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Uniqueness of charged static asymptotically flat black holes in dynamical Chern-Simons gravity

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Making use of the conformal positive energy theorem, we prove the uniqueness of four-dimensional static *electrically* charged black holes being the solution of Chern-Simons modified gravity equations of motion. We assume that black hole spacetime contains an asymptotically flat spacelike hypersurface with compact interior and nondegenerate components of the event horizon.

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I. INTRODUCTION

Gravitational collapse and emergence of black holes is one of the most essential research problem of general relativity and its generalizations. The problem of classification of nonsingular black hole solutions was first discussed by Israel [1], Müller zum Hagen *et al.* [2], and Robinson [3], while the most complete results were proposed in Refs. [4–8]. The classification of static vacuum black hole solutions was finished in [9], where the condition of nondegeneracy of the event horizon was removed. For Einstein-Maxwell black holes, it was proved that for the static electrovacuum black holes all degenerate components of the event horizon should have charges of the same sign [10].

The construction of the uniqueness black hole theorem for stationary axisymmetric spacetime turned out to be a far more complicated task [11]. However, the complete proof was presented by Mazur [12] and Bunting [13] (for a review of the uniqueness of black hole solutions, see [14] and references therein).

A different issue, related to the problem of gravitational collapse in generalization of Einstein theory to higher dimensions and emergence of higher dimensional black objects (like black rings, black Saturns) and multidimensional black holes, was widely studied. The complete classification of n-dimensional charged black holes both with nondegenerate and degenerate components of the event horizon was proposed in Ref. [15], while partial results for the very nontrivial case of the n-dimensional rotating black hole uniqueness theorem were provided in [16]. The problem of the behavior of matter fields in the spacetime of higher dimensional black holes was studied in Ref. [17].

Because of the attempts of building a consistent quantum gravity theory, there was also a resurgence of works treating the mathematical aspects of the low-energy string theory black holes. This research comprised also the case of the low-energy limit of the string theory, such as dilaton gravity, Einstein-Maxwell-axion-dilaton gravity, and supergravity theories [18]. On the other hand, the strictly stationary static vacuum spacetimes in Einstein-Gauss-Bonnet theory were discussed in [19].

Black holes and their properties as key ingredients of the AdS/CFT attitude [20] to superconductivity have also acquired much attention. Questions about possible matter configurations in AdS spacetime arose naturally during the aforementioned research. In Ref. [21] it was revealed that strictly stationary AdS spacetime could not allow for the existence of nontrivial configurations of complex scalar fields or form fields. The generalization of the aforementioned problem, i.e., strict stationarity of spacetimes with complex scalar fields in Einstein-Maxwell-axion-dilaton gravity with negative cosmological constant, was given in [22].

The Chern-Simons (CS) modified gravity, where the Einstein action is modified by the addition of a parity violating Pontryagin term [23], has its roots in particle physics. Namely, the imbalance between left-handed and right-handed fermions induces gravitational anomaly in a fermion number current, proportional to the aforementioned Pontryagin term [24]. It also emerges in string theory as an anomaly-canceling term in the Green-Schwarz mechanism [25]. Moreover, CS gravity was elaborated in the context of cosmology, gravitational waves, and Lorentz invariance [26] (see also references therein). In Ref. [27] it was revealed that a static asymptotically flat black hole solution is unique as a Schwarzschild spacetime in CS modified gravity.

Motivated by the aforementioned problems, we shall consider the problem of the uniqueness of static asymptotically flat black holes in CS modified gravity with $U(1)$-gauge field. The basic idea in our treatment of the problem in question will be to implement the conformal positive energy theorem [28].

The paper is organized as follows. In Sec. II we review some basic facts concerning dynamical CS modified gravity. Then, applying the conformal positive energy theorem, we perform the uniqueness proof of static asymptotically flat electrically charged black holes in CS modified gravity.

