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$OSp(1|4)$ supergravity and its noncommutative extension

Leonardo Castellani*

Dipartimento di Scienze e Innovazione Tecnologica, INFN Gruppo collegato di Alessandria, Università del Piemonte Orientale, Viale T. Michel 11, 15121 Alessandria, Italy

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We review the $OSp(1|4)$-invariant formulation of $N = 1$, $D = 4$ supergravity and present its noncommutative extension, based on a $\star$ product originating from an Abelian twist with deformation parameter θ. After the use of a geometric generalization of the Seiberg-Witten map, we obtain an extended (higher-derivative) supergravity theory, invariant under usual $OSp(1|4)$ gauge transformations. Gauge fixing breaks the $OSp(1|4)$ symmetry to its Lorentz subgroup and yields a Lorentz-invariant extended theory for which the classical limit $\theta \to 0$ is the usual $N = 1$, $D = 4$ anti–de Sitter supergravity.

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I. INTRODUCTION

We present a noncommutative (NC) extension of the $OSp(1|4)$-invariant action of $N = 1$, $D = 4$ anti–de Sitter supergravity, obtained by the use of a twisted $\star$ product, and a geometric generalization [1] of the Seiberg-Witten map [2] for Abelian twists. We thus find a higher-derivative extension of $OSp(1|4)$ supergravity where the higher-order couplings are dictated by the noncommutative structure of the original NC action. The resulting extended theory is geometric (diffeomorphic invariant) and gauge invariant under usual $OSp(1|4)$ gauge transformations.

Noncommutativity of spacetime coordinates

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \tag{1.1}$$

is a recurrent theme in physics, being advocated already by Heisenberg in the hope that uncertainty relations between spacetime coordinates could resolve UV divergences arising in quantum field theory [3]. This motivation still holds, in particular, for nonrenormalizable theories of gravity in which finiteness is the only option for consistency. The issue was explored initially by Snyder in Ref. [4], and since then noncommutative geometry has found applications in many branches of physics mainly in the last two decades. Some comprehensive reviews can be found in Refs. [5–11].

Relations (1.1) provide a (kinematical) way to encode quantum properties directly in the texture of spacetime. Field theories on noncommuting spacetime can be reformulated as field theories on ordinary (commuting) spacetime but with a deformed $\star$ product between fields. When the deformation originates from a twist, as in the present paper, the resulting $\star$ product is a twisted product, associative and noncommutative.

This product between fields generates infinitely many derivatives and introduces a dimensionful noncommutativity parameter θ. The prototypical example of a twisted product is the Moyal-Groenewold product [12] (historically arising in phase space after Weyl quantization [13]):

$$\begin{aligned} f(x) \star g(x) &\equiv \exp\left(\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\right) f(x)g(y)|_{y\to x} \\ &= f(x)g(x) + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f \partial_\nu g + \cdots \\ &\quad + \frac{1}{n!}\left(\frac{i}{2}\right)^n \theta^{\mu_1\nu_1}\cdots\theta^{\mu_n\nu_n}(\partial_{\mu_1}\cdots\partial_{\mu_n} f) \\ &\quad \times (\partial_{\nu_1}\cdots\partial_{\nu_n} g) + \cdots, \end{aligned} \tag{1.2}$$

with a constant θ. Using this deformed product, one finds $x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}$, realizing the commutation relations (1.1).

A straightforward generalization is provided by the twisted $\star$ product, in which the partial derivatives in Eq. (1.2) are replaced by a set of commuting tangent vectors $X_A \equiv X_A^\mu \partial_\mu$. Dealing with (super)gravity theories, it is desirable to extend the twisted $\star$ product to forms. This can be done simply by replacing the tangent vectors X_A, acting on functions, with Lie derivatives along X_A, acting on forms.

Replacing products between fields with $\star$ products yields nonlocal actions (called twisted or NC actions), containing an infinite number of new interactions and higher-derivative terms. In this way, twisted Yang-Mills theories in flat space have been constructed (see, for example, Refs. [14–16]) as well as twisted metric gravity [11,17]. Noncommutative $D = 4$ vielbein gravity has been treated in Refs. [18,19], in which deformations of conformal gravity and complex vielbein gravity were considered, and in Ref. [20], where a $U(2,2)$ $\star$-gauge-invariant NC action with constraints was proposed as a NC deformation of Einstein gravity. More recently, twisted vielbein gravity and its couplings to fermions [21], gauge fields [22], and scalars [23] have been constructed as well as a NC deformation of $D = 4$, $N = 1$ supergravity [24].

These twisted theories are invariant under deformations of the original symmetries. For example, the NC action for gauge fields is

*leonardo.castellani@unipmn.it

$$S = \frac{1}{4g^2} \int \mathrm{Tr}(F_{\mu\nu} \star F_{\mu\nu}), \tag{1.3}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - (A_\mu \star A_\nu - A_\nu \star A_\mu) \tag{1.4}$$

$$A_\mu = A^I_\mu T_I, \qquad \mathrm{Tr}(T^I T^J) = \delta^{IJ}. \tag{1.5}$$

The noncommutative gauge transformations

$$\delta_\varepsilon A_\mu = \partial_\mu \varepsilon - (A_\mu \star \varepsilon - \varepsilon \star A_\mu) \tag{1.6}$$

$$\delta_\varepsilon F_{\mu\nu} = -(F_{\mu\nu} \star \varepsilon - \varepsilon \star F_{\mu\nu}) \tag{1.7}$$

leave the action invariant because of the cyclicity of the trace and of the property

$$\int f \star g = \int g \star f \tag{1.8}$$

(cyclicity of integral) holding up to boundary terms.

Noncommutativity apparently comes with a price, i.e., a proliferation of new degrees of freedom. This can be understood by considering the $\star$ deformation of the Yang-Mills field strength:

$$F^I_{\mu\nu} T_I = \partial_\mu A^I_\nu T_I - \partial_\nu A^I_\mu T_I - (A^I_\mu \star A^J_\nu - A^I_\nu \star A^J_\mu) T_I T_J. \tag{1.9}$$

Because of the noncommutativity of the $\star$ product, anticommutators as well as commutators of group generators appear in the right-hand side, and therefore the T_I must be a basis for the whole universal enveloping algebra of G. Thus, I runs, in principle, on the infinite set of universal enveloping algebra elements (all symmetrized products of the original gauge generators), and the number of independent A^I_μ field components increases to infinity. This proliferation can be drastically reduced by choosing a specific representation for the generators T_I. For example, if the gauge group is $SU(2)$ and we take its generators to be in the defining 2×2 representation, these are just the Pauli matrices, and a basis for the enveloping algebra only requires an additional matrix proportional to the unit matrix.

We may get rid even of these additional degrees of freedom if we use the Seiberg-Witten map, which allows us to express all the fields appearing in the NC action (usually called the NC fields) in terms of series expansions in θ containing only the original fields of the undeformed theory, the so-called classical fields. The map is engineered so that the classical gauge transformations on the classical fields induce the NC gauge transformations on the NC fields. In the $SU(2)$ example, the map relates the four noncommutative fields to the three classical $SU(2)$ gauge fields.

Substituting in the action the NC fields with their expressions in terms of the classical fields yields an infinite series in powers of θ, for which the zeroth-order term is the classical action. This higher-derivative action is *invariant under the classical gauge variations* since these by construction induce the NC symmetries of the NC action. Every higher-order term in the θ expansion is actually separately invariant because the classical symmetries do not involve θ.

With this procedure, the NC deformation of vielbein gravity, found in Ref. [21], has been reexpressed in Ref. [1] in terms of the classical vielbein and spin connection, and its Lorentz-invariant (and higher-derivative) geometric action has been computed up to second order in the noncommutativity parameter [25]. The Seiberg-Witten map was also used in Ref. [20] to compute the first-order correction of the deformed $U(2,2)$ gauge-invariant and constrained theory and in Refs. [26–28] for the MacDowell-Mansouri gauge theory of gravity. We also mention the NC extension of $SO(2,3)$ anti–de Sitter (AdS) gravity of Ref. [29], which contains its expansion to order θ^2, and Ref. [30], in which the SW map for pure gravity is examined at second order.

