Quantum energy inequality for the massive Ising model

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(Received 29 April 2013; published 10 July 2013)

A quantum energy inequality is derived for the massive Ising model, giving a state-independent lower bound on suitable averages of the energy density, the first quantum energy inequality to be established for an interacting quantum field theory with nontrivial S-matrix. It is shown that the Ising model has one-particle states with locally negative energy densities and that the energy density operator is not additive with respect to the combination of one-particle states into multiparticle configurations.

I. INTRODUCTION

In relativistic quantum field theory, the Hamiltonian is a positive operator in all inertial frames of reference; this is the content of the spectrum condition. By contrast, it is impossible for nontrivial local averages of the energy density to be positive operators in any quantum field theory obeying standard assumptions [1]; typically, the expectation value of the energy density at any point is unbounded from below with respect to the state [2]. This state of affairs demonstrates a fundamental incompatibility between quantum fields and the energy conditions usually assumed in classical general relativity and which are the essential input for results such as the singularity theorems of Penrose and Hawking [3–5], the positive mass theorems [6–9], and Hawking's chronology protection results [10], among many others.

Nonetheless, various models of quantum field theory obey local remnants of the spectrum condition called quantum energy inequalities (QEIs), which provide lower bounds on expectation values of the energy density when averaged along a timelike curve or over a spacetime region. Their study originates from Ford's insight [11] that the quantum field theory could produce observable deviations from the second law of thermodynamics unless there were mechanisms to constrain negative energy densities (or fluxes). QEI bounds place severe constraints on the extent to which quantum fields can support exotic space-time geometries [12,13]; moreover, weakened classical energy conditions inspired by QEIs can be used to prove singularity theorems [14].

QEIs have been established for free (minimally coupled) Klein-Gordon [15–21], Dirac [22–25], Maxwell [26,27], and Proca fields [27] in both flat and curved spacetimes, for the Rarita–Schwinger field in Minkowski space [28], and for the whole class of unitary positive-energy conformal field theories in two-dimensional Minkowski space [29] (generalizing a special case [16]). In all these cases, the

energy density is bounded from below on the class of physically acceptable states if it is smeared against a positive test function over a region or curve of nonzero temporal extent. Even within the setting of free fields, however, the nonminimally coupled Klein-Gordon field provides an example in which only a weaker type of QEI holds [30]—the lower bound is no longer state-independent but exhibits a dependence on the energy scale—and it is expected that this behavior would be typical for interacting quantum field theories [31]. Analogues of these energy-dependent QEIs exist in general quantum field theories for observables arising in operator product expansions of "classically positive" expressions [32]; however, a direct connection to the energy density is lacking.

In short, QEIs are known to hold in interaction-free situations and for fields interacting with a gravitational background, but their status remains open in models with self-interaction in the sense of a nontrivial scattering matrix. This is unsatisfactory since self-interaction is clearly expected to influence the energy density of a physical system. However, QEIs are an inherently nonperturbative concept, and the rigorous nonperturbative description of interacting quantum field theories remains challenging: it is still out of reach in physical space-time, and even in simplified low-dimensional models that are under full mathematical control, such as $P(\phi)_2$ [33] or integrable models [34], the local observables—including the energy density—are of considerable complexity.

We will bypass this problem here by restricting ourselves to the very simplest interacting example of an integrable quantum field theory, the massive Ising model, which has a two-particle S-matrix of $S_2 = -1$. Our object is to investigate the phenomenon of negative energy density in this model. Despite its simple scattering theory, we find that the Ising model shows clear signs of interaction; its energy density differs in essential ways from that of the free scalar field. As we will show (Sec. III), there are single-particle states of the Ising model that have locally negative energy density, and, when passing to multiparticle states, the energy density is not additive. Yet a state-independent QEI bound holds (Sec. IV) and is given as

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the $(-)$ case of Eq. (8) below; the $(+)$ case applies to the free scalar field and is given for comparison.

