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Continuous wavelet transform in quantum field theory

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We describe the application of the continuous wavelet transform to calculation of the Green functions in quantum field theory: scalar ϕ^4 theory, quantum electrodynamics, and quantum chromodynamics. The method of continuous wavelet transform in quantum field theory, presented by Altaisky [Phys. Rev. D **81**, 125003 (2010)] for the scalar ϕ^4 theory, consists in substitution of the local fields $\phi(x)$ by those dependent on both the position x and the resolution a. The substitution of the action $S[\phi(x)]$ by the action $S[\phi_a(x)]$ makes the local theory into a nonlocal one and implies the causality conditions related to the scale a, the *region causality* [J. D. Christensen and L. Crane, J. Math. Phys. (N.Y.) **46**, 122502 (2005)]. These conditions make the Green functions $G(x_1, a_1, \ldots, x_n, a_n) = \langle \phi_{a_1}(x_1) \ldots \phi_{a_n}(x_n) \rangle$ finite for any given set of regions by means of an effective cutoff scale $A = \min(a_1, \ldots, a_n)$.

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I. INTRODUCTION

The fundamental problem of quantum field theory and statistical mechanics is the problem of divergences of Feynman integrals emerging in Green functions. The formal infinities appearing in perturbation expansion of Feynman integrals are tackled with different regularization methods, from Pauli-Villars regularization to renormalization methods for gauge theories; see, e.g., [1] for a review. A special class of regularizations are the lattice regularizations tailored for the precise numerical simulations in gauge theories [2,3].

There are a few basic ideas connected with those regularizations. First, a certain minimal scale $L = \frac{2\pi}{\Lambda}$, where Λ is the cutoff momentum, is introduced into the theory, with all the fields $\phi(x)$ being substituted by their Fourier transforms truncated at momentum Λ. The physical quantities are then demanded to be independent on the rescaling of the cutoff parameter Λ. The second thing is the Kadanoff blocking procedure [4], which averages the small-scale fluctuations up to a certain scale—this makes a kind of effective interaction.

Physically, all these methods imply the self-similarity assumption: Blocks interact with each other similarly to the sub-blocks [5]. Similarly, but not necessarily having the same interaction strength—the latter can be dependent on scale $\lambda = \lambda(a)$. However, there is no place for such dependence if the fields are described solely in terms of their Fourier transform—except for the cutoff momentum dependence. The latter representation, being based on the representation of the translation group, is rather restrictive: It determines the effective interaction of *all fluctuations up to a certain scale* but says nothing about the interaction of the fluctuations at a given scale.

That is why the functional methods capable of taking into account the interaction at a specific scale are required. Wavelet analysis, the multiscale alternative to the Fourier transform, emerged in geophysics [6] and is the most known of such methods. Its application to quantum field theory has been suggested by many authors [7–11]. The other side of the problem is that the quantum nature of the fields considered in quantum field theory is constrained by the Heisenberg uncertainty principle. To localize a particle in an interval Δx, the measuring device requests a momentum transfer of the order of $\Delta p \sim \hbar/\Delta x$. If Δx is too small, the field $\phi(x)$ at a fixed point x has no experimentally verifiable meaning. What is meaningful is the vacuum expectation of the product of fields in a certain region centered around x, the width of which (Δx) is constrained by the experimental conditions of the measurement [12]. That is why, at least from the physical point of view, any such field should be designated by the resolution of observation $\phi_{\Delta x}(x)$.

In the present paper, we exploit the observation that quantum field theory models, which yield divergent Feynman graphs, can be studied analytically if we project original fields $\phi(x)$ into the fields $\phi_a(x)$, subscribed by the scale of measurement a. The Green functions $\langle \phi_{a_1}(x_1) \ldots \phi_{a_n}(x_n) \rangle$ become finite under certain causality assumptions, which stand for the fact that any n-point correlation function can be dependent only on space-time regions, rather than points, and thus cannot be infinite [12]. These Green functions describe the effect of propagation of a perturbation from a region of size a, centered at a point x, to a region of size a', centered at a point x'. The standard quantum field theory models can be reformulated by

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expressing the (point-dependent) local fields $\phi(x)$, the distributions, in terms of the region-dependent fields $\phi_a(x)$. The integration over all scales a will of course drive us back to the known divergent results, but the physical observables are always those measured with finite resolution, and their correlations are always finite. Therefore, the idea of wavelet transform of quantum fields, which will be considered below, is very similar to the idea of the renormalization group (this similarity, being studied in the lattice framework [10,13], is beyond the scope of the present paper).

The remainder of this paper is organized as follows. In Sec. II, we recall the basics of the continuous wavelet transform and its application to the multiresolution analysis of quantum fields. The definitions of scale-dependent fields and Green functions and the modifications of the Feynman diagram technique are presented. The ϕ^4 scalar field model examples of calculations are given. Section III considers the case of operator-valued scale-dependent fields. The operator ordering and commutation relations are presented. The relations between the theory of scale-dependent fields in Euclidean and Minkowski spaces are discussed. Section IV presents the examples of calculation of one-loop Feynman graphs in QED and QCD. The conclusion gives a few remarks on the perspectives and applicability of the multiscale field theory approach based on continuous wavelet transform.

II. CONTINUOUS WAVELET TRANSFORM IN QUANTUM FIELD THEORY

A. Basics of the continuous wavelet transform

Let $\mathcal{H}$ be a Hilbert space of states for a quantum field $|\phi\rangle$. Let G be a locally compact Lie group acting transitively on $\mathcal{H}$, with $d\mu(\nu)$, $\nu \in G$ being a left-invariant measure on G. Then, similarly to representation of a vector $|\phi\rangle$ in a Hilbert space of states $\mathcal{H}$ as a linear combination of an eigenvectors of momentum operator $|\phi\rangle = \int |p\rangle dp \langle p|\phi\rangle$, any $|\phi\rangle \in \mathcal{H}$ can be decomposed with respect to a representation $U(\nu)$ of G in $\mathcal{H}$ [14,15]:

$$|\phi\rangle = \frac{1}{C_g} \int_G U(\nu)|g\rangle d\mu(\nu) \langle g|U^*(\nu)|\phi\rangle, \tag{1}$$

where $|g\rangle \in \mathcal{H}$ is referred to as an admissible vector, or *basic wavelet*, satisfying the admissibility condition

$$C_g = \frac{1}{\|g\|^2} \int_G |\langle g|U(\nu)|g\rangle|^2 d\mu(\nu) < \infty.$$

The coefficients $\langle g|U^*(\nu)|\phi\rangle$ are referred to as wavelet coefficients.

If the group G is Abelian, the wavelet transform (1) with G: $x' = x + b'$ coincides with Fourier transform.

Next to the Abelian group is the group of the affine transformations of the Euclidean space $\mathbb{R}^d$:

$$\begin{aligned} G\colon x' &= aR(\theta)x + b, \qquad x, b \in \mathbb{R}^d, \\ a &\in \mathbb{R}_+, \qquad \theta \in SO(d), \end{aligned} \tag{2}$$

where $R(\theta)$ is the rotation matrix. We define unitary representation of the affine transform (2) with respect to the basic wavelet $g(x)$ as follows:

$$U(a, b, \theta)g(x) = \frac{1}{a^d} g\left(R^{-1}(\theta)\frac{x-b}{a}\right). \tag{3}$$

(We use the L^1 norm [16,17] instead of the usual L^2 to keep the physical dimension of wavelet coefficients equal to the dimension of the original fields.)

Thus the wavelet coefficients of the function $\phi(x) \in L^2(\mathbb{R}^d)$ with respect to the basic wavelet $g(x)$ in Euclidean space $\mathbb{R}^d$ can be written as

$$\phi_{a,\theta}(b) = \int_{\mathbb{R}^d} \frac{1}{a^d} \overline{g\left(R^{-1}(\theta)\frac{x-b}{a}\right)} \phi(x) d^d x. \tag{4}$$

The wavelet coefficients (4) represent the result of the measurement of function $\phi(x)$ at the point b at the scale a with an aperture function g rotated by the angle(s) θ [18].

The function $\phi(x)$ can be reconstructed from its wavelet coefficients (4) by using the formula (1):

$$\phi(x) = \frac{1}{C_g} \int \frac{1}{a^d} g\left(R^{-1}(\theta)\frac{x-b}{a}\right) \phi_{a\theta}(b) \frac{da d^d b}{a} d\mu(\theta). \tag{5}$$

The normalization constant C_g is readily evaluated by using the Fourier transform:

$$\begin{aligned} C_g &= \int_0^\infty |\tilde{g}(aR^{-1}(\theta)k)|^2 \frac{da}{a} d\mu(\theta) \\ &= \int |\tilde{g}(k)|^2 \frac{d^d k}{|k|^d} < \infty. \end{aligned}$$

For isotropic wavelets

$$C_g = \int_0^\infty |\tilde{g}(ak)|^2 \frac{da}{a} = \int |\tilde{g}(k)|^2 \frac{d^d k}{S_d |k|^d},$$

where $S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the area of the unit sphere in $\mathbb{R}^d$.