In the present paper, we apply this method to $OSp(1|4)$ supergravity. For reviews on the $OSp(1|4)$ formulation of supergravity, see, for example, Refs. [31–33]. The classical theory contains the vielbein V^a, the spin connection ω^{ab}, the gravitino ψ, and nondynamical auxiliary fields (a scalar, a pseudoscalar, a vector, and a spin-1/2 fermion) necessary to ensure the full off-shell invariance (and closure) under local $OSp(1|4)$ gauge transformations. The auxiliary fields satisfy $OSp(1|4)$-invariant constraints. The $OSp(1|4)$ symmetry can be exploited to reach a gauge (the soldering gauge) in which the auxiliary fields take constant values. This gauge choice breaks the supergroup $OSp(1|4)$ to its Lorentz $SO(1,3)$ subgroup and reproduces the MacDowell-Mansouri action [34], equivalent up to boundary terms to the action of usual $N=1$, $D=4$ anti–de Sitter supergravity. For this action, supersymmetry is not a *gauge* symmetry any more since it gets broken along with the translations [the $SO(2,3)$ boosts]. However, supersymmetry is still "alive" in the gauge fixed theory. This can be seen in two distinct ways:

(i) by solving the supertorsion constraint and passing to second-order formalism (expressing the spin connection in terms of the vielbein and the gravitino fields) [35];

(ii) or, remaining in first-order formalism, by an appropriate modification of the spin connection supersymmetry variation [36].

Note that the supersymmetry transformations leaving the gauge fixed action invariant do not close off-shell [whereas the $OSp(4|1)$ gauge variations close off-shell by construction].

After $\star$ deforming the product in the $OSp(1|4)$ supergravity action and using the geometric Seiberg-Witten map, the resulting higher-derivative theory contains the same fields as the classical theory and is invariant under the same local $OSp(1|4)$ symmetries.

The reason we start from the $OSp(1|4)$ gauge-invariant theory resides in that all local symmetries (except general coordinate invariance) are contained in a gauge supergroup. The derivation of the Seiberg-Witten map in Ref. [2] is purely algebraic, and nothing changes in the derivation if groups are replaced by supergroups, connections by superconnections, etc. In the present paper, we apply the map to $OSp(1|4)$ superconnections and supermatrix (adjoint) auxiliary fields, containing all the fields of $N = 1$, $D = 4$ supergravity. Thus, we are guaranteed that supersymmetry [part of the $OSp(1|4)$ symmetry] survives in the extended theory.

By choosing the same gauge as in the classical theory [the gauge group $OSp(1|4)$ is the same], we obtain an extended theory containing only the vielbein, spin connection, and gravitino fields, reducing in the commutative limit to $N = 1$, $D = 4$ AdS supergravity.

The "mother," non-gauge-fixed extended theory is $OSp(1|4)$ invariant and as such is a locally supersymmetric higher-derivative theory. The price to pay for realizing this local gauge supersymmetry (closing off-shell) is the presence of constrained auxiliary fields.

The plan of the paper is as follows. In Sec. II, we briefly review $OSp(1|4)$ supergravity. In Sec. III, we recall its manifestly $OSp(1|4)$-invariant action. The noncommutative deformation is presented in Sec. IV. Section V deals with the geometric Seiberg-Witten map, applied in Sec. VI to obtain the extended $OSp(1|4)$ supergravity action to second order in θ. Section VII contains some conclusions.

II. CLASSICAL $OSp(1|4)$ SUPERGRAVITY

A. Geometric MacDowell-Mansouri action

The MacDowell-Mansouri action [34] for $N = 1$, $D = 4$ supergravity can be recast in an index-free form:

$$S = 2i \int \mathrm{Tr}(R \wedge R\gamma_5 + 2\Sigma \wedge \bar{\Sigma}\gamma_5), \qquad (2.1)$$

where the trace is taken on spinor indices, and the two-form curvatures R (bosonic) and Σ (fermionic) originate from the one-form $OSp(1|4)$ connection supermatrix:

$$\mathbf{\Omega} \equiv \begin{pmatrix} \Omega & \psi \\ \bar{\psi} & 0 \end{pmatrix}, \qquad \Omega \equiv \frac{1}{4}\omega^{ab}\gamma_{ab} - \frac{i}{2}V^a\gamma_a, \qquad (2.2)$$

for which the corresponding $OSp(1|4)$ curvature supermatrix is

$$\mathbf{R} = d\mathbf{\Omega} - \mathbf{\Omega} \wedge \mathbf{\Omega} \equiv \begin{pmatrix} R & \Sigma \\ \bar{\Sigma} & 0 \end{pmatrix}. \qquad (2.3)$$

Immediate matrix algebra yields[1]

$$R = \frac{1}{4}R^{ab}\gamma_{ab} - \frac{i}{2}R^a\gamma_a \qquad (2.4)$$

$$\Sigma = d\psi - \frac{1}{4}\omega^{ab}\gamma_{ab}\psi + \frac{i}{2}V^a\gamma_a\psi \qquad (2.5)$$

$$\bar{\Sigma} = d\bar{\psi} - \frac{1}{4}\bar{\psi}\omega^{ab}\gamma_{ab} + \frac{i}{2}\bar{\psi}V^a\gamma_a, \qquad (2.6)$$

with

$$R^{ab} \equiv d\omega^{ab} - \omega^{ac}\omega^{cb} + V^aV^b + \frac{1}{2}\bar{\psi}\gamma^{ab}\psi \qquad (2.7)$$

$$R^a \equiv dV^a - \omega^{ab}V^b - \frac{i}{2}\bar{\psi}\gamma^a\psi. \qquad (2.8)$$

We have also used the Fierz identity for one-form Majorana spinors:

$$\psi\bar{\psi} = \frac{1}{4}\left(\bar{\psi}\gamma^a\psi\gamma_a - \frac{1}{2}\bar{\psi}\gamma^{ab}\psi\gamma_{ab}\right) \qquad (2.9)$$

(to prove it, just multiply both sides by γ_c or γ_{cd}, and take the trace on spinor indices). The one-forms V^a, ω^{ab}, and ψ are, respectively, the vielbein, the spin connection, and the gravitino field (a Majorana spinor, i.e., $\bar{\psi} = \psi^T C$, for which C is the charge-conjugation matrix).

Carrying out the spinor trace in the action (2.1) yields the familiar MacDowell-Mansouri action:

$$S = 2\int \frac{1}{4}R^{ab} \wedge R^{cd}\varepsilon_{abcd} - 2i\bar{\Sigma} \wedge \gamma_5\Sigma. \qquad (2.10)$$

After inserting the curvature definitions, the action takes the form

$$S = \int \mathcal{R}^{ab}V^cV^d\varepsilon_{abcd} + 4\bar{\rho}\gamma_a\gamma_5\psi V^a + \frac{1}{2}(V^aV^bV^cV^d + 2\bar{\psi}\gamma^{ab}\psi V^cV^d)\epsilon_{abcd}, \qquad (2.11)$$

with

$$\mathcal{R}^{ab} \equiv d\omega^{ab} - \omega^{ac}\omega^{cb}, \qquad \rho \equiv d\psi - \frac{1}{4}\omega^{ab}\gamma_{ab}\psi \equiv \mathcal{D}\psi. \qquad (2.12)$$

We have dropped the topological term $\mathcal{R}^{ab}\mathcal{R}^{cd}\epsilon_{abcd}$ (Euler form) and used the gravitino Bianchi identity,

$$\mathcal{D}\rho = -\frac{1}{4}\mathcal{R}^{ab}\gamma_{ab}, \qquad (2.13)$$

and the gamma-matrix identity $2\gamma_{ab}\gamma_5 = i\epsilon_{abcd}\gamma^{cd}$ to recognize that $\frac{1}{2}\mathcal{R}^{ab}\bar{\psi}\gamma^{cd}\psi\epsilon_{abcd} - 4i\bar{\rho}\gamma_5\rho$ is a total derivative. Bianchi identities are easily obtained by taking the exterior derivative of the curvature definitions in Eq. (2.3) or in Eq. (2.12). The action (2.11) describes $N = 1$, $D = 4$ anti–de Sitter supergravity, the last term being the supersymmetric cosmological term. After

[1]We omit wedge products between forms, and all index contractions involve the Minkowski metric η_{ab}.

rescaling the vielbein and the gravitino as $V^a \to \lambda V^a$, $\psi \to \sqrt{\lambda}\psi$ and dividing the action by λ^2, the usual (Minkowski) $N = 1$, $D = 4$ supergravity is retrieved by taking the limit $\lambda \to 0$. This corresponds to the Inonü-Wigner contraction of $OSp(1|4)$ to the super-Poincaré group.