The simplicity of the Ising model will allow us to obtain our results in a surprisingly short and elegant manner. Specifically, although it is a theory of interacting bosons, the Ising model has a free Fermi (Majorana) field closely associated with it (Sec. II). This will allow us to adapt arguments originally developed in the context of the free Dirac field [22–24] in order to derive our QEI.

II. ISING MODEL

In this paper, we understand the Ising model of mass $\mu > 0$ as a quantum field theory on $1 + 1$-dimensional Minkowski space. This is usually derived as the continuum limit of a two-dimensional Ising lattice spin system above the critical temperature, which yields a (not conformally invariant) statistical theory on two-dimensional Euclidean space, from which a quantum field theory on $1 + 1$-dimensional Minkowski space can be defined by the analytic continuation of the n-point functions [35]. We will not follow this route explicitly here but define the model directly on Minkowski space in terms of wedge-local interacting fields, in the spirit of Ref. [34].

Let us recall the mathematical setting of the Ising model in a way that makes parallels to the $1 + 1$-dimensional massive scalar free field clear. We work on the single particle space $\mathcal{K} = L^2(\mathbb{R}, d\theta)$, where the rapidity θ is related to two-momentum p by $\mathsf{p}(\theta) = \mu(\cosh\theta, \sinh\theta)$. The two models [with $(+)$ for the free field and $(-)$ for the Ising model] are defined on the symmetric, respectively, antisymmetric, Fock space $\mathcal{H}_\pm$ over $\mathcal{K}$, with a vacuum vector denoted as Ω. On $\mathcal{H}_\pm$, we have the usual action of creators and annihilators $a_\pm^\dagger(\theta)$, $a_\pm(\eta)$, which fulfill canonical (anti)commutation relations,

$$a_\pm(\eta)a_\pm^\dagger(\theta) \mp a_\pm^\dagger(\theta)a_\pm(\eta) = \delta(\theta - \eta)\mathbf{1}.$$

Spacetime symmetries, i.e., translations $\mathsf{x} = (t, x)$, boosts λ, and the space-time reflection j, act on $\mathcal{H}_\pm$ by

$$U_\pm(\mathsf{x}, \lambda)a_\pm^\dagger(\theta_1)\cdots a_\pm^\dagger(\theta_n)\Omega$$
$$= e^{i(\mathsf{p}(\theta_1)+\cdots+\mathsf{p}(\theta_n))\cdot\mathsf{x}}a_\pm^\dagger(\theta_1 + \lambda)\cdots a_\pm^\dagger(\theta_n + \lambda)\Omega, \quad (1)$$

$$U_\pm(j)a_\pm^\dagger(\theta_1)\cdots a_\pm^\dagger(\theta_n)\Omega = a_\pm^\dagger(\theta_n)\cdots a_\pm^\dagger(\theta_1)\Omega, \quad (2)$$

with $U_\pm(j)$, the PCT operator, extended antilinearly.

We now describe the basic observables of the model. Following Lechner [34] (although with slightly different conventions), we define quantum fields $\phi_\pm$, $\phi_\pm'$ as

$$\phi_\pm(\mathsf{x}) = \frac{1}{\sqrt{4\pi}}\int d\theta(e^{i\mathsf{p}(\theta)\cdot\mathsf{x}}a_\pm^\dagger(\theta) + e^{-i\mathsf{p}(\theta)\cdot\mathsf{x}}a_\pm(\theta)),$$

$$\phi_\pm'(\mathsf{x}) = U_\pm(j)\phi_\pm(-\mathsf{x})U_\pm(j).$$

These are covariant under the symmetry operations (1) and (2), with the space-time reflection exchanging $\phi_\pm$ and $\phi_\pm'$.