B. Resolution-dependent fields

If the ordinary quantum field theory defines the field function $\phi(x)$ as a scalar product of the state vector of the system and the state vector corresponding to the localization at the point x:

$$\phi(x) \equiv \langle x|\phi\rangle, \tag{6}$$

the modified theory [12,19] should respect the resolution of the measuring equipment. Namely, we define the *resolution-dependent fields*

$$\phi_{a\theta}(x) \equiv \langle x, \theta, a; g|\phi\rangle, \tag{7}$$

also referred to as the scale components of ϕ, where $\langle x, \theta, a; g|$ is the bra vector corresponding to localization of the measuring device around the point x with the spatial resolution a and the orientation $\theta \in SO(d)$; g labels the apparatus function of the equipment, an *aperture* [18]. The field theory of extended objects with the basis g defined on the spin variables was considered in Refs. [20,21].

The introduction of resolution into the definition of the field function has a clear physical interpretation. If the particle, described by the field $\phi(x)$, has been initially prepared in the interval $(x - \frac{\Delta x}{2}, x + \frac{\Delta x}{2})$, the probability of registering this particle in this interval is generally less than unity: for the probability of registration depends on the strength of interaction and the ratio of typical scales of the measured particle and the measuring equipment. The maximum probability of registering an object of typical scale Δx by the equipment with typical resolution a is achieved when these two parameters are comparable. For this reason, the probability of registering an electron by visual range photon scattering is much higher than by that of long radio-frequency waves. As a mathematical generalization, we should say that if measuring equipment with a given spatial resolution a fails to register an object, prepared on a spatial interval of width Δx with certainty, then tuning the equipment to *all* possible resolutions a' would lead to the registration. This certifies the fact of the existence of the measured object.

In terms of the resolution-dependent field (7), the unit probability of registering the object ϕ anywhere in space at any resolution and any orientation of the measuring device is expressed by normalization:

$$\int |\phi_{a,\theta}(x)|^2 d\mu(a, \theta, x) = 1, \tag{8}$$

where $d\mu(a, \theta, x)$ is an invariant measure on $\mathbb{R}_+ \times SO(d) \times \mathbb{R}^d$, which depends on the position x, the resolution a, and the orientation θ of the aperture g.

If the measuring equipment has the resolution A—i.e., all states $\langle g; a \geq A, x|\phi\rangle$ are registered with significant probability, but those with $a < A$ are not—the regularization of the field theory in momentum space with the cutoff momentum $\Lambda = 2\pi/A$ corresponds to the UV-regularized functions

$$\phi^{(A)}(x) = \frac{1}{C_g} \int_{a\geq A} \langle x|g; a, b\rangle d\mu(a, b)\langle g; a, b|\phi\rangle. \tag{9}$$

The regularized n-point Green functions are $\mathcal{G}^{(A)}(x_1, \ldots, x_n) \equiv \langle \phi^{(A)}(x_1), \ldots, \phi^{(A)}(x_n)\rangle_c$.

However, the momentum cutoff is merely a technical trick: The physical analysis, performed by the renormalization group method [1,22,23], demands the independence of physical results from the cutoff at $\Lambda \to \infty$.

C. Scalar field example

To illustrate the method, following Refs. [12,19], we start with Euclidean scalar field theory. The widely known example which fairly illustrates the problem is the ϕ^4 interaction model in $\mathbb{R}^d$ (see, e.g., [1,24]), determined by the generating functional

$$W[J] = \mathcal{N} \int e^{-\int d^d x[\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 - J\phi]} \mathcal{D}\phi, \tag{10}$$

where $\mathcal{N}$ is a formal normalization constant. The connected Green functions are given by variational derivatives of the generating functional:

$$G^{(n)} = \frac{\delta^n \ln W[J]}{\delta J(x_1) \ldots \delta J(x_n)}\Bigg|_{J=0}. \tag{11}$$

In the statistical sense, these functions have the meaning of the n-point correlation functions [25]. The divergences of Feynman graphs in the perturbation expansion of the Green functions (11) with respect to the coupling constant λ emerge at coinciding arguments $x_i = x_k$. For instance, the bare two-point correlation function

$$G_0^{(2)}(x - y) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{-\imath p(x-y)}}{p^2 + m^2} \tag{12}$$

is divergent at $x = y$ for $d \geq 2$.

For simplicity let us assume the basic wavelet g to be isotropic; i.e., we can drop the rotation matrix $R(\theta)$. Substitution of the continuous wavelet transform (5) into field theory (10) gives the generating functional for the scale-dependent fields $\phi_a(x)$ [19]:

$$\begin{aligned} W_W[J_a] = \mathcal{N} \int \mathcal{D}\phi_a(x) \exp\Bigg[-\frac{1}{2} \int \phi_{a_1}(x_1) \\ \times D(a_1, a_2, x_1 - x_2)\phi_{a_2}(x_2) \frac{da_1 d^d x_1}{a_1} \frac{da_2 d^d x_2}{a_2} \\ - \frac{\lambda}{4!} \int V^{a_1,\ldots,a_4}_{x_1,\ldots,x_4} \phi_{a_1}(x_1) \ldots \phi_{a_4}(x_4) \\ \times \frac{da_1 d^d x_1}{a_1} \frac{da_2 d^d x_2}{a_2} \frac{da_3 d^d x_3}{a_3} \frac{da_4 d^d x_4}{a_4} \\ + \int J_a(x)\phi_a(x) \frac{da d^d x}{a} \Bigg], \end{aligned} \tag{13}$$

with $D(a_1, a_2, x_1 - x_2)$ and $V^{a_1,\ldots,a_4}_{x_1,\ldots,x_4}$ denoting the wavelet images of the inverse propagator and that of the interaction potential, respectively. The Green functions for scale component fields are given by functional derivatives

$$\langle \phi_{a_1}(x_1) \ldots \phi_{a_n}(x_n)\rangle_c = \frac{\delta^n \ln W_W[J_a]}{\delta J_{a_1}(x_1) \ldots \delta J_{a_n}(x_n)}\Bigg|_{J=0}.$$

Surely the integration in (13) over all scale variables $\int_0^\infty \frac{da_i}{a_i}$ turns us back to the divergent theory (10).

This is the point to restrict the functional integration in (13) only to the field configurations $\{\phi_a(x)\}_{a\geq A}$. The restriction is imposed at the level of the Feynman diagram technique. Indeed, applying the Fourier transform to the right-hand side of (4) and (5) yields

$$\phi(x) = \frac{1}{C_g}\int_0^\infty \frac{da}{a}\int \frac{d^dk}{(2\pi)^d} e^{-\imath kx}\tilde{g}(ak)\tilde{\phi}_a(k),$$
$$\tilde{\phi}_a(k) = \overline{\tilde{g}(ak)}\,\tilde{\phi}(k).$$

Doing so, we have the following modification of the Feynman diagram technique [11]:

(i) Each field $\tilde{\phi}(k)$ will be substituted by the scale component $\tilde{\phi}(k) \to \tilde{\phi}_a(k) = \overline{\tilde{g}(ak)}\,\tilde{\phi}(k)$.

(ii) Each integration in the momentum variable is accompanied by a corresponding scale integration:

$$\frac{d^dk}{(2\pi)^d} \to \frac{d^dk}{(2\pi)^d}\frac{da}{a}.$$

(iii) Each interaction vertex is substituted by its wavelet transform; for the Nth power interaction vertex, this gives multiplication by a factor of $\prod_{i=1}^N \overline{\tilde{g}(a_ik_i)}$.

According to these rules, the bare Green function in wavelet representation takes the form

$$G_0^{(2)}(a_1, a_2, p) = \frac{\tilde{g}(a_1p)\tilde{g}(-a_2p)}{p^2+m^2}.$$

The finiteness of the loop integrals is provided by the following rule: *There should be no scales a_i in internal lines smaller than the minimal scale of all external lines.* Therefore the integration in a_i variables is performed from the minimal scale of all external lines up to infinity.

To understand how the method works, one can look at the one-loop contributions to the two-point Green function $G^{(2)}(a_1, a_2, p)$ shown in Fig. 1(a) and to the vertex shown in Fig. 1(b). The best choice of the wavelet function $g(x)$ would be the apparatus function of the measuring device; however, any well localized function with $\tilde{g}(0) = 0$ will suit. The tadpole integral, to keep with the notation of Ref. [19], is written as

$$T_1^d(Am) = \frac{1}{C_g^2}\int_{a_3,a_4\geq A}\frac{d^dq}{(2\pi)^d}\frac{|\tilde{g}(a_3q)|^2|\tilde{g}(-a_4q)|^2}{q^2+m^2}\frac{da_3}{a_3}\frac{da_4}{a_4}$$
$$= \frac{S_dm^{d-2}}{(2\pi)^d}\int_0^\infty f^2(Amx)\frac{x^{d-1}dx}{x^2+1},$$

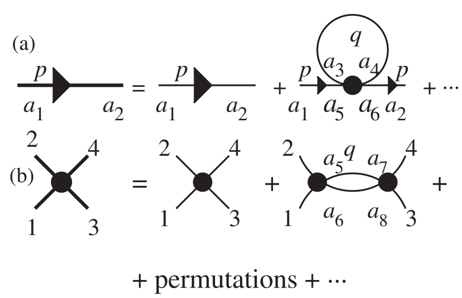


FIG. 1. Feynman diagrams for the Green functions $G^{(2)}$ and $G^{(4)}$ for the resolution-dependent fields. Redrawn from Ref. [12].

where the integration over the scale variables resulted in the effective cutoff function

$$f(x) \equiv \frac{1}{C_g}\int_x^\infty |\tilde{g}(a)|^2\frac{da}{a}, \qquad f(0) = 1, \tag{14}$$

which depends on the squared modulus of the Fourier image of the basic wavelet and, thus, is even with respect to reflections.