The action (2.1) can be rewritten even more compactly using the $OSp(1|4)$ curvature supermatrix $\mathbf{R}$:

$$S = 4\int \mathrm{STr}\left(\mathbf{R}\left(\mathbf{1} + \frac{\mathbf{\Gamma}^2}{2}\right)\mathbf{R}\mathbf{\Gamma}\right), \tag{2.14}$$

where STr is the supertrace and $\mathbf{\Gamma}$ is the following constant matrix:

$$\mathbf{\Gamma} \equiv \begin{pmatrix} i\gamma_5 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.15}$$

All boldface quantities are 5×5 supermatrices.

B. $OSp(1|4)$ gauge variations

The gauge transformation of the connection $\mathbf{\Omega}$

$$\delta_{\boldsymbol{\epsilon}}\mathbf{\Omega} = d\boldsymbol{\epsilon} - \mathbf{\Omega}\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{\Omega}, \tag{2.16}$$

where $\boldsymbol{\epsilon}$ is the $OSp(1|4)$ gauge parameter,

$$\boldsymbol{\epsilon} \equiv \begin{pmatrix} \frac{1}{4}\varepsilon^{ab}\gamma_{ab} - \frac{i}{2}\varepsilon^a\gamma_a & \epsilon \\ \bar{\epsilon} & 0 \end{pmatrix}, \tag{2.17}$$

becomes, on the component fields entering $\mathbf{\Omega}$,

$$\delta\omega^{ab} = d\varepsilon^{ab} - \omega^{ac}\varepsilon^{cb} + \omega^{bc}\varepsilon^{ca} - \varepsilon^a V^b + \varepsilon^b V^a - \bar{\epsilon}\gamma^{ab}\psi \tag{2.18}$$

$$\delta V^a = d\varepsilon^a - \omega^{ab}\varepsilon^b + \varepsilon^{ab}V^b + i\bar{\epsilon}\gamma^a\psi \tag{2.19}$$

$$\delta\psi = d\epsilon - \frac{1}{4}\omega^{ab}\gamma_{ab}\epsilon + \frac{i}{2}V^a\gamma_a\epsilon + \frac{1}{4}\varepsilon^{ab}\gamma_{ab}\psi - \frac{i}{2}\varepsilon^a\gamma_a\psi. \tag{2.20}$$

Similarly, from the gauge variation of the curvature $\mathbf{R}$,

$$\delta_{\boldsymbol{\epsilon}}\mathbf{R} = -\mathbf{R}\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{R}, \tag{2.21}$$

we find the gauge transformations of the curvature components:

$$\delta R^{ab} = -R^{ac}\varepsilon^{cb} + R^{bc}\varepsilon^{ca} - \varepsilon^a R^b + \varepsilon^b R^a - \bar{\epsilon}\gamma^{ab}\Sigma \tag{2.22}$$

$$\delta R^a = -R^{ab}\varepsilon^b + \varepsilon^{ab}R^b + i\bar{\epsilon}\gamma^a\Sigma \tag{2.23}$$

$$\delta\Sigma = -\frac{1}{4}R^{ab}\gamma_{ab}\epsilon + \frac{i}{2}R^a\gamma_a\epsilon + \frac{1}{4}\varepsilon^{ab}\gamma_{ab}\Sigma - \frac{i}{2}\varepsilon^a\gamma_a\Sigma. \tag{2.24}$$

As is well known, the action (2.14), although a bilinear in the $OSp(1|4)$ curvature, is *not* invariant under the $OSp(1|4)$ gauge transformations. In fact, it is not a Yang-Mills action (involving the exterior product of $\mathbf{R}$ with its Hodge dual) nor a topological action of the form $\int \mathbf{R}\mathbf{R}$; the constant supermatrix $\mathbf{\Gamma}$ ruins the $OSp(1|4)$ gauge invariance, and breaks it to its Lorentz subgroup. This can be seen easily by noting that the gauge parameter in Eq. (2.17) commutes with $\mathbf{\Gamma}$ only when restricted to Lorentz rotations ($\varepsilon^a = \epsilon = 0$) so that Lorentz rotations indeed leave the action invariant since the supertrace is cyclic. On the other hand, a gauge parameter supermatrix containing also translation and/or supersymmetry parameters does not commute with $\mathbf{\Gamma}$, and therefore the action is not invariant under $OSp(1|4)$ translations or supersymmetry transformations.

However, supersymmetry is still there; to see it, one needs to modify the ω^{ab} supersymmetry transformation.

C. Supersymmetry

The (nonvanishing) variation of the action (2.14) under gauge supersymmetry can be computed rather quickly by using $\delta\mathbf{R} = [\boldsymbol{\epsilon}, \mathbf{R}]$ with $\boldsymbol{\epsilon}$ containing only the off-diagonal fermionic supersymmetry parameter ϵ. The result is

$$\delta S = -4\int R^a\bar{\rho}\gamma_a\gamma_5\epsilon. \tag{2.25}$$

Now, consider instead the variation of the action under an *arbitrary* variation of the spin connection ω^{ab}, i.e., the variation that defines the ω^{ab} field equation. To compute it with a minimum of algebra, first vary Eq. (2.14) with respect to $\mathbf{\Omega}$, and then set $\delta V^a = \delta\psi = 0$ in $\delta\mathbf{\Omega}$ as defined by Eq. (2.2). The result is

$$\delta S = 16\int R^a V^b \delta\omega^{cd}\epsilon_{abcd}. \tag{2.26}$$

Requesting this variation to vanish for arbitrary $\delta\omega^{ab}$ yields the spin connection field equation $R^a = 0$.

Thus, if we consider a supersymmetry variation of the action, where the variation of ω^{ab} is modified by an extra piece (in addition to its gauge variation),

$$\delta\omega^{ab} = \delta_{\text{gauge}}\omega^{ab} + \delta_{\text{extra}}\omega^{ab}, \tag{2.27}$$

the corresponding variation of the action (2.14) will be

$$\delta S = -4\int R^a(\bar{\rho}\gamma_a\gamma_5\epsilon - 2\delta_{\text{extra}}\omega^{bc}V^d\epsilon_{abcd}). \tag{2.28}$$

This variation can be made to vanish in two distinct ways:

(1) The first is by enforcing the constraint $R^a = 0$, which is really equivalent to the field equation of ω^{ab}. As is well known, $R^a = 0$ allows us to express the spin connection ω^{ab} in terms of the vielbein and gravitino fields. Substituting back $\omega^{ab}(V, \psi)$ in the action leads to the supersymmetric action of AdS supergravity in second-order formalism. In this

formalism, one never needs to vary the fields inside the "package" $\omega^{ab}(V,\psi)$ since any variation of S due to $\delta\omega^{ab}$ vanishes identically, being proportional to R^a (then one works in the so-called "1.5 order formalism").

(2) The second is by choosing $\delta_{\text{extra}}\omega^{ab}$ so that

$$\bar{\rho}\gamma_a\gamma_5\epsilon - 2\delta_{\text{extra}}\omega^{bc}V^d\epsilon_{abcd} = 0. \qquad (2.29)$$

This equation can be solved for $\delta_{\text{extra}}\omega^{ab}$ in the same way one solves $R^a = 0$ for ω^{ab}. The result is

$$\delta_{\text{extra}}\omega^{ab} = \frac{1}{2}\epsilon^{abcd}(\bar{\rho}_{de}\gamma_c\gamma_5\epsilon + \bar{\rho}_{ec}\gamma_d\gamma_5\epsilon - \bar{\rho}_{cd}\gamma_e\gamma_5\epsilon)V^e, \qquad (2.30)$$

where $\bar{\rho}_{cd}$ are the components along the vielbein basis of the gravitino curvature, i.e., $\bar{\rho} \equiv \bar{\rho}_{cd}V^cV^d$.

Thus, the first-order action (2.14) is invariant under the supersymmetry transformations, given by Eq. (2.20) for the vielbein and the gravitino,

$$\delta V^a = -i\bar{\epsilon}\gamma^a\psi, \qquad \delta\psi = d\epsilon - \frac{1}{4}\omega^{ab}\gamma_{ab}\epsilon \equiv \mathcal{D}\epsilon, \qquad (2.31)$$

and by the modified rule for ω^{ab}:

$$\delta\omega^{ab} = \delta_{\text{gauge}}\omega^{ab} + \delta_{\text{extra}}\omega^{ab} = -\bar{\epsilon}\gamma^{ab}\psi + \frac{1}{2}\epsilon^{abcd}(\bar{\rho}_{de}\gamma_c\gamma_5\epsilon + \bar{\rho}_{ec}\gamma_d\gamma_5\epsilon - \bar{\rho}_{cd}\gamma_e\gamma_5\epsilon)V^e. \qquad (2.32)$$

More details can be found, for example, in Refs. [32,33].

