A_\mu \partial_\nu A_\alpha\right.\right. \\ & \quad \left.\left.- 2i \epsilon_{\mu\nu\alpha} \left(a_\mu - \frac{1}{\sqrt{8}} s_\mu\right) \partial_\nu A_\alpha\right)\right) \\ &= \exp\left(-\frac{i}{2} \int d^3x \epsilon_{\mu\nu\alpha} \left(a_\mu \partial_\nu a_\alpha - \frac{2}{\sqrt{8}} s_\mu \partial_\nu a_\alpha\right.\right. \\ & \quad \left.\left.- \frac{1}{8} s_\mu \partial_\nu s_\alpha\right)\right) \end{aligned} \quad (24)$$

so that we end with a generating functional of the form

$$\begin{aligned} Z_{\text{bos}}[s] &= \int Da_\mu \exp\left(-\int d^3x \left(\frac{1}{2} a_\mu T a_\mu + \frac{i}{2} \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda\right)\right) \\ & \quad \times \exp\left(-\int d^3x \left(\frac{1}{\sqrt{2}} s_\mu \partial_\nu a_\alpha + \frac{1}{8} s_\mu \partial_\nu s_\alpha\right)\right) \\ & \equiv Z_{\text{TSD}}[a; s]. \end{aligned} \quad (25)$$

In this way we have proven at the quantum level an exact duality identity between the bosonic actions S_{TMCS} and S_{TSD} defined as:

$$\begin{aligned} S_{\text{TMCS}} &= \frac{1}{2\pi} \int d^3x (F_{\mu\nu} T^{-1} F_{\mu\nu} - 2i \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda) \\ S_{\text{TSD}} &= \int d^3x \left(\frac{1}{2} a_\mu T a_\mu + \frac{i}{2} \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda\right), \end{aligned} \quad (26)$$

to be compared with the quantum duality between self-dual (SD) and Maxwell–Chern-Simons (MCS) models established in Ref. [14] within a canonical approach and in Refs. [2,28] within the path-integral approach in the massive case,

$$\begin{aligned} S_{\text{MCS}} &= \frac{1}{2\pi} \int d^3x \left(\frac{1}{|m|} F_{\mu\nu} F_{\mu\nu} - 2i \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda\right) \\ S_{\text{SD}} &= \int d^3x \left(\frac{1}{2} m a_\mu a_\mu + \frac{i}{\pi} \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda\right). \end{aligned} \quad (27)$$

Let us end this section by describing a different dualization that is possible due to the fact that one can always trade antisymmetric Kalb–Ramond fields (in this case just a vector field) to scalars with a higher-derivative Lagrangian [16,17]. Following Ref. [15] we start from Eq. (5) and add a gauge fixing term for b_μ ,

$$Z_{\text{fer}}[s_\mu] = \int Db_\mu \det(i\not{\partial} + \not{s} + \not{b}) \delta(f_\mu(b)) e^{\frac{1}{2\zeta} \int d^3x (\partial \cdot b)^2}. \quad (28)$$

To enforce the constraint we introduce as before the vector field A_μ ,

$$\begin{aligned} Z_{\text{fer}}[s_\mu] &= \int DA_\mu Db_\mu \det(i\not{\partial} + \not{s} + \not{b}) \\ & \quad \times e^i \int d^3x A_\mu f_\mu(b) e^{\frac{1}{2\zeta} \int d^3x (\partial \cdot b)^2}. \end{aligned} \quad (29)$$

The first exponential factor in the rhs can be written as $\exp(i \int d^3x b_\mu f_\mu(A))$ making evident that the field A_μ is also a gauge field. Shifting $b_\mu \rightarrow b_\mu - s_\mu$ we absorb the source in the determinant obtaining

$$\begin{aligned} Z_{\text{fer}}[s_\mu] &= \int DA_\mu Db_\mu \det(i\not{\partial} + \not{b}) \\ & \quad \times e^i \int d^3x A_\mu (f_\mu(b) - f_\mu(s)) e^{\frac{1}{2\zeta} \int d^3x (\partial \cdot (b-s))^2}. \end{aligned} \quad (30)$$