Of course, $\phi_+ = \phi_+'$ is just the usual local free scalar field. But ϕ_- is an interacting and nonlocal field; rather than local commutation relations, we have $[\phi_-(\mathsf{x}), \phi_-'(\mathsf{y})] = 0$ if $(\mathsf{x} - \mathsf{y})^2 < 0$ and $\mathsf{x}^1 < \mathsf{y}^1$ (we say x is *to the left* of y). A general operator A is then considered to be localized at x (or in a region O) if

$$[\phi_-(\mathsf{y}), A] = 0 = [A, \phi_-'(\mathsf{z})] \quad (3)$$

whenever y is to the left of x (or O) and z is to the right. By abstract arguments, a large class of such local observables exists [34].

It may seem surprising, although entirely expected by general results [36], that this abstract notion of local observables fixes the scattering theory of the models completely [34]. The case ϕ_+ is trivial, of course. In the case of ϕ_-, despite its formulation on a fermionic Fock space, one finds that both the incoming and outgoing states are *bosonic*, i.e., we can identify $\mathcal{H}_{\text{in}}$ and $\mathcal{H}_{\text{out}}$ with $\mathcal{H}_+$. The incoming Møller operator is then given by $V_{\text{in}}: \mathcal{H}_- \to \mathcal{H}_+$,

$$V_{\text{in}}a_-^\dagger(\theta_1)\cdots a_-^\dagger(\theta_n)\Omega$$
$$= \left(\prod_{i<j}\epsilon(\theta_i - \theta_j)\right)a_+^\dagger(\theta_1)\cdots a_+^\dagger(\theta_n)\Omega$$

and the outgoing Møller operator V_{out} by a similar formula, but with the argument of each ϵ negated. The S-matrix is $S = V_{\text{in}}V_{\text{out}}^* = (-1)^{N_+(N_+-1)/2}$, where N_+ is the bosonic number operator on $\mathcal{H}_+$. This confirms our interpretation of the case $(-)$ as the interacting massive Ising model.

A crucial observation, for our purposes, is that a free Majorana field $\psi = (\psi_1, \psi_2)^T$ can be defined on the fermionic Fock space $\mathcal{H}_-$ by

$$\psi_{1,2}(\mathsf{x}) = \sqrt{\frac{\mu}{4\pi}}\int d\theta e^{i\pi(-2\pm1)/4}e^{i\mathsf{p}(\theta)\cdot\mathsf{x}\pm\theta/2}a_-^\dagger(\theta) + \text{h.c.}$$

(here $+$ for the case 1 and $-$ for the case 2). This Majorana field is covariant under $U_-(\mathsf{x}, \lambda)$ as defined above but *not* under $U_-(j)$; its associated PCT operator is fundamentally different. By analogy with Ref. [37], Sec. 6, ψ fulfills

$$\{\phi_-(\mathsf{y}), \psi(\mathsf{x})\} = 0 = [\psi(\mathsf{x}), \phi_-'(\mathsf{z})] \quad (4)$$

if y is to the left and z is to the right of x. That is, $\psi(\mathsf{x})$ is not a local observable of the interacting theory [cf. Eq. (3)]. However, from Eq. (4), all *even polynomials* in $\psi_{1,2}$ have this property. This applies in particular to the energy density of the Majorana field,

$$T_-^{00}(\mathsf{x}) = \frac{i}{4}:\psi(\mathsf{x})^T\partial_t\psi(\mathsf{x}) - (\partial_t\psi(\mathsf{x}))^T\psi(\mathsf{x}):.$$

Since the energy-momentum operators of the Majorana field coincide with those on $\mathcal{H}_-$ by Eq. (1), this T_-^{00} is also the energy density of the interacting Ising model. Nonetheless, we emphasize that the Ising model is distinct from the free Majorana theory; there are many local

observables in the Ising model (including local fields [38]) that do not arise from bilocal expressions in the $\psi_{1,2}$. Also, as we have mentioned, the PCT operators of the two theories are distinct.

