Duality-symmetric actions for non-Abelian tensor fields

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We construct the duality-symmetric actions for a large class of six-dimensional models describing hierarchies of non-Abelian scalar, vector and tensor fields related to one another by first-order (self-)duality equations that follow from these actions. In particular, this construction provides a Lorentz invariant action for non-Abelian self-dual tensor fields. The class of models includes the bosonic sectors of the $6d$ $(1,0)$ superconformal models of interacting non-Abelian self-dual tensor, vector, and hypermultiplets.

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I. INTRODUCTION

Understanding the detailed structure of the effective $6d$ theory of multiple $M5$-branes remains one of the important long-standing issues of string/M theory which, in particular, hampers the development of $\mathrm{AdS}_7/\mathrm{CFT}_6$ correspondence. On general grounds this should be a (2, 0) superconformal theory of non-Abelian chiral tensor supermultiplets [1]. The theory does not have a free dimensionless parameter to make it weakly coupled and this casts doubts on the very existence of its action. However, the action for a single $M5$-brane does exist [2–4][1] and produces the $M5$-brane equations of motion [8] first derived in [9] and analyzed in detail in [10,11] using the superembedding techniques (see [12] and e.g. [13–16] for a review and references). Various other aspects of the theory of M-branes are reviewed e.g. in [17–20]. One may hope that also for the multiple $M5$-branes an action may exist at least for a certain branch of the theory in which a dimensionless coupling constant appears and makes perturbative Lagrangian description possible.

To make progress in the construction of the theory of multiple $M5$-branes one should first of all solve the problem of consistently endowing the chiral tensor field with non-Abelian gauge structure, which itself is a highly nontrivial problem. If one succeeds, one can then look for equations of motion and eventually for the action. Different ways of tackling these problems have been pursued. Several approaches have been aimed at rewriting and reinterpreting the $6d$ theory (compactified on a circle) in terms of a $5d$ super-Yang-Mills theory [21–35]. In this way one gains a dimensional parameter (the radius of the compactification circle) which allows for perturbative description. Other approaches use more sophisticated mathematical tools such as higher gauge theories, twistor spaces and gerbes [36–41]. Each of the approaches has its advantages, but also issues and limitations. However, one may hope that all these approaches should be related to one another and can give us, from different perspectives, hints on what is a detailed structure of the multiple $M5$-brane theory.

A more traditional field-theoretical approach based on the hierarchy of non-Abelian vector-tensor systems [42] has been put forward in [43] (see also [44] for a particular case) and further considered in [45–47]. It aims at the construction of superconformal models including non-Abelian tensor multiplets directly in 6-dimensional space-time showing that a non-Abelian deformation of $6d$ chiral tensor fields is indeed possible upon further introducing higher rank p-forms. Supersymmetrization of this construction and on-shell closure of the (1, 0) supersymmetry algebra produces unique equations of motion of the fields. It has further been shown that a subclass of these models can be promoted to have a pseudoaction in the sense that it reproduces all equations of motion except for the self-duality equations of the tensor fields. While such a pseudoaction may be considered as an efficient bookkeeping device for checking supersymmetry (and other symmetries) of the field equations, it does not provide a reliable starting point for the quantization of the theory. In particular, one may wonder to which extent some of the bothersome features of these models such as the apparent presence of ghosts in the scalar sector and the complicated vector field dynamics are an artifact of a pseudoaction.

The aim of this paper is to complete the construction of the actions for (1, 0) superconformal theories initiated in [43,45,47] by integrating the equations of motion to a fully fledged self-dual Lagrangian for the non-Abelian chiral 2-form fields and a duality-symmetric Lagrangian for vector gauge fields and their 3-rank tensor duals. This construction yields a non-Abelian generalization of the covariant actions for $6d$ Abelian chiral tensor fields [48] and of their gauge-fixed counterparts [49–51]. This paper also generalizes and extends to $D = 6$ the results of [52] in which duality-symmetric but nonmanifestly covariant

[1]For an alternative construction based on the Bagger-Lambert-Gistavsson model with the gauge symmetry of volume preserving diffeomorphisms see [5–7]. The equivalence of these models to [2–4] is still to be proved.

actions for $D = 4$ models with non-Abelian twisted self-duality were constructed.

The paper is organized as follows: in Sec. II, we review the general system of non-Abelian p-forms and their gauge transformations in six dimensions describing the hierarchy of non-Abelian scalar, vector and tensor fields. The bosonic field equations for this system are given in Sec. III as dictated by (1, 0) superconformal symmetry. They contain the non-Abelian self-duality equations for the tensor fields, while the vector field dynamics may be expressed in terms of a first-order duality equation relating their non-Abelian field strength to the field strength of the 3-form gauge potentials. The main part of the paper is Sec. IV in which we present an action that gives rise to a general set of non-Abelian (self-)duality equations in six dimensions, including the bosonic sector of the (1, 0) models as a special case. Space-time covariance is ensured by the presence of an auxiliary scalar field. We carefully analyze the Euler-Lagrange equations of the duality-symmetric action and show that their various bits cascade down to a combination of the first-order duality equations and derivatives thereof (which can be integrated as in the Abelian case). Together, the various parts of the equations of motion assemble into the full set of first-order (self-)duality equations. In Sec. V, we work out some illustrative example and collect our conclusions in Sec. VI.

II. NON-ABELIAN p-FORMS IN SIX DIMENSIONS

In this section we will briefly review the system of non-Abelian p-forms ($p = 1, \dots, 4$) and their gauge transformations in six dimensions describing the hierarchy of non-Abelian scalar, vector and tensor fields. For more details the reader is referred to [43,45,47].

The tensor hierarchy is formed by the p-forms $(A_1^r, B_2^I, C_{3r}, C_{4\alpha})$ or in component notation $(A_\mu^r, B_{\mu\nu}^I, C_{\mu\nu\rho r}, C_{\mu\nu\rho\lambda\alpha})$, with six-dimensional space-time indices $\mu, \nu, \dots$. They couple to the scalars (Y^{ijr}, ϕ^I), where Y^{ijr} are part of the off-shell vector multiplets, while ϕ^I together with B_2^I and their fermionic partners form the chiral tensor supermultiplet. Later we will also add the non-Abelian hypermultiplets. To avoid the proliferation of indices, we will work with differential forms on which the external derivative will act from the right. In what follows we will be only interested in a subclass of the models which have a Lagrangian description. This requires the introduction of an (indefinite) constant metric η^{IJ} and its inverse η_{IJ} ($\eta^{IJ}\eta_{JK} = \delta^I_K$) which raise, lower and contract the indices I, J, K.

The non-Abelian field strengths of the p-form gauge potentials are given by

$$\mathcal{F}^r := \frac{1}{2} dx^\nu \wedge dx^\mu \mathcal{F}^r_{\mu\nu} = dA^r + \frac{1}{2} f_{st}{}^r A^s \wedge A^t + g^r_I B_2^I, \tag{2.1}$$

$$\begin{aligned}\mathcal{H}_3^I :=& \frac{1}{3!} dx^\rho \wedge dx^\nu \wedge dx^\mu \mathcal{H}^I_{\mu\nu\rho} \\ =& DB_2^I + d^I_{st} A^s \wedge dA^t + \frac{1}{3} f_{pq}{}^s d^I_{rs} A^r \wedge A^p \wedge A^q + g^{Ir} C_{3r} \\ =& dB_2^I + d^I_{st} A^s \wedge (\mathcal{F}^t + g^t_J B_2^J) - \frac{1}{6} f_{pq}{}^s d^I_{rs} A^r \wedge A^p \wedge A^q \\ &+ g^{Ir}(C_{3r} - 2 d_{Jrs} B_2^J \wedge A^s),\end{aligned} \tag{2.2}$$

$$\begin{aligned}\mathcal{H}_{4r} :=& \frac{1}{4!} dx^\sigma \wedge dx^\rho \wedge dx^\nu \wedge dx^\mu \mathcal{H}_{\mu\nu\rho\sigma r} \\ =& DC_{3r} - 2B_2^J \wedge \left(dA^s + \frac{1}{2} f_{pq}{}^s A^p \wedge A^q \right) d_{Jrs} \\ &- B_2^J \wedge B_2^I d_{Irs} g^s_J + \frac{1}{3} dA^s \wedge A^t \wedge A^u d_{Js[t} d^J_{u]r} \\ &+ k^\alpha_r C_{4\alpha} + \propto A \wedge A \wedge A \wedge A.\end{aligned} \tag{2.3}$$

They are constructed with the use of the antisymmetric "structure constants" $f_{st}{}^r = f_{[st]}{}^r$, the constant tensors $d^I_{rs} = d^I_{(rs)}$ inducing Chern-Simons couplings, and the constant tensors g^{Ir} and k^α_r that induce Stückelberg-type couplings among forms of different degree. These tensors satisfy certain algebraic relations which we have collected in the Appendix. Notably, they satisfy the orthogonality relations

$$g^{Ir} g^s_I := g^{Ir} \eta_{IJ} g^{Js} = 0, \qquad g^{Ir} k^\alpha_r = 0. \tag{2.4}$$

The covariant derivatives D are defined as follows:

$$\begin{aligned}D\mathcal{F}^r :=& d\mathcal{F}^r + \mathcal{F}^t \wedge A^s X_{st}{}^r \\ =& d\mathcal{F}^r - \mathcal{F}^t \wedge A^s f_{st}{}^r + \mathcal{F}^t \wedge A^s d^I_{st} g^r_I,\end{aligned} \tag{2.5}$$

$$\begin{aligned}D\mathcal{H}_3^I :=& d\mathcal{H}_3^I + \mathcal{H}_3^J \wedge A^s X_{sJ}{}^I \\ =& d\mathcal{H}_3^I + 2\mathcal{H}_3^J \wedge A^s d^I_{st} g_J{}^t - 2\mathcal{H}_3^J \wedge A^r g^{Is} d_{Jsr},\end{aligned} \tag{2.6}$$

$$D\mathcal{H}_{4r} := d\mathcal{H}_{4r} - \mathcal{H}_{4t} \wedge A^s X_{sr}{}^t. \tag{2.7}$$