III. MANIFESTLY $OSp(1|4)$-INVARIANT ACTION

Can we reformulate supergravity in an explicit $OSp(1|4)$-invariant way? The answer is yes [37–40] and generalizes the $SO(2,3)$ formulation of AdS gravity of Refs. [40–43]. Indeed, looking at Eq. (2.14), we see that, promoting the constant matrix $\mathbf{\Gamma}$ to a field supermatrix $\mathbf{\Phi}$ transforming under $OSp(1|4)$ as

$$\delta\mathbf{\Phi} = -\mathbf{\Phi}\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{\Phi}, \qquad (3.1)$$

the action S becomes

$$S = \int \mathrm{STr}\left(\mathbf{R}\left(\mathbf{1} + \frac{\mathbf{\Phi}^2}{2}\right)\mathbf{R}\mathbf{\Phi}\right) \qquad (3.2)$$

and is manifestly $OSp(1|4)$ invariant. By doing so, we are introducing new, auxiliary fields contained in $\mathbf{\Phi}$. We have to ensure, however, that a particular gauge choice exists such that $\mathbf{\Phi}$ reduces to the constant supermatrix $\mathbf{\Gamma}$; only if this gauge choice exists, the theory is equivalent to the one described by Eq. (2.14). To satisfy this requirement, we choose $\mathbf{\Phi}$ in the symmetric (traceless) five-dimensional representation of $OSp(1|4)$ [37]:

$$\mathbf{\Phi}(x) \equiv \begin{pmatrix} \frac{1}{4}\pi(x) + i\phi(x)\gamma_5 + \phi^a(x)\gamma_a\gamma_5 & \zeta(x) \\ -\bar{\zeta}(x) & \pi(x) \end{pmatrix}. \qquad (3.3)$$

Now, translations and supersymmetries of $OSp(1|4)$ can be used to set ϕ^a and ζ to zero [37]. Moreover, the $OSp(1|4)$-invariant constraint

$$\mathbf{\Phi}^3 + \mathbf{\Phi} = 0 \qquad (3.4)$$

enforces $\pi = 0$ and $\phi = \pm 1$, reducing $\mathbf{\Phi}$ to the constant supermatrix $\pm\mathbf{\Gamma}$ (ignoring the trivial solution $\mathbf{\Phi} = 0$. If we want to exclude it, we can instead impose the $OSp(1|4)$-invariant constraints $\mathrm{STr}(\mathbf{\Phi}^2) = 4(\text{const})^2$, $\mathrm{STr}(\mathbf{\Phi}^3) = 0$; see Ref. [38]). The simplest way to implement the constraint (3.4) is to add a [$OSp(1|4)$-invariant] Lagrange multiplier term in the action,

$$S_{\boldsymbol{\lambda}} = \int \mathrm{STr}(\boldsymbol{\lambda}\mathbf{\Phi}(\mathbf{\Phi}^2 + \mathbf{1})\mathbf{\Phi}D\mathbf{\Phi}D\mathbf{\Phi}D\mathbf{\Phi}D\mathbf{\Phi}), \qquad (3.5)$$

where the Lagrange multiplier $\boldsymbol{\lambda}(x)$ is proportional to the unit matrix, i.e., $\boldsymbol{\lambda}(x) = \lambda(x)\mathbf{1}$, generalizing the analogous term in the $SO(2,3)$-invariant formulation of gravity (see, for example, Refs. [42,43]).

Another interesting possibility is to give dynamics (cf. Ref. [37]) to the fields $\pi(x)$ and $\phi(x)$ with a potential admitting a stable minimum for the values $\pi = 0$ and $\phi = \text{const}$. In this paper, the constrained auxiliary fields are considered as *background* fields, on the same footing of the background vector fields that define the $\star$ product (see the next section). We do not introduce Higgs fields to break spontaneously the $OSp(1|4)$ invariance. The breaking of $OSp(1|4)$, and contact with AdS $D = 4$ supergravity, is made by explicit gauge fixing.

The $OSp(1|4)$ gauge-invariant formulation of $N = 1$, $D = 4$ anti–de Sitter supergravity is our starting point for a noncommutative supersymmetric extension.

IV. NONCOMMUTATIVE $OSp(1|4)$ SUPERGRAVITY

A. NC action

The NC theory is obtained by a $\star$ deformation of the action in Eq. (3.2):

$$S = \int \mathrm{STr}\left(\mathbf{R} \star \left(\mathbf{1} + \frac{\mathbf{\Phi}\star\mathbf{\Phi}}{2}\right)\wedge_\star \mathbf{R}\star\mathbf{\Phi}\right), \qquad (4.1)$$

where the curvature two-form $\mathbf{R}$ is now

$$\mathbf{R} = d\mathbf{\Omega} - \mathbf{\Omega}\wedge_\star\mathbf{\Omega}, \qquad (4.2)$$

and the $\star$-exterior product between forms is defined as

$$\tau \wedge_\star \tau' \equiv \sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n \theta^{A_1B_1} \cdots \theta^{A_nB_n} (\ell_{A_1} \cdots \ell_{A_n} \tau) \wedge (\ell_{B_1} \cdots \ell_{B_n} \tau') = \tau \wedge \tau' + \frac{i}{2}\theta^{AB}(\ell_A \tau) \wedge (\ell_B \tau') + \frac{1}{2!}\left(\frac{i}{2}\right)^2 \theta^{A_1B_1}\theta^{A_2B_2}(\ell_{A_1}\ell_{A_2}\tau) \wedge (\ell_{B_1}\ell_{B_2}\tau') + \cdots, \tag{4.3}$$

where ℓ_A are Lie derivatives along commuting vector fields X_A. This noncommutative product is associative due to $[X_A, X_B] = 0$. If the vector fields X_A are chosen to coincide with the partial derivatives ∂_μ, and if τ, τ' are zero-forms, then $\tau \star \tau'$ reduces to the well-known Moyal-Groenewold product [12].

The $\star$-gauge transformations of the NC fields are

$$\delta_{\boldsymbol{\epsilon}} \boldsymbol{\Omega} = d\boldsymbol{\epsilon} - \boldsymbol{\Omega} \star \boldsymbol{\epsilon} + \boldsymbol{\epsilon} \star \boldsymbol{\Omega} \tag{4.4}$$

$$\delta_{\boldsymbol{\epsilon}} \boldsymbol{\Phi} = -\boldsymbol{\Phi} \star \boldsymbol{\epsilon} + \boldsymbol{\epsilon} \star \boldsymbol{\Phi}. \tag{4.5}$$

Recalling the $\star$-gauge transformation of the curvature induced by Eq. (4.4),

$$\delta_{\boldsymbol{\epsilon}} \mathbf{R} = -\mathbf{R} \star \boldsymbol{\epsilon} + \boldsymbol{\epsilon} \star \mathbf{R}, \tag{4.6}$$

and the cyclicity of the supertrace and of the integral,[2] the action (4.1) is manifestly invariant under the $\star$-gauge symmetry.

Because of noncommutativity, the $\star$-symmetry group is enhanced to $U(1, 3|1)$ so as to contain all enveloping algebra generators. Thus, the NC one-form connection is given by

$$\boldsymbol{\Omega} = \begin{pmatrix} \Omega & \psi \\ \bar{\psi} & w \end{pmatrix}, \quad \Omega \equiv \frac{1}{4}\omega^{ab}\gamma_{ab} + i\omega I + \tilde{\omega}\gamma_5 - \frac{i}{2}V^a\gamma_a - \frac{i}{2}\tilde{V}^a\gamma_a\gamma_5, \tag{4.7}$$

and correspondingly the gauge parameter supermatrix $\boldsymbol{\epsilon}$ becomes

$$\boldsymbol{\epsilon} = \begin{pmatrix} \varepsilon & \epsilon \\ \bar{\epsilon} & \eta \end{pmatrix}, \quad \epsilon \equiv \frac{1}{4}\varepsilon^{ab}\gamma_{ab} + i\varepsilon I + \tilde{\varepsilon}\gamma_5 - \frac{i}{2}\varepsilon^a\gamma_a - \frac{i}{2}\tilde{\varepsilon}^a\gamma_a\gamma_5, \tag{4.8}$$

containing all the gauge parameters of the superalgebra $U(1, 3|1)$.