II. SYSTEM

We commence this section with the action of the CS modified gravity with matter fields provided by the action

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$$I = \kappa \int d^4x\sqrt{-g}R + \frac{\alpha}{4}\int d^4x\sqrt{-g}\theta * R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - \frac{\beta}{2}\int d^4x\sqrt{-g}\, \nabla_\alpha\theta\nabla^\alpha\theta + \int d^4x\sqrt{-g}\,\mathcal{L}_{\rm mat}, \quad (1)$$

where α, β are the dimensional coupling constants, while θ (CS coupling field) is a scalar field, which is a function parametrizing deformation from ordinary Einstein theory. $*R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is the Pontryagin density, while $\mathcal{L}_{\rm mat}$ stands for some matter Lagrangian density which does not depend on the scalar field in question. In what follows we assume that $\mathcal{L}_{\rm mat}$ will constitute the matter Lagrangian for $U(1)$-gauge fields, given by $\mathcal{L}_{\rm mat} = -F_{\mu\nu}F^{\mu\nu}$. The dual to the Riemannian tensor is defined as

$$*R_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\gamma\delta}{}^{\psi\zeta}R_{\alpha\beta\psi\zeta}. \quad (2)$$

The field equations obtained by variation of the action (2) imply

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{\alpha}{\kappa}C_{\mu\nu} = \frac{1}{\kappa}(T_{\mu\nu}(\theta) + T_{\mu\nu}(F)), \quad (3)$$

$$\nabla_\alpha\nabla^\alpha\theta = \frac{\alpha}{4\beta}R_{\alpha\beta\gamma\delta} * R^{\alpha\beta\gamma\delta}, \quad (4)$$

where we have denoted by $C^{\alpha\beta}$ the following relation:

$$C^{\alpha\beta} = \nabla_\gamma\theta\epsilon^{\gamma\psi\mu(\alpha}\nabla_\mu R^{\beta)}_\psi + \nabla_\gamma\nabla_\delta\theta * R^{\delta(\alpha\beta)\gamma}. \quad (5)$$

On the other hand, the energy-momentum tensor $T_{\mu\nu} = -\frac{\delta S}{\sqrt{-g}\delta g^{\mu\nu}}$ of matter fields in question yields

$$T_{\alpha\beta}(\theta) = \frac{\beta}{2}\Big(\nabla_\alpha\theta\nabla_\beta\theta - \frac{1}{2}g_{\alpha\beta}\nabla_\gamma\theta\nabla^\gamma\theta\Big), \quad (6)$$

$$T_{\alpha\beta}(F) = 2F_{\alpha\gamma}F^\gamma_\beta - \frac{1}{2}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}. \quad (7)$$

The line element of static spacetime subject to the asymptotically timelike Killing vector field $k_\alpha = (\frac{\partial}{\partial t})_\alpha$ and $V^2 = -k_\mu k^\mu$ can be provided by the relation

$$ds^2 = -V^2dt^2 + g_{ij}dx^idx^j, \quad (8)$$

where V and g_{ij} are independent of the t coordinate as the quantities of the hypersurface Σ of constant t. We assume that on the hypersurface Σ the electromagnetic potential will be of the form $A_0 = \psi dt$; i.e., one deals with an electrically charged black hole.

In our consideration we shall take into account the asymptotically flat spacetime. Namely, the spacetime in question will contain a data set $(\Sigma_{\rm end}, g_{ij}, K_{ij})$ with gauge fields of $F_{\alpha\beta}$ such that $\Sigma_{\rm end}$ is diffeomorphic to $\mathbf{R^3}$ minus a ball. The fields (g_{ij}, K_{ij}) will satisfy the falloff condition of the form

$$| g_{ij} - \delta_{ij} | + r | \partial_a g_{ij} | + \cdots + r^m | \partial_{a_1\ldots a_m} g_{ij} | + r | K_{ij} | + \cdots + r^m | \partial_{a_1\ldots a_m} K_{ij} | \leq \mathcal{O}\Big(\frac{1}{r}\Big). \quad (9)$$

Likewise, we require that in the local coordinates as above, the defined $U(1)$-gauge field fulfills the following falloff demand:

$$| F_{\alpha\beta} | + r | \partial_a F_{\alpha\beta} | + \cdots + r^m | \partial_{a_1\ldots a_m} F_{\alpha\beta} | \leq \mathcal{O}\Big(\frac{1}{r^2}\Big). \quad (10)$$

In light of these stipulations, the hypersurface will be said to be asymptotically flat if it contains an asymptotically flat end.