Let us also note that the algebraic constraints among the constant tensors parametrizing the gauge system in particular imply that the gauge generators appearing in these covariant derivatives are related to the Stückelberg coupling k^α_r via

$$\begin{aligned}X_{rs}{}^t &\equiv d^I_{rs} g^t_I - f_{rs}{}^t = -k^\alpha_r c^t_{\alpha s}, \\ X_{rIJ} &\equiv 4 g^s_{[I} d_{J]rs} = 2k^\alpha_r c_{\alpha IJ},\end{aligned} \tag{2.8}$$

with tensors $c^t_{\alpha s}$ and $c_{\alpha IJ}$ whose role will be clarified below. From (2.1)–(2.7) one gets the Bianchi identities

$$D\mathcal{F}^r = g^r_I \mathcal{H}_3^I, \tag{2.9}$$

$$D\mathcal{H}_3^I = d^I_{st} \mathcal{F}^s \wedge \mathcal{F}^t + g^{Ir} \mathcal{H}_{4r}, \tag{2.10}$$

$$D\mathcal{H}_{4r} = -2\mathcal{H}_3^I \wedge \mathcal{F}^s d_{Isr} + k_r^\alpha \mathcal{H}_{5\alpha}. \quad (2.11)$$

Equation (2.11) defines the 5-form field strength $\mathcal{H}_{5\alpha} = DC_{4\alpha} + \cdots$ (at least under projection with k_r^α). We will not need its explicit form in our construction. We only notice that $\mathcal{H}_{5\alpha}$ contains the tensors $c_{\alpha s}^t$ and $c_{\alpha IJ}$ which enter Eqs. (2.8), so that its Bianchi identities read [43]

$$D\mathcal{H}_{5\alpha} = -c_{\alpha IJ}\mathcal{H}_3^I \wedge \mathcal{H}_3^J - c_{\alpha s}^r \mathcal{F}^s \wedge \mathcal{H}_{4r} + \ldots . \quad (2.12)$$

Actually, also neither explicit form of $\mathcal{H}_{4r}$ nor $\mathcal{H}_3^I$ is needed for our calculations. The expressions for the general variation of the covariant field strengths (2.1)–(2.3) can be reproduced formally from the Bianchi identities. These are

$$\begin{aligned}
\delta\mathcal{F}^r &= D\delta A^r + g_I^r \Delta B_2^I,\\
\delta\mathcal{H}_3^I &= D\Delta B_2^I + 2d_{rs}^I \mathcal{F}^r \wedge \delta A^s + g^{Ir}\Delta C_{3r},\\
\delta\mathcal{H}_{4r} &= D\Delta C_{3r} - 2d_{Irs}\mathcal{F}^s \wedge \Delta B_2^I\\
&\quad - 2d_{Irs}\mathcal{H}_3^I \wedge \delta A^s + k_r^\alpha \Delta C_{4\alpha}, \quad (2.13)
\end{aligned}$$

where we have introduced the compact notation

$$\begin{aligned}
\Delta B_2^I &\equiv \delta B_2^I + d_{rs}^I A^r \wedge \delta A^s,\\
\Delta C_{3r} &\equiv \delta C_{3r} - 2d_{Irs}B_2^I \wedge \delta A^s - \frac{1}{3}d_{Irs}d_{pq}^I A^s \wedge A^p \wedge \delta A^q,\\
k_r^\alpha \Delta C_{4\alpha} &\equiv k_r^\alpha \delta C_{4\alpha} + \cdots \quad (2.14)
\end{aligned}$$

The non-Abelian gauge transformations with $(p-1)$-form parameters $(\Lambda^r, \Lambda_1^I, \Lambda_{2r}, \Lambda_{3\alpha})$ are given by

$$\begin{aligned}
\delta A^r &= D\Lambda^r - g_I^r \Lambda_1^I,\\
\Delta B_2^I &= D\Lambda_1^I - 2d_{rs}^I \Lambda^r \mathcal{F}^s - g^{Ir}\Lambda_{2r},\\
\Delta C_{3r} &= D\Lambda_{2r} + 2d_{Irs}\mathcal{F}^s \wedge \Lambda_1^I + 2d_{Irs}\mathcal{H}_3^I \Lambda^s - k_r^\alpha \Lambda_{3\alpha}, \quad (2.15)
\end{aligned}$$

$$\begin{aligned}
k_r^\alpha \Delta C_{4\alpha} &= k_r^\alpha D\Lambda_{3\alpha} - 4X_{rIJ}\mathcal{H}_3^I \wedge \Lambda_1^J\\
&\quad - X_{rs}{}^t(\mathcal{F}^s \wedge \Lambda_{2t} + \Lambda^s \mathcal{H}_{4t}). \quad (2.16)
\end{aligned}$$

Under these transformations, the field strengths (2.1) transform covariantly as $\delta\mathcal{F}^r = -\mathcal{F}^t \Lambda^s X_{st}{}^r$, $\delta\mathcal{H}_3^I = -\mathcal{H}_3^J \Lambda^s X_{sJ}{}^I$, etc. Notice, in particular, that the left- and the right-hand sides of (2.16) vanish when contracted with g^{Kr} in virtue of the identities (A1). For completeness, we also note that the connection in (2.16) is given by $k_r^\alpha A^s X_{s\alpha}{}^\beta \equiv A^s X_{rs}{}^t k_t^\beta$. Consequently, also the 4-form field strengths transform covariantly as

$$\begin{aligned}
\delta\mathcal{H}_{4r} &= 2\Lambda^s d_{rs}^I g_I^t \mathcal{H}_{4t} + \mathcal{F}^s \wedge X_{rs}{}^t \Lambda_{2t}\\
&\quad + k_r^\alpha c_{\alpha s}^t(\mathcal{F}^s \wedge \Lambda_{2t} + \Lambda^s \mathcal{H}_{4t}) = \Lambda^s X_{sr}{}^t \mathcal{H}_{4t}. \quad (2.17)
\end{aligned}$$

III. BOSONIC PART OF THE (1, 0) SUPERCONFORMAL FIELD EQUATIONS

So far, we have introduced the non-Abelian system of p-forms in six dimensions on a purely kinematical level. Its supersymmetric dynamics may be deduced from closure of the (1, 0) supersymmetry algebra [43]. In particular, this fixes the couplings of the p-forms to the scalar fields ϕ^I and Y^{ij} completing the (1, 0) vector and tensor multiplets, respectively.

In absence of hypermultiplets, and when all the fermions are set to zero, the resulting bosonic field equations are

$$\mathcal{H}_3^I + *\mathcal{H}_3^I = 0, \quad (3.1)$$

$$\begin{aligned}
&D * D\phi_I - 2d_{Irs}\mathcal{F}^r \wedge *\mathcal{F}^s - d^6x(2d_{Irs}Y^{ijr}Y_{ij}^s\\
&\quad + 3g_{(J}^r g_K^s d_{I)rs}\phi^J\phi^K) = 0, \quad (3.2)
\end{aligned}$$

for the tensor multiplets and

$$d_{Irs}Y_{ij}^s \phi^I = 0, \quad (3.3)$$

$$2d_{Irs}\phi^I * \mathcal{F}^s + \mathcal{H}_{4r} = 0, \quad (3.4)$$

for the vector multiplets. Equation (3.3) reflects the auxiliary nature of the fields Y^{ijr}. Equation (3.1) tells us that the 3-form field strength is self-dual and Eq. (3.4) is the first-order duality equation that relates the vector field strengths to the field strengths of the 3-form tensors. Its derivative together with the Bianchi identities (2.11) yields the standard second-order Yang-Mills equation for the vector fields. In turn, the 4-form tensors are related by their field strength to the scalar fields of the theory by means of the duality equation

$$k_r^\alpha * \mathcal{H}_{5\alpha} = \frac{1}{2}\mathcal{J}_r \equiv \frac{1}{2}X_{rIJ}\phi^I D\phi^J, \quad (3.5)$$

with the scalar matter current $\mathcal{J}_r$. In presence of hypermultiplets, the rhs of this duality equation receives an additional contribution from the hyperscalar current [47].

In the next section we will construct an action that reproduces the first-order (self-)duality equations (3.1), (3.4), and (3.5), by extending the construction of [48] to the non-Abelian case.

IV. THE ACTION

In this section we will present an action from which the field equations of the previous section are derived. In particular, this includes an action for the non-Abelian chiral gauge field B_2^I. More generally, we will construct an action which reproduces the general set of six-dimensional non-Abelian (self-)duality equations for the p-forms

$$\begin{aligned}
\mathcal{H}_3^I + *\mathcal{H}_3^I = 0, \qquad \mathcal{H}_{4r} + \mathcal{M}_{rs} * \mathcal{F}^s = 0,\\
2k_r^\alpha * \mathcal{H}_{5\alpha} - \mathcal{J}_r = 0. \quad (4.1)
\end{aligned}$$

The particular choice of

$$\mathcal{M}_{rs} = 2\phi^I d_{Irs}, \qquad \mathcal{J}_r = X_{rIJ}\phi^I D\phi^J, \tag{4.2}$$

for the vector kinetic matrix and the scalar current corresponds to the bosonic sector of the (1, 0) superconformal models discussed in Sec. III above, but our results apply to any six-dimensional system of the form (4.1). In particular, they include the coupling of the vector and tensor multiplets to the (1, 0) hypermultiplets considered in [47].