In the one-loop contribution to the vertex, shown in Fig. 1(b), the value of the loop integral is

$$X_d = \frac{\lambda^2}{2}\frac{1}{(2\pi)^d}\int\frac{d^dq}{(2\pi)^d}\frac{f^2(qA)f^2((q-s)A)}{[q^2+m^2][(q-s)^2+m^2]}, \tag{15}$$

where $s = p_1 + p_2$ and $A = \min(a_1, a_2, a_3, a_4)$. The integral (15) can be calculated by symmetrization of loop momenta $q \to q + \frac{s}{2}$ in Fig. 1(b), introducing dimensionless variable $\mathbf{y} = \mathbf{q}/s$; after a simple algebra we get

$$X_d = \frac{\lambda^2}{2}\frac{S_{d-1}s^{d-4}}{(2\pi)^{2d}}\int_0^\pi d\theta\sin^{d-2}\theta\int_0^\infty dyy^{d-3}$$
$$\times\frac{f^2\left(As\sqrt{y^2+y\cos\theta+\frac{1}{4}}\right)f^2\left(As\sqrt{y^2-y\cos\theta+\frac{1}{4}}\right)}{\left[\frac{y^2+\frac{1}{4}+\frac{m^2}{s^2}}{y}+\cos\theta\right]\left[\frac{y^2+\frac{1}{4}+\frac{m^2}{s^2}}{y}-\cos\theta\right]},$$

where θ is the angle between the loop momentum q and the total momentum s. For the simple choice of the basic wavelet g_1 [12,19]

$$g(x) = -\frac{xe^{-x^2/2}}{(2\pi)^{d/2}}, \qquad \tilde{g}(k) = \imath ke^{-k^2/2}$$

in four dimensions, we get a finite result

$$T_1^4(\alpha^2) = \frac{-4\alpha^4e^{2\alpha^2}\mathrm{Ei}_1(2\alpha^2) + 2\alpha^2}{64\pi^2\alpha^4}m^2,$$
$$\lim_{s^2\gg 4m^2} X_4(\alpha^2) = \frac{\lambda^2}{256\pi^6}\frac{e^{-2\alpha^2}}{2\alpha^2}[e^{\alpha^2} - 1 - \alpha^2e^{2\alpha^2}\mathrm{Ei}_1(\alpha^2)$$
$$+ 2\alpha^2e^{2\alpha^2}\mathrm{Ei}_1(2\alpha^2)],$$

depending on dimensionless scale factor $\alpha \equiv Am$, where A is the minimal scale of all external lines.

These results display an evident fact that for the massive scalar field all length scales are to be measured in units of inverse mass.

III. CAUSALITY AND COMMUTATION RELATIONS

A. Operator ordering

Up to now, we have considered the calculation of the Feynman diagrams for the scale-dependent fields $\phi_{a,\cdot}(x)$ treated as c-valued functions. In quantum field theory, adjusted to high energy physics applications, the fields $\phi_{a,\cdot}(x)$ are operator-valued functions. So, as was already emphasized in the context of the wavelet application to quantum chromodynamics [9,12], the operator ordering and the commutation relations are to be defined.

In standard quantum field theory, the operator ordering is performed according to the nondecreasing of the time argument in the product of the operator-valued functions acting on vacuum state

$$\underbrace{A(t_n)A(t_{n-1})\ldots A(t_2)A(t_1)}_{t_n \geq t_{n-1} \geq \cdots \geq t_2 \geq t_1}|0\rangle.$$

In the infinite momentum frame, which simplifies algebraic structure of the current algebra, the time ordering is performed in the proper time argument x_+ [26]. The quantization is performed by separating the Fourier transform of quantum fields into the positive- and the negative-frequency parts

$$\phi = \phi^+(x) + \phi^-(x),$$

defined as follows:

$$\phi(x) = \int \frac{d^d k}{(2\pi)^d}[e^{\imath kx}u^+(k) + e^{-\imath kx}u^-(k)], \qquad (16)$$

where the operators $u^{\pm}(k) = u(\pm k)\theta(k_0)$ are subjected to canonical commutation relations

$$[u^+(k), u^-(k')] = \Delta(k, k').$$

In the case of the scale-dependent fields, because of the presence of the scale argument in new fields $\phi_{a,\eta}(x)$, where a and η label the size and the shape, respectively, of the region centered at x, the problem arises how to order the operators supported by different regions. This problem was solved in Refs. [12,27] on the base of the *region causality assumption* [28]. If two regions $(\Delta x, x)$ and $(\Delta y, y)$ do not intersect, the standard time ordering procedure is applied. Alternatively, if one of the regions is *inside* another (see Fig. 2), the operator standing for the bigger region acts on vacuum first [12]. This causal ordering, drawn in Euclidean space, is presented in Fig. 2. The time ordering in Euclidean space, as an analytic continuation of time ordering in Minkowski space, has been already considered in Ref. [29]. The diagram in Fig. 2 shows spacelike regions in Euclidean space. For Minkowski space, corresponding

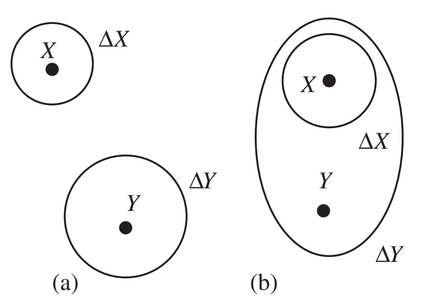


FIG. 2. Causal ordering of scale-dependent fields. Spacelike regions are drawn in Euclidean space: (a) The event regions do not intersect; (b) event X is inside event Y.

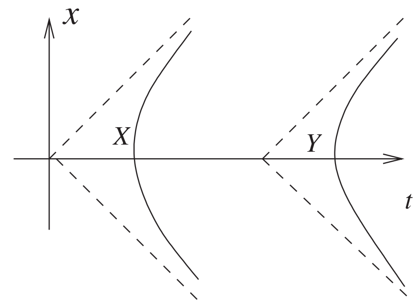


FIG. 3. Disjoint events in (t, x) plane in Minkowski space.

diagrams can be obtained by analytic continuation of the Euclidean ball of imaginary radius $\imath\Delta$ into Minkowski space, where we can restrict ourselves with forward light cone $t \geq 0$, $|x| \leq t$. The disjoint events in Minkowski space are shown in Fig. 3. The correspondence to the other case of one Euclidean event inside another, shown in Fig. 2(b), looks more complex after analytic continuation to Minkowski space. The forward light-cone part of such an intersection is shown in Fig. 4.

We consider partial intersection of regions ($A \cap B = C$, $C \neq A$, $C \neq B$, $C \neq \varnothing$) as unphysical. For this reason, corresponding ordering of operator-valued fields is not defined. Since a region is identified with a possibility of measurement, a simultaneous measurement of a part within and not within the parent entity is inconsistent. The "partial intersection" just implies that when doing the functional integration one has to go to the finer scale, so that the regions do not intersect. The same happens in p-adic models of quantum gravity: Two p-adic balls are either disjoint or one within another [30].

Mathematically, when we make the functional measure of a Feynman integral into a discrete product of wavelet fields on a lattice $\mathcal{D}u_a(b) \to \prod_{j,k} dd^j_k$, we get rid of the partial intersection, as can be seen in the example of a binary tree, shown in Table I.

Phenomenologically, the principle "the coarse acts first" is related to the definition of the measurement procedure, possibly generalized, where the state of a part can be measured or affected only after and relative to the

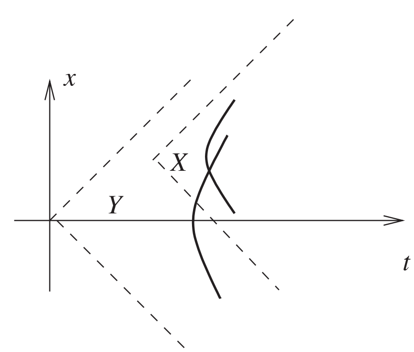


FIG. 4. Nontrivial intersection of two events $X \subset Y$ in the (t, x) plane in Minkowski space.