The curvature supermatrix $\mathbf{R}$,

$$\mathbf{R} \equiv \begin{pmatrix} R & \Sigma \\ \bar{\Sigma} & r \end{pmatrix}, \tag{4.9}$$

[2]Twisted differential geometry is treated, for example, in Ref. [11]; see the appendix of Ref. [24] for a summary.

defined in Eq. (4.2), is now given by

$$R = d\Omega - \Omega \wedge_\star \Omega - \psi \wedge_\star \bar{\psi} \tag{4.10}$$

$$\Sigma = d\psi - \Omega \wedge_\star \psi - \psi \wedge_\star w \tag{4.11}$$

$$\bar{\Sigma} = d\bar{\psi} - \bar{\psi} \wedge_\star \Omega - w \wedge_\star \bar{\psi} \tag{4.12}$$

$$r = dw - \bar{\psi} \wedge_\star \psi - w \wedge_\star w, \tag{4.13}$$

where R has components along the complete Dirac basis.

As usual in NC theories, the algebra of gauge transformations closes as follows:

$$[\delta_{\boldsymbol{\epsilon}_1}, \delta_{\boldsymbol{\epsilon}_2}] = \delta_{\boldsymbol{\epsilon}_1 \star \boldsymbol{\epsilon}_2 - \boldsymbol{\epsilon}_2 \star \boldsymbol{\epsilon}_1}. \tag{4.14}$$

Consistency with the $\star$-gauge transformations requires for the zero-form $\boldsymbol{\Phi}$ a similar expansion:

$$\boldsymbol{\Phi} = \begin{pmatrix} \Phi & \zeta \\ -\bar{\zeta} & \pi \end{pmatrix}, \quad \Phi \equiv \frac{i}{4}\phi^{ab}\gamma_{ab} + \frac{1}{4}\pi I + i\phi\gamma_5 + \phi^a\gamma_a + \tilde{\phi}^a\gamma_a\gamma_5. \tag{4.15}$$

The crucial difference between the two supermatrix fields $\boldsymbol{\Omega}$ and $\boldsymbol{\Phi}$ (besides their different form degree) is their commutative limit. We will see in Sec. VI how the Seiberg-Witten map ensures that in the $\theta \to 0$ limit, $\boldsymbol{\Omega}$ contains only C-antisymmetric gamma matrices [cf. Eq. (2.2)] and $\boldsymbol{\Phi}$ only C-symmetric gamma matrices [cf. Eq. (3.3)].

In analogy with the classical case, we also require the $U(1, 3|1)$-invariant constraint,

$$\boldsymbol{\Phi} \star \boldsymbol{\Phi} \star \boldsymbol{\Phi} + \boldsymbol{\Phi} = 0, \tag{4.16}$$

reducing to Eq. (3.4) for $\theta \to 0$. In an alternative, we can require $\mathrm{STr}(\boldsymbol{\Phi} \star \boldsymbol{\Phi}) = 4(\mathrm{const})^2$, $\mathrm{STr}(\boldsymbol{\Phi} \star \boldsymbol{\Phi} \star \boldsymbol{\Phi}) = 0$.

B. Hermiticity conditions and reality of the NC action

In the expansions (4.7) and (4.15), all fields are taken to be real. This is equivalent to the relations

$$\boldsymbol{\Omega}^\dagger = -\boldsymbol{\Gamma}_0 \boldsymbol{\Omega} \boldsymbol{\Gamma}_0, \qquad \boldsymbol{\Phi}^\dagger = \boldsymbol{\Gamma}_0 \boldsymbol{\Phi} \boldsymbol{\Gamma}_0, \qquad \boldsymbol{\Gamma}_0 \equiv \begin{pmatrix} \gamma_0 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.17}$$

due to γ_{ab} and γ_5 being γ_0 anti-Hermitian (i.e., $\gamma_{ab}^\dagger = -\gamma_0\gamma_{ab}\gamma_0$, etc.), while 1, γ_a, and $\gamma_a\gamma_5$ are γ_0-Hermitian. Noting that $\boldsymbol{\Gamma}_0^2 = \mathbf{1}$ and that the $\boldsymbol{\Gamma}_0$ anti-Hermiticity of $\boldsymbol{\Omega}$ implies $\boldsymbol{\Gamma}_0$ anti-Hermiticity of $\mathbf{R}$, one easily proves that the NC action is real.

C. Charge-conjugation invariance

The NC action is also invariant under substitution of the fields by their charge conjugates,

$$\mathbf{\Omega}^c \equiv -\mathbf{C}^{-1}\mathbf{\Omega}^T\mathbf{C} \Rightarrow \mathbf{R}^c = -\mathbf{C}^{-1}\mathbf{R}^T\mathbf{C}, \qquad \mathbf{\Phi}^c \equiv \mathbf{C}^{-1}\mathbf{\Phi}^T\mathbf{C}, \qquad \mathbf{C} \equiv \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.18}$$

simultaneously changing θ into $-\theta$ in the $\star$ products. Indeed,

$$\begin{aligned} S^c &= \int \mathrm{Str}\Big(\mathbf{C}^{-1}\mathbf{R}^T\mathbf{C}\Big(\mathbf{1} + \frac{1}{2}\mathbf{C}^{-1}\mathbf{\Phi}^T\mathbf{C}\mathbf{C}^{-1}\mathbf{\Phi}^T\mathbf{C}\Big) \\ &\qquad \times \mathbf{C}^{-1}\mathbf{R}^T\mathbf{C}\mathbf{C}^{-1}\mathbf{\Phi}^T\mathbf{C}\Big)_{-\theta} \\ &= \int \mathrm{Str}\Big(\mathbf{R}^T\Big(\mathbf{1} + \frac{1}{2}\mathbf{\Phi}^T\mathbf{\Phi}^T\Big)\mathbf{R}^T\mathbf{\Phi}^T\Big)_{-\theta} \\ &= \int \mathrm{Str}\Big(\mathbf{\Phi}\mathbf{R}\Big(\mathbf{1} + \frac{1}{2}\mathbf{\Phi}\mathbf{\Phi}\Big)\mathbf{R}\Big)^T_{\theta} \\ &= \int \mathrm{Str}\Big(\mathbf{R}\Big(\mathbf{1} + \frac{1}{2}\mathbf{\Phi}\mathbf{\Phi}\Big)\mathbf{R}\mathbf{\Phi}\Big)^T_{\theta} = S, \end{aligned} \tag{4.19}$$

using cyclicity of the integral and of the supertrace and invariance of the supertrace under matrix transposition. We have defined $(ABC\ldots)_\theta$ to be the $\star$ (exterior) product between the forms $A, B, C\ldots$ and $(ABC\ldots)_{-\theta}$ to be the same product with opposite θ. Note that, for example, $(AB)^T_\theta = \pm(B^TA^T)_{-\theta}$ for $A(x)$, $B(x)$ matrix valued fields (the minus sign when A and B are both forms of odd degree), i.e., the transposition acts only on the matrix structure of A and B. To interchange the ordering of A and B as functions of x, one needs $\theta \to -\theta$ since $(f \star g)_\theta = (g \star f)_{-\theta}$, as follows from the definition (1.2).

V. GEOMETRIC SEIBERG-WITTEN MAP

The results of this section hold for any gauge group. Here, we denote by $\hat{\Omega}$ the NC gauge field and by $\hat{\varepsilon}$ the NC gauge parameter. The Seiberg-Witten map relates $\hat{\Omega}$ to the ordinary Ω and $\hat{\varepsilon}$ to the ordinary ε so as to satisfy

$$\hat{\Omega}(\Omega) + \hat{\delta}_{\hat{\varepsilon}}\hat{\Omega}(\Omega) = \hat{\Omega}(\Omega + \delta_\varepsilon\Omega), \tag{5.1}$$

with

$$\delta_\varepsilon\Omega_\mu = \partial_\mu\varepsilon + \varepsilon\Omega_\mu - \Omega_\mu\varepsilon, \tag{5.2}$$

$$\hat{\delta}_{\hat{\varepsilon}}\hat{\Omega}_\mu = \partial_\mu\hat{\varepsilon} + \hat{\varepsilon} \star \hat{\Omega}_\mu - \hat{\Omega}_\mu \star \hat{\varepsilon}. \tag{5.3}$$

In words, the dependence of the noncommutative gauge field on the ordinary gauge field is fixed by requiring that ordinary gauge variations of Ω inside $\hat{\Omega}(\Omega)$ produce the noncommutative gauge variation of $\hat{\Omega}$.