We will proceed in two steps. First, in Sec. IV A we construct an action that gives rise to the non-Abelian self-duality equation (3.1) for the tensor fields together with the standard second-order field equations for the remaining fields. It is of the form

$$\mathcal{S} = \int_{\mathcal{M}^6} \mathcal{L} = \int_{\mathcal{M}^6} (\mathcal{L}^{\rm scal} + \mathcal{L}^{\rm vec} + \mathcal{L}^{\rm top} + \mathcal{L}^{\mathcal{H}\mathcal{H}}). \tag{4.3}$$

The first three terms in (4.3) which include kinetic terms of the scalars and the vector gauge field have been constructed in [43]. The last term $\mathcal{L}^{\mathcal{H}\mathcal{H}}$ is the Lagrangian for the non-Abelian chiral gauge field B_2^I whose construction is one of the main results of this paper. In the second step, in Sec. IV B, we generalize this action to a duality-symmetric action that also treats vector fields and 3-form gauge potentials on the same footing and produces their first-order duality equation (3.4) among the proper field equations. This is achieved by extending (4.3) to

$$\mathcal{S}_{\rm ext} = \int_{\mathcal{M}^6} \mathcal{L}_{\rm ext} \equiv \int_{\mathcal{M}^6} (\mathcal{L}^{\rm scal} + \mathcal{L}^{\rm vec} + \mathcal{L}^{\rm top} + \mathcal{L}^{\mathcal{H}\mathcal{H}} + \mathcal{L}^{\mathcal{H}_4/\mathcal{F}}), \tag{4.4}$$

with the new term $\mathcal{L}^{\mathcal{H}_4/\mathcal{F}}$ carrying the field strength of the 3-form gauge potentials.

In the differential form notation the first term in the actions (4.3) and (4.4) has the following generic form:

$$\mathcal{L}^{\rm scal} = -\frac{1}{2} D\phi^I \wedge *D\phi^J \eta_{IJ} - V_{\rm scal} d^6x, \tag{4.5}$$

with covariant derivatives D and where d^6x stands for the 6-form $dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_6} = \varepsilon^{\mu_1\cdots\mu_6} d^6x$. The scalar potential $V_{\rm scal}$ is *a priori* arbitrary. In the case of the (1, 0) superconformal models of [43] it takes the following form:

$$V_{\rm scal} \equiv -d_{Irs}(2\phi^I Y^{ijr} Y^s_{ij} + g^r_J g^s_K \phi^I \phi^J \phi^K), \tag{4.6}$$

with additional contributions in the presence of hypermultiplets [47]. The kinetic term for the vector fields in (4.3) and (4.4) is of the standard form

$$\mathcal{L}^{\rm vec} = \mathcal{M}_{rs} \mathcal{F}^r \wedge *\mathcal{F}^s \tag{4.7}$$

where the matrix $\mathcal{M}_{rs}$ is constructed from the scalars. In the case of the (1, 0) superconformal models it is defined by (4.2) in terms of the tensor multiplet scalars.

The presence of the topological term $\mathcal{L}^{\rm top}$ in the action (as in the other cases of this kind) is due to the presence of Chern-Simons-like terms in the covariant field strengths (2.1)–(2.3). It is constructed as follows. The $6d$ space-time $\mathcal{M}^6$ is formally extended to a $7d$ manifold $\mathcal{M}^7$ assuming $\mathcal{M}^6$ to be the boundary of $\mathcal{M}^7$ ($\mathcal{M}^6 = \partial\mathcal{M}^7$). Then using the field strengths (2.1)–(2.3) one constructs the $7d$ form

$$\begin{aligned} d\mathcal{L}^{\rm top} &:= -2d_{Ist}\mathcal{F}^s \wedge \mathcal{F}^t \wedge \mathcal{H}^I_3 + \mathcal{H}^I_3 \wedge D\mathcal{H}^J_3 \eta_{IJ} \\ &= -d_{Ist}\mathcal{F}^s \wedge \mathcal{F}^t \wedge \mathcal{H}^I_3 + \mathcal{H}^I_3 \wedge g^r_I \mathcal{H}_{4r}, \end{aligned} \tag{4.8}$$

which is (identically) closed $dd\mathcal{L}^{\rm top} \equiv 0$, as can easily be checked using the Bianchi identities (2.9)–(2.11). Then the topological action is

$$\begin{aligned} \mathcal{S}^{\rm top} &= \int_{\mathcal{M}^7} (\mathcal{H}^I_3 \wedge g^r_I \mathcal{H}_{4r} - d_{Ist}\mathcal{F}^s \wedge \mathcal{F}^t \wedge \mathcal{H}^I_3) \\ &= \int_{\mathcal{M}^6 = \partial\mathcal{M}^7} \mathcal{L}^{\rm top}. \end{aligned} \tag{4.9}$$

For performing the variation of the action we do not need the explicit form of $\mathcal{L}^{\rm top}$, since $\delta\mathcal{L}^{\rm top} = i_\delta(d\mathcal{L}^{\rm top}) + d(i_\delta\mathcal{L}^{\rm top})$ and the second term does not contribute to the integral when the $6d$ space is assumed to have no boundaries.[2] Actually we also use this property for other Lagrangian forms and omit total derivative terms in their variation.

The following construction applies to arbitrary scalar and vector couplings $\mathcal{L}^{\rm scal}$, $\mathcal{L}^{\rm vec}$ and, in the following, we will not make use of the specific form of the scalar potential (4.6) and the kinetic matrix (4.2) dictated by superconformal invariance. The topological term on the other hand is universal with its form determined by the non-Abelian tensor hierarchy of Sec. II.

A. Action for chiral tensor fields

Let us now describe in detail the chiral tensor field Lagrangian entering the actions (4.3) and (4.4). It has the following form:

$$\begin{aligned} \mathcal{L}^{\mathcal{H}\mathcal{H}} &= -(i_v * \mathcal{H}^I_3 + i_v \mathcal{H}^I_3) \wedge \mathcal{H}_{3I} \wedge v \\ &= \frac{1}{2} d^6x v^\rho (*\mathcal{H}^I_{\mu\nu\rho} + \mathcal{H}^I_{\mu\nu\rho})(*\mathcal{H}_I^{\mu\nu\lambda}) v_\lambda, \end{aligned} \tag{4.10}$$

where the 1-form

$$v := dx^\mu v_\mu = \frac{dx^\mu \partial_\mu a(x)}{\sqrt{\partial_\mu a \partial^\mu a}}, \qquad v_\mu v^\mu = 1, \tag{4.11}$$

is the normalized derivative of the auxiliary scalar field $a(x)$, whose presence in the action ensures its space-time covariance (see [48] for the Abelian chiral field case in $D = 6$). Consistency of the construction requires that the action (4.3) is invariant under a local symmetry which

[2] i_δ is the contraction operation with the variation δ considered as a vector field, so that $i_\delta d = \delta$, $i_\delta dA^r = \delta A^r$ etc. In our conventions this operation acts from the right, e.g. $i_\delta(d\phi^I \wedge d\phi^J) = d\phi^I \delta\phi^J - \delta\phi^I d\phi^J$.

allows one to gauge-fix v_μ to a constant value and moreover that the variation of the action produces the desired equations of motion. To this end, let us consider a generic variation of (4.3) with respect to the scalar and tensor fields. The variation of the Lagrangian (4.10) reads

$$\delta\mathcal{L}^{\mathcal{H}\mathcal{H}} = 2i_v(*\mathcal{H}_3^I + \mathcal{H}_3^I)\wedge v \wedge\left(\delta\mathcal{H}_3^I - \frac{1}{2}\delta v\wedge(i_v*\mathcal{H}_3^I + i_v\mathcal{H}_3^I)\right) - \mathcal{H}_{3I}\wedge\delta\mathcal{H}_3^I, \tag{4.12}$$

where $\delta\mathcal{H}_3^I$ was defined in (2.13) and

$$\delta v = dx^\mu\delta v_\mu, \quad \delta v_\mu = \frac{(\eta_{\mu\nu} - v_\mu v_\nu)\partial^\nu\delta a(x)}{\sqrt{\partial_\lambda a\partial^\lambda a}}. \tag{4.13}$$

To obtain (4.12), the following identities are useful:

$$F_p \equiv i_v F_p\wedge v + *(i_v * F_p\wedge v), \qquad F_6 = i_v F_6\wedge v,$$
$$i_v * \mathcal{H}_3^I \equiv *(\mathcal{H}_3^I\wedge v), \tag{4.14}$$

and $F_p\wedge *G_p = G_p\wedge *F_p$. Introducing the notation

$$\mathcal{G}_2^I := \frac{i_v(*\mathcal{H}_3^I + \mathcal{H}_3^I)}{\sqrt{\partial a\partial a}}, \tag{4.15}$$

we can write (4.12) as

$$\delta\mathcal{L}^{\mathcal{H}\mathcal{H}} = 2\mathcal{G}_2^J\eta_{IJ}\wedge da\wedge\left(\delta\mathcal{H}_3^I - \frac{1}{2}d(\delta a)\wedge\mathcal{G}_2^I\right) - \mathcal{H}_{3I}\wedge\delta\mathcal{H}_3^I, \tag{4.16}$$

Now, using Eqs. (2.13) and the Bianchi identities (2.9) and (2.10), one gets

$$\begin{aligned}\delta\mathcal{L}^{\mathcal{H}\mathcal{H}} =& 2\mathcal{G}_{2I}\wedge da\wedge\Big(D\Delta B_2^I - \frac{1}{2}d(\delta a)\wedge\mathcal{G}_2^I \\ &+ 2d_{st}^I\mathcal{F}^s\wedge\delta A^t + g^{Ir}\Delta C_{3r}\Big) \\ &+ (g_I^r\mathcal{H}_{4r} + d_{Ist}\mathcal{F}^s\wedge\mathcal{F}^t)\wedge\Delta B_2^I - g_I^r\mathcal{H}_3^I\wedge\Delta C_{3r} \\ &- 2d_{Ist}\mathcal{H}_3^I\wedge\mathcal{F}^s\wedge\delta A^t.\end{aligned} \tag{4.17}$$

Terms similar to those in the second line of (4.17) enter the variation of the topological term $\mathcal{L}^{\rm top}$ (4.8)

$$\delta\mathcal{L}^{\rm top} = (-d_{Ist}\mathcal{F}^s\wedge\mathcal{F}^t + g_I^r\mathcal{H}_{4r})\wedge\Delta B_2^I + g_I^r\mathcal{H}_3^I\wedge\Delta C_{3r} - 2d_{Ist}\mathcal{H}_3^I\wedge\mathcal{F}^s\wedge\delta A^t. \tag{4.18}$$