TABLE I. Binary tree of operator-valued functions. Vertical direction corresponds to the scale variable. The causal sequence of the operator-valued functions shown in the table is d_0^0, d_{00}^1, d_{01}^1, d_1^0, d_{10}^1, d_{11}^1. As shown, the underlined regions of different scales do not intersect.

	$\underline{d_0^0}$		d_1^0
d_{00}^1	d_{01}^1	$\underline{d_{10}^1}$	d_{11}^1

state of the whole. A similar reason underlies the restriction on the scales in internal loops by the minimal scales of all external lines of the Feynman diagram: If we measure a quantum system from outside, we cannot excite modes finer than the minimal available scale of measurement. Thus the functional integration over the trajectories in the space of square-integrable functions $\mathcal{D}\phi(x)$ is substituted by functional integration over all causal paths, or tubes of all different thicknesses, in the space of scale-dependent functions $\mathcal{D}\phi_a(x)$. Referring the reader to the original works of Refs. [28,31] for the topological aspects of causal paths, we ought to mention that the Bogolioubov microcausality condition holds for causal paths in the same way as it holds for trajectories [12]. It is also easy to show that, if the domain Y is inside the domain X, the corresponding Green function is not singular at coinciding arguments—it is a projection from a coarser scale to a finer scale:

$$G_0^{(2)}(a_1, a_2, b_1 - b_2 = 0) = \int \frac{d^4p}{(2\pi)^4} \frac{\tilde{g}(a_1 p)\tilde{g}(-a_2 p)}{p^2 + m^2} e^{-\imath p \cdot 0},$$

since $|\tilde{g}(p)|$ vanish at $p \to \infty$.

B. Commutation relations

In the case of wavelet transform the positive- and negative-frequency part operators (16) can be expressed by using wavelet transform:

$$u_i^\pm(k) = \frac{1}{C_{g_i}} \int_{-\infty}^{\infty} d\eta \int_0^\infty \frac{da}{a} \tilde{g}_i(aM^{-1}(\eta)k) u_{ia\eta}^\pm(k), \quad (17)$$

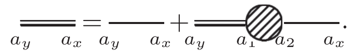
from where we can set [12]

$$[u_{ia\eta}^+(k), u_{ja'\eta'}^-(k')] = \delta_{ij} C_{g_i} a\delta(a - a')\delta(\eta - \eta') \times [u^+(k), u^-(k')] \quad (18)$$

to ensure canonical commutation relations for $[u^+(k), u^-(k')]$.

C. The Dyson-Schwinger equation

Ordering in the scale argument results in the modification of the Dyson-Schwinger equation in the theory of scale-dependent functions. Let $G(x - y, a_x, a_y)$ be the bare field propagator, describing the propagation of the field from the region (y, a_y) to the region (x, a_x). Let $\mathcal{G}(x - y, a_x, a_y)$ be the full propagator for the same regions. The Dyson-Schwinger equation relating the full propagator with the bare propagator is symbolically drawn in the diagram



(19)

The integral equation depicted in diagram (19) can be written as

$$\mathcal{G}(x - y, a_x, a_y) = G(x - y, a_x, a_y) + \int \frac{da_1}{a_1} \int \frac{da_2}{a_2} \times \int dx_1 dx_2 G(x - x_2, a_x, a_2) \times \mathcal{P}(x_2 - x_1, a_2, a_1) \times \mathcal{G}(x_1 - y, a_1, a_y), \quad (20)$$

where the full vertex

 $= \mathcal{P}(x_2 - x_1, a_2, a_1)$

denotes the vacuum polarization operator if G is the massless boson, or the self-energy diagram otherwise. The Fourier counterpart of Eq. (20) can be written as

$$\tilde{\mathcal{G}}_{a_x,a_y}(p) = \tilde{G}_{a_x,a_y}(p) + \int \frac{da_1}{a_1} \int \frac{da_2}{a_2} \tilde{G}_{a_x,a_2}(p) \times \tilde{\mathcal{P}}_{a_2,a_1}(p) \tilde{\mathcal{G}}_{a_1,a_y}(p).$$

D. Wavelet transform in Minkowski space

The straightforward application of wavelet transform (4), defined in Euclidean space $\mathbb{R}^d$, to the Minkowski space M_1^4 would be the analytic continuation of the results into the imaginary time $x_4 = \imath x_0$, making the Euclidean rotations into Lorentz boosts. The construction of such wavelets with respect to the representations of the Poincaré group have been studied by several authors [32,33]. From the physical point of view, there exists a simple and an elegant way of making the wavelet transform in Minkowski space.

In quantum field theory problems related to relativistic particle collisions, we can always change the coordinate frame to the comoving frame of a relativistic projectile moving at the utmost speed of light. Because of the Lorentz contraction of the projectile, the longitudinal and the transversal degrees of freedom behave essentially differently in such a system. Without loss of generality, we can always assume the projectile to move along the z axis.

The Lorentz contraction, i.e., the hyperbolic rotation in the (t, z) plane, is determined by the hyperbolic rotation angle—the rapidity η. The rotations in the transverse plane are not affected by the Lorentz contraction and are determined by the $SO(2)$ rotation angle ϕ. If the problem is axially symmetric, the latter can be dropped.

Therefore, it is convenient to change from the space-time coordinates (t, x, y, z) to the *light-cone coordinates* (x_+, x_-, x, y):

$$x_\pm = \frac{t \pm z}{\sqrt{2}}, \qquad \mathbf{x}_\perp = (x, y). \tag{21}$$

This is the so-called infinite momentum frame. The infinite momentum frame is not a Lorentzian system but a limit of that at $v \to c$. The advantage of the coordinates (21) for the calculations, say, in QED or QCD, is significant simplification of the vacuum structure [26,34]. The metrics in the light-cone coordinates becomes

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The rotation matrix has a block-diagonal form:

$$M(\eta, \phi) = \begin{pmatrix} e^\eta & 0 & 0 & 0 \\ 0 & e^{-\eta} & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{pmatrix},$$

so that $M^{-1}(\eta, \phi) = M(-\eta, -\phi)$.

We can define the wavelet transform in light-cone coordinates in the same way as in Euclidean space by using the representation of the affine group

$$x' = aM(\eta, \phi)x + b,$$

defined in L^1 norm as

$$U(a, b, \eta, \phi)u(x) = \frac{1}{a^4} u\left(M^{-1}(\eta, \phi)\frac{x-b}{a}\right).$$

We have the definition of wavelet coefficients of a function $f(x)$ with respect to the basic wavelet g as follows:

$$W_{a,b,\eta,\phi}[f] = \int dx_+ dx_- d^2\mathbf{x}_\perp \frac{1}{a^4} \overline{g\left(M^{-1}(\eta, \phi)\frac{x-b}{a}\right)} \times f(x_+, x_-, \mathbf{x}_\perp). \tag{22}$$

In contrast to wavelet transform in Euclidean space, where the basic wavelet g can be defined globally on $\mathbb{R}^d$, the basic wavelet in Minkowski space is to be defined separately in four domains impassible by Lorentz rotations:

$$A_1\colon k_+ > 0, k_- < 0; \qquad A_2\colon k_+ < 0, k_- > 0;$$
$$A_3\colon k_+ > 0, k_- > 0; \qquad A_4\colon k_+ < 0, k_- < 0,$$

where k is wave vector $k_\pm = \frac{\omega \pm k_z}{\sqrt{2}}$. Hence we have four separate wavelets in these four domains [35]:

$$g_i(x) = \int_{A_i} e^{\imath kx} \tilde{g}(k) \frac{d^4k}{(2\pi)^4}. \tag{23}$$

We assert the following definition of the Fourier transform in light-cone coordinates:

$$f(x_+, x_-, \mathbf{x}_\perp) = \int e^{\imath k_- x_+ + \imath k_+ x_- - \imath \mathbf{k}_\perp \mathbf{x}_\perp} \tilde{f}(k_-, k_+, \mathbf{k}_\perp) \times \frac{dk_+ dk_- d^2\mathbf{k}_\perp}{(2\pi)^4}.$$

Substituting the Fourier images into the definition (22), we get

$$W^i_{ab\eta\phi} = \int_{A_i} e^{\imath k_- b_+ + \imath k_+ b_- - \imath \mathbf{k}_\perp \mathbf{b}_\perp} \tilde{f}(k_-, k_+, \mathbf{k}_\perp) \times \bar{\tilde{g}}(ae^\eta k_-, ae^{-\eta} k_+, aR^{-1}(\phi)\mathbf{k}_\perp) \frac{dk_+ dk_- d^2\mathbf{k}_\perp}{(2\pi)^4}. \tag{24}$$

In Fourier space the relation between Fourier coefficients and wavelet coefficients is therefore the same as in $\mathbb{R}^d$:

$$\tilde{W}_{a\eta\phi}(k) = \tilde{f}(k)\bar{\tilde{g}}(aM^{-1}(\eta, \phi)k).$$

Similarly, the reconstruction formula is [36]

$$f(x) = \sum_{i=1}^4 \frac{1}{C_{g_i}} \int_{-\infty}^\infty d\eta \int_0^{2\pi} d\phi \int_0^\infty \frac{da}{a} \int_{M_1^4} db_+ db_- d^2\mathbf{b}_\perp \frac{1}{a^4} g_i\left(M^{-1}(\eta)\frac{\xi - b}{a}\right) W^i_{ab\eta\phi}$$
$$= \sum_{i=1}^4 \frac{1}{C_{g_i}} \int_{-\infty}^\infty d\eta \int_0^{2\pi} d\phi \int_0^\infty \frac{da}{a} \int_{A_i} \frac{dk_+ dk_- d^2\mathbf{k}_\perp}{(2\pi)^4} e^{\imath k_- x_+ + \imath k_+ x_- - \imath \mathbf{k}_\perp \mathbf{x}_\perp} \tilde{W}_{a\eta\phi}(k) \tilde{g}(ak_- e^\eta, ak_+ e^{-\eta}, aR^{-1}(\phi)\mathbf{k}_\perp).$$

If the problem is axially symmetric, the azimuthal part of integration (ϕ) can be dropped. It is also convenient to use logarithmic scale $a = e^u$ to study different rapidity domains.