For completeness, we recall that the energy density of the free scalar Bose field ϕ_+ on $\mathcal{H}_+$ is

$$T_+^{00}(\mathbf{x}) = \frac{1}{2}{:}(\partial_t \phi_+(\mathbf{x}))^2 + (\partial_x \phi_+(\mathbf{x}))^2 + \mu^2 \phi_+(\mathbf{x})^2{:}.$$

After some computation, the energy densities of both models take the form

$$\begin{aligned} T_\pm^{00}(\mathbf{x}) = \frac{1}{2}\int d\theta d\eta (&F_\pm(\theta, \eta, \mathbf{x}) a_\pm^\dagger(\theta) a_\pm^\dagger(\eta) \\ &+ 2F_\pm(\theta, \eta + \pi i, \mathbf{x}) a_\pm^\dagger(\theta) a_\pm(\eta) \\ &+ F_\pm(\theta + \pi i, \eta + \pi i, \mathbf{x}) a_\pm(\theta) a_\pm(\eta)), \end{aligned} \quad (5)$$

with

$$F_+(\boldsymbol{\zeta}, \mathbf{x}) = -\frac{\mu^2}{2\pi} \sinh^2 \frac{\zeta_1 + \zeta_2}{2} e^{i(\mathsf{p}(\zeta_1) + \mathsf{p}(\zeta_2)) \cdot \mathbf{x}},$$

$$F_-(\boldsymbol{\zeta}, \mathbf{x}) = i \sinh \frac{\zeta_1 - \zeta_2}{2} F_+(\boldsymbol{\zeta}, \mathbf{x}).$$

(See Refs. [39,40] for the general theory of these expansions.) Using methods from Ref. [40], Sec. 9.1, one can see from the analyticity structure of $F_\pm$ that $T_\pm^{00}(h) = \int dt\, h(t) T_\pm^{00}(t, x)$ is a closable operator, and a local observable, for any $x \in \mathbb{R}$ and Schwartz test function h.

III. STATES WITH NEGATIVE ENERGY DENSITY

Our first aim is to construct states with locally negative energy density; i.e., we are looking for $\Phi \in \mathcal{H}_\pm$ such that $\langle \Phi, T_\pm^{00}(\mathbf{x})\Phi \rangle < 0$ for some $\mathbf{x}$.

In the bosonic free field situation, it is well known (see, e.g., Ref. [41]) that T_+^{00} has a non-negative expectation value in all single particle states, or more generally, states with a sharp particle number, while more general superpositions (such as that of the vacuum and a two-particle state) can yield negative expectation values (see, e.g., Ref. [1]). By contrast, we will now exhibit single-particle states in the Ising model with negative energy density at the origin, reminiscent of the situation for free Dirac fields [41].

These states are essentially superpositions of two plane waves. More specifically, let us choose a non-negative real-valued Schwartz function h with $\int h(\theta)d\theta = 1$. We set

$$\Phi := a_-^\dagger(\varphi_{\alpha,\beta,\gamma})\Omega = \int d\theta \varphi_{\alpha,\beta,\gamma}(\theta) a_-^\dagger(\theta)\Omega,$$

where $\varphi_{\alpha,\beta,\gamma}(\theta) := c_{\alpha,\beta,\gamma}(h_\alpha(\theta) + \beta h_\alpha(\theta - \gamma))$,

$$h_\alpha(\theta) := \alpha^{-1} h(\alpha^{-1}\theta),$$

with parameters $\alpha > 0$, $\beta, \gamma \in \mathbb{R}$ and normalization constant $c_{\alpha,\beta,\gamma} > 0$. We will show that $\langle \Phi, T_-^{00}(0)\Phi \rangle < 0$ for a

suitable choice of the parameters. To that end, we compute from Eq. (5)

$$\langle \Phi, T_-^{00}(0)\Phi \rangle = \frac{\mu^2}{2\pi} c_{\alpha,\beta,\gamma}^2 (I_\alpha + J_{\alpha,\gamma}\beta + K_{\alpha,\gamma}\beta^2), \quad (6)$$

where we denoted

$$I_\alpha = \int d\theta d\eta\, h_\alpha(\theta) h_\alpha(\eta) \cosh^2 \frac{\theta + \eta}{2} \cosh \frac{\theta - \eta}{2},$$