Thus

$$\begin{aligned}\delta(\mathcal{L}^{\mathcal{H}\mathcal{H}} + \mathcal{L}^{\rm top}) =& 2\mathcal{G}_2^J\eta_{JI}\wedge da\wedge\Big(D\Delta B_2^I - \frac{1}{2}(\delta a)\mathcal{G}_2^I \\ &+ 2d_{st}^I\mathcal{F}^s\wedge\delta A^t + g^{Ir}\Delta C_{3r}\Big) \\ &+ 2g_I^r\mathcal{H}_{4r}\wedge\Delta B_2^I \\ &- 4d_{Ist}\mathcal{H}_3^I\wedge\mathcal{F}^s\wedge\delta A^t.\end{aligned} \tag{4.19}$$

Combining this with the variation of the matter Lagrangian $\mathcal{L}^{\rm scal} + \mathcal{L}^{\rm vec}$, we finally obtain the variation of the full Lagrangian (4.3)

$$\begin{aligned}\delta\mathcal{L} =& 2\mathcal{G}_2^J\eta_{JI}\wedge da\wedge\Big(D\Delta B_2^I - \frac{1}{2}(\delta a)\mathcal{G}_2^I \\ &+ 2d_{st}^I\mathcal{F}^s\wedge\delta A^t + g^{Ir}\Delta C_{3r}\Big) \\ &+ 2g_I^r(\mathcal{M}_{rs}*\mathcal{F}^s + \mathcal{H}_{4r})\wedge\Delta B_2^I \\ &+ (\mathcal{J}_t + 2D(\mathcal{M}_{ts}*\mathcal{F}^s) - 4d_{Ist}\mathcal{H}_3^I\wedge\mathcal{F}^s)\wedge\delta A^t \\ &+ \delta_Y\mathcal{L} + \delta_\phi\mathcal{L},\end{aligned} \tag{4.20}$$

with the matter current $\mathcal{J}_r$ defined by the variation of the matter Lagrangian as

$$\delta\mathcal{L}^{\rm scal} = \mathcal{J}_r\wedge\delta A^r \equiv k_r^\alpha\mathcal{J}_\alpha\wedge\delta A^r. \tag{4.21}$$

The form of Eq. (4.20) suggests that the action (4.3) is invariant under the following local transformations of B_2^I and C_{3r}:

$$\Delta_{\varphi_1}B_2^I = \varphi_1^I\wedge da, \qquad \Delta_{\varphi_2}C_{3r} = \varphi_{2r}\wedge da, \tag{4.22}$$

where the 2-form parameter $\varphi_{2r}(x)$ is arbitrary and the 1-form parameter $\varphi_1^I(x)$ satisfies the condition $g_I^s\varphi_1^I = 0$. Note that in general these symmetries are not included in the tensor gauge symmetries of (2.15) whose parameters Λ_{2r} and $\Lambda_{3\alpha}$ appear only under the projection with the tensors g^{Ir} and k_r^α, respectively.

Another local symmetry of the action is the one which exposes the auxiliary nature of the scalar field $a(x)$

$$\delta a = \varphi(x), \quad \Delta_\varphi B_2^I = \delta a\mathcal{G}_2^I, \quad \Delta_\varphi C_{3r} = \delta a\mathcal{G}_{3r}, \tag{4.23}$$

where $\varphi(x)$ is an arbitrary scalar parameter and

$$\mathcal{G}_{3r} := \frac{i_v(\mathcal{H}_{4r} + \mathcal{M}_{rs}*\mathcal{F}^s)}{\sqrt{\partial a\partial a}}. \tag{4.24}$$

One can use this symmetry to gauge-fix v_μ to be e.g. the constant unit timelike vector

$$v_\mu = \delta_\mu^0. \tag{4.25}$$

If in (4.10) we substitute v_μ with its gauge-fixed value (4.25), the manifest space-time invariance of the action will be broken and it reduces to the non-Abelian generalization of the Henneaux-Teitelboim action [49,50] for a single chiral 2-form in $D = 6$. However, the gauge-fixed action is still invariant under modified Lorentz transformations, which preserve the gauge (4.25). They are the combination of Lorentz rotations with the parameters ${l_\mu}^\nu$ and local transformations (4.23) and (4.13) such that

$$\begin{aligned}\Delta_L v_\mu &= \delta_l v_\mu + \delta_\varphi v_\mu = \Delta_L(\delta_\mu^0) = 0 \\ &= {l_\mu}^0 + \partial_\mu\varphi - \delta_\mu^0\partial_0\varphi\end{aligned} \tag{4.26}$$

from which it follows that

$$\varphi(x) = -x^\mu {l_\mu}^0 \tag{4.27}$$

and the modified Lorentz transformations under which the gauge-fixed action is invariant are

$$\Delta_L B_2^I = \delta_l B_2^I - x^\mu l_\mu{}^0 \mathcal{G}_2^I, \qquad \Delta_L C_{3r} = \delta_l C_{3r} - x^\mu l_\mu{}^0 \mathcal{G}_{3r}. \tag{4.28}$$

In (4.28) it is implied that in the quantities $\mathcal{G}_2^I$ and $\mathcal{G}_{3r}$, defined in (4.15) and (4.24), v_μ takes its gauge-fixed value (4.25).

1. Derivation of the field equations

Let us now discuss the derivation of the field equations from the variation (4.20) of the action. It demonstrates in an instructive manner how the tensor hierarchy intertwines equations of motion of different tensor fields. For the analysis of the equations of motion it is useful to introduce a constant projector $\mathbb{P}^I{}_J$ of minimal rank satisfying

$$g_I^r \mathbb{P}^I{}_J = g_J^r, \qquad \mathbb{P}^I{}_J \mathbb{P}^J{}_K = \mathbb{P}^I{}_K \tag{4.29}$$

and the complementary (orthogonal) projector

$$\bar{\mathbb{P}} = \mathrm{I} - \mathbb{P}, \qquad \bar{\mathbb{P}}\bar{\mathbb{P}} = \bar{\mathbb{P}}. \tag{4.30}$$

which obeys $g_I^r \bar{\mathbb{P}}^I{}_J = 0$. We stress that the introduction of this projector is an auxiliary structure in order to derive the different parts of the equations of motion, whereas eventually the combined set of equations of motion does not carry any reference to this projector.

We start with the equation of motion produced by the variation of C_{3r}

$$\mathcal{G}_2^I g_I^r \wedge da := g_I^r i_v(\mathcal{H}_3^I + *\mathcal{H}_3^I) \wedge v = 0. \tag{4.31}$$

Due to the properties of the projector (4.29) and its complementary (4.30), we see that this equation is equivalent to

$$\mathbb{P}^I{}_J i_v(\mathcal{H}_3^J + *\mathcal{H}_3^J) \wedge v = 0, \tag{4.32}$$

since (4.31) is satisfied if and only if (4.32) holds. In view of the identities Eq. (4.14), Eq. (4.32) amounts to the anti-self-duality of the part of $\mathcal{H}_3^I$ projected with $\mathbb{P}^I{}_J$,

$$\mathbb{P}^I{}_J(\mathcal{H}_3^J + *\mathcal{H}_3^J) = 0. \tag{4.33}$$

Moreover, we can use the second symmetry in (4.22) to put

$$i_v(\mathcal{H}_{3I} + *\mathcal{H}_{3I}) \wedge v \mathbb{P}^I{}_J = 0 \Rightarrow (\mathcal{H}_{3I} + *\mathcal{H}_{3I})\mathbb{P}^I{}_J = 0. \tag{4.34}$$

Indeed, under the second symmetry in (4.22) $\delta\mathcal{G}_{2I} = g_I^r \varphi_{2r} \Rightarrow \delta\mathcal{G}_{2J}\mathbb{P}^J{}_I = g_I^r\varphi_{2r}$, which can be used to fix $\mathcal{G}_{2J}\mathbb{P}^J{}_I = 0$. Now, the variation of ΔB_2^I gives

$$D(i_v(\mathcal{H}_3^I + *\mathcal{H}_3^I) \wedge v) - g^{Ir}(\mathcal{H}_{4r} + \mathcal{M}_{rs} * \mathcal{F}^s) = 0. \tag{4.35}$$

Projecting the above equation with $\mathbb{P}^I{}_J$ and $\bar{\mathbb{P}}^I{}_J$, in view of (4.34) we get

$$g_J^r(\mathcal{H}_{4r} + \mathcal{M}_{rs} * \mathcal{F}^s) = 0, \tag{4.36}$$

$$\bar{\mathbb{P}}^I{}_J D(i_v(\mathcal{H}_{3I} + *\mathcal{H}_{3I}) \wedge v) = 0. \tag{4.37}$$

Equation (4.36) is a projected version of the duality relation between $\mathcal{H}_{4r}$ and $\mathcal{F}^r$. As for Eq. (4.37), it reduces to

$$\bar{\mathbb{P}}^I{}_J d(i_v(\mathcal{H}_3^J + *\mathcal{H}_3^J) \wedge v) = 0, \tag{4.38}$$

since, by virtue of (4.33), the nontrivial connection part of the covariant derivative D in (4.37) is

$$\bar{\mathbb{P}}^K{}_I X_{rLK}(i_v(\mathcal{H}_3^L + *\mathcal{H}_3^L) \wedge v) = \bar{\mathbb{P}}^K{}_I \bar{\mathbb{P}}^L{}_J X_{rLK}(i_v(\mathcal{H}_3^J + *\mathcal{H}_3^J) \wedge v), \tag{4.39}$$

and thus vanishes since

$$\bar{\mathbb{P}}^K{}_I \bar{\mathbb{P}}^L{}_J X_{rLK} = 0, \tag{4.40}$$

according to the definition of $X_{rL}{}^K$ in (2.6) and of the projectors in (4.29) and (4.30). Then, Eq. (4.38) can be solved in the same way as in the case of the Abelian chiral tensor fields [48] with the general solution (at least locally or in the topologically trivial cases) being

$$i_v(\mathcal{H}_3^I + *\mathcal{H}_3^I) \wedge v = d(\phi_1^I \wedge da), \tag{4.41}$$

where the 1-form $\phi_1^I(x)$ is such that $\phi_1^I g_I^r = 0$, or equivalently $\mathbb{P}^K{}_I \phi_1^I = 0$. One can now use the local symmetry (4.22) with the parameter $\varphi_1^I(x)$ (also obeying $\varphi_1^I g_I^r = 0$) to annihilate the right-hand side of Eq. (4.41) and, in view of (4.33), arrive at the anti-self-duality condition for all $\mathcal{H}_3^I$

$$\mathcal{H}_3^I + *\mathcal{H}_3^I = 0. \tag{4.42}$$

When (4.42) is satisfied, the variation (4.20) with respect to δa vanishes identically, thus confirming that the scalar $a(x)$ is entirely auxiliary, while the variation of A^r provides us with the vector field equations of motion. The complete set of the field equations obtained from varying the action (4.3) with respect to the p-forms is

$$\mathcal{H}_3^I + *\mathcal{H}_3^I = 0, \qquad g^{Ir}(\mathcal{H}_{4r} + \mathcal{M}_{rs} * \mathcal{F}^s) = 0, \qquad 2D(\mathcal{M}_{st} * \mathcal{F}^s) + \mathcal{J}_t - 4d_{Ist}\mathcal{H}_3^I \wedge \mathcal{F}^s = 0. \tag{4.43}$$