E. Choice of the basic wavelet

The choice of the basis of wavelet decomposition is always a subtle question, especially in quantum field theory. (The best choice, as was already emphasized in [12], would be the apparatus function of a classical measuring device interacting with a quantum system.) Some basis is always tacitly assumed. Even by describing the massless photons, which are not localized anywhere, by plane waves, the possibility of photon registration by photomultiplier implies its interaction with the electron and, hence, some scale and some localization.

If the continuous wavelet transform is used in place of the Fourier transform, the choice of the basic function is constrained by the feasibility of the analytical integration in Feynman diagrams, on one hand, and by the possibility to understand this basic function as a localized (quasi) particle. The latter has been claimed by some authors to be important for Minkowski space [37], which seems questionable for Euclidean space calculations. If the wavelet transform is performed on a lattice, there is a bias that only the similarity properties are important rather than the shape of wavelet [10,13]. The question of whether or not the basic wavelet should satisfy some equation of motion is still not clear. We are also not aware of the effect of the discrete symmetries of the basic wavelet.

To justify our choice of the derivatives of the Gaussian as the basic wavelets, we present the following heuristic consideration, inferred from the coherent states theory [33]. Let us introduce a localized wave packet in Fourier space:

$$\tilde{g}(t,k) = e^{-\imath t k - k^2/2}. \tag{25}$$

If the wave packet is considered in Minkowski space, then $k^2 = 0$ can be assumed for the photon and the whole equation (25) turns to be a plane wave. Otherwise, it is a localized wave. If t is time, the packet (25) is a Gaussian wave packet at initial time $t = 0$. At finite but small instants of time the wave packet can be approximated by its Taylor expansion

$$\tilde{g}(t,k) = \tilde{g}_0(k) + \frac{t}{1!}\tilde{g}_1(k) + \frac{t^2}{2!}\tilde{g}_2(k) + O(t^3),$$

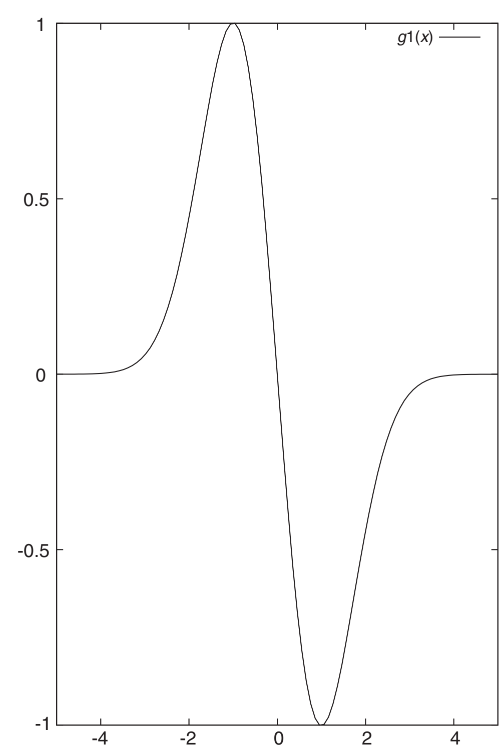


FIG. 5. Graph of g_1 wavelet: $g_1(x) = -x e^{-x^2/2}$.

where the expansion coefficients

$$\tilde{g}_n(k) = \frac{d^n}{dt^n}\tilde{g}(t,k)\Big|_{t=0}$$

are responsible for the shape of the packet at the time scales at which 1, 2, or n excitations are significant. Clearly, $g_n(x)$ are the excitations of the harmonic oscillator, with g_1 being the first excitation; see Fig. 5.

IV. GAUGE THEORIES

A. QED

Quantum electrodynamics is the simplest case of gauge theory. The local $U(1)$ invariance of the fermion field

$$\psi(x) \to e^{-\imath e \Lambda(x)}\psi(x)$$

is accompanied by the gradient invariance of the vector field $A_\mu(x)$, the electromagnetic field

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x), \tag{26}$$

to keep the total action $S(\bar{\psi}, \psi, A) = \int L d^4x$ invariant under the local $U(1)$ transform generated by $\Lambda(x)$. The interaction of the charged fermion field ψ with electromagnetic field A_μ is introduced by substitution of ordinary derivatives ∂_μ to covariant derivatives

$$D_\mu = \partial_\mu + \imath e A_\mu(x),$$

with e being the charge of the fermion.

The Lagrangian of QED has the (Euclidean) form

$$L = \bar{\psi}(x)(\not{D} + \imath m)\psi(x) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \underbrace{\frac{1}{2\alpha}(\partial_\mu A_\mu)^2}_{\text{gauge fixing}},$$

$$\text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{27}$$

being the field strength tensor of the electromagnetic field A and γ_μ being the Dirac γ matrices.

The straightforward application of the Feynman integral to the gauge theory with the Lagrangian (27) would be inefficient, for the integration over the field $A(x)$ would contain an infinite set of physically equivalent field configurations. For this purpose the gauge fixing, which restricts the integration only to gauge-nonequivalent field configurations, was introduced by Faddeev and Popov [38].

Quantum electrodynamics is the most firmly established and most verified field theory model in the physics of elementary particles. The probability amplitude of scattering obtained at tree level are in fairly good agreement with classical scattering theory. Starting from one-loop level, the Feynman integrals are formally divergent, and the physical results are derived by using the renormalization invariance of QED. The most accurate tests for the renormalized calculations of the electron-photon interaction are the Lamb shift of the hydrogenlike ion energy levels and the anomalous magnetic momentum of the electron [39–42].

In one-loop approximation, the radiation corrections in QED come from three primitive Feynman graphs: fermion self-energy $\Sigma(p)$, vacuum polarization operator $\Pi_{\mu\nu}(p)$, and the vertex function $\Gamma_\rho(p,q)$. In Euclidean space the equations for the above three graphs have the following form:

Electron self-energy

$$\Sigma(p) = -e^2 \int \frac{d^4q}{(2\pi)^4} \gamma_\mu \frac{-\imath}{\not{p} - \not{q} + m} \gamma_\nu \frac{\delta_{\mu\nu}}{q^2} \tag{28}$$

gives the corrections to the bare electron mass m_0 related to irradiation of virtual photons.

Vacuum polarization diagram

$$\Pi_{\mu\nu}(p) = -e^2 \int \frac{d^4q}{(2\pi)^4} \mathrm{Sp}\left[\gamma_\mu \frac{1}{\not{p}+\not{q}+m} \gamma_\nu \frac{1}{\not{q}+m}\right] \tag{29}$$

could be expected to give the nonzero corrections to the photon mass, but due to gauge invariance the photon mass remains zero; instead, the one-loop contribution (29) renormalizes the electron charge at small distances and, therefore, modifies the Coulomb interaction by screening the bare electron charge e_0 by virtual electron-positron pairs polarizing the vacuum at small distances. This diagram contributes to the Lamb shift of the atom energy levels.

One-loop vertex function

$$\Gamma_\rho(p,q) = -\imath e^3 \int \frac{d^4f}{(2\pi)^4} \gamma_\tau \frac{1}{\not{p}+\not{f}+m} \gamma_\rho \frac{1}{\not{f}+\not{q}+m} \gamma_\sigma \frac{\delta_{\tau\sigma}}{f^2} \tag{30}$$

determines the anomalous magnetic moment of the electron.

All three integrals (28)–(30) are divergent. Their evaluation involves regularization procedures. The most common is the dimensional regularization with all integrals taken in formal $d = 2\omega$ dimension with physical value $\omega = 2$. In this way the divergences come as poles in $\epsilon = 2 - \omega$; see, e.g., [22,24,43].

B. One-loop corrections in wavelet-based theory

The evaluation of Feynman diagrams with fermions and gauge fields in wavelet-based Euclidean theory is similar to that of scalar theory (15). The evaluation of the one-loop radiative corrections for the scale-dependent fields gives finite results by construction with no regularization procedure required.