The next step is to integrate the field b_μ . To this end we shall work to lowest order in the fields so that the fermion determinant becomes

$$\det(i\not{\partial} + \not{b}) = \exp\left\{-\int d^3x b_\mu \Pi_{\mu\nu} b_\nu\right\}, \quad (31)$$

where $\Pi_{\mu\nu}$, is the $d = 3$ vacuum polarization,

$$\Pi_{\mu\nu} = \frac{1}{\pi}(\partial_\nu T^{-1} \partial_\mu - \delta_{\mu\nu} \partial_\alpha T^{-1} \partial_\alpha) + \frac{1}{\pi} \epsilon_{\mu\nu\alpha} \partial_\alpha. \quad (32)$$

As it should be, the vacuum polarization is transverse, $\partial_\mu \Pi_{\mu\nu} = 0$.

Substituting the value of the fermion determinant in the partition function, we get

$$Z_{\text{fer}}[s_\mu] = \int DA_\mu Db_\mu e^{-S[b,A,s]}, \quad (33)$$

where

$$S[b, A, s] = \int d^3x \left\{ b_\mu \hat{\Pi}_{\mu\nu} b_\nu + b_\mu \left(-if_\mu(A) + \frac{1}{\zeta} \partial_\mu \partial \cdot s \right) + is_\mu f_\mu(A) - \frac{1}{2\zeta^2} s_\mu \partial_\mu \partial_\nu s_\nu \right\} \quad (34)$$

and $\hat{\Pi}_{\mu\nu} = \Pi_{\mu\nu} - \frac{1}{2\zeta} \partial_\mu \partial_\nu$. Since this action is quadratic, the path integral over b_μ can be performed in closed form leading to

$$Z_{\text{fer}}[s_\mu] = \int DA_\mu e^{-S_2[A,s]}, \quad (35)$$

where

$$S_2[A, s] = \int d^3x \left\{ \frac{1}{4} (f_\mu(A) \hat{\Pi}_{\mu\nu}^{-1} f_\nu(A)) + if_\mu(A) \left(\frac{1}{2\zeta} \hat{\Pi}_{\mu\nu}^{-1} \partial_\nu (\partial \cdot s) + s_\mu \right) - \frac{1}{4\zeta^2} (\partial_\mu (\partial \cdot s) \hat{\Pi}_{\mu\nu}^{-1} \partial_\nu (\partial \cdot s)) - \frac{1}{2\zeta^2} (s_\mu \partial_\mu \partial_\nu s_\nu) \right\}. \quad (36)$$

Now we trade the integration over A_μ by an integration over $f_\mu(A)$ using the identity

$$\int DAF[f_\mu(A)] = \int DfF[f] \delta(\partial \cdot f). \quad (37)$$

Introducing a scalar field φ to enforce the δ function, we then get

$$Z_{\text{fer}}[s_\mu] = \int Df_\mu D\varphi e^{-S_3[f,\varphi,s]}, \quad (38)$$

with

$$S_3[f, s] = \int d^3x \left\{ \frac{1}{4} (f_\mu \hat{\Pi}_{\mu\nu}^{-1} f_\nu) + if_\mu \left(\frac{1}{2\zeta} \hat{\Pi}_{\mu\nu}^{-1} \partial_\nu (\partial \cdot s) + s_\mu + \partial_\mu \varphi \right) - \frac{1}{4\zeta^2} (\partial_\mu (\partial \cdot s) \hat{\Pi}_{\mu\nu}^{-1} \partial_\nu (\partial \cdot s)) - \frac{1}{2\zeta} (s_\mu \partial_\mu \partial_\nu s_\nu) \right\}. \quad (39)$$