Notice that the field $a(x)$ does not enter these equations. This once again manifests the fact that $a(x)$ is completely auxiliary and is only required for ensuring the space-time covariance of the action.

The (bosonic limit of the) (1, 0) models of [43] are recovered with the particular choice of $\mathcal{M}_{rs}$ and J_t as in Eq. (4.2) and the scalar potential as in (4.6) dictated by supersymmetry. In this case the variation of the action (4.3) with respect to the scalar fields yields the equations of motion

$$D*D\phi_I - 2d_{Irs}\mathcal{F}^r \wedge *\mathcal{F}^s - d^6x(2d_{Irs}Y^{ijr}Y^s_{ij} + 3g^r_{(J}g^s_K d_{I)rs}\phi^J\phi^K) = 0, \qquad \phi^I d_{Irs}Y^{ijr} = 0. \tag{4.44}$$

Comparing (4.43) to the full system of first-order duality equations (4.1), we see that the action (4.3) gives rise to the full self-duality equation (3.1) but only to a projection of the duality equation between the vector and the 3-form gauge potentials. This will be rectified in the next section.

Finally, before concluding this section let us note that, using the Bianchi identities (2.11) one can rewrite the general variation (4.20) as follows:

$$\delta\mathcal{L} = 2\mathcal{G}_2^J \eta_{JI}\wedge da\wedge\Big(D\Delta B_2^I - \frac{1}{2}d(\delta a)\wedge \mathcal{G}_2^I + 2d^I_{st}\mathcal{F}^s\wedge\delta A^t + g^{Ir}\Delta C_{3r}\Big) + 2g_I^r\mathcal{K}_{4r}\wedge\Delta B_2^I + 2D\mathcal{K}_{4r}\wedge\delta A^r + k_r^\alpha(*\mathcal{J}_\alpha - 2\mathcal{H}_{5\alpha})\wedge\delta A^r + \delta_Y\mathcal{L} + \delta_\phi\mathcal{L}, \quad (4.45)$$

where

$$\mathcal{K}_{4r} := \mathcal{H}_{4r} + \mathcal{M}_{rs} * \mathcal{F}^s. \quad (4.46)$$

The first term in the fourth line of (4.45) infers that the 4-form gauge potential $k_r^\alpha C_{4\alpha}$ can be dual to the scalars ϕ^I [see Eq. (3.5)]. This duality condition, however, does not follow from the above action [note that the action (4.3) does not even contain the 4-form field $C_{4\alpha}$]. In the next section, we will construct an action for the extended tensor hierarchy system, that explicitly includes the 4-form $C_{4\alpha}$ and also treats the vector and 3-form fields A_1^r and C_{3r} in a duality-symmetric fashion. Equation (3.5) will then appear as a full-fledged equation of motion.

B. Action with manifest vector-tensor duality symmetry

In this section we will extend the action (4.3) to the form (4.4) in such a way that it treats the vector and 3-form tensor fields on an equal footing and yields their first-order duality relation (4.46) among the proper equations of motion. The corresponding action includes all the p-form fields (Y^{ijr}, ϕ^I, A_1^r, B_2^I, C_{3r}, $C_{4\alpha}$) and is obtained by adding to the action (4.3) the following term:

$$\mathcal{L}^{\mathcal{H}_4/\mathcal{F}} = -\frac{1}{4}\tilde{\mathcal{M}}^{rs}(i_v * \mathcal{K}_{4r})\wedge *(i_v * \mathcal{K}_{4s}), \quad (4.47)$$

where $\mathcal{K}_{4r}$ has been defined in (4.46), and the matrix $\tilde{\mathcal{M}}^{rs}$ is such that

$$\tilde{\mathcal{M}}^{rs}\mathcal{M}_{st} = P^r{}_s, \qquad \tilde{\mathcal{M}}^{rs}\mathcal{M}_{st}\tilde{\mathcal{M}}^{tq} = \tilde{\mathcal{M}}^{rq}, \qquad \mathcal{M}_{st}\tilde{\mathcal{M}}^{tq}\mathcal{M}_{qr} = \mathcal{M}_{sr}, \quad (4.48)$$

where $P^r{}_s$ is the projector of the same rank as $\mathcal{M}_{st}$, i.e.

$$\mathcal{M}P = \mathcal{M}, \qquad P\tilde{\mathcal{M}} = \tilde{\mathcal{M}}.$$

If $\mathcal{M}_{rs}$ is invertible, which is the case we shall mostly deal with, $\tilde{\mathcal{M}}^{rs}$ is inverse of $\mathcal{M}_{rs}$, i.e.

$$P^r{}_s = \tilde{\mathcal{M}}^{rt}\mathcal{M}_{ts} = \delta^r_s. \quad (4.49)$$

The full duality-symmetric Lagrangian is given by (4.4). The terms $\mathcal{L}^{\rm vec}$ from (4.7) and $\mathcal{L}^{\mathcal{H}_4/\mathcal{F}}$ from (4.47) together form the duality-symmetric Lagrangian for the fields A_1^r and C_{3r}. Indeed, their sum can be rewritten in the following manifestly duality-symmetric form:

$$\mathcal{L}^{\rm vec} + \mathcal{L}^{\mathcal{H}_4/\mathcal{F}} = \frac{1}{2}\mathcal{M}_{rs}\mathcal{F}^r\wedge *\mathcal{F}^s - \frac{1}{2}\tilde{\mathcal{M}}^{rs}\mathcal{H}_{4r}\wedge *\mathcal{H}_{4s} - (\tilde{\mathcal{M}}\mathcal{M})^r{}_s\mathcal{H}_{4r}\wedge\mathcal{F}^s + \frac{1}{4}\tilde{\mathcal{M}}^{rs}(i_v * \mathcal{K}_{4r})\wedge *(i_v * \mathcal{K}_{4s}) + \frac{1}{2}\tilde{\mathcal{M}}^{rs}(i_v\mathcal{K}_{4r})\wedge *(i_v\mathcal{K}_{4s}). \quad (4.50)$$

We should now check that the addition of the Lagrangian (4.47) to the action (4.3) does not spoil the local symmetries (4.22) and (4.23). Using the relation $*(i_v * \mathcal{K}_{4r}\wedge v) = \mathcal{K}_{4r} - i_v\mathcal{K}_{4r}\wedge v$ we find that the generic variation of (4.47) is

$$\delta\mathcal{L}^{\mathcal{H}_4/\mathcal{F}} = -\frac{1}{2}\tilde{\mathcal{M}}^{rs}\mathcal{G}_{3r}\wedge\mathcal{G}_{1s}\wedge da\wedge d\delta a + \frac{1}{2}\tilde{\mathcal{M}}^{rs}\mathcal{G}_{1s}\wedge da\wedge\delta\mathcal{H}_{4r} - \frac{1}{2}\tilde{\mathcal{M}}^{rt}\mathcal{M}_{ts}\mathcal{G}_{3r}\wedge da\wedge\delta\mathcal{F}^s + \frac{1}{2}\tilde{\mathcal{M}}^{rt}\mathcal{M}_{ts}\mathcal{K}_{4r}\wedge\delta\mathcal{F}^s + \frac{1}{4}\delta\tilde{\mathcal{M}}^{rs}i_v(*\mathcal{K}_{4r})\wedge\mathcal{K}_{4s}\wedge v_1 + \frac{1}{2}\tilde{\mathcal{M}}^{rt}\delta\mathcal{M}_{ts}\mathcal{F}^s\wedge\mathcal{K}_{4r}, \quad (4.51)$$

where we introduced the definitions

$$\mathcal{G}_{1r} := \frac{i_v(*\mathcal{K}_{4r})}{\sqrt{\partial a\partial a}}, \qquad \mathcal{G}_{3r} := \frac{i_v\mathcal{K}_{4r}}{\sqrt{\partial a\partial a}}, \quad (4.52)$$

in accordance with (4.24). Adding this variation to (4.45) and making use of the explicit form of $\delta\mathcal{F}^r$ and $\delta\mathcal{H}_{4r}$ given in (2.13) we get

$$\delta\mathcal{L}_{\rm ext} = 2\mathcal{G}_2^J\eta_{JI}\wedge da\wedge\Big(D\Delta B_2^I - \frac{1}{2}d(\delta a)\wedge\mathcal{G}_2^I + 2d^I_{st}\mathcal{F}^s\wedge\delta A^t + g^{Ir}\Delta C_{3r}\Big) + 2\tilde{\mathcal{M}}^{rs}\mathcal{G}_{3r}\wedge\mathcal{G}_{1s}\wedge da\wedge d\delta a - 2\tilde{\mathcal{M}}^{rs}\mathcal{G}_{1s}\wedge da\wedge D\Delta C_{3r} + 2g_I^s(\delta - \tilde{\mathcal{M}}\mathcal{M})^r{}_s\mathcal{K}_{4r}\wedge\Delta B_2^I + 2(\delta - \tilde{\mathcal{M}}\mathcal{M})^r{}_s\mathcal{K}_{4r}\wedge D\delta A^s + k_r^\alpha(*\mathcal{J}_\alpha - 2\mathcal{H}_{5\alpha})\wedge\delta A^r + 4\tilde{\mathcal{M}}^{rt}\mathcal{G}_{1t}\wedge da\wedge\mathcal{F}^s d_{Irs}\wedge\Delta B_2^I + 2\tilde{\mathcal{M}}^{rt}\mathcal{M}_{ts}\mathcal{G}_{3r}\wedge da\wedge g_I^s\Delta B_2^I + 4\tilde{\mathcal{M}}^{rt}\mathcal{G}_{1t}\wedge da\wedge\mathcal{H}_3^I\wedge\delta A^s d_{Irs} + 2\tilde{\mathcal{M}}^{rt}\mathcal{M}_{ts}\mathcal{G}_{3r}\wedge da\wedge D\delta A^s - 2\tilde{\mathcal{M}}^{rs}\mathcal{G}_{1s}\wedge da\wedge k_r^\alpha\Delta C_{4\alpha} + \delta_Y\mathcal{L} + \delta_\phi\mathcal{L}. \quad (4.53)$$