Taking the form of the static metric into account, the corresponding equations of motion yield

$$V^{(g)}\nabla_i{}^{(g)}\nabla^i V = \frac{1}{\kappa}{}^{(g)}\nabla_i\psi^{(g)}\nabla^i\psi, \quad (11)$$

$${}^{(g)}\nabla_i{}^{(g)}\nabla^i\psi = \frac{1}{V}{}^{(g)}\nabla_i\psi^{(g)}\nabla^i V, \quad (12)$$

$$V^{(g)}\nabla_i{}^{(g)}\nabla^i\theta + {}^{(g)}\nabla_i\theta^{(g)}\nabla^i V = 0, \quad (13)$$

$${}^{(g)}R_{ij} - \frac{{}^{(g)}\nabla_i{}^{(g)}\nabla_j V}{V} = \frac{1}{\kappa}\Big(\frac{\beta}{2}{}^{(g)}\nabla_i\theta^{(g)}\nabla_j\theta + g_{ij}\frac{{}^{(g)}\nabla_i\psi^{(g)}\nabla^i\psi}{V^2} - 2\frac{{}^{(g)}\nabla_i\psi^{(g)}\nabla_j\psi}{V^2}\Big). \quad (14)$$

In the above relations, the covariant derivative with respect to the metric tensor g_{ij} is denoted by ${}^{(g)}\nabla$, while ${}^{(g)}R_{ij}(g)$ is the Ricci tensor defined on the hypersurface Σ. Furthermore, let us suppose that for each of the quantities in question, i.e., V, ψ, ϕ, there is a standard coordinate system in which they have the usual asymptotic expansion.

To proceed further, let us introduce the definitions of the crucial quantities in the proof of the uniqueness. Namely, they can be written as follows:

$$\Phi_1 = \frac{1}{2}\Big[V + \frac{1}{2V}\Big], \quad (15)$$

$$\Phi_0 = i\sqrt{\frac{\beta}{2\kappa}}\theta, \quad (16)$$

$$\Phi_{-1} = \frac{1}{2}\Big[V - \frac{1}{2V}\Big], \quad (17)$$

and

$$\Psi_1 = \frac{1}{2}\left[V + \frac{1}{2V} - \sqrt{\frac{2}{\kappa}}\frac{\psi^2}{V}\right], \tag{18}$$

$$\Psi_0 = \frac{2}{\kappa}\frac{\psi}{V}, \tag{19}$$

$$\Psi_{-1} = \frac{1}{2}\left[V - \frac{1}{2V} - \sqrt{\frac{2}{\kappa}}\frac{\psi^2}{V}\right]. \tag{20}$$

It is worth pointing out that by defining the metric tensor $\eta_{AB} = \mathrm{diag}(1, -1, -1)$, it can be achieved that $\Phi_A\Phi^A = \Psi_A\Psi^A = -1$, where $A = -1, 0, 1$. Having in mind the conformal transformation provided by

$$\tilde{g}_{ij} = V^2 g_{ij}, \tag{21}$$

one can introduce the symmetric tensors written in terms of Φ_A in the following form:

$$\tilde{G}_{ij} = \tilde{\nabla}_i\Phi_{-1}\tilde{\nabla}_j\Phi_{-1} - \tilde{\nabla}_i\Phi_0\tilde{\nabla}_j\Phi_0 - \tilde{\nabla}_i\Phi_1\tilde{\nabla}_j\Phi_1, \tag{22}$$

and similarly for the potential Ψ_A

$$\tilde{H}_{ij} = \tilde{\nabla}_i\Psi_{-1}\tilde{\nabla}_j\Psi_{-1} - \tilde{\nabla}_i\Psi_0\tilde{\nabla}_j\Psi_0 - \tilde{\nabla}_i\Psi_1\tilde{\nabla}_j\Psi_1, \tag{23}$$

where by $\tilde{\nabla}_i$ we have denoted the covariant derivative with respect to the metric $\tilde{g}_{ij}$. Consequently, according to the relations (22) and (23), the field equations may be cast in the forms

$$\tilde{\nabla}^2\Phi_A = \tilde{G}_i{}^i\Phi_A, \qquad \tilde{\nabla}^2\Psi_A = \tilde{H}_i{}^i\Psi_A. \tag{24}$$