1. Electron self-energy

For the scale components of the electron self-energy diagram, we get

$$\frac{\Sigma^{(A)}(p)}{\tilde{g}(ap)\tilde{g}(-a'p)} = -\imath e^2 \int \frac{d^4q}{(2\pi)^4} \frac{F_A(p,q)\gamma_\mu[\frac{\not{p}}{2} - \not{q} - m]\gamma_\mu}{[(\frac{p}{2}-q)^2+m^2][\frac{p}{2}+q]^2}, \tag{31}$$

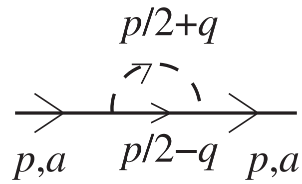


FIG. 6. Electron self-energy diagram.

where A is the minimal scale of two external lines shown in Fig. 6: $A = \min(a, a')$. The regularizing function $F_A(p,q)$ is the result of integration over the scales of two internal lines. For the isotropic basic wavelet g, the regularizing function is given by (14):

$$F_A(p,q) = f^2(A(p/2-q))f^2(A(p/2+q)). \tag{32}$$

Introducing the dimensionless variable $\mathbf{y} = \mathbf{q}/|\mathbf{p}|$, after straightforward algebra, we can perform the integration in Euclidean space:

$$\frac{\Sigma^{(A)}(p)}{\tilde{g}(ap)\tilde{g}(-a'p)} = -\imath e^2 \int \frac{d^4y}{(2\pi)^4} F_A(p,|p|y) \times \frac{\not{p} + 4m - 2|p|\not{y}}{\left[y^2 + \frac{1}{4} - y\cos\theta - \frac{m^2}{p^2}\right]\left[y^2 + \frac{1}{4} + y\cos\theta\right]}, \tag{33}$$

where θ is the Euclidean angle between the p and the q directions. In high energy limit $p^2 \gg 4m^2$, the contribution of the last term in the numerator of (33) vanishes for the symmetry, and Eq. (33) can be easily integrated in angle variable (C1):

$$\frac{\Sigma^{(A)}(p)}{\tilde{g}(ap)\tilde{g}(-a'p)} = -\frac{\imath e^2}{4\pi^2} R_1(p)(\not{p} + 4m),$$

$$\text{where } R_1(p) = \int_0^\infty dy y F_A(p,|p|y)\left[1 - \sqrt{1 - \frac{1}{\beta^2(y)}}\right],$$

$$\beta(y) = y + \frac{1}{4y}.$$

The integral $R_1(p)$ is finite for any wavelet cutoff function (14). For the g_1 wavelet we get

$$R_1(p) = e^{-A^2p^2} \int_0^\infty dy y e^{-4A^2p^2y^2}\left[1 - \sqrt{1 - \frac{1}{\beta^2}}\right].$$

After a simple algebra this gives

$$R_1(p) = \frac{1}{8A^2p^2}(2A^2p^2\mathrm{Ei}_1(A^2p^2) - 4A^2p^2\mathrm{Ei}_1(2A^2p^2) - e^{-A^2p^2} + 2e^{-2A^2p^2}). \tag{34}$$

2. Vacuum polarization diagram

The vacuum polarization diagram in quantum electrodynamics of scale-dependent fields is obtained by integration over the scale variables of the fermion loop shown in Fig. 7:

$$\frac{\Pi^{(A)}_{\mu\nu}(p)}{\tilde g(ap)\tilde g(-a'p)} = -e^2\int\frac{d^4q}{(2\pi)^4}F_A(p,q)\frac{\mathrm{Sp}(\gamma_\mu(\not q+\not p/2-m)\gamma_\nu(\not q-\not p/2-m))}{[(q+p/2)^2+m^2][(q-p/2)^2+m^2]}$$
$$= -4e^2\int\frac{d^4q}{(2\pi)^4}F_A(p,q)\frac{2q_\mu q_\nu-\frac12 p_\mu p_\nu+\delta_{\mu\nu}(\frac{p^2}{4}-q^2-m^2)}{[(q+\frac{p}{2})^2+m^2][(q-\frac{p}{2})^2+m^2]}. \quad (35)$$

Similarly to the previous diagram, we use the dimensionless variable y and integrate over the angle variable. The momentum integration in Eq. (35) is straightforward: Having expressed all momenta in units of electron mass m, we express the loop momentum in terms of the photon momentum and perform the integration over the angle variable:

$$\frac{\Pi^{(A)}_{\mu\nu}}{\tilde g(ap)\tilde g(-a'p)} = -\frac{e^2}{\pi^3}(m^2p^2)\int_0^\infty dyyF_A(mp,mpy)\int_0^\pi d\theta\sin^2\theta\frac{2y_\mu y_\nu-\frac12\frac{p_\mu p_\nu}{p^2}+\delta_{\mu\nu}(\frac14-y^2-\frac{1}{p^2})}{\left[\frac{\frac14+y^2+\frac{1}{p^2}}{y}+\cos\theta\right]\left[\frac{\frac14+y^2+\frac{1}{p^2}}{y}-\cos\theta\right]},$$

where p is dimensionless, i.e., is expressed in units of m. Introducing the notation $\beta(y)\equiv\frac{\frac14+y^2+\frac{1}{p^2}}{y}$ and using the substitution $y_\mu y_\nu\to Ay^2\delta_{\mu\nu}+By^2\frac{p_\mu p_\nu}{p^2}$, we get

$$\frac{\Pi^{(A)}_{\mu\nu}}{\tilde g(ap)\tilde g(-a'p)} = -\frac{e^2}{\pi^3}(m^2p^2)\int_0^\infty dyyF_A(mp,mpy)\int_0^\pi d\theta\sin^2\theta\frac{\delta_{\mu\nu}((2A-1)y^2+\frac14-\frac{1}{p^2})+\frac{p_\mu p_\nu}{p^2}(2By^2-\frac12)}{\beta^2(y)-\cos^2\theta},$$

where A and B depend only on the modulus of y, but not on the direction, and can be expressed in terms of angle integrals (C1).

Finally, writing the polarization operator as a sum of transversal and longitudinal parts, we have the equations

$$\frac{\Pi^{(A)}_{\mu\nu}(p)}{\tilde g(ap)\tilde g(-a'p)} \equiv \delta_{\mu\nu}\pi_T^{(A)}+\frac{p_\mu p_\nu}{p^2}\pi_L^{(A)}$$
$$= \left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\pi_T^{(A)}+X^{(A)}\frac{p_\mu p_\nu}{p^2},$$
$$\pi_T^{(A)} = -\frac{e^2}{3\pi^2}m^2p^2\int_0^\infty dyyF_A(mp,mpy)\left[y^2+\left(1-\sqrt{\frac{\frac{1}{16}+y^4+\frac{1}{p^4}-\frac{y^2}{2}+\frac{1}{2p^2}+\frac{2y^2}{p^2}}{(\frac14+y^2+\frac{1}{p^2})^2}}\right)\times\left(\frac58-\frac{4}{p^2}-\frac{2}{p^4}-2y^2\left(1+\frac{2}{p^2}\right)-2y^4\right)\right],$$
$$\pi_L^{(A)} = -\frac{e^2}{3\pi^2}m^2p^2\int_0^\infty dyyF_A(mp,mpy)\left[-4y^2+\left(1-\sqrt{\frac{\frac{1}{16}+y^4+\frac{1}{p^4}-\frac{y^2}{2}+\frac{1}{2p^2}+\frac{2y^2}{p^2}}{(\frac14+y^2+\frac{1}{p^2})^2}}\right)\times\left(8y^4+2y^2\left(1+\frac{8}{p^2}\right)+\frac{4}{p^2}+\frac{8}{p^4}-1\right)\right], \quad (36)$$

where

$$X^{(A)} = \pi_L^{(A)}+\pi_T^{(A)} = \frac{e^2m^2p^2}{\pi^2}\int_0^\infty dyyF_A(mp,mpy)\left[y^2-\left(1-\sqrt{\frac{\frac{1}{16}+y^4+\frac{1}{p^4}-\frac{y^2}{2}+\frac{1}{2p^2}+\frac{2y^2}{p^2}}{(\frac14+y^2+\frac{1}{p^2})^2}}\right)\left(2y^4+4\frac{y^2}{p^2}+\frac{2}{p^4}-\frac18\right)\right].$$

The integrals above are finite and can be easily evaluated in large momenta limit, $p^2\gg 4$, introducing the dimensionless scale $a=Am$.