One can check that this variation vanishes for the local symmetry transformations (4.23) provided that A_1^r and $C_{4\alpha}$ transform as follows:

$$\delta A^r = \delta a \tilde{\mathcal{M}}^{rs} \mathcal{G}_{1s}, \tag{4.54}$$

$$\Delta C_{4\alpha} k_r^\alpha = \frac{\delta a}{\sqrt{(\partial a)^2}} (k_r^\alpha i_v \mathcal{H}_{5\alpha} - 2 g^s_{[I} d_{J]sr} \mathcal{M}^I i_v * D\mathcal{M}^J + *(\mathcal{G}_{1t} \wedge da) i_v D(\tilde{\mathcal{M}}\mathcal{M})^t{}_r) - (\delta - \tilde{\mathcal{M}}\mathcal{M})^s{}_r X_{4s}, \tag{4.55}$$

where the 4-form X_{4s} is such that

$$X_{4s}(\delta - \tilde{\mathcal{M}}\mathcal{M})^s{}_r g^{Ir} = \frac{\delta a}{\sqrt{(\partial a)^2}} g^{Ir} *(\mathcal{G}_{1t} \wedge da) i_v D(\tilde{\mathcal{M}}\mathcal{M})^t{}_r.$$

This relation has solutions when $(\tilde{\mathcal{M}}\mathcal{M})^t{}_r g^{Ir} = 0$. It is trivially satisfied in the case of nondegenerate $\mathcal{M}_{rs}$, i.e. when $(\tilde{\mathcal{M}}\mathcal{M})^t{}_r = \delta^t{}_r$. It is this case that we shall consider in detail in the following. If on the other hand $\mathcal{M}_{rs}$ is degenerate, some vector gauge fields do not have the kinetic terms in the Lagrangian and are therefore nondynamical. In [47] it has been shown that for the (1, 0) superconformal models, invertibility of $\mathcal{M}_{rs}$ from (4.2) can always be achieved by including Abelian factors in the gauge group.

1. Derivation of the field equations

Let us now discuss the derivation of the field equations from the variation (4.53) of the extended Lagrangian (4.4), assuming that the kinetic matrix $\mathcal{M}_{rs}$ of the vector fields is invertible (4.49). In this case, the variation (4.53) reduces to

$$\begin{aligned}\delta\mathcal{L}_{\rm ext} &= 2\mathcal{G}_2^J \eta_{JI} \wedge da \wedge \left(D\Delta B_2^I - \frac{1}{2} d(\delta a) \wedge \mathcal{G}_2^I + 2 d^I_{st} \mathcal{F}^s \wedge \delta A^t + g^{Ir} \Delta C_{3r}\right) + 2\tilde{\mathcal{M}}^{rs} \mathcal{G}_{3r} \wedge \mathcal{G}_{1s} \wedge da \wedge d\delta a \\ &\quad - 2\tilde{\mathcal{M}}^{rs} \mathcal{G}_{1s} \wedge da \wedge D\Delta C_{3r} + 2\mathcal{G}_{3r} \wedge da \wedge g_I^r \Delta B_2^I + 4\tilde{\mathcal{M}}^{rt} \mathcal{G}_{1t} \wedge da \wedge \mathcal{F}^s d_{Irs} \wedge \Delta B_2^I \\ &\quad + 4\tilde{\mathcal{M}}^{rt} \mathcal{G}_{1t} \wedge da \wedge \mathcal{H}_3^I \wedge \delta A^s d_{Irs} + 2\mathcal{G}_{3r} \wedge da \wedge D\delta A^r + k_r^\alpha (*\mathcal{J}_\alpha - 2\mathcal{H}_{5\alpha}) \wedge \delta A^r \\ &\quad - 2\tilde{\mathcal{M}}^{rs} \mathcal{G}_{1s} \wedge da \wedge k_r^\alpha \Delta C_{4\alpha} + \delta_Y \mathcal{L}_{\rm ext} + \delta_\phi \mathcal{L}_{\rm ext}.\end{aligned} \tag{4.56}$$

It follows that this variation vanishes under an extension of the local symmetry transformations (4.22) and under

$$\delta A^r = \varphi^r da, \qquad \Delta_{\varphi_1} B_2^I = \varphi_1^I \wedge da, \qquad \Delta_{\varphi_2} C_{3r} = \varphi_{2r} \wedge da, \qquad \Delta C_{4\alpha} = \varphi_{3\alpha} \wedge da, \tag{4.57}$$

where the parameters φ_1^I, φ_{2r}, and $\varphi_{3\alpha}$ are arbitrary and φ^r satisfies $k_r^\alpha \varphi^r = 0$. Similar to (4.29) it turns out to be useful to introduce two projectors $\mathbb{P}_r{}^s$ and $\mathcal{P}^r{}_s$ of minimal rank satisfying

$$k_r^\alpha \mathcal{P}^r{}_s = k_s^\alpha, \qquad g_I^r \mathbb{P}_r{}^s = g_I^s, \tag{4.58}$$

respectively, together with their respective complementary projectors defined according to (4.30). The orthogonality $g_I^r k_r^\alpha = 0$ implies that

$$\mathcal{P}^r{}_t \mathbb{P}_s{}^t = 0, \tag{4.59}$$

whereas the opposite contraction of the two projectors is not necessarily vanishing. The equations of motion which follow from the $\Delta C_{4\alpha}$ variation of (4.56) are

$$k_r^\alpha \tilde{\mathcal{M}}^{rs} \mathcal{G}_{1s} \wedge da = 0. \tag{4.60}$$

By construction $\mathcal{G}_{1s} \propto i_v \mathcal{K}_4$ does not contain any contribution proportional to da, which means that (4.60) implies

$$\mathcal{P}^r{}_t \tilde{\mathcal{M}}^{ts} \mathcal{G}_{1s} = 0. \tag{4.61}$$

Let us turn to the equations appearing as the coefficient for ΔC_{3r} in the variation (4.56):

$$D(\tilde{\mathcal{M}}^{rs} \mathcal{G}_{1s} \wedge da) = g^{Ir} \mathcal{G}_{2I} \wedge da. \tag{4.62}$$

Upon projection with $\bar{\mathbb{P}}$, we find

$$0 = \bar{\mathbb{P}}_s{}^r D(\tilde{\mathcal{M}}^{st} \mathcal{G}_{1t} \wedge da) = \bar{\mathbb{P}}_s{}^r \bar{\mathcal{P}}^s{}_v d(\tilde{\mathcal{M}}^{vt} \mathcal{G}_{1t} \wedge da), \tag{4.63}$$

where the second equality uses (4.60) and the fact that the connection part vanishes due to

$$\bar{\mathbb{P}}_s{}^r \bar{\mathcal{P}}^t{}_v X_{ut}{}^s = \bar{\mathbb{P}}_s{}^r \bar{\mathcal{P}}^t{}_v (k_t^\alpha c^s_{\alpha u} + 2 d^I_{tu} g_I^s) = 0. \tag{4.64}$$

Similar to the Abelian case, we thus conclude that locally

$$(\bar{\mathbb{P}}_s{}^r \tilde{\mathcal{M}}^{st} \mathcal{G}_{1t}) \wedge da = d(\bar{\mathbb{P}}_s{}^r \phi^s da), \tag{4.65}$$

with ϕ^s satisfying $k_s^\alpha \phi^s = 0$. We can thus use the local symmetry (4.57) with the parameter φ^r (also obeying $k_s^\alpha \varphi^s = 0$) to obtain $\bar{\mathbb{P}}_s{}^r \tilde{\mathcal{M}}^{st} \mathcal{G}_{1t} = 0$. Finally, the local symmetry (4.57) with properly chosen parameter φ_1^I can be used to extend this equation to the full duality equation

$$\tilde{\mathcal{M}}^{rs} \mathcal{G}_{1s} = 0. \tag{4.66}$$

We note that this fixes the local symmetry with parameter φ_1^I up to parameters satisfying $g_I^r \varphi_1^I = 0$ which do not contribute to the variation of $\tilde{\mathcal{M}}^{rs} \mathcal{G}_{1s}$. We are thus left with the local symmetries of (4.22) above, while Eq. (4.62) reduces to (4.31). Thus we can proceed as in Sec. IV A 1 for the minimal case and obtain

$$g^{Ir}(\mathcal{H}_{3I} + *\mathcal{H}_{3I}) = 0. \tag{4.67}$$

Let us turn to the equations produced by the variation ΔB_2^I in (4.56). In view of (4.66) we get

$$D(\mathcal{G}_{2I} \wedge da) = g_I^r \mathcal{G}_{3r} \wedge da. \tag{4.68}$$

Equations (4.67) and (4.68) are precisely analogous to Eqs. (4.33) and (4.35) which have been our starting point in the discussion of field equations in the minimal case in Sec. IV A 1. Proceeding as above, we may thus further gauge-fix the remaining local symmetries of (4.22) and arrive at the field equations

$$\mathcal{H}_{3I} + *\mathcal{H}_{3I} = 0, \qquad g_I^r \mathcal{G}_{3r} = 0. \tag{4.69}$$

Finally, let us turn to the equations appearing as the coefficient for the vector fields δA^r in (4.56). Upon using all field equations that we have already derived, these equations reduce to

$$2D(\mathcal{G}_{3r} \wedge da) = k_r^\alpha(*\mathcal{J}_\alpha - 2\mathcal{H}_{5\alpha}), \tag{4.70}$$

and can be solved with the same strategy: projection with $\bar{\mathcal{P}}$ yields

$$0 = \bar{\mathcal{P}}^r{}_s D(\mathcal{G}_{3r} \wedge da) = \bar{\mathcal{P}}^r{}_s \bar{\mathbb{P}}_r{}^t d(\mathcal{G}_{3t} \wedge da), \tag{4.71}$$

where again we have used (4.64) together with (4.69) to show that the connection part of the covariant derivative vanishes. As in the Abelian case we conclude that locally

$$(\bar{\mathcal{P}}^r{}_s \mathcal{G}_{3r}) \wedge da = d(\bar{\mathcal{P}}^r{}_s \phi_{3r} \wedge da). \tag{4.72}$$

As above, proper combinations of the remaining local symmetries from (4.57) allow us to obtain $\mathcal{G}_{3r} = 0$. Together with (4.66) we thus obtain $\mathcal{K}_{4r} = 0$. The rhs of (4.70) eventually gives the last equation of (4.1).

Summarizing, we have shown that the extended Lagrangian (4.4) gives rise to the set of non-Abelian duality equations (4.1). Again, the field $a(x)$ does not enter these equations, showing that $a(x)$ is completely auxiliary and is only required for ensuring the space-time covariance of the action. Via the Bianchi identities (2.9) these equations give rise to the second-order field equations for the vector fields in (4.43). The last equation in (4.1) is a projection of the duality relation (3.5) between the scalar fields and the 4-form gauge fields. In addition, the variation of the Lagrangian (4.4) with respect to the scalar fields gives rise to their standard second-order field equations.

V. EXAMPLE

Let us now consider an example of a minimal Lagrangian model given in [45]. In this model the vector fields split into two sets

$$A^r = (A^a, \mathcal{A}^{\hat{I}}) \tag{5.1}$$

and the constant tensors $f_{rs}{}^t$ and d_{rs}^I reduce as follows:

$$f_{rs}{}^t \to \left(f_{ab}{}^c, -\frac{1}{2}(T_a)_{\hat{I}}{}^{\hat{J}}\right), \qquad d_{rs}^I \to \frac{1}{2}(T_a)_{\hat{I}}{}^{\hat{J}}, \tag{5.2}$$

i.e., e.g. $f_{a\hat{I}}{}^{\hat{J}} = -f_{\hat{I}a}{}^{\hat{J}} = -\frac{1}{2}(T_a)_{\hat{I}}{}^{\hat{J}}$, where the indices a, b, c label the adjoint representation of a gauge group G whose algebra is defined by the structure constants $f_{ab}{}^c$, and the indices $\hat{I}$, $\hat{J}$ label representations $\mathcal{R}$ (upper indices) and $\mathcal{R}'$ (lower indices) of G generated by $(T_a)_{\hat{I}}{}^{\hat{J}}$.

The scalars ϕ^I and the 2-form fields B^I split into two sets taking values in $\mathcal{R}'$ and $\mathcal{R}$

$$\phi^I = (\hat{\phi}_{\hat{I}}, \phi^{\hat{J}}), \qquad B_2^I = (\hat{B}_{2\hat{I}}, B_2^{\hat{J}}). \tag{5.3}$$

It is important to note that the fields with lower and upper indices $\hat{I}$ are *different* fields, and that the metric η_{IJ} is antidiagonal:

$$\eta^{IJ} = \begin{pmatrix} 0 & \delta_{\hat{I}}^{\hat{J}} \\ \delta_{\hat{J}}^{\hat{I}} & 0 \end{pmatrix}. \tag{5.4}$$

To be more explicit, in the case under consideration

$$g^{Is} = \delta^{I\hat{J}} \delta_{\hat{J}}^s, \qquad g_J^r = \delta_{\hat{J}}^r \delta_I^{\hat{J}}, \tag{5.5}$$

$$\begin{aligned} d_{rs}^I &= (0, d_{rs}^{\hat{I}}) = (0, \delta_{(r}^a \delta_{s)}^{\hat{J}} T_{a\hat{J}}{}^{\hat{I}}), \\ d_{Irs} &= (d_{rs}^{\hat{I}}, 0) = (\delta_{(r}^a \delta_{s)}^{\hat{J}} T_{a\hat{J}}{}^{\hat{I}}, 0), \end{aligned} \tag{5.6}$$

$$f_{st}{}^r = \delta_s^b \delta_t^c f_{bc}{}^a \delta_a^r - \delta_{[s}^a \delta_{t]}^{\hat{J}} T_{a\hat{J}}{}^{\hat{I}} \delta_{\hat{I}}^r, \tag{5.7}$$

$$\begin{aligned} X_{st}{}^r &= -\delta_s^b \delta_t^c f_{bc}{}^a \delta_a^r + 2\delta_{[s}^a \delta_{t]}^{\hat{J}} T_{a\hat{J}}{}^{\hat{I}} \delta_{\hat{I}}^r, \\ X_{rJ}{}^I &= \delta_a^r T_{a\hat{J}}{}^{\hat{I}} (\delta_{\hat{I}}^I \delta_J^{\hat{J}} - \delta^{I\hat{I}} \delta_{J\hat{J}}). \end{aligned} \tag{5.8}$$

Notice that both of Eqs. (5.8) give $X_{a\hat{J}}{}^{\hat{I}} = T_{a\hat{J}}{}^{\hat{I}}$.

Finally, the 3-form fields take values in the $\mathcal{R}'$ representation only, i.e.

$$C_{3r} = (C_{3\hat{I}}, 0). \tag{5.9}$$

For simplicity, in the further consideration we shall not take into account tensor fields which are singlets with respect to the non-Abelian symmetries.

The field strengths (2.1)–(2.3) take the following form:

$$\begin{aligned} \mathcal{F}^r\colon\ \mathcal{F}^a &= dA^a + \frac{1}{2} f_{bc}{}^a A^b \wedge A^c, \\ \mathcal{F}^{\hat{I}} &= d\mathcal{A}^{\hat{I}} + \frac{1}{2}\mathcal{A}^{\hat{I}} \wedge A^a T_{a\hat{J}}{}^{\hat{I}} + B_2^{\hat{I}} \equiv \mathcal{B}_2^{\hat{I}}, \end{aligned} \tag{5.10}$$

$$\begin{aligned} \mathcal{H}_3^I\colon\ \mathcal{H}_3^{\hat{I}} &= D\mathcal{B}_2^{\hat{I}}, \\ \mathcal{H}_{3\hat{I}} &= d\hat{B}_{2\hat{I}} - \frac{1}{2} T_{a\hat{J}}{}^{\hat{I}} A^a \wedge \hat{B}_{\hat{I}} + C_{3\hat{I}} \equiv \mathcal{C}_{3\hat{I}}, \end{aligned} \tag{5.11}$$

$$\mathcal{H}_{4\hat{r}}\colon\ \mathcal{H}_{4\hat{I}} = DC_{3\hat{I}}, \tag{5.12}$$

where the covariant derivative D contains the vector potential A^a only,

$$D = d + A^a T_a$$

or more explicitly

$$D\mathcal{B}_2^{\hat{I}} = d\mathcal{B}_2^{\hat{I}} + \mathcal{B}_2^{\hat{J}} \wedge A^a T_{a\hat{J}}{}^{\hat{I}},$$
$$D\mathcal{C}_{3\hat{I}} = d\mathcal{C}_{3\hat{I}} - \mathcal{C}_{3\hat{J}} \wedge A^a T_{a\hat{I}}{}^{\hat{J}}. \quad (5.13)$$

Note that the fields $\mathcal{A}^{\hat{I}}$ and $\hat{B}_{2\hat{I}}$ are of a Stückelberg type and thus can be absorbed, respectively, by $B_2^{\hat{I}}$ and $C_{3\hat{I}}$, which is indicated in (5.10) and (5.11) by renaming $\mathcal{F}^{\hat{I}} \equiv \mathcal{B}_2^{\hat{I}}$ and $\mathcal{H}_{3\hat{I}} \equiv \mathcal{C}_{3\hat{I}}$. It is these latter fields that transform covariantly under the gauge-group representations generated by $(T_a)_{\hat{J}}{}^{\hat{I}}$ and enter the action.