$$\begin{aligned} J_{\alpha,\gamma} = 2\int d\theta d\eta\, h_\alpha(\theta) h_\alpha(\eta) &\cosh^2 \frac{\theta + \eta + \gamma}{2} \\ &\times \cosh \frac{\theta - \eta + \gamma}{2}, \end{aligned}$$

$$K_{\alpha,\gamma} = \int d\theta d\eta\, h_\alpha(\theta) h_\alpha(\eta) \cosh^2 \frac{\theta + \eta + 2\gamma}{2} \cosh \frac{\theta - \eta}{2}.$$

The right-hand side of Eq. (6) is negative for some β if the polynomial $I_\alpha + J_{\alpha,\gamma}\beta + K_{\alpha,\gamma}\beta^2$ has two real zeros, that is, if $J_{\alpha,\gamma}^2 > 4I_\alpha K_{\alpha,\gamma}$. This inequality holds for small α if it holds in the limit $\alpha \to 0$. Noting that $h_\alpha(\theta) \to \delta(\theta)$ in this limit, we obtain

$$I_0 = 1, \qquad J_{0,\gamma} = 2\cosh^3 \frac{\gamma}{2}, \qquad K_{0,\gamma} = \cosh^2 \gamma.$$

The condition $J_{0,\gamma}^2 > 4I_0 K_{0,\gamma}$ then becomes

$$(1 + \cosh\gamma)^3 > 8\cosh^2\gamma,$$

which is in fact fulfilled for sufficiently large γ. With these choices, we achieve $\langle \Phi, T_-^{00}(0)\Phi \rangle < 0$. For an example, see Fig. 1, where $\rho(t, x) := \langle \Phi, T_-^{00}(t, x)\Phi \rangle$ is plotted for a suitable choice of the parameters.
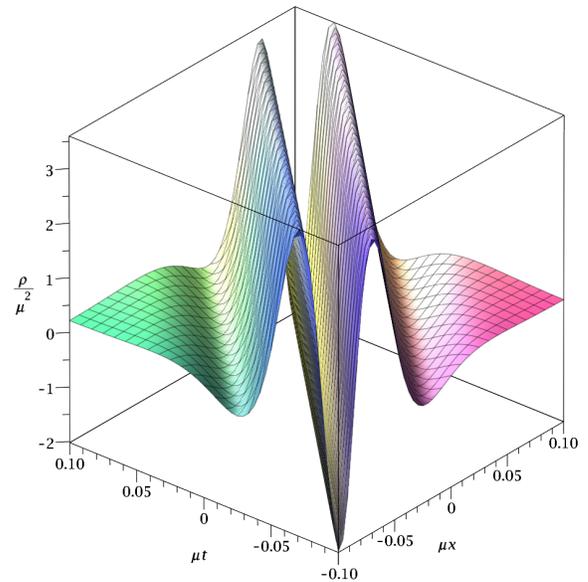
FIG. 1 (color online). Energy density in the single-particle state $a_-^\dagger(\varphi_{\alpha,\beta,\gamma})\Omega$ over space-time, with parameters $\alpha = 0.5$, $\beta = -0.04$, $\gamma = 5$, and $h(\theta) = \exp(-\theta^2)/\sqrt{\pi}$.

Let us now proceed to states with more than one particle. Given any normalized single-particle state wave function $\varphi \in \mathcal{K}$, we may form multiparticle states by taking tensor products in the "in" Hilbert space and applying the inverse Møller operator. For the free model, this yields n-particle vectors $\Phi_{n+} := (n!)^{-1/2} a_+^\dagger(\varphi)^n \Omega \in \mathcal{H}_+$, while the corresponding states in the Ising model are $\Phi_{n-} := V_{\text{in}}^* \Phi_{n+} \in \mathcal{H}_-$.