As an example we can evaluate the vacuum polarization operator for the g_1 wavelet. For the g_1 wavelet the regularizing function

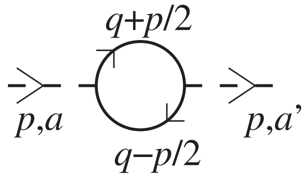


FIG. 7. Vacuum polarization diagram in (Euclidean) scale-dependent QED.

$$F_A(p,q) = \exp(-A^2p^2 - 4A^2q^2).$$

Hence for large $p^2 \gg 4$ the integral (36) can be evaluated by substitution $y^2 = t$ [19]:

$$\pi_T^{(A)} = -\frac{e^2}{6\pi^2}m^2p^2\Big\{\frac{e^{-a^2p^2}}{8a^6p^6}(4a^4p^4 - a^2p^2 - 1) + \frac{e^{-2a^2p^2}}{8a^6p^6}(1 - 4a^4p^4 + 2a^2p^2) - \frac{1}{2}\,\mathrm{Ei}_1(a^2p^2) + \mathrm{Ei}_1(2a^2p^2)\Big\}.$$

Similarly, the longitudinal term X^A evaluated with the g_1 wavelet in the limit $p^2 \gg 4$ is equal to

$$X^A = \frac{e^2m^2p^2}{16\pi^2}\frac{e^{-a^2p^2}(a^2p^2 - 1 + e^{-a^2p^2})}{a^6p^6}. \tag{37}$$

In the limit of small scales $ap \ll 1$, Eq. (37) does not depend on p: $X^A \propto \frac{1}{a^2}$. Therefore the whole equation (36) is similar to the result obtained by Pauli-Villars regularization of the vacuum polarization

$$\Pi^M_{\mu\nu}(p) = cM^2\delta_{\mu\nu} + (p^2\delta_{\mu\nu} - p_\mu p_\nu)F\left(\frac{p^2}{4m^2}, \frac{m}{M}\right),$$

where $M \to \infty$ is a regularizing mass [43]. The gauge invariance is restored if the multiscale diagram (35) is integrated over all scales. In this limit the theory can be subjected to dimensional regularization [12].

3. Vertex function

The one-loop contribution to the QED vertex function for the theory with scale-dependent matter fields is shown in Fig. 8. The equation, which corresponds to the vertex diagram in Fig. 8, can be cast in the form

$$-\imath e\frac{\Gamma^{(A)}_{\mu,r}}{\tilde{g}(-pa')\tilde{g}(-qr)\tilde{g}(ka)} = (-\imath e)^3\int\frac{d^4l}{(2\pi)^4}\gamma_\alpha G(p-f) \times \gamma_\mu G(k-f) \times \gamma_\beta D_{\alpha\beta}F_A(p-f) \times F_A(k-f)F_A(f).$$

The explicit substitution with photon propagator taken in the Feynman gauge gives

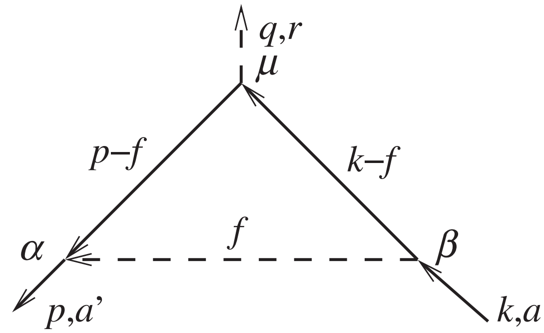


FIG. 8. One-loop vertex function in scale-dependent QED.

$$\imath e\frac{\Gamma^{(A)}_{\mu,r}}{\tilde{g}(-pa')\tilde{g}(-qr)\tilde{g}(ka)} = (-\imath e)^3\int\frac{d^4f}{(2\pi)^4}\gamma_\alpha\frac{\not{p}-\not{f}-m}{(p-f)^2+m^2} \times\gamma_\mu\frac{\not{k}-\not{f}-m}{(k-f)^2+m^2}\gamma_\alpha\frac{1}{f^2} \times F_A(p-f)F_A(k-f)F_A(f). \tag{38}$$

By representing the numerator of the latter equation in the form

$$A_\mu = \gamma_\alpha(\not{p}-\not{f})\gamma_\mu(\not{k}-\not{f})\gamma_\alpha - m[(\not{p}-\not{f})\gamma_\mu + \gamma_\mu(\not{k}-\not{f})] + 2m^2\gamma_\mu,$$

it can be seen that the right-hand side of Eq. (38) can be represented as a linear combination of three finite integrals ($J^{(0)}$, $J^{(1)}_\mu$, $J^{(2)}_{\mu\nu}$) presented in Appendix C, in analog to their divergent counterparts in Minkowski space [44]. After some algebra, the vertex (38) turns to be

$$\frac{\Gamma^{(A)}_{\mu,r}}{\tilde{g}(-pa')\tilde{g}(-qr)\tilde{g}(ka)} = e^2\gamma_\alpha[(\not{p}\gamma_\mu\not{k} - m\not{p}\gamma_\mu - m\gamma_\mu\not{k})J^{(0)} - (\gamma_\nu\gamma_\mu\not{k} + \not{p}\gamma_\mu\gamma_\nu + 2m\delta_{\mu\nu})J^{(1)}_\nu + \gamma_\lambda\gamma_\mu\gamma_\nu J^{(2)}_{\lambda\nu}]\gamma_\alpha. \tag{39}$$

C. Ward-Takahashi identities

The Ward-Takahashi identity in spinor electrodynamics relates the vertex function to the difference of fermion propagators:

$$q_\mu\Gamma_\mu(p, q, p+q) = G^{-1}(p+q) - G^{-1}(p), \tag{40}$$

where $G(p)$ is the complete fermion propagator. The identity (40) is a helpful constraint which ensures the gauge invariance of the renormalized QED in any order of perturbation theory [45,46]. The constraint (40) makes the perturbation expansion gauge invariant in the presence of the gauge-fixing terms in the QED generating functional.

In wavelet-based theory, where the fields explicitly depend on scale, the divergence does not appear in the Feynman diagrams, but the evaluation of integrals in internal lines with the integration scales constrained by the minimal scale of external lines may spoil the gauge invariance of the complete propagator. To prevent this, the Ward-Takahashi identities are required.

In the absence of gauge-fixing terms in the Lagrangian (27), the generating functional

$$e^{-Z[J,\bar{\chi},\chi]} = \int \mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-\int d^4x(L(\psi,\bar{\psi},A)+JA+\imath\bar{\chi}\psi+\imath\bar{\psi}\chi)}, \tag{41}$$

with $L(\psi, \bar{\psi}, A)$ given by Eq. (27), would be invariant under the gauge transformations (26) if no source term $-\int d^4x(J_\mu A_\mu + \imath\bar{\chi}\psi + \imath\bar{\psi}\chi)$ is present.

In the framework of scale-dependent functions, the gauge field $A_\mu(x)$ is expressed in terms of its wavelet coefficients $A_{\mu a}(b)$:

$$A_\mu(x) = \frac{1}{C_g}\int_{\mathbb{R}_+\times\mathbb{R}^d}\frac{1}{a^d}g\left(\frac{x-b}{a}\right)A_{\mu a}(b)\frac{dad^db}{a}$$

[with the angular part of wavelet transform (5) dropped for simplicity]. In view of linearity of the wavelet transform, we may infer the gauge transform of the scale components to have the form

$$A'_{\mu a}(x) = A_{\mu a}(x) + \frac{\partial\Lambda_a(x)}{\partial x_\mu},$$

where

$$\Lambda_a(x) = \int_{\mathbb{R}^d}\frac{1}{a^d}\bar{g}\left(\frac{y-x}{a}\right)\Lambda(y)d^dy$$

is the scale component of the gauge function (26). That is, the gauge transform of the Abelian gauge field $A_{\mu a}(x)$ is a projection of the (no-scale) gauge field $A_\mu(x)$ onto the space of resolution a.

Since the free Lagrangian of QED is gauge invariant by construction, the derivative of the Ward-Takahashi identities turns into evaluation of the functional overage of the variation of source and gauge-fixing terms under infinitesimal gauge transform

$$\delta A_\mu = \partial_\mu\Lambda, \qquad \delta\psi = -\imath e\Lambda\psi, \qquad \delta\bar{\psi} = \imath e\Lambda\bar{\psi},$$

where $\Lambda = \Lambda(x)$ is considered to be small. Under this variation the integrand in the functional integral (41), after integration by parts, acquires a multiplicative factor e^{δ_Λ}, with

$$\delta_\Lambda \equiv \int d^dx\left[-\frac{1}{\alpha}\partial^2(\partial_\mu A_\mu) + \partial_\mu J_\mu + e(\bar{\psi}\chi - \bar{\chi}\psi)\right]\Lambda(x). \tag{42}$$

Considering δ_Λ as small, we can approximate $e^{\delta_\Lambda} \approx 1 + \delta_\Lambda$ and proceed with the derivation procedure from

$$\delta = \langle\delta_\Lambda\rangle = 0. \tag{43}$$

The standard procedure of the variation of action with a gauge-fixing term [47] with respect to $\Lambda_a(x)$ (43) leads to the equations [48]

$$q_\mu\Gamma_{\mu a_4 a_3 a_1}(p,q,p+q) = \int\frac{da_2}{a_2}G^{-1}_{a_1a_2}(p+q)\tilde{M}_{a_2a_3a_4}(p+q,q,p) - \int\frac{da_2}{a_2}\tilde{M}_{a_1a_3a_2}(p+q,q,p)G^{-1}_{a_2a_4}(p),$$

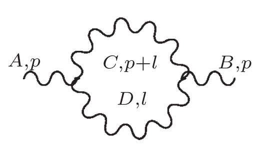
where $\tilde{M}_{a_1a_2a_3}(k_1,k_2,k_3)$

$$= (2\pi)^d\delta^d(k_1-k_2-k_3)\bar{\tilde{g}}(a_1k_1)\tilde{g}(a_2k_2)\tilde{g}(a_3k_3). \tag{44}$$

Equation (44) is exactly the wavelet transform of the standard Ward-Takahashi identity (40).