Similarly noncommutative "matter fields" are related to the commutative ones by requiring

$$\hat{\Phi}(\Phi, \Omega) + \hat{\delta}_{\hat{\varepsilon}}\hat{\Phi}(\Phi, \Omega) = \hat{\Phi}(\Phi + \delta_\varepsilon\Phi, \Omega + \delta_\varepsilon\Omega). \tag{5.4}$$

The conditions (5.1) and (5.4) are satisfied if the following differential equations in the noncommutativity parameter θ^{AB} hold [1,2]:

$$\frac{\partial}{\partial\theta^{AB}}\hat{\Omega} = \frac{i}{4}\{\hat{\Omega}_{[A}, \ell_{B]}\hat{\Omega} + \hat{R}_{B]}\}_\star, \tag{5.5}$$

$$\frac{\partial}{\partial\theta^{AB}}\hat{\Phi} = \frac{i}{4}\{\hat{\Omega}_{[A}, \mathbb{L}_{B]}\hat{\Phi}\}_\star, \tag{5.6}$$

$$\frac{\partial}{\partial\theta^{AB}}\hat{\varepsilon} = \frac{i}{4}\{\hat{\Omega}_{[A}, \ell_{B]}\hat{\varepsilon}\}_\star, \tag{5.7}$$

where we have the following:

(i) $\hat{\Omega}_A$, $\hat{R}_A$ are defined as the contraction i_A along the tangent vector X_A of the exterior forms $\hat{\Omega}$, $\hat{R}$, i.e., $\hat{\Omega}_A \equiv i_A\hat{\Omega}$, $\hat{R}_A \equiv i_A\hat{R}$.

(ii) The bracket $[AB]$ denotes antisymmetrization of the indices A and B with weight 1, so that, for example, $\hat{\Omega}_{[A}\hat{R}_{B]} = \frac{1}{2}(\hat{\Omega}_A\hat{R}_B - \hat{\Omega}_B\hat{R}_A)$. The bracket $\{,\}_\star$ is the usual $\star$ anticommutator, for example, $\{\Omega_A, R_B\}_\star = \Omega_A \star R_B + R_B \star \Omega_A$.

(iii) The second differential equation holds for fields transforming in the adjoint representation. Notice that $\hat{\Phi}$ can also be an exterior form. The "fat" Lie derivative $\mathbb{L}_B$ is defined by $\mathbb{L}_B \equiv \ell_B + L_B$ where L_B is the covariant Lie derivative along the tangent vector X_B; it acts on the field $\hat{\Phi}$ as

$$L_B\hat{\Phi} = \ell_B\hat{\Phi} - [\hat{\Omega}_B, \hat{\Phi}]_\star,$$

with $[\hat{\Omega}_B, \hat{\Phi}]_\star = \hat{\Omega}_B \star \hat{\Phi} - \hat{\Phi} \star \hat{\Omega}_B$. In fact, the covariant Lie derivative L_B can be written in Cartan form:

$$L_B = i_BD + Di_B, \tag{5.8}$$

where D is the covariant derivative.

The differential equations (5.5), (5.6), and (5.7) hold for any Abelian twist defined by arbitrary commuting vector fields X_A [1]. They reduce to the usual Seiberg-Witten differential equations [2] in the case of a Moyal-Groenewold twist, i.e., when $X_A \to \partial_\mu$.

We can solve these differential equations order by order in θ by expanding $\hat{\Omega}$, $\hat{\varepsilon}$, and $\hat{\Phi}$ in power series of θ:

$$\hat{\Omega} = \Omega + \Omega^1 + \Omega^2 \ldots + \Omega^n \ldots \tag{5.9}$$

$$\hat{\varepsilon} = \varepsilon + \varepsilon^1 + \varepsilon^2 \ldots + \varepsilon^n \ldots \tag{5.10}$$

$$\hat{\Phi} = \Phi + \Phi^1 + \Phi^2 \ldots + \Phi^n \ldots, \tag{5.11}$$

where the fields Ω^n, ε^n, and Φ^n are homogeneous polynomials in θ of order n. By multiplying the differential equations by θ^{AB} and using the identities $\theta^{AB}\frac{\partial}{\partial\theta^{AB}}\Omega^{n+1} = (n+1)\Omega^{n+1}$ and similarly for ε^{n+1} and Φ^{n+1}, we obtain the recursive relations:

$$\Omega^{n+1} = \frac{i\theta^{AB}}{4(n+1)}\{\hat{\Omega}_A, \ell_B\hat{\Omega} + \hat{R}_B\}^n_\star, \tag{5.12}$$

$$\Phi^{n+1} = \frac{i\theta^{AB}}{4(n+1)}\{\hat{\Omega}_A, \mathbb{L}_B\hat{\Phi}\}^n_\star, \tag{5.13}$$

$$\varepsilon^{n+1} = \frac{i\theta^{AB}}{4(n+1)}\{\hat{\Omega}_A, \ell_B\hat{\varepsilon}\}^n_\star, \tag{5.14}$$

where, for any field P (also composite, as, for example, $\{\hat{\Omega}_A, \mathbb{L}_B\hat{\Phi}\}_\star$), P^n denotes its component of order n in θ. These recursion relations reduce to the ones found in Ref. [44] in the special case of a Moyal twist.

In the following, we omit the hat denoting noncommutative fields, the $\star$ and $\wedge_\star$ products, and simply write $\{,\}$, $[,]$ for $\{,\}_\star$, $[,]_\star$.

If P and Q are forms in the adjoint representation of the gauge group (i.e., if $\delta_\varepsilon P = -P\varepsilon + \varepsilon P$, etc.), the following recursion relation for the product PQ holds [25]:

$$(PQ)^{n+1} = \frac{i\theta^{AB}}{4(n+1)}(\{\Omega_A, \mathbb{L}_B(PQ)\} + 2L_A P L_B Q)^n. \tag{5.15}$$

Some other useful identities are [25]

$$\theta^{AB}L_A L_B P = -\frac{1}{2}\theta^{AB}\{R_{AB}, P\} \tag{5.16}$$

$$\theta^{AB}\mathbb{L}_A\Omega_B = \theta^{AB}R_{AB} \tag{5.17}$$

$$\theta^{AB}\int \mathrm{Tr}(\{\Omega_A, \mathbb{L}_B(PQ)\} + 2L_A P L_B Q) = \theta^{AB}\int \mathrm{Tr}(\{R_{AB}, P\}Q), \tag{5.18}$$

where $R_{AB} \equiv i_B i_A R$. Finally, using Eq. (5.15), one can find the recursion relation for the curvature:

$$R^{n+1} = \frac{i\theta^{AB}}{4(n+1)}(\{\Omega_A, \mathbb{L}_B R\} - [R_A, R_B])^n. \tag{5.19}$$

Some basic formulas of Cartan calculus, used in deriving the above identities, are listed in Appendix A.

We list below the first-order corrections to the classical $OSp(1|4)$ fields and curvatures, obtained by using the general recursion formulas (5.12), (5.13), and (5.19) for $n=0$. On the right-hand sides, all products are ordinary exterior products, and all fields are classical.