Just the Ricci curvature tensor with respect to the conformally rescaled metric $\tilde{g}_{ij}$ implies

$$\tilde{R}_{ij} = \tilde{G}_{ij} + \tilde{H}_{ij}. \tag{25}$$

In general, as far as the conformal positive energy theorem is concerned, one assumes that we have to deal with two asymptotically flat Riemannian $(n-1)$-dimensional manifolds $(\Sigma^\Phi, {}^\Phi g_{ij})$ and $(\Sigma^\Psi, {}^\Psi g_{ij})$. Moreover, the conformal transformation of the form ${}^\Psi g_{ij} = \Omega^{2\Phi} g_{ij}$. Then, it implies that the corresponding masses satisfy ${}^\Phi m + \beta {}^\Psi m \geq 0$ if ${}^\Phi R + \beta\Omega^2 {}^\Psi R \geq 0$, for some positive constant β. The aforementioned inequalities are fulfilled if the $(n-1)$-dimensional Riemannian manifolds are flat [28].

To proceed further, due to the requirement of the conformal positive energy theorem, we introduce conformal transformations obeying the relations

$${}^\Phi g^{\pm}_{ij} = {}^\phi\omega^2_\pm \tilde{g}_{ij}, \qquad {}^\Psi g^{\pm}_{ij} = {}^\psi\omega^2_\pm \tilde{g}_{ij}. \tag{26}$$

On the other hand, their conformal factors are subject to the relations

$${}^\Phi\omega_\pm = \frac{\Phi_1 \pm 1}{2}, \qquad {}^\Psi\omega_\pm = \frac{\Psi_1 \pm 1}{2}. \tag{27}$$

Thus, the above conformal transformations enable one to obtain four manifolds: $(\Sigma^\Phi_+, {}^\Phi g^+_{ij})$, $(\Sigma^\Phi_-, {}^\Phi g^-_{ij})$, $(\Sigma^\Psi_+, {}^\Psi g^+_{ij})$, and $(\Sigma^\Psi_-, {}^\Psi g^+_{ij})$. The standard procedure of pasting $(\Sigma^\Phi_\pm, {}^\Phi g^\pm_{ij})$ and $(\Sigma^\Psi_\pm, {}^\Psi g^\pm_{ij})$ across the surface $V = 0$ enables to construct regular hypersurfaces $\Sigma^\Phi = \Sigma^\Phi_+ \cup \Sigma^\Phi_-$ and $\Sigma^\Psi = \Sigma^\Psi_+ \cup \Sigma^\Psi_-$. If $(\Sigma, g_{ij}, \Phi_A, \Psi_A)$ are an asymptotically flat solution of (24) and (25) with a nondegenerate black hole event horizon, our next task will be to check that the total gravitational mass on hypersurfaces Σ^Φ and Σ^Ψ is equal to zero. In order to do this, we shall implement the conformal positive mass theorem presented in Ref. [28]. Hence, using another conformal transformation given by

$$\hat{g}^\pm_{ij} = [({}^\Phi\omega_\pm)^2({}^\Psi\omega_\pm)^2]^{\frac{1}{2}}\tilde{g}_{ij}, \tag{28}$$

it follows that the Ricci curvature tensor on the space yields

$$\begin{aligned} 2\hat{R} &= [{}^\Phi\omega^2_\pm {}^\Psi\omega^{2\lambda}_\pm]^{-\frac{1}{2}}({}^\Phi\omega^2_\pm {}^\Phi R + {}^\Psi\omega^2_\pm {}^\Psi R) \\ &\quad + (\hat{\nabla}_i \ln {}^\Phi\omega_\pm - \hat{\nabla}_i \ln {}^\Psi\omega_\pm) \\ &\quad \times (\hat{\nabla}^i \ln {}^\Phi\omega_\pm - \hat{\nabla}^i \ln {}^\Psi\omega_\pm). \end{aligned} \tag{29}$$

The close inspection of the first term in relation (29) reveals that it is non-negative. Namely, one can establish that it may be written in the form as follows:

$$\begin{aligned} {}^\Phi\omega^2_\pm {}^\Phi R + {}^\Psi\omega^2_\pm {}^\Psi R &= 2\left|\frac{\Phi_0\tilde{\nabla}_i\Phi_{-1} - \Phi_{-1}\tilde{\nabla}_i\Phi_0}{\Phi_1 \pm 1}\right|^2 \\ &\quad + 2\left|\frac{\Psi_0\tilde{\nabla}_i\Psi_{-1} - \Psi_{-1}\tilde{\nabla}_i\Psi_0}{\Psi_1 \pm 1}\right|^2. \end{aligned} \tag{30}$$