The integral over f_μ is Gaussian and can be performed, leading to

$$Z_{\text{fer}}[s_\mu] = \int D\phi e^{-S_{\text{eff}}[\varphi,s]}, \quad (40)$$

with

$$S_{\text{eff}} = \int d^3x \left\{ \left(s_\mu + \frac{1}{2\zeta} \hat{\Pi}_{\mu\alpha}^{-1} \partial_\alpha \partial \cdot s + \partial_\mu \varphi \right) \times \hat{\Pi}_{\mu\nu} \left(s_\nu + \frac{1}{2\zeta} \hat{\Pi}_{\nu\beta}^{-1} \partial_\beta \partial \cdot s + \partial_\nu \varphi \right) + \frac{1}{2\zeta^2} s_\mu \partial_\mu \partial_\nu s_\nu + \frac{1}{4\zeta^2} s_\mu \partial_\mu \partial_\alpha \hat{\Pi}_{\alpha\beta}^{-1} \partial_\beta \partial_\nu s_\nu \right\}. \quad (41)$$

Expanding this last expression, and using that $\Pi_{\mu\nu}$ is transverse, we finally have

$$Z_{\text{fer}}[s_\mu] = \int D\varphi \exp \left\{ -\int d^3x \left(s_\mu \Pi_{\mu\nu} s_\nu - \frac{1}{2\zeta} \varphi \square \square \varphi \right) \right\}. \quad (42)$$

The source is decoupled from the field φ which has a higher-derivative propagator, a result which is identical to the one found in Ref. [15] in any number of dimensions including $d = 3$. One should note that the s_μ term reproduces the correct result for the fermionic current algebra so that, as observed in Ref. [15], its functional integral may be completely ignored, contributing as it does just a field-independent overall constant.

IV. BOSONIZATION IN TERMS OF A CANONICAL FREE FIELD ϕ

It has been proven in Ref. [6] that the field equations and the commutation relations of the massive quantum Maxwell–Chern–Simons theory are solved by a canonical, free, massive spin 1 field (with the mass coinciding with the Chern–Simons action coefficient). Following the approach developed in Ref. [6], we will show the equivalence between the bosonic model with action $S_{\text{TMCS}}[A]$ defined in Eq. (12) and a massless free field model. Our results can be seen as establishing a fermion-boson duality which agrees, in the $U(1)$ case, with that discussed in Ref. [10].

Let us start from our the bosonized Lagrangian containing Maxwell-like and a Chern–Simons actions:

$$S_{\text{TMCS}}[A] = \int d^3x \left(\frac{1}{2} F_{\mu\nu} T^{-1} F_{\mu\nu} - \alpha \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \right). \quad (43)$$

Here we have reintroduced the regularization parameter α previously fixed to a particular value. The Gauss law derived from action (43) takes the form

$$\partial_i (T^{-1} E_i) = -2\alpha B, \quad (44)$$

where as before $E_i = F_{i0}$ and $B = \epsilon_{ij} \partial_i A_j$.

Following Ref. [6] we now propose to solve this constraint in terms of a canonical free field ϕ ,

$$E_i = \epsilon_{ij} \mathbf{A} \partial_j \Pi + \mathbf{B} \partial_i \phi \quad B = \mathbf{C} \phi, \quad (45)$$

where \mathbf{A} , \mathbf{B} , and \mathbf{C} , are three kernels to be determined. Using this expression to write the Gauss law, we obtain

$$\partial_i (T^{-1} * E_i) = \nabla^2 T^{-1} * \mathbf{B} * \phi$$

so

$$\nabla^2 T^{-1} * \mathbf{B} = -2\alpha \mathbf{C}. \quad (46)$$

Given action (43), the corresponding Hamiltonian reads

$$H = \frac{1}{2} \int d^2x (E_i T^{-1} E_i + B T^{-1} B).$$

Writing $\mathbf{B} = a T^{3/2} \hat{T}^{-1}$ and choosing $\hat{T} = (-\nabla^2)^{-1/2}$ with a an arbitrary parameter, one can see that the magnetic term in the Hamiltonian can be written as

$$\int d^2x B T^{-1} B = -\frac{a^2}{4\alpha^2} \int d^2x \phi \nabla^2 \phi. \quad (47)$$

Note that \hat{T} contains solely a spacial derivative. To obtain this result we have used that T and \hat{T} commute (which can be proven following Refs. [24,25]).

Concerning the electric contribution to the Hamiltonian,

$$\begin{aligned} \int d^2x E_i T^{-1} E_i &= \int d^2x (\mathbf{B} \partial_i \phi) (T^{-1} \mathbf{B} \partial_i \phi) \\ &+ 2 \int d^2x (\epsilon_{ij} \mathbf{A} \partial_j \Pi) (T^{-1} \mathbf{B} \partial_i \phi) \\ &+ \int d^2x (\epsilon_{ij} \mathbf{A} \partial_j \Pi) (T^{-1} \epsilon_{ik} \mathbf{A} \partial_k \Pi), \end{aligned} \quad (48)$$

one has

$$\begin{aligned} \int d^2x (\mathbf{B} \partial_i \phi) (T^{-1} \mathbf{B} \partial_i \phi) &= a^2 \int d^2x \phi \square \phi \\ \int d^2x (\epsilon_{ij} \mathbf{A} \partial_j \Pi) (T^{-1} \mathbf{B} \partial_i \phi) &= 0 \\ \int d^2x (\epsilon_{ij} \mathbf{A} \partial_j \Pi) (T^{-1} \epsilon_{ik} \mathbf{A} \partial_k \Pi) &= \int d^2x \Pi^2. \end{aligned} \quad (49)$$

To get the third line result we have chosen $\mathbf{A} = T^{1/2} * \hat{T}^{-1}$.

With all this one has for the dual Hamiltonian

$$\begin{aligned} H &= \frac{1}{2} \int d^2x (E_i * T^{-1} * E_i + B * T^{-1} * B) \\ &= \frac{1}{2} \int d^2x \left(\Pi^2 + a^2 \phi \square \phi - \frac{a^2}{4\alpha^2} \phi \nabla^2 \phi \right) \\ &= \frac{1}{2} \int d^2x \left(\Pi^2 - a^2 \dot{\phi}^2 + a^2 \left(1 - \frac{1}{4\alpha^2} \right) \phi \nabla^2 \phi \right) \end{aligned} \quad (50)$$

so that, choosing the arbitrary parameter a as $a = 2\alpha$, we get

$$H = \frac{1}{2} \int d^2x \left(\Pi^2 - 4\alpha^2 \dot{\phi}^2 + a^2 \left(1 - \frac{1}{4\alpha^2} \right) \phi \nabla^2 \phi \right).$$

Using $\partial H / \partial \Pi = \dot{\phi}$ one gets $\dot{\phi} = \Pi$ leading to

$$H = \frac{(1 - 4\alpha^2)}{2} \int (\Pi^2 + \phi (-\nabla^2) \phi), \quad (51)$$

which is the Hamiltonian for a massless boson. A field ϕ redefinition together with a consistent choice of regularization parameter α leads to the free field massless Lagrangian

$$L_\phi = \int d^3x (\partial_\mu \phi)^2. \quad (52)$$

With the choice of \mathbf{A} , \mathbf{B} y \mathbf{C} that we made, the electric and magnetic fields, as given by Eq. (45), become

$$E_i = T^{1/2} (\epsilon_{ij} \hat{T}^{-1} \partial_j \Pi + 2\alpha T \hat{T}^{-1} \partial_i \phi) \quad (53)$$

$$B = T^{1/2} \hat{T} \phi. \quad (54)$$

We can also express the generator of spatial translations of the theory in terms of ϕ finding that it coincides with that of a scalar field,

$$P^i = \int d^2x \epsilon^{ij} E^j T^{-1} B = \int d^2x \Pi \partial_i \phi. \quad (55)$$

Massless representations of the three-dimensional Poincaré group describe particles with no spin [29]. This implies that the parity-violating massless Dirac theory as well as its dual bosonic theory with Hamiltonian (51) describes spinless fermions and bosons respectively (see Ref. [6] for a discussion on this issue).

V. DISCUSSION

We summarize in Fig. 1 the duality relations that we established in this work between generating functionals for current correlation functions of a free massless fermion model in $d = 3$ space-time dimensions and several bosonic models. We have not included in the graph the connection found at the end of Sec. III with a higher-derivative theory involving purely scalars since in that case the scalar field completely decouples from the external source. Although in that case the scalar can be ignored, the dependence of the

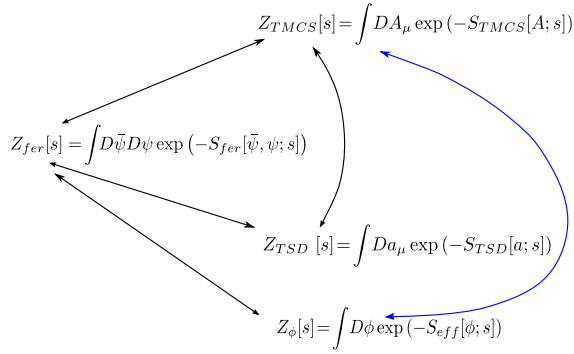


FIG. 1 (color online). Dualities between the free fermion model and the bosonic dual models.

complete bosonic generating function on the source (which can be thought of as an electromagnetic potential) leads to the correct massless fermions current algebra.

Concerning the dualities represented in the graph, they extend, for the massless case, those obtained in the massive case. Indeed, for $m \neq 0$ fermions in $d = 3$ dimensions, the connection with the topologically massive gauge theory (i.e., Maxwell–Chern–Simons theory) was established in Refs. [2–4], the mass in the fermionic theory being related to the Chern–Simons coefficient. As already shown in Ref. [14], the Maxwell–Chern–Simons theory is in turn dual to a self-dual theory, and this closes the triangle in Fig. 1 joining the three generating functionals. Interestingly enough, the role of the mass in the bosonic models dual to massive fermions is played in the massless case by the square root of the $d = 3$ Laplacian, an operator that can be rigorously defined leading to nonlocal bosonic theories that are the natural extension of the Maxwell–Chern–Simons and self-dual theories. As mentioned at the end of Sec. III, canonical quantization of these kinds of nonlocal theories can be formulated preserving causality, as discussed in detail in Ref. [26].

An important point in the dualities discussed in the previous paragraph concerns the fact that, although the dual bosonic actions were calculated within a quadratic approximation, the parity-violating part, which is the one leading to the correct current algebra, does not receive higher order corrections so that the bosonization rule (1) is in this sense an exact one.

Section IV was dedicated to find still another duality between $d = 3$ fermion and boson models following a

procedure similar to that developed in Refs. [6,14] to prove that a topologically massive gauge theory and self-dual theory can be written in terms of a canonical, free, massive field ϕ with a Lagrangian that looks like that of a spinless field. In that case a careful analysis of the Lorentz generators as functionals of ϕ shows that the spin is in fact ± 1 . Interestingly enough, the square root of the (spatial) Lagrangian plays a central role in this massive case in finding the connection between the vector field A_μ and the ϕ field [6].

We were able to extend the procedure described above to the case of the massless bosonic theories that we prove to be equivalent to the massless free fermion theory, and also in this case the square root of two-dimensional and three-dimensional Laplacians enter into the game. In this way we were able to write the massless bosonic model with a vector field A_μ and dynamics governed by a Maxwell-like Chern–Simons action in terms of single field ϕ , which in this case, due to peculiarities of the Lorentz group in $d = 3$ dimensions, has spin 0, as it is the case for $d = 3$ massless fermions.

The duality between the $d = 3$ massless fermion Lagrangian and the bosonic one given in Eq. (52) can be seen as the $N = 1$ version of the bosonization $1/N$ results obtained in Ref. [10] based in the AdS/CFT conjecture. Since the path-integral approach is particularly adequate to treat $d = 3$ non-Abelian bosonization, one could expect that the extension of the results for massive fermions [5] should lead, in the massless case, to results analogous to those presented in Ref. [10] concerning the equivalence, for large N , of the theory of N massless scalars coupled to a level k $U(N)$ Chern–Simons term with the Legendre transform of a theory of k massless fermions coupled to a level N $U(k)$ Chern–Simons term. Moreover, the authors of Ref. [10] conjecture that the fermionic/bosonic duality is valid for finite values of N , making our results for $N = 1$ consistent with that claim. We hope to discuss this issue in a forthcoming work.

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