In this case the action (4.3)–(4.10) reduces to the following form:

$$S = \int_{\mathcal{M}_6} (-D\hat{\phi}_{\hat{I}} \wedge *D\phi^{\hat{I}} + d^6x \hat{\phi}_{\hat{I}}(T_a)_{\hat{J}}{}^{\hat{I}} Y^{aij} Y_{ij}^{\hat{J}}$$
$$+ 2\hat{\phi}_{\hat{I}}(T_a)_{\hat{J}}{}^{\hat{I}} \mathcal{B}_2^{\hat{J}} \wedge *\mathcal{F}^a)$$
$$- \int_{\mathcal{M}_6} (i_v(\mathcal{H}_3^{\hat{I}} + *\mathcal{H}_3^{\hat{I}}) \wedge \mathcal{C}_{3\hat{I}} \wedge v$$
$$+ i_v(\mathcal{C}_{3\hat{I}} + *\mathcal{C}_{3\hat{I}}) \wedge \mathcal{H}_3^{\hat{I}} \wedge v - \mathcal{H}_3^{\hat{I}} \wedge \mathcal{C}_{3\hat{I}}). \quad (5.14)$$

The first term in (5.14) can be rewritten in the following form:

$$i_v(\mathcal{H}_3^{\hat{I}} + *\mathcal{H}_3^{\hat{I}}) \wedge \mathcal{C}_{3\hat{I}} \wedge v = i_v(\mathcal{C}_{3\hat{I}} + *\mathcal{C}_{3\hat{I}}) \wedge \mathcal{H}_3^{\hat{I}} \wedge v + \mathcal{C}_{3\hat{I}} \wedge \mathcal{H}_3^{\hat{I}}. \quad (5.15)$$

So the action (5.14) takes the form

$$S = \int_{\mathcal{M}_6} (-D\hat{\phi}_{\hat{I}} \wedge *D\phi^{\hat{I}} + d^6x \hat{\phi}_{\hat{I}}(T_a)_{\hat{J}}{}^{\hat{I}} Y^{aij} Y_{ij}^{\hat{J}}$$
$$+ 2\hat{\phi}_{\hat{I}}(T_a)_{\hat{J}}{}^{\hat{I}} \mathcal{B}_2^{\hat{J}} \wedge *\mathcal{F}^a)$$
$$+ 2\int_{\mathcal{M}_6} \mathcal{H}_3^{\hat{I}} \wedge (\mathcal{C}_{3\hat{I}} - i_v(\mathcal{C}_{3\hat{I}} + *\mathcal{C}_{3\hat{I}}) \wedge v). \quad (5.16)$$

Now note that the combination of the $\mathcal{C}_{3\hat{I}}$ terms is anti-self-dual. Indeed, in view of the identity (4.14)

$$C_{3\hat{I}}^{-} := \mathcal{C}_{3\hat{I}} - i_v(\mathcal{C}_{3\hat{I}} + *\mathcal{C}_{3\hat{I}}) \wedge v$$
$$= -i_v * \mathcal{C}_{3\hat{I}} \wedge v + *(i_v * \mathcal{C}_{3\hat{I}} \wedge v) = - * C_{3\hat{I}}^{-}. \quad (5.17)$$

The generic identity (4.14) applied to an anti-self-dual tensor reads

$$C_{3\hat{I}}^{-} = i_v C_{3\hat{I}}^{-} \wedge v - *(i_v C_{3\hat{I}}^{-} \wedge v). \quad (5.18)$$

From (5.17) and (5.18) it follows that

$$*\mathcal{C}_{3\hat{I}} = -C_{3\hat{I}}^{-} + *(\varphi_{2\hat{I}} \wedge v) \Rightarrow \mathcal{C}_{3\hat{I}} = C_{3\hat{I}}^{-} + \varphi_{2\hat{I}} \wedge v,$$
$$\varphi_{2\hat{I}} = i_v(\mathcal{C}_{3\hat{I}} + *\mathcal{C}_{3\hat{I}}). \quad (5.19)$$

The $\varphi_{2\hat{I}} \wedge v$ part of $\mathcal{C}_{3\hat{I}}$ does not contribute to the action, so without loss of generality, in (5.16) we can replace $\mathcal{C}_{3\hat{I}}$ with $C_{3\hat{I}}^{-}$ and the action reduces to

$$S = \int_{\mathcal{M}_6} (-D\hat{\phi}_{\hat{I}} \wedge *D\phi^{\hat{I}} + d^6x \hat{\phi}_{\hat{I}}(T_a)_{\hat{J}}{}^{\hat{I}} Y^{aij} Y_{ij}^{\hat{J}}$$
$$+ 2\hat{\phi}_{\hat{I}}(T_a)_{\hat{J}}{}^{\hat{I}} \mathcal{B}_2^{\hat{J}} \wedge *\mathcal{F}^a) + 2\int_{\mathcal{M}_6} \mathcal{H}_3^{\hat{I}} \wedge C_{3\hat{I}}^{-}. \quad (5.20)$$

We see that the auxiliary 1-form field $v(x)$ completely disappears from the action and the anti-self-dual field $C_{3\hat{I}}^{-} = - * C_{3\hat{I}}^{-}$ is the Lagrange multiplier which ensures the anti-self-duality of $\mathcal{H}_3^{\hat{I}}$. On the other hand, the variation of this action with respect to $\mathcal{B}_2^{\hat{I}}$ produces the duality relation between the field strengths of $C_{3\hat{I}}^{-}$ and A^a

$$DC_{3\hat{I}}^{-} + *\mathcal{F}^a \hat{\phi}_{\hat{J}}(T_a)_{\hat{I}}{}^{\hat{J}} = 0. \quad (5.21)$$

The variation with respect to A^a gives the equation of motion

$$\Big(\hat{\phi}_{\hat{J}} D * \mathcal{B}_2^{\hat{I}} + *\mathcal{B}_2^{\hat{I}} \wedge D\hat{\phi}_{\hat{J}} - 2\mathcal{B}_2^{\hat{I}} \wedge C_{3\hat{J}}^{-} + \frac{1}{2}\phi^{\hat{I}} * D\hat{\phi}_{\hat{J}}$$
$$+ \frac{1}{2}\phi_{\hat{J}} * D\hat{\phi}^{\hat{I}}\Big) T_{a\hat{I}}{}^{\hat{J}} = 0. \quad (5.22)$$

Yang-Mills-type equations for the vector gauge fields can be obtained as a self-consistency condition for Eq. (5.21),

$$(\hat{\phi}_{\hat{J}} D * \mathcal{F}^a + *\mathcal{F}^a \wedge D\hat{\phi}_{\hat{J}} - 1/2\mathcal{F}^a \wedge C_{3\hat{J}}^{-}) T_{a\hat{I}}{}^{\hat{J}} = 0. \quad (5.23)$$

Finally, the scalar field equations are

$$D * D\hat{\phi}_{\hat{I}} = 0,$$
$$D * D\phi^{\hat{I}} = T_{a\hat{J}}{}^{\hat{I}}(d^6x Y^{aij} Y_{ij}^{\hat{J}} + 2\mathcal{B}_2^{\hat{J}} \wedge *\mathcal{F}^a), \quad (5.24)$$

$$\hat{\phi}_{\hat{I}}(T_a)_{\hat{J}}{}^{\hat{I}} Y^{aij} = 0, \qquad \hat{\phi}_{\hat{I}}(T_a)_{\hat{J}}{}^{\hat{I}} Y_{ij}^{\hat{J}} = 0. \quad (5.25)$$

VI. CONCLUSION

We have constructed the duality-symmetric actions for a large class of six-dimensional models describing hierarchies of non-Abelian scalar, vector and tensor fields related to each other by (self-)duality equations that follow from these actions. This class includes the bosonic sectors of the 6d (1, 0) superconformal models of interacting non-Abelian vector, tensor and hypermultiplets constructed in [43,45,47]. The supersymmetrization of the actions of this paper by the inclusion of fermionic sectors will be considered elsewhere. A generic feature of the supersymmetric manifestly duality-invariant actions is that the off-shell supersymmetry transformations of fermionic fields get augmented by terms which vanish when the bosonic fields satisfy the (self-)duality conditions (see e.g. [51,53]).

We have first obtained the action (4.3) that gives rise to non-Abelian self-duality equations for the tensor fields. In the second step, we have extended this action to the action (4.4) that also yields the non-Abelian first-order duality equations between vector and 3-form tensor gauge potentials. Continuing this line of thought, a natural next step in the construction would be the extension of (4.4) to an action that also yields the first-order duality equations between scalar and 4-form tensor gauge potentials. This would correspond to a truly democratic formulation of the six-dimensional models, in which *all* p-forms enter on equal footing with the forms of different degree interlocked by the non-Abelian structure of the tensor hierarchy. This final extension to include the duality equations for the scalar fields will proceed straightforwardly along the pattern put forward in Sec. IV. On the technical side it will require us to extend the six-dimensional tensor hierarchy of Sec. II by the inclusion of 5-form gauge potentials, cf. [54].

In connection with the issues of the (2, 0) superconformal theory of multiple $M5$-branes, further study is required for understanding whether in some of these (1, 0) supersymmetric models the redundant degrees of freedom associated with propagating vector fields can be removed and (1, 0) supersymmetry can be enhanced to (2, 0). Another important issue to be resolved is the presence (in general) of ghosts in the action due to the nonpositive definiteness of the metric η_{IJ} [see e.g. Eq. (4.5)]. Clearly, it would also be of interest to study the relation of these systems to other proposals of non-Abelian $6d$ chiral tensor models and, by dimensional reduction, to $5d$ and $4d$ super-Yang-Mills theories.

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APPENDIX: ALGEBRAIC CONSTRAINTS ON THE CONSTANT TENSORS

The algebraic consistency conditions for the tensors $f_{st}{}^r$, d^I_{rs}, g^{Ir}, k^α_r defining the six-dimensional tensor hierarchy are given by

$$\begin{aligned}
d_{Ir(u}d^I_{vs)} &= 0,\\
(d^J_{r(u}d^I_{v)s} - d^J_{uv}d^I_{rs} + d_{Krs}d^K_{uv}\eta^{IJ})g^s_J &= f_{r(u}{}^s d^I_{v)s},\\
3f_{[pq}{}^u f_{r]u}{}^s - g^s_I d^I_{u[p}f_{qr]}{}^u &= 0,\\
X_{rs}{}^t \equiv d^I_{rs}g^t_I - f_{rs}{}^t &= -k^\alpha_r c^t_{\alpha s}\\
X_{rIJ} \equiv 4g^s_{[I}d_{J]rs} &= 2k^\alpha_r c_{\alpha IJ}\\
f_{rs}{}^t g^r_I - d^J_{rs}g^t_J g^r_I &= 0,\\
g^r_K g^s_{[I}d_{J]sr} = 0, \qquad g^r_I g^{Is} = 0, \qquad k^\alpha_r g^{Ir} &= 0.
\end{aligned} \tag{A1}$$

In particular, the third equation shows that the violation of the Jacobi identities of the "structure constants" $f_{rs}{}^t$ is related to the Stückelberg coupling g^r_I. The general structure of solutions to these constraints has been analyzed in [45].

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