Now, the total energy, given by the Hamiltonian $H_\pm$ (the generator of time translations), is additive in the sense that $\langle \Phi_{n\pm}, H_\pm \Phi_{n\pm} \rangle = n \langle \Phi_{1\pm}, H_\pm \Phi_{1\pm} \rangle$. (For H_+, this is evident from Eq. (1), and, for H_-, it follows since $V_{\text{in}} H_- V_{\text{in}}^* = H_+$.) Furthermore, the energy density in the free model is also additive:

$$\langle \Phi_{n+}, T_+^{00}(\mathsf{x}) \Phi_{n+} \rangle = n \langle \Phi_{1+} T_+^{00}(\mathsf{x}) \Phi_{1+} \rangle.$$

However, the same relation does not hold in the interacting situation. In general,

$$\langle \Phi_{n-}, T_-^{00}(\mathsf{x}) \Phi_{n-} \rangle \neq n \langle \Phi_{1-}, T_-^{00}(\mathsf{x}) \Phi_{1-} \rangle. \tag{7}$$

We can deduce this from our other results; we saw above that $\langle \Phi_{1-}, T_-^{00}(\mathsf{x}) \Phi_{1-} \rangle$ is negative in some examples, and, therefore, at large n, equality in Eq. (7) would be in contradiction to the QEI (8) that we will establish below.

But let us give a direct argument for Eq. (7). It is useful to note that $V_{\text{in}} a_-^\dagger(\theta) V_{\text{in}}^* = a_+^\dagger(\theta) M(\theta)$, where $M(\theta)$ is the multiplication operator,

$$M(\theta) a_+^\dagger(\eta_1) \cdots a_+^\dagger(\eta_n) \Omega$$
$$= \left(\prod_{j=1}^n \epsilon(\theta - \eta_j) \right) a_+^\dagger(\eta_1) \cdots a_+^\dagger(\eta_n) \Omega.$$
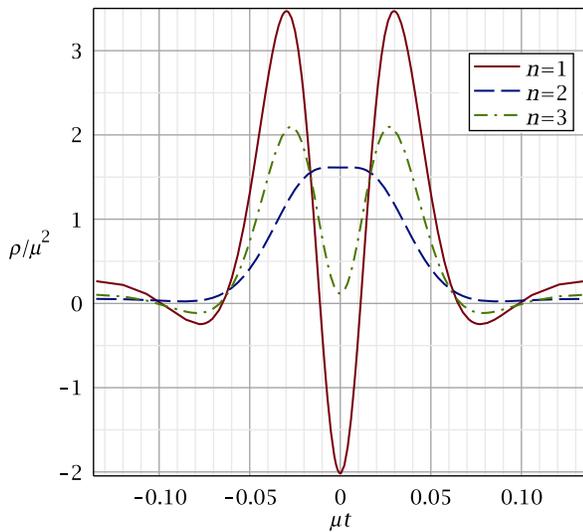
FIG. 2 (color online). Energy density per particle in the Ising model, computed in the multiparticle states Φ_{n-} along the time axis, with parameters as in Fig. 1.

Using this relation and its adjoint, it is straightforward to compute from Eq. (5) that

$$\langle \Phi_{n-}, T_-^{00}(\mathsf{x}) \Phi_{n-} \rangle = n \frac{\mu^2}{2\pi} \int d\theta \, d\eta \cosh^2 \frac{\theta + \eta}{2} \cosh \frac{\theta - \eta}{2}$$
$$\times \overline{\varphi(\theta)} \varphi(\eta) L_\varphi(\theta, \eta)^{n-1} e^{i(\mathsf{p}(\theta) - \mathsf{p}(\eta)) \cdot \mathsf{x}},$$

where

$$L_\varphi(\theta, \eta) := \int d\lambda |\varphi(\lambda)|^2 \epsilon(\theta - \lambda) \epsilon(\eta - \lambda).$$

The factor L_φ, which does not occur in the free case, prevents additivity of the energy density. This nonlinearity is also apparent in Fig. 2, where the energy density per particle, $\rho(t) := \frac{1}{n} \langle \Phi_{n-}, T_-^{00}(t, 0) \Phi_{n-} \rangle$, is plotted along the time axis for the same choice of $\varphi = \varphi_{\alpha, \beta, \gamma}$ as above. This behavior reflects the interaction in the Ising model.

IV. QUANTUM ENERGY INEQUALITY

Finally, we turn to the derivation of the QEI bounds on the energy density. In fact, the free scalar field $(+)$ and the Ising model $(-)$ obey closely related QEIs, namely,

$$\int dt g(t)^2 \langle \Phi, T_\pm^{00}(t, x) \Phi \rangle$$
$$\geq -\frac{1}{4\pi^2} \int_\mu^\infty d\omega \, \omega^2 |\tilde{g}(\omega)|^2 Q_\pm \left(\frac{\omega}{\mu} \right), \tag{8}$$

for any real-valued smooth compactly supported function g and all sufficiently regular normalized states Φ, where the dimensionless functions $Q_\pm : [1, \infty) \to \mathbb{R}^+$ are

$$Q_\pm(u) = \sqrt{1 - u^{-2}} \pm u^{-2} \log(u + \sqrt{u^2 - 1}) \tag{9}$$

and obey $Q_\pm(1) = 0$, $\lim_{u \to +\infty} Q_\pm(u) = 1$, while the Fourier transform is $\tilde{g}(\omega) = \int dt \, g(t) e^{i\omega t}$.

The QEI for the free Bose field $(+)$ has been known for some time [19] and holds rigorously for all Hadamard states Φ [21]; our goal is to establish the QEI for the Ising model $(-)$. As we have already seen that its energy density coincides with that of the free Majorana field, the QEI is exactly the same as for the latter theory. This has not been computed before, but the argument is sufficiently similar to treatments in Refs. [23,24] that we only sketch it here. For $\nu \in \mathbb{R}$, let R_ν and S_ν be continuous one-parameter families in $\mathcal{K} = L^2(\mathbb{R}, d\theta)$, and define for each ν the (bounded) operator

$$\mathcal{O}_\nu = \int d\theta (R_\nu(\theta) a_-(\theta) + S_\nu(\theta) a_-^\dagger(\theta))$$

on $\mathcal{H}_-$. Elementary use of the canonical anticommutation relations shows that

$$\mathcal{O}_\nu^\dagger \mathcal{O}_\nu - \|S_\nu\|^2 \mathbf{1} = -\mathcal{O}_\nu \mathcal{O}_\nu^\dagger + \|R_\nu\|^2 \mathbf{1} =: X_\nu,$$

where $\|\cdot\|$ is the norm in $\mathcal{K}$. As $\nu \mathcal{O}_\nu^\dagger \mathcal{O}_\nu$ (respectively, $-\nu \mathcal{O}_\nu \mathcal{O}_\nu^\dagger$) is positive semidefinite for $\nu \geq 0$ (respectively, $\nu \leq 0$), we have

$$\int_{-\infty}^{\infty} \frac{d\nu}{\pi} \nu \langle \Phi, X_\nu \Phi \rangle \geq -\int_0^\infty \frac{d\nu}{\pi} \nu (\|R_{-\nu}\|^2 + \|S_\nu\|^2) \tag{10}$$

for all normalized quantum states Φ. For our application, we put

$$R_\nu(\theta) = \sqrt{\frac{\mu}{4\pi}} \tilde{g}(\nu - \mu \cosh \theta) \cosh \frac{\theta}{2},$$

$$S_\nu(\theta) = -i\sqrt{\frac{\mu}{4\pi}} \tilde{g}(\nu + \mu \cosh \theta) \sinh \frac{\theta}{2}$$

for any real-valued $g \in C_0^\infty(\mathbb{R})$. Using the identity

$$(\omega + \omega') \tilde{g}^2(\omega' - \omega) = -\int_{-\infty}^\infty \frac{d\nu}{\pi} \nu \overline{\tilde{g}(\nu + \omega)} \tilde{g}(\nu + \omega')$$