A. $OSp(1|4)$ fields and curvatures at first order in θ

1. Ω connection components

$$\Omega^1 = \frac{i}{4}\theta^{AB}(\{\Omega_A, \ell_B\Omega + R_B\} + \psi_A\ell_B\bar{\psi} + \ell_B\psi\bar{\psi}_A + \psi_A\bar{\Sigma}_B + \Sigma_B\bar{\psi}_A) \tag{5.20}$$

$$\psi^1 = \frac{i}{4}\theta^{AB}(\Omega_A\ell_B\psi + \ell_B\Omega\psi_A + \Omega_A\Sigma_B + R_B\psi_A) \tag{5.21}$$

$$w^1 = \frac{i}{4}\theta^{AB}(\bar{\psi}_A\ell_B\psi + \ell_B\bar{\psi}\psi_A + \bar{\psi}_A\Sigma_B + \bar{\Sigma}_B\psi_A). \tag{5.22}$$

2. Φ field components

$$\Phi^1 = -\frac{1}{4}\theta^{AB}(\{\Omega_A, \gamma_5\Omega_B - \Omega_B\gamma_5\} + \psi_A\bar{\psi}_B\gamma_5) \tag{5.23}$$

$$\zeta^1 = -\frac{1}{4}\theta^{AB}(2\Omega_A\gamma_5\psi_B + \gamma_5\Omega_B\psi_A) \tag{5.24}$$

$$\pi^1 = -\frac{1}{2}\theta^{AB}(\bar{\psi}_a\gamma_5\psi_B). \tag{5.25}$$

3. R curvature components

$$R^1 = \frac{i}{4}\theta^{AB}(\{\Omega_A, \mathbb{L}_B R\} - [R_A, R_B] + \{\Omega_A, -\psi_B\bar{\Sigma} + \sigma\bar{\psi}_B\} + \psi_A\mathbb{L}_B\bar{\Sigma} + \mathbb{L}_B\Sigma\bar{\psi}_A - \psi_A\bar{\psi}_B R + R\psi_B\bar{\psi}_A - 2\Sigma_A\bar{\Sigma}_B) \tag{5.26}$$

$$\Sigma^1 = \frac{i}{4}\theta^{AB}(\Omega_A(\mathbb{L}_B\Sigma + R\psi_B) - \psi_A(\bar{\psi}_B\Sigma + \bar{\Sigma}\psi_B) + (\Sigma\bar{\psi}_B - \psi_B\bar{\Sigma})\psi_A + \mathbb{L}_B R\psi_A - 2R_A\Sigma_B) \tag{5.27}$$

$$r^1 = \frac{i}{4}\theta^{AB}(\bar{\psi}_A\mathbb{L}_B\Sigma + \mathbb{L}_B\bar{\Sigma}\psi_A + 2\bar{\psi}_A R\psi_B - 2\bar{\Sigma}_A\Sigma_B). \tag{5.28}$$

VI. EXTENDED $OSp(1|4)$ SUPERGRAVITY ACTION

We now discuss the θ expansion of the NC action (4.1), where the NC supermatrix fields $\mathbf{\Omega}$ and $\mathbf{\Phi}$ have been substituted by their SW expansion in terms of the classical fields.

A. Action is even in θ

We first note that the SW map is such that

$$\mathbf{\Omega}^c_\theta \equiv -\mathbf{C}^{-1}\mathbf{\Omega}^T_\theta\mathbf{C} = \mathbf{\Omega}_{-\theta}, \quad \Rightarrow \mathbf{R}^c_\theta = -\mathbf{C}^{-1}\mathbf{R}^T_\theta\mathbf{C} = \mathbf{R}_{-\theta} \tag{6.1}$$

$$\mathbf{\Phi}^c_\theta \equiv \mathbf{C}^{-1}\mathbf{\Phi}^T_\theta\mathbf{C} = \mathbf{\Phi}_{-\theta}, \tag{6.2}$$

where the θ dependence is explicitly indicated as a subscript. The proof by induction, using Eqs. (5.12) and (5.13), is straightforward. Suppose that relations (6.1) hold up to order θ^n. Then,

$$
\begin{aligned}
-\mathbf{C}^{-1}(\mathbf{\Omega}^T)^{n+1}_\theta \mathbf{C} &= \frac{-i\theta^{AB}}{4(n+1)}(\mathbf{C}^{-1}\mathbf{\Omega}^T_\theta \mathbf{C}\mathbf{C}^{-1}(\ell_B\mathbf{\Omega}+\mathbf{R})^T_\theta \mathbf{C} \\
&\quad + \mathbf{C}^{-1}(\ell_B\mathbf{\Omega}+\mathbf{R})^T_\theta \mathbf{C}\mathbf{C}^{-1}\mathbf{\Omega}^T_\theta \mathbf{C})^n \\
&= \frac{-i\theta^{AB}}{4(n+1)}(\mathbf{\Omega}_{-\theta}(\ell_B\mathbf{\Omega}+\mathbf{R})_{-\theta} \\
&\quad + (\ell_B\mathbf{\Omega}+\mathbf{R})_{-\theta}\mathbf{\Omega}_{-\theta})^n = \mathbf{\Omega}^{n+1}_{-\theta}.
\end{aligned} \tag{6.3}
$$

Similarly, one proves Eq. (6.2). Exploiting now the invariance of the NC action S under charge conjugation, proven in Sec. IV, and using the relations (6.1) and (6.2), one finally finds

$$S_\theta = S^c_\theta = S_{-\theta}, \tag{6.4}$$

i.e., the NC action S *is even* in θ. Therefore, the θ expansion of S has the form

$$S = S^0 + S^2 + S^4 + \cdots, \tag{6.5}$$

and the first nonvanishing correction to the classical action S^0 is at order θ^2.

Note that the relations (6.1) and (6.2) imply the following conditions on the NC component fields:

$$\omega^{ab}_\theta = \omega^{ab}_{-\theta}, \qquad V^a_\theta = V^a_{-\theta}, \qquad C\bar{\psi}^T_\theta = \psi_{-\theta} \tag{6.6}$$

$$\omega_\theta = -\omega_{-\theta}, \qquad \tilde{\omega}_\theta = -\tilde{\omega}_{-\theta}, \qquad \tilde{V}^a_\theta = -\tilde{V}^a_{-\theta} \tag{6.7}$$

and

$$\pi_\theta = \pi_{-\theta}, \qquad \phi_\theta = \phi_{-\theta}, \qquad \tilde{\phi}^a_\theta = \tilde{\phi}^a_{-\theta}, \qquad C\bar{\zeta}^T_\theta = \zeta_{-\theta} \tag{6.8}$$

$$\phi^{ab}_\theta = -\phi^{ab}_{-\theta}, \qquad \phi^a_\theta = -\phi^a_{-\theta}, \tag{6.9}$$

where the θ dependence of the NC fields is indicated with a subscript. Thus, in the limit $\theta \to 0$, we see that only ω^{ab}, V^a, and ψ survive in $\mathbf{\Omega}$, and only π, ϕ, $\tilde{\phi}^a$, and ζ survive in $\mathbf{\Phi}$, in agreement with the classical fields in Eqs. (2.2) and (3.3). Finally, we recall that $C\bar{\psi}^T_\theta = \psi_{-\theta}$ (and similarly for ζ) is the noncommutative definition for a Majorana spinor [21,24], consistent with the NC gauge transformations and reducing to the usual definition for $\theta = 0$.

B. Action at order θ^2

We can compute the θ^2 correction with the help of the recursion relations (5.15) for composite fields and the identities at the end of Sec. V. The result reads

$$S^2 = S^2_{RR\Phi} + S^2_{R\Phi\Phi R\Phi}, \tag{6.10}$$

with

$$
\begin{aligned}
S^2_{RR\Phi} = -\frac{1}{16}\theta^{AB}\theta^{CD}\int \mathrm{STr}\Big(& R_{AB}R_{CD}RR\Phi + \frac{1}{2}\{R_{CD}, RR\}R_{AB}\Phi - 2R_{AC}R_{BD}\{RR, \Phi\} + \{R_{AB}, L_C R\}L_D R\Phi \\
&+ \{R_{AB}, \Phi\}L_C R L_D R\Phi + 2\{R_{AC}, L_D R\}[L_B R, \Phi] - \{R_{CD}, R_A R_B\}R\Phi - \{R_{CD}, R\}R_A R_B\Phi \\
&- R_{AB}\{R_C R_D, R\}\{\Phi, R_{AB}\} + R_{AB}L_C(RR)L_D\Phi + RRL_C R_{AB}L_D\Phi - 2L_A(R_C R_D)\{L_B R.\Phi\} \\
&+ 2L_A R(L_C L_B R)L_D\Phi - L_C R_A L_D R_B R\Phi - 2RL_C(R_A R_B)L_D\Phi - 2RL_C R_A L_D R_B\Phi \\
&+ 2i_A(R_C R_D)(\{R_B, R\Phi\} + [R_B, \Phi R]) + 2R_A R_B L_C R L_D\Phi + 4R_A R_B R_C R_D\Phi\Big),
\end{aligned} \tag{6.11}
$$