Applying the conformal energy theorem, we draw a conclusion that $(\Sigma^\Phi, {}^\Phi g_{ij})$, $(\Sigma^\Psi, {}^\Psi g_{ij})$, and $(\hat{\Sigma}, \hat{g}_{ij})$ are flat, and it in turn implies about the conformal factors that ${}^\Phi\omega = {}^\Psi\omega$ and $\Phi_1 = \Psi_1$. Furthermore, $\Phi_0 = \mathrm{const}\Phi_{-1}$ and $\Psi_0 = \mathrm{const}\Psi_{-1}$. Just the above potentials are functions of a single variable. Moreover, the manifold (Σ, g_{ij}) is conformally flat. We can rewrite $\hat{g}_{ij}$ in a conformally flat form; i.e., we define a function

$$\hat{g}_{ij} = \mathcal{U}^{4\Phi} g_{ij}, \tag{31}$$

where one sets $\mathcal{U} = ({}^\Phi\omega_\pm V)^{-1/2}$. Because of the fact that the Ricci scalar in the $\hat{g}_{ij}$ metric is equal to zero, equations of motion of the system in question reduce to the Laplace equation on the three-dimensional Euclidean manifold

$$\nabla_i\nabla^i\mathcal{U} = 0, \tag{32}$$

where ∇ is the connection on a flat manifold. The above equation implies that the following expression for the flat base space is valid. Namely, one gets

$${}^\Phi g_{ij}dx^i dx^j = \tilde{\rho}^2 d\mathcal{U}^2 + \tilde{h}_{AB}dx^A dx^B. \tag{33}$$

Then, the event horizon will be located at some constant value of $\mathcal{U}$. The radius of the black hole event horizon can be terminated at a fixed value of the ρ coordinate [29], which in turn can be introduced on the hypersurface Σ by the relation

$$\hat{g}_{ij}dx^i dx^j = \rho^2 dV^2 + h_{AB}dx^A dx^B.$$

Moreover, a connected component of the event horizon can be identified at a fixed value of ρ.

Proceeding further, let us assume that $\mathcal{U}_1$ and $\mathcal{U}_2$ consist of two solutions of the boundary value problem of the system in question. Using the Green identity and integrating over the volume element, we arrive at the relation

$$\left(\int_{r\to\infty} - \int_{\mathcal{H}}\right)(\mathcal{U}_1 - \mathcal{U}_2)\frac{\partial}{\partial r}(\mathcal{U}_1 - \mathcal{U}_2)dS = \int_{\Omega} |\nabla(\mathcal{U}_1 - \mathcal{U}_2)|^2 d\Omega. \tag{34}$$

In view of the last equation, the surface integrals disappear due to the imposed boundary conditions. On the other hand, by virtue of the above relation one finds that the volume integral must be identically equal to zero. To summarize, we have established the conclusion of our investigations.

Theorem: Let us consider a static four-dimensional solution to the equation of motion in Chern-Simons modified gravity with $U(1)$-gauge field. Suppose that one has an asymptotically timelike Killing vector field k_μ orthogonal to the connected and simply connected spacelike hypersurface Σ. The topological boundary $\partial\Sigma$ of Σ is a nonempty topological manifold with $g_{ij}k^i k^j = 0$ on $\partial\Sigma$. It yields the following: if $\partial\Sigma$ is connected, then there exists a neighborhood of the hypersurface Σ which is diffeomorphic to an open set of Reissner-Nordsröm nonextreme solutions with *electric* charge.

III. CONCLUSIONS

In our paper we prove the uniqueness of four-dimensional static black holes being the solution of Chern-Simons modified gravity with $U(1)$-gauge field. Assuming the existence of an asymptotically timelike Killing vector field orthogonal to the simply connected spacelike hypersurface with topological boundary, it turns out that if the boundary in question is connected, then there is a neighborhood of the hypersurface which is diffeomorphic to an open set Reissner-Nordsröm nonextreme solution with *electric* charge.

It may be interesting to generalize the proof to the case of both degenerate and nondegenerate components of the event horizon of the black hole in question. On the other hand, the stationary axisymmetric case and the Chern-Simons modified gravity with cosmological constant are challenges for future investigations. We hope to return to these problems elsewhere.

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