(a mild rewriting of Eq. (2.17) in Ref. [23]), the left-hand side of Eq. (10) becomes, after a computation,

$$\int_{-\infty}^\infty \frac{d\nu}{\pi} \nu \langle \Phi, X_\nu \Phi \rangle = \int dt g(t)^2 \langle \Phi, T_\pm^{00}(t, 0) \Phi \rangle.$$

It is expected that this is rigorously valid at least for all $\Phi \in \mathcal{H}_-$ that are Hadamard states of the Majorana field (cf. the treatment of the Dirac equation in four-dimensional curved spacetimes [25]). We also compute

rhs of Eq. (10)

$$= -\frac{\mu}{4\pi^2} \int_0^\infty d\nu \nu \int_{-\infty}^\infty d\theta \cosh \theta |\tilde{g}(\mu \cosh \theta + \nu)|^2$$

$$= -\frac{1}{4\pi^2} \int_\mu^\infty d\omega \, \omega^2 |\tilde{g}(\omega)|^2 Q_-(\omega/\mu), \tag{11}$$

where Q_- was defined in Eq. (9). In more detail, the second equality in Eq. (11) is obtained by using evenness of the integrand to alter the inner integration region to $[0, \infty)$, then changing variables from (ν, θ) to (ω, θ), where $\omega = \nu + \mu \cosh \theta$ (with a consequent change of integration region), and finally evaluating the θ integral. For $x = 0$, the (common) energy density of the Majorana and Ising models thus satisfies the QEI given as the $(-)$ case of Eq. (8), for all real-valued test functions g and for a large domain of $\Phi \in \mathcal{H}_-$; the result for general x follows by translation invariance. This bound is precisely half of the bound for the free massive Dirac field in two dimensions obtained in Ref. [24] using similar arguments, as might be expected.

It is also worth considering the limit of these QEIs as $\mu \to 0$, for fixed g, corresponding to the short-distance scaling limit of the theory [42]. In both cases,

$$\text{rhs of Eq. (8)} \to -\frac{1}{4\pi} \int_{-\infty}^\infty dt |g'(t)|^2,$$

where we have used the fact that $|\tilde{g}(\omega)|$ is even and applied the Plancherel theorem. But both the massless free scalar field and the massless Majorana field are conformal field theories and obey sharp QEIs [see Ref. [29], Eq. (4.25)],

$$\int dt g(t)^2 \langle \Phi, T_\pm^{00}(t, x) \Phi \rangle \geq -\frac{C_\pm}{6\pi} \int_{-\infty}^\infty dt |g'(t)|^2,$$

for suitable normalized Φ, where $C_\pm$ are the central charges of the left- and right-moving components: $C_+ = 1$ (free scalar field) and $C_- = \frac{1}{2}$ (free Majorana), so the sharp bound is, therefore, tighter by a factor of $3/2$, respectively 3, in these two cases. Accordingly, we do not expect our QEI to be sharp for $\mu > 0$.

V. CONCLUSION

In this paper, for the first time, a QEI has been derived for a quantum field theory with nontrivial S-matrix. Even if the S-matrix is rather simple in the case at hand, this underpins the expectation that QEIs are a general consistency property of relativistic quantum physics, which is stable against the introduction of self-interaction.

We also saw that the interacting model allows for negative energy densities in single-particle states. In a free theory, one could combine such single-particle states to obtain multiparticle configurations with arbitrarily large negative energy densities (cf. [30]), thus excluding the existence of a state independent QEI. However, the energy density of the Ising model is not additive with respect to tensor products of single-particle states, a signature of the interacting nature of the theory. It is intriguing that the interaction is responsible for maintaining the QEI in this sense.

We hope the present methods will prove to be a foundation for similar results on other integrable theories in two spacetime dimensions, such as the sinh-Gordon model. Intriguingly, the interaction of the Ising model conspires to yield a QEI with a state-independent lower bound; it will be interesting to see whether this persists in other models.

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