D. QCD example

The same as in QED, we can evaluate the gluon vacuum polarization operator by using g_1 as the basic wavelet. The corresponding one-loop diagram is shown in Eq. (45):



$$\equiv \Pi^{(\mathcal{A})}_{AB,\mu\nu}(p) = -\frac{g^2}{2}f^{ABC}f^{BDC}\int\frac{d^4l}{(2\pi)^4}\frac{N_{\mu\nu}(l,p)F_{\mathcal{A}}(l+p,l)}{l^2(l+p)^2}, \tag{45}$$

where

$$N_{\mu\nu}(l,p) = 10l_\mu l_\nu + 5(l_\mu p_\nu + l_\nu p_\mu) - 2p_\mu p_\nu + (p-l)^2\delta_{\mu\nu} + (2p+l)^2\delta_{\mu\nu}$$

is the tensor structure of the vacuum polarization diagram (45) in $\mathbb{R}^4$ Euclidean space. $\mathcal{A}$ is the minimal scale of two external lines. The regularizing function, if calculated with the g_1 wavelet, has the form (32)

$$F_{\mathcal{A}}(l+p,l) = \exp(-2\mathcal{A}^2(l+p)^2 - 2\mathcal{A}^2l^2).$$

Symmetrizing the loop momenta in Eq. (45) by substitution $l = q - \frac{p}{2}$, we obtain

$$\Pi^{(\mathcal{A})}_{AB,\mu\nu}(p) = -\frac{g^2}{2}f^{ACD}f^{BDC}\int\frac{d^4q}{(2\pi)^4}F_{\mathcal{A}}(p,q) \times\frac{10q_\mu q_\nu - \frac{9}{2}p_\mu p_\nu + \delta_{\mu\nu}(\frac{9}{2}p^2+2q^2)}{[q^2-\frac{p^2}{4}]^2}. \tag{46}$$

For the g_1 wavelet the regularizing function $F_{\mathcal{A}}(p,q)$ is given by Eq. (32).

The integral (46) can be easily evaluated in the infrared limit where ordinary QCD is divergent:

$$\Pi^{(\mathcal{A},g_1)}_{AB,\mu\nu}(p\to 0) = -g^2 f^{ACD} f^{BDC} \int \frac{d^4q}{(2\pi)^4} \frac{e^{-4\mathcal{A}^2q^2}}{q^4} \times [5q_\mu q_\nu + q^2\delta_{\mu\nu}]. \quad (47)$$

Making use of the isotropy

$$d^4q \to 2\pi^2 q^3 dq, \qquad q_\mu q_\nu \to \delta_{\mu\nu}\frac{q^2}{4},$$

we get

$$\Pi^{(\mathcal{A},g_1)}_{AB,\mu\nu}(p\to 0) = -\frac{9g^2 f^{ACD} f^{BDC}\delta_{\mu\nu}}{32}\int_0^\infty q dq e^{-4\mathcal{A}^2q^2} = -\frac{9g^2 f^{ACD} f^{BDC}\delta_{\mu\nu}}{256\mathcal{A}^2}.$$

A similar contribution comes from the ghost loop.

V. CONCLUSION

In this paper, we developed a regularization method for quantum field theory based on a continuous wavelet transform. Regardless of the significant amount of work devoted to wavelet-based regularization in different quantum field theory models [9,10,49], all those are basically the lattice theories. The novelty of the present approach, developed by the authors [11,12,50], consists in using continuous wavelet transform to substitute the local fields $\phi(x)$ by the scale-dependent fields $\phi_a(x)$, defined as wavelet coefficients of the physical field. Substitution of such fields into the action, supplied by appropriate causality assumptions and operator ordering [12,27,28], results in effective regularization of Feynman graphs, which makes each internal line decay as an effective factor $\propto e^{-p^2A^2}$, where A is the minimal scale of all internal lines and p is momentum.

Regularization factors, that are technically similar to our approach, were already known in QCD. They are related to the modification of the gluon vacuum state to the instanton vacuum, with the parameter A understood as the size of the instanton [51,52]. The difference between the instanton vacuum model and our model is that the scattered quark fields are local fields in the instanton model and only the interaction with instanton vacuum is smeared. In our approach, the incident particles are nonlocal wave packets and only the integration over all scales makes the theory local.

The physics of using scale-dependent fields $\phi_a(x)$ instead of local fields $\phi(x)$ lies in the fact that no physical quantity can be measured in a point but in a region of nonzero size $a > 0$. Thus, only the finite resolution projections $\phi_a(x)$ of a quantum field ϕ are physically meaningful. The n-point Green functions for such fields constructed by our method are finite by construction and do not require regularization. The gauge invariance of the theory results in appropriate Ward-Takahashi identities, which are the projections of ordinary Ward-Takahashi identities onto finite resolution spaces.

The practical applications of our approach can be found in such physical settings where the separation of the field from the components of different scales is physically meaningful. Such models have been presented in QED calculations of the dependence of the Casimir force on the size of displacement in measurement [50] and also in application of quantum field theory methods to the calculation of correlations of the turbulent velocity fluctuations of different scales [53]. We strongly hope that, regardless of the yet unsolved problem of deriving the renormalization group equation in the continuous limit of wavelet-based theory, this method can be also applied for QCD calculations, where it was originally proposed [9].

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APPENDIX A: DIRAC γ MATRICES IN EUCLIDEAN SPACE

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = -2\delta_{\mu\nu}, \quad (A1)$$

$$\gamma_\mu\gamma_\mu = -4, \qquad \gamma_\mu \not{p} \gamma_\mu = 2\not{p}. \quad (A2)$$

Slashed vectors denote convolution with Dirac gamma matrices $\not{k} = \gamma_\mu k_\mu$, $\not{k}\not{k} = -k^2$.

APPENDIX B: FEYNMAN RULES IN EUCLIDEAN SPACE

The photon propagator is taken in the Feynman gauge:

$$D(k) = \frac{\delta_{\mu\nu}}{k^2}.$$

Fermion propagator:

$$G^{(2)}_E(p) = \frac{-\imath}{\not{p}+m} = \imath\frac{\not{p}-m}{p^2+m^2}.$$

Electron-fermion vertex:

$$-\imath e\gamma_\rho.$$

Besides that, each fermion vertex results in an extra sign $-$ of the whole diagram.

APPENDIX C: FUNCTIONS AND INTEGRALS

Exponential integral of the first type:

$$\mathrm{Ei}_1(z) = \int_1^\infty \frac{e^{-xz}}{x} dx.$$

Integrals for angle integration in Euclidean Green functions [19]:

$$I_k(y) \equiv \int_0^\pi d\theta \frac{\sin^2\theta \cos^{2k}\theta}{\beta^2(y) - \cos^2\theta}, \quad I_0(y) = \pi\left(1 - \sqrt{1 - \beta^{-2}(y)}\right), \quad I_1(y) = -\frac{\pi}{2} + \beta^2(y) I_0(y), \dots \tag{C1}$$

The constants A and B for the vacuum polarization diagram (35) are given by $4A + B = 1$ and $A + B = I_1/I_0$, from which we get

$$A = \frac{1}{3} + \frac{\pi}{6} I_0^{-1}(y) - \frac{1}{3}\beta^2(y),$$

$$B = -\frac{1}{3} - \frac{2\pi}{3} I_0^{-1}(y) + \frac{4}{3}\beta^2(y).$$

Integrals in the one-loop fermion-photon vertex:

$$J^{(0)} = \int \frac{d^4 f}{(2\pi)^4} \frac{F_A(p-f)F_A(k-f)F_A(f)}{[(p-f)^2 + m^2][(k-f)^2 + m^2] f^2}, \tag{C2}$$

$$J^{(1)}_\mu = \int \frac{d^4 f}{(2\pi)^4} \frac{f_\mu F_A(p-f)F_A(k-f)F_A(f)}{[(p-f)^2 + m^2][(k-f)^2 + m^2] f^2}, \tag{C3}$$

$$J^{(2)}_{\mu\nu} = \int \frac{d^4 f}{(2\pi)^4} \frac{f_\mu f_\nu F_A(p-f)F_A(k-f)F_A(f)}{[(p-f)^2 + m^2][(k-f)^2 + m^2] f^2}. \tag{C4}$$

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