$$
\begin{aligned}
S^2_{R\Phi\Phi R\Phi} = -\frac{1}{16}\theta^{AB}\theta^{CD}\int \mathrm{STr}\Big(\Big(& \frac{1}{2}\{R_{CD}, \{R_{AB}, R\Phi\Phi\}\} - \{R_{CD}, \{R_A R_B\Phi, \Phi\}\} + \{R_{CD}, L_A R\{\Phi, L_B\Phi\}\} \\
&+ \{R_{CD}, RL_A\Phi L_B\Phi\} + \{R_{CD}, \Phi L_A R L_B\Phi\} - \{\{R_{AC}, R_{BD}\}, R\Phi\Phi\} + [L_C R_{AB}, L_D(R\Phi\Phi)] \\
&+ \{R_{AB}, L_C R L_D(\Phi\Phi) - R_C R_D\Phi\Phi + RL_C\Phi L_D\Phi\} - [L_C(R_A R_B), L_D\Phi] - \{L_C R_A L_D R_B\Phi, \Phi\} \\
&+ \{[i_A(R_C R_D), R_B\Phi], \Phi\} + L_C L_A(R\Phi)L_D L_B\Phi + ((L_C L_A R)L_D\Phi + \{R_{AC}, L_D R\}\Phi - L_A(R_C R_D\Phi))L_B\Phi \\
&+ L_A(R\Phi)\{R_{BC}, L_D\Phi\} + L_C R L_D L_A\Phi L_B\Phi + R\{R_{AC}, L_D\Phi\}L_B\Phi + [L_C(L_A R L_B\Phi), L_D\Phi] \\
&+ \{L_C L_A R L_D L_B\Phi + \{R_{AC}, L_D R\}L_B\Phi - L_A(R_C R_D)L_B\Phi + L_A R\{R_{BC}, L_D\Phi\}, \Phi\}\Big)R\Phi \\
&+ 2\Big(\frac{1}{2}\{R_{AB}, R\Phi\Phi\} - \{R_A R_B\Phi, \Phi\} + L_A(R\Phi)L_B\Phi + \{L_A R L_B\Phi, \Phi\}\Big)(L_C R L_D\Phi - R_C R_D\Phi)\Big).
\end{aligned} \tag{6.12}
$$

Here, all products are ordinary exterior products between classical fields. These corrections to the classical $OSp(1|4)$ action are invariant under local (ordinary) $OSp(1|4)$ gauge variations, as is manifest since all quantities appearing in S^2 are gauge covariant and transform in the adjoint (i.e., as commutators with the gauge parameter). The SW map is designed to ensure this invariance; to find explicitly gauge invariant corrections, order by order in θ, is a powerful check on the computations.

To recover the usual $N = 1$, $D = 4$ AdS supergravity (without auxiliary fields) in the $\theta \to 0$ limit, one still needs to break $OSp(1|4)$ to its Lorentz subgroup. This is done exactly as in the classical case by choosing the gauge for which $\mathbf{\Phi}$ becomes the constant supermatrix $\mathbf{\Gamma}$ defined in Eq. (2.15) (the constrained auxiliary fields take constant values). This gauge breaks translations and supersymmetry. We have seen how supersymmetry can be uncovered in the classical ($\theta = 0$) gauge fixed action. The question whether a hidden supersymmetry is present also in the gauge fixed extended ($\theta \neq 0$) action is left to future investigations.

VII. CONCLUSIONS

The fascinating idea that (super)gravity has some kind of conformal phase, before breaking occurs and dimensionful constants emerge, is rather old, and the $OSp(4|1)$ actions we have been discussing are part of this idea.

The result we have presented here is a noncommutative extension of $OSp(4|1)$ supergravity, the novelty being on one side a $D = 4$ *supergravity* action S invariant under local $\star$ supersymmetry (part of the supergroup noncommutative gauge symmetry) and on the other side explicit invariance of S under diffeomorphisms, thanks to a geometrical formulation of Abelian twists. Previous works have addressed noncommutative extensions of MacDowell-Mansouri gravity actions, but without treating their supersymmetric versions.

We have then used a generalization of the Seiberg-Witten map (adapted to Abelian twists and suitably "geometrized"), obtaining a higher-derivative $D = 4$ supergravity, with constrained auxiliary fields, invariant under the *usual* gauge transformations of the whole supergroup $OSp(1|4)$. Recursion formulas for the SW higher-order corrections have been applied to compute the θ^2 correction to the classical $OSp(1|4)$ action.

In short, noncommutativity has been used as a guide to construct an extended, locally supersymmetric higher-derivative theory with the same symmetries of its classical $\theta \to 0$ limit.

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APPENDIX A: CARTAN FORMULAS

The usual Cartan calculus formulas simplify if we consider commuting vector fields X_A and read

$$\ell_A = i_A d + d i_A, \qquad L_A = i_A D + D i_A \tag{A1}$$

$$[\ell_A, \ell_B] = 0, \qquad [L_A, L_B] = i_A i_B R \tag{A2}$$

$$[\ell_A, i_B] = 0, \qquad [L_A, i_B] = 0 \tag{A3}$$

$$i_A i_B + i_B i_A = 0, \qquad d \circ d = 0, \qquad D \circ D = R. \tag{A4}$$

APPENDIX B: GAMMA MATRICES IN $D = 4$

We summarize in this Appendix our gamma-matrix conventions in $D = 4$:

$$\eta_{ab} = (1, -1, -1, -1), \qquad \{\gamma_a, \gamma_b\} = 2\eta_{ab}, \qquad [\gamma_a, \gamma_b] = 2\gamma_{ab}, \tag{B1}$$

$$\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3, \qquad \gamma_5\gamma_5 = 1, \qquad \varepsilon_{0123} = -\varepsilon^{0123} = 1, \tag{B2}$$

$$\gamma_a^\dagger = \gamma_0\gamma_a\gamma_0, \qquad \gamma_5^\dagger = \gamma_5 \tag{B3}$$

$$\gamma_a^T = -C\gamma_a C^{-1}, \qquad \gamma_5^T = C\gamma_5 C^{-1}, \qquad C^2 = -1, \qquad C^\dagger = C^T = -C. \tag{B4}$$

1. Useful identities

$$\gamma_a\gamma_b = \gamma_{ab} + \eta_{ab} \tag{B5}$$

$$\gamma_{ab}\gamma_5 = \frac{i}{2}\epsilon_{abcd}\gamma^{cd} \tag{B6}$$

$$\gamma_{ab}\gamma_c = \eta_{bc}\gamma_a - \eta_{ac}\gamma_b - i\varepsilon_{abcd}\gamma_5\gamma^d \tag{B7}$$

$$\gamma_c\gamma_{ab} = \eta_{ac}\gamma_b - \eta_{bc}\gamma_a - i\varepsilon_{abcd}\gamma_5\gamma^d \tag{B8}$$

$$\gamma_a\gamma_b\gamma_c = \eta_{ab}\gamma_c + \eta_{bc}\gamma_a - \eta_{ac}\gamma_b - i\varepsilon_{abcd}\gamma_5\gamma^d \tag{B9}$$

$$\gamma^{ab}\gamma_{cd} = -i\varepsilon^{ab}{}_{cd}\gamma_5 - 4\delta^{[a}_{[c}\gamma^{b]}{}_{d]} - 2\delta^{ab}_{cd} \tag{B10}$$

$$\mathrm{Tr}(\gamma_a\gamma^{bc}\gamma_d) = 8\,\delta^{bc}_{ad} \tag{B11}$$

$$\mathrm{Tr}(\gamma_5\gamma_a\gamma_{bc}\gamma_d) = -4i\,\varepsilon_{abcd}, \tag{B12}$$

where $\delta^{ab}_{cd} \equiv \frac{1}{2}(\delta^a_c\delta^b_d - \delta^b_c\delta^a_d)$, $\delta^{rse}_{abc} \equiv \frac{1}{3!}(\delta^r_a\delta^s_b\delta^e_c + 5\,\text{terms})$, and indices' antisymmetrization in square brackets has total weight 1.

2. Charge conjugation and Majorana condition

$$\text{Dirac conjugate } \bar{\psi} \equiv \psi^\dagger\gamma_0 \tag{B13}$$

$$\text{Charge conjugate spinor } \psi^C = C(\bar{\psi})^T \tag{B14}$$

$$\text{Majorana spinor } \psi^C = \psi \Rightarrow \bar{\psi} = \psi^T C. \tag{B15}$$

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