PHYSICAL REVIEW D **88,** 024045 (2013)

Gravitational energy, gravitational pressure, and the thermodynamics of a charged black hole in teleparallel gravity

K. H. C. Castello-Branco[1,*] and J. F. da Rocha-Neto[2,†]

[1]*Universidade Federal do Oeste do Pará, Avenida Marechal Rondon, 68040-070 Santarém, Pará, Brazil*
[2]*Instituto de Física, Universidade de Brasília, 70910-900 Brasília, Distrito Federal, Brazil*

(Received 29 May 2013; published 25 July 2013)

We investigate, in the case of a Reissner–Nordström black hole, the definitions of gravitational energy and gravitational pressure that naturally arise in the framework of the teleparallel equivalent of general relativity. In particular, we calculate the gravitational energy enclosed by the event horizon of the black hole, E, and the radial pressure over it, p. With these quantities, we then analyze the thermodynamic relation $dE + pdV$ (as p turns out to be a density, and dV is actually given by $dV = drd\theta d\phi$, in spherical-type coordinates). We compare the latter with the standard first law of black hole dynamics. Also, by identifying $TdS = dE + pdV$, we comment on a possible modification of the standard, Bekenstein–Hawking entropy-area relation due to the gravitational energy and gravitational pressure of the black hole. The infinitesimal variations in question refer to the Penrose process for a Reissner–Nordström black hole.

DOI: 10.1103/PhysRevD.88.024045 PACS numbers: 04.20.−q, 04.20.Cv, 04.70.Dy

I. INTRODUCTION

As is well known, the behavior of black holes as thermodynamic systems deeply connects gravitation, quantum mechanics, and thermodynamics. The black hole surface gravity κ plays the role of temperature, its horizon area A that of entropy, and its mass M that of internal energy. This striking connection initially flourished from a close analogy between the laws of black hole dynamics and the laws of thermodynamics. It was only later that it was put on a firm basis, due to the discovery of Hawking that quantum mechanical effects permit a black hole to create and emit particles like a hot body with temperature $\kappa/2\pi$ (in units with $G = c = \hbar = \kappa_B = 1$) [1]. Nevertheless, such a connection is considered to be still poorly understood presently [2].

It is also known that a notion of gravitational energy can be ascribed to black holes, not to mention the gravitational energy transported by gravitational waves. By means, for instance, of the quasilocal energy approach of Brown and York, which is based on a Hamilton–Jacobi formulation of general relativity, one can compute the gravitational energy enclosed by the event horizon of a black hole [3]. We note that the old attempts to define gravitational energy by means of pseudotensors are not appropriate, nor are definitions based on space-time symmetries (see, for instance, item (1) in the introduction of Ref. [4]). Since a black hole encloses gravitational energy, one can then naturally consider that such energy plays a role on the thermodynamical behavior of black holes as internal energy (see Ref. [3]).

The notion of gravitational energy has also been shown to be well defined in the framework of the teleparallel equivalent of general relativity (TEGR). The TEGR [5–16] is not a new theory of gravity but an alternative geometric formulation of general relativity, which (in its simplest formulation) has as basic field variables only *tetrad fields*. The space-time of the theory is endowed only with torsion, rather than curvature. In this setting, it is then possible to define a distant parallelism or *teleparallelism* of vectors at different points of space-time, provided that they have identical components with respect to the local tetrads at the points considered. The equivalence of the theory with general relativity is at the level of field equations [7]. For a recent review on the TEGR, we refer the reader to Ref. [17]. In the TEGR, the notion of gravitational energy, E, has been defined from the Hamiltonian formulation of the theory [9], and later it was shown that it derives directly from the field equations of the theory [18]. Recently, the notion of *gravitational pressure*, p, over the event horizon of a black hole has also been shown to be well defined in the realm of the TEGR. The gravitational pressure naturally arises from the field equations of the TEGR and from the gravitational energy-momentum tensor defined in the theory. The spatial components of such an energy-momentum tensor yield the standard definition of the gravitational pressure in the TEGR [19] (see also Ref. [18], in which the definition was first established). In Sec. II, we review such definitions of gravitational energy and gravitational pressure.

In this work, by considering a Reissner–Nordström black hole, we further extend the investigation of the concept of gravitational pressure that arises in the context of the TEGR and that has recently been studied in the case of a Kerr black hole [19]. It is important to better understand the nature of the gravitational pressure and its effects on the thermodynamic behavior of black holes. One of our main goals is to make the comparison of the relation

*khccb@yahoo.com.br
†rocha@fis.unb.br

$dE + pdV$, obtained entirely in the TEGR, with the standard first law of black hole mechanics. The variations in the latter quantity are considered to be related to the Penrose process.[1] The variation dV is basically obtained by means of the variation of the radius of the event horizon, r_+, when the parameters M (mass) and Q (charge) of the black hole vary by infinitesimal amounts dM and dQ (as p turns out to be a density, then dV is actually given by $dV = drd\theta d\phi$). Analogously to Ref. [19], in which a Kerr black hole was considered, we remark that our analysis is essentially restricted to the event horizon of a Reissner–Nordström black hole, without considering any property of its horizon area A. It is only after we derive, entirely in the framework of the TEGR, our main result, which is the quantity $dE + pdV$, that, in order to compare it with the standard first law of black hole mechanics, namely,

$$\frac{\kappa}{8\pi} dA = dM - \Phi_H dQ, \tag{1}$$

in which $\Phi_H = Q/r_+$ is the electrostatic potential at r_+, we will consider the area A and its property according to which by no continuous process can it be decreased (i.e., $dA \geq 0$), the latter being, as is well known, simply a consequence of the fact that the irreducible mass of a black hole cannot be decreased by any continuous process, as a Penrose process [20,21]. We note that, as far as we know, the concept of gravitational pressure in the first law of black hole (thermo)dynamics (for static, spherically symmetric black holes) has been first introduced by Brown and York [3], who defined a *surface* pressure, whereas, recently, the use of the concept of gravitational pressure has been made by Dolan [22], by considering that the cosmological constant plays the role of pressure.

Another important question is how the gravitational pressure affects the efficiency of the Penrose process. A comparison of the effect of the gravitational pressure on the efficiency of the Penrose process for a Kerr black hole (obtained in Ref. [19]) with that for a Reissner–Nordström black hole (which we investigate in this paper) may be important in order to achieve a better understanding of the concept of gravitational pressure. For the Kerr case, according to Ref. [19], it is shown that the efficiency of the Penrose process in the context of the TEGR is lower than in the ordinary thermodynamic formulation in general relativity.

II. GRAVITATIONAL ENERGY MOMENTUM AND GRAVITATIONAL PRESSURE IN THE TEGR

The equivalence of the TEGR with Einstein's general relativity is obtained by means of an identity that relates the scalar curvature $R(e)$ constructed out of the tetrad field and a combination of quadratics terms of the torsion tensor [5,8,23,24],

$$eR(e) \equiv -e\left(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^aT_a\right) + 2\partial_\mu(eT^\mu), \tag{2}$$

where $e = \det(e^a{}_\mu)$, $T_a = T^b{}_{ba}$, $T_{abc} = e_b{}^\mu e_c{}^\nu T_{a\mu\nu}$, and $T_{a\mu\nu}$ is the torsion tensor, defined by $T_{a\mu\nu} = \partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}$.

In the framework of the TEGR, the Lagrangian density is given in terms of the combinations of the quadratic terms in the equation above, i.e.,

$$\begin{aligned} L &= -ke\left(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^aT_a\right) - \frac{1}{c}L_m \\ &\equiv -ke\Sigma^{abc}T_{abc} - \frac{1}{c}L_m, \end{aligned} \tag{3}$$

in which $k = c^3/16\pi G$, and Σ^{abc} is defined by

$$\Sigma^{abc} = \frac{1}{4}(T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2}(\eta^{ac}T^b - \eta^{ab}T^c), \tag{4}$$

and L_m is the Lagrangian density for matter fields.

The field equations derived from Eq. (3) for the tetrad field are equivalent to Einstein's equations, and they read

$$\begin{aligned} &e_{a\lambda}e_{b\mu}\partial_\nu(e\Sigma^{b\lambda\nu}) - e\left(\Sigma^{b\nu}{}_aT_{b\nu\mu} - \frac{1}{4}e_{a\mu}T_{bcd}\Sigma^{bcd}\right) \\ &= \frac{1}{4kc}eT_{a\mu}, \end{aligned} \tag{5}$$

in which $eT_{a\mu} = \delta L_m/\delta e^{a\mu}$. In fact, one can show that the left-hand side of the latter equation may be written exactly as $\frac{1}{2}e[R_{a\mu}(e) - \frac{1}{2}e_{a\mu}R(e)]$. Therefore, it turns out that Eq. (5) is Einstein's equation of general relativity in terms of tetrad fields. From now on, we will set $G = c = 1$, unless we say otherwise.

As shown in Ref. [18], Eq. (5) may be simplified as

$$\partial_\nu(e\Sigma^{a\lambda\nu}) = \frac{1}{4k}ee^a{}_\mu(t^{\lambda\mu} + T^{\lambda\mu}), \tag{6}$$

where $T^{\lambda\mu} = e_a{}^\lambda T^{a\mu}$ and $t^{\lambda\mu}$ is defined by

$$t^{\lambda\mu} = k(4\Sigma^{bc\lambda}T_{bc}{}^\mu - g^{\lambda\mu}\Sigma^{bcd}T_{bcd}). \tag{7}$$

In view of the property $\Sigma^{a\mu\nu} = -\Sigma^{a\nu\mu}$, it follows that

$$\partial_\lambda[ee^a{}_\mu(t^{\lambda\mu} + T^{\lambda\mu})] = 0. \tag{8}$$

[1]As one knows, the Penrose process occurs not only with rotating black holes but also with a charged static black hole. In the case with rotation, if a particle with nonzero angular momentum has negative energy inside the ergosphere of a Kerr black hole, then an extraction of energy and angular momentum from the black hole will take place. Although no ergoregion like that of the Kerr case exists for a Reissner–Nordström black hole, there is something like it since it is possible for a particle to arrive at the horizon with negative energy, provided its electric charge is opposite to that of the black hole. If such a particle falls down into the black hole, this process will lead to an extraction of mass and electric charge from the black hole [20]. The extracted energy comes at the expense of some of the mass and charge of the black hole.

This equation then yields the following *continuity (or balance) equation*:

$$\frac{d}{dt}\int_V d^3x e e^a{}_\mu (t^{0\mu} + T^{0\mu}) = -\oint_S dS_j [e e^a{}_\mu (t^{j\mu} + T^{j\mu})]. \tag{9}$$

Thus, $t^{\lambda\mu}$ can be identified as the *gravitational energy-momentum tensor* [18,25],[2]

$$P^a = \int_V d^3x e e^a{}_\mu (t^{0\mu} + T^{0\mu}) \tag{10}$$

as the total energy momentum contained within a volume V of the three-dimensional space,

$$\Phi^a_g = \oint dS_j (e e^a{}_\mu t^{j\mu}) \tag{11}$$

as the energy-momentum flux of the gravitational field, and

$$\Phi^a_m = \oint dS_j (e e^a{}_\mu T^{j\mu}) \tag{12}$$

as the energy-momentum flux of matter.

In view of Eq. (6), the Eq. (10) may be written simply as

$$P^a = -\int_V d^3x \partial_i \Pi^{ai}, \tag{13}$$

where $\Pi^{ai} = -4ke\Sigma^{a0i}$ is the momentum canonically conjugated to e_{ai}. This expression was first obtained in the context of Hamiltonian formulation of the TEGR in vacuum (see Ref. [28]). It is invariant under coordinate transformations of the three-dimensional space and under time reparametrizations. The gravitational energy enclosed by a three-dimensional volume, limited by a surface S, is defined by the $a = (0)$ component of Eq. (13), i.e.,

$$P^{(0)} = \oint_S dS_i 4ke\Sigma^{(0)0i}. \tag{14}$$

This definition has been successfully applied to several important space-times, as for determining the energy enclosed by the event horizon of a Kerr black hole [9], the energy (mass) loss described by the Bondi metric [29], and the energy of gravitational waves [14], for instance.

Let us now see how pressure naturally arises from some of the latter equations. It follows from Eqs. (6), (9), and (10) that

$$\frac{dP^a}{dt} = -4k \oint_S dS_j \partial_\nu (e\Sigma^{aj\nu}). \tag{15}$$

If one now makes the Lorentz index a to be restricted to $a = (i) = (1), (2), (3)$, then Eq. (15) can be written as

$$\frac{dP^{(i)}}{dt} = \oint_S dS_j (-\phi^{(i)j}), \tag{16}$$

in which

$$\phi^{(i)j} = 4k\partial_\nu (e\Sigma^{(i)j\nu}). \tag{17}$$

We note that Eq. (16) is precisely the Eq. (39) presented in Ref. [18]. As remarked by Maluf in Ref. [18], the left-hand side of Eq. (16) represents the momentum divided by time, what implies it has the dimension of force. And since, on the right-hand side of Eq. (16), dS_j is an element of area, one sees that $-\phi^{(i)j}$ can be understood as force per unit area, i.e., a *pressure density*; it represents the pressure along the (i) direction over an element of area oriented along the j direction. If one considers, for instance, Cartesian coordinates, then the index $j = 1, 2, 3$ represents the directions x, y, z, respectively. To compute the radial pressure over the event horizon of a black hole, in spherical-type coordinates, we set $j = r, \theta, \varphi$. In this case we need to consider only the index $j = 1$, which is associated with the radial direction. Therefore, in spherical-type coordinates, the density $\phi^{(r)1}$ is given by

$$-\phi^{(r)1} = -(\sin\theta\cos\varphi\phi^{(1)1} + \sin\theta\sin\varphi\phi^{(2)1} + \cos\theta\phi^{(3)1}), \tag{18}$$

from which we define the *radial pressure* p as

$$p(r) = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta [-\phi^{(r)1}]. \tag{19}$$

In the next two sections, we will compute both the gravitational energy enclosed by the event horizon of a Reissner–Nordström black hole and the radial pressure over its surface.

III. GRAVITATIONAL ENERGY OF A REISSNER–NORDSTRÖM BLACK HOLE

In standard spherical-type coordinates, the line element for a Reissner–Nordström black hole is given by

$$ds^2 = -\alpha^2 dt^2 + \alpha^{-2} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2), \tag{20}$$

in which

$$\alpha = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{1/2}. \tag{21}$$

The parameters M and Q are the mass and charge of the black hole, in geometrized units, respectively. The roots of $\alpha = 0$ are

$$r_\pm = M \pm \sqrt{M^2 - Q^2}, \tag{22}$$

with r_+ and r_- being the radius of the (external) event horizon and the (internal) Cauchy horizon, respectively.

Let us now choose a set of tetrad fields related to Eq. (20). Tetrad fields, which are the basic field variables

[2]We note that a pseudotensor for gravitational energy momentum in the realm of the TEGR was proposed in Ref. [26] but is different from our Eq. (7). The mentioned expression of Ref. [26] is shown therein to be equivalent to Möller's pseudotensor expression in his formulation of gravity by means of tetrad fields [27].

of the TEGR, can naturally be interpreted as *reference frames* adapted to observers in space-time [30], an interpretation that has been explored in investigations on both the energy and angular momentum of the gravitational field in the TEGR [13]. To each observer in space-time, one can adapt a tetrad field in the following way [30]. If $x^\mu(s)$ denotes the world line C of an observer in space-time, where s is the observer's proper time, the observer's four-velocity along C, defined by $u^\mu(s) = dx^\mu/ds$, is identified with the $a = (0)$ component of $e_a{}^\mu$, that is, $u^\mu(s) = e_{(0)}{}^\mu$ along C. In this way, each set of tetrad fields defines a class of reference frames in space-time [30]. In what follows, we will consider a set of tetrad fields adapted to a static observer in space-time [13]. Given a metric $g_{\mu\nu}$, the tetrad field related to it can be easily obtained through $g_{\mu\nu} = \eta^{ab} e_{b\mu} e_{a\nu}$. The realization of tetrad fields adapted to static observers is achieved by imposing on $e_{a\mu}$ the following conditions: (i) $e_{(0)}{}^i = 0$, which implies that $e_{(k)0} = 0$, and (ii) $e_{(0)i} = 0$, which implies that $e_{(k)}{}^0 = 0$. While the physical meaning of condition (i) is straightforward (the translational velocity of the observer is null, i.e., the three components of the frame velocity in the three-dimensional space are null), for condition (ii), it is not so. The latter is a condition on the rotational state of motion of the observer. It implies that the observer (more precisely, the three spatial axes of the observer's local spatial frame) is (are) not rotating with respect to a nonrotating frame (for details, we refer the reader to Ref. [13] and references therein). Therefore, conditions (i) and (ii) are six conditions one can impose on the tetrad field in order to completely fix its structure.

By applying the above-mentioned conditions (i) and (ii), one can easily construct the set of tetrad fields related to Eq. (20) and that corresponds to static observers (we note that for this class of observers the components of $T_{\mu\nu}$ that correspond to the magnetic field vanish). It is given by

$$e_{a\mu} = \begin{pmatrix} -\alpha & 0 & 0 & 0 \\ 0 & \alpha^{-1}\sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ 0 & \alpha^{-1}\sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ 0 & \alpha^{-1}\cos\theta & -r\sin\theta & 0 \end{pmatrix}. \tag{23}$$

From Eq. (14), the energy enclosed by a spherical surface of fixed radius r is given by

$$P^{(0)} = 4k \int d\theta d\varphi e \Sigma^{(0)01}. \tag{24}$$

In order to evaluate the quantity $\Sigma^{(0)01}$, we resort to Eq. (4). After a somewhat long but straightforward algebra, it yields

$$\Sigma^{(0)01} = \frac{1}{2} \alpha g^{00} g^{11} (g^{22} T_{212} + g^{33} T_{313}). \tag{25}$$

The computation of the components of the torsion tensor in the latter expression is straightforward. They read

$$\begin{aligned} T_{212} &= -r\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1/2} + r, \\ T_{313} &= -r\sin^2\theta\left[\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1/2} - 1\right]. \end{aligned} \tag{26}$$

Now, inserting the determinant $e = r^2 \sin\theta$ and Eqs. (26) into Eq. (25), we obtain

$$e\Sigma^{(0)01} = r\sin\theta\left[\frac{1}{2} - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{1/2}\right] + \frac{1}{2} r\sin\theta. \tag{27}$$

From Eq. (24), the energy enclosed by a spherical surface of constant radius r is then given by

$$E(r) \equiv P^{(0)} = r\left[1 - \sqrt{1 - \frac{2M}{r} + \frac{Q^2}{r^2}}\right]. \tag{28}$$

This is precisely the expression that is obtained by means of the quasilocal energy approach of Brown and York [31]. For $Q = 0$, Eq. (28) gives the distribution of gravitational energy in the space-time of a Schwarzschild black hole.

From Eq. (28), it follows that the energy enclosed by the event horizon of a Reissner–Nordström black hole is simply given by

$$E \equiv E(r_+) = r_+. \tag{29}$$

It is interesting to express the result (29) in terms of the *irreducible mass*, $M_{\rm irr}$, of the black hole [20,21]. When a charged or rotating black hole is subject to the Penrose process, this leads to changes in its mass and charge or its mass and angular momentum, respectively [for a general stationary (i.e., Kerr–Newman) black hole, the Penrose process will lead to the extraction of charge as well as angular momentum from the black hole]. In any case, the Penrose process is such that it cannot make the initial mass M less than $M_{\rm irr}$. For a Reissner–Nordström black hole, the irreducible mass is given by $M_{\rm irr} = (1/2)r_+$. Hence, from Eq. (29), one sees that the gravitational energy inside the event horizon of a Reissner–Nordström black hole can be simply written as

$$E = 2M_{\rm irr}. \tag{30}$$

For a Schwarzschild black hole, one simply has $M_{\rm irr} = M$, what corresponds to the fact that there is neither electric nor rotational energy to be extracted from the black hole in this case. We remark that for a Kerr black hole the gravitational energy inside its event horizon is strikingly close to the value $2M_{\rm irr}$, as computed in the framework of the TEGR [9]. We stress that this has been shown to be valid to any value of the rotation parameter. On the other hand, by applying the Brown–York quasilocal approach, Martinez [32] has shown that the gravitational energy enclosed by the horizon is given by $2M_{\rm irr}$, in the regime of *slow rotation*. He conjectured that this would hold for any value of the rotation parameter. However, by means of a generalization of the quasilocal method of Brown and York, Deghani and Mann have numerically shown that such a conjecture is not valid [33]. As far as we know, the computation of the energy enclosed by the event horizon, for any value of the rotation parameter, via the original quasilocal approach of Brown and York has not been performed. This is due to the technical difficulty in applying it for any regime of rotation [32].

The results obtained in the context of the TEGR suggest that one considers the case of the Kerr–Newman black hole in order to see if the value $2M_{\rm irr}$ still holds for the energy enclosed by the event horizon of such a black hole. In what concerns the use of the Brown–York method, it has been shown that, in the slow-rotation approximation, such a value still holds for a Kerr–Newman black hole [34]. The same result has been found by a computation done in the framework of the TEGR [35]. Anyway, for the case of a Kerr–Newman black hole, this issue deserves to be further investigated in the TEGR itself [36].

IV. RADIAL PRESSURE OVER THE EVENT HORIZON OF A REISSNER–NORDSTRÖM BLACK HOLE

In order to evaluate the radial pressure over the event horizon of a Reissner–Nordiström black hole, we need to compute the components of $\phi^{(i)1}$ [see Eq. (18)]. After a long but straightforward calculation, we obtain, considering the tetrad field given by Eq. (23), that

$$\begin{aligned}\phi^{(1)1} &= 4k\sin^2\theta\cos\varphi(\alpha\alpha' r + \alpha^2 - \alpha),\\ \phi^{(2)1} &= 4k\sin^2\theta\sin\varphi(\alpha\alpha' r + \alpha^2 - \alpha),\\ \phi^{(3)1} &= 4k\sin\theta\cos\theta\sin\varphi(\alpha\alpha' r + \alpha^2 - \alpha).\end{aligned} \tag{31}$$

By inserting now the above relations into Eq. (18) and performing the integration in Eq. (19), we obtain that the radial pressure over a spacelike spherical surface of radius r in the space-time of a Reissner–Nordström black hole is given by

$$p(r) = -(r\alpha\alpha' + \alpha^2 - \alpha), \tag{32}$$

in which the prime denotes the derivative with respect to r. By making use of Eq. (21) in Eq. (32), one obtains

$$p(r) = \frac{M}{r} + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{1/2} - 1, \tag{33}$$

from which it follows that the radial pressure over the event horizon ($r = r_+$) of a Reissner–Nordström black hole is given by

$$p \equiv p(r_+) = \frac{M}{r_+} - 1 = -\frac{(M^2 - Q^2)^{1/2}}{r_+}. \tag{34}$$

In particular, for $Q = 0$, r_+ reduces to $2M$, and one thus is left with $p = -1/2$ (or $p = -c^3/2G$, by restoring the physical constants), which is precisely the value of the radial pressure over the horizon of a Schwarzschild black hole, a result that has recently been obtained by Maluf *et al.* in Ref. [19].

It is instructive to compare the magnitude of the pressure over the event horizons of Schwarzschild and Reissner–Nordström black holes. As the radius of the event horizon of a Reissner–Nordström black hole is less than for a Schwarzschild one, i.e., $(r_+)_{\rm RN} < (r_+)_{\rm Sch}$, from Eq. (34), one sees that the pressure over the event horizon of a Reissner–Nordström black hole is, in modulus, greater than for a Schwarzschild one. This is physically reasonable since a Reissner–Nordström black hole is more compact than a Schwarzschild one.

V. THERMODYNAMICS IN THE TEGR AND THE STANDARD FIRST LAW OF BLACK HOLE MECHANICS

In the standard formulation of the thermodynamics of black holes the gravitational energy is not taken into account, and the internal energy of a black hole is considered to be given only by the black hole mass, which is parametrized in terms of its area, charge, and angular momentum. Nevertheless, from the point of view of the conservation of energy and thermodynamics, it is quite natural that gravitational energy should be taken into account if a black hole is to be considered as a thermodynamic system. That is, the internal energy of a black hole should be considered not only as its rest mass and other, nongravitational forms of energy (as electrostatic energy), but one should also take into account the gravitational energy as part of the total internal energy ascribed to a black hole. Also, taking into account the concept of gravitational pressure, it is natural to consider its role in black hole thermodynamics. As a result, in this section, we will analyze the role played by gravitational energy and gravitational pressure on the thermodynamics of a Reissner–Nordström black hole. Our aim in this section is basically to compute the quantity $dE + pdV$ and compare it with the standard first law of black hole dynamics.

Let us first compute pdV. Since $\phi^{(r)1}$ is a density, the differential pdV is evaluated as

$$pdV = \left[\int_S (-\phi^{(r)1})d\theta d\varphi\right]dr_+ = pdr_+, \tag{35}$$

in which S is the surface of constant radius $r = r_+$, with p given by Eq. (34). The differential dr_+ is obtained from Eq. (22), and it reads

$$dr_+ = \frac{r_+}{\sqrt{M^2 - Q^2}}\left(dM - \frac{Q}{r_+}dQ\right). \tag{36}$$

It must be noted that, as one is assuming that dr_+, dM, and dQ are infinitesimals, the present analysis is not valid when

$\sqrt{M^2 - Q^2}$ approaches zero, i.e., when Q is very close to M.

From Eqs. (34) and (36), one is then left with

$$pdV = -\left(dM - \frac{Q}{r_+}dQ\right). \tag{37}$$

The differential dE is easily obtained from Eq. (29) as

$$dE = dr_+. \tag{38}$$

Of course, this result also derives from the variation

$$dE = \frac{\partial E}{\partial M}dM + \frac{\partial E}{\partial Q}dQ, \tag{39}$$

as it should be.

From Eqs. (37) and (38), we have

$$dE + pdV = \frac{M}{r_+}dr_+. \tag{40}$$

By replacing now Eq. (36) in the equation above, we are left with

$$dE + pdV = \frac{M}{\sqrt{M^2 - Q^2}}\left(dM - \frac{Q}{r_+}dQ\right). \tag{41}$$

We will now come back to the expression given by Eq. (40). Its right-hand side can be written in terms of the surface gravity of the black hole, i.e., in terms of

$$\kappa = \frac{\sqrt{M^2 - Q^2}}{r_+^2}. \tag{42}$$

As the horizon area is $A = 4\pi r_+^2$, it follows that Eq. (40) can be rewritten as

$$dE + pdV = \frac{1}{8\pi}\frac{M}{r_+^2}dA, \tag{43}$$

which, by virtue of Eq. (42), can be written as

$$dE + pdV = \left(\frac{M}{\sqrt{M^2 - Q^2}}\right)\frac{\kappa}{8\pi}dA. \tag{44}$$

In particular, one sees that, for a Schwarzschild black hole ($Q = 0$), the latter result reduces to

$$dE + pdV = \frac{\kappa}{8\pi}dA. \tag{45}$$

Hence, for a Schwarzschild black hole, the expression $dE + pdV$, obtained entirely in the context of the teleparallelism, coincides with the standard expression for the first law of black hole dynamics.

Let us now compare the result given by Eq. (44) with the standard one, in what concerns a Reissner–Nordström black hole. We first recall that, for the latter, the first law of black hole dynamics is given by

$$\frac{\kappa}{8\pi}dA = dM - \Phi_H dQ, \tag{46}$$

where $\Phi_H = Q/r_+$ is the Coulombian potential at the black hole event horizon (the zero of the electric potential is taken at infinity). Considering now Eq. (44), and since $M \geq |Q|$, it follows that the factor that multiplies the term $(\kappa/8\pi)dA$ is greater than one. This implies that, for a Reissner–Nordström black hole, the following inequality holds:

$$dE + pdV > \frac{\kappa}{8\pi}dA. \tag{47}$$

If one now defines

$$TdS = dE + pdV \tag{48}$$

as the first law of black hole dynamics, established entirely in the framework of the TEGR, it follows that Eq. (44) can be rewritten as

$$TdS = \frac{M}{\sqrt{M^2 - Q^2}}\left(\frac{\kappa}{8\pi}dA\right). \tag{49}$$

Although the area A appears on the right-hand side of Eq. (49), we stress it does not play any role in arriving at an expression for $dE + pdV(= TdS)$, but rather the latter is given by Eq. (41), which has been obtained without any need to resort to A. It is only after one arrives at an expression for $dE + pdV(= TdS)$ that it has been expressed, for convenience, in terms of A. We also remark that, up to now, we have not assumed that S and A are related by the standard, Bekenstein–Hawking relation.

Assuming now that T in Eq. (49), which is the temperature of the black hole, is the Hawking temperature $\kappa/2\pi$, it follows from Eq. (49) that

$$dS = \frac{M}{\sqrt{M^2 - Q^2}}\frac{dA}{4}. \tag{50}$$

In this way, one is led to the result that, in the TEGR (due to both the gravitational energy and gravitational pressure so defined), the variation of the entropy of a Reissner–Nordström black hole is greater than the variation of the standard, Bekenstein–Hawking entropy, $S_{BH} = A/4$ (in natural units). For a Schwarzschild black hole ($Q = 0$), the entropy (50) so derived in the TEGR coincides with the standard one, even though the gravitational pressure is not null in this case. On the other hand, we recall that the Bekenstein–Hawking postulate, according to which the entropy of a black hole is given by the entropy-area relation $S_{BH} = A/4$, follows from the *classical* laws of black hole dynamics together with the (quantum) Hawking temperature $\kappa/2\pi$. Hence, the result given by Eq. (50) can be viewed as a possible modification of the entropy of a Reissner–Nordström black hole, as a result of considering both gravitational energy and gravitational pressure in formulating the classical laws of black hole dynamics. In this direction, as pointed out by York [37], we recall that the constant of proportionality in the relation $S_{BH} = A/4$ was originally obtained from a mechanical-thermodynamical analogy based on the relation

$$dM = \frac{\kappa}{8\pi} dA, \tag{51}$$

which is derived from $M = (\kappa/8\pi)A$, which, in its turn, is valid for neutral nonrotating black holes. It is only upon the identification by Hawking that (the black hole temperature is) $T = \kappa/2\pi$; it would follow from the hypothesis that Eq. (51) can be written in thermodynamic form, with $dS_{\mathrm{BH}} = (1/4)dA$, if and only if one assumes that the thermodynamic law for uncharged nonrotating black holes is given by

$$dM = TdS_{\mathrm{BH}}, \tag{52}$$

from which one sees that there is no term corresponding to "pdV" in the standard formulation of black hole thermodynamics. The point is that, given $T = \kappa/2\pi$, the expressions for M and dM do not by themselves imply uniquely a value for the entropy. As York remarked [37], Eq. (52) gives the simplest possibility that leads to an entropy-area relation.

VI. CONCLUDING REMARKS

The plausibility of a pdV "work" term in the first law of black hole thermodynamics is perhaps best summarized in the following remark by York [37]: "it is quite plausible that if 'heat' TdS is slowly added to a black hole in equilibrium, thereby causing it to expand, that it should do 'work' in lifting itself in its own gravitational 'potential well' ". Besides, as the electromagnetic field exerts pressure, one might expect that the gravitational field would behave in the same way. In fact, this has been shown to be to the case for gravitational waves [14].

We have obtained the thermodynamic relation $TdS = dE + pdV$ (which is the first law of black hole thermodynamics) entirely within the framework of the TEGR, without identifying dS with the variation dA of the area of the event horizon of the black hole [see Eq. (41)]. However, in order to compare $TdS = dE + pdV$, as given by the TEGR result (41), with the (standard) TdS_{BH}, as given by the standard first law of black hole dynamics (1), we have written the right-hand side of Eq. (40), which is a more compacted, preliminary form of Eq. (41), in terms of $(\kappa/8\pi)dA$. The result is given by Eq. (44), which implies that, in the framework of the TEGR, $TdS \geq (\kappa/8\pi)dA = TdS_{\mathrm{BH}}$, where the inequality becomes an equality only for the particular case of a Schwarzschild black hole, whereas the inequality holds for a Reissner–Nordström black hole. This result imply that (i) for a Schwarzschild black hole, the expression $dE + pdV$, obtained entirely in the context of the teleparallelism, coincides with the standard expression for the first law of black hole dynamics, while (ii) for a Reissner–Nordström black hole, it leads to the fact that the efficiency of the Penrose process is less than in standard black hole thermodynamics. We note that the same conclusion has been achieved in the case of a Kerr black hole [19].

The fact that the entropy given by Eq. (50) (which has been derived entirely in the framework of the TEGR) is different from the standard, Bekenstein–Hawking entropy, $S_{\mathrm{BH}} = A/4$, is not a surprise, of course. It should be noted that, in the standard first law of black hole thermodynamics, the role of internal energy is ascribed to the black hole mass, M, while, in the TEGR, it is played by the total energy enclosed by the event horizon of the black hole, E, which includes M and other possible forms of energy. Also, the role of gravitational pressure, which is crucial in establishing in the TEGR the first law as $TdS = dE + pdV$, is a concept that is absent in the standard formulation of classical black hole dynamics. An exception is the consideration of a gravitational *surface* pressure ascribed to the horizon of a black hole, as defined by Brown and York in the context of their quasilocal analysis [3]. In this way, they arrive at the first law of black hole thermodynamics for a spherically symmetric black hole but with the black hole temperature blueshifted from infinity to a fixed distance R. Nevertheless, when the surface (of radius R) is taken as the horizon ($R = r_+$), the surface pressure diverges as well as the temperature (see Eqs. (6.19) and (6.20) of Ref. [3]). In the context of the TEGR, the gravitational pressure that enters into the first law of black hole dynamics is not a surface pressure but a radial pressure over the event horizon, where it has a finite value [see Eq. (34)]. Such a pressure is negative, which means it is directed toward the center of the black hole. Physically, one can view this as similar as in the first law of ordinary thermodynamics since, corresponding to the fact that the Penrose process leads to the extraction of energy from the black hole, the black hole pressure is over the horizon, (radially) directed to its center. In the case of a Kerr black hole, the radial pressure over the event horizon is also negative, as shown in Ref. [19], taking into account also the Penrose process.

Since by no continuous process (such as the Penrose process) can the irreducible mass of a black hole be decreased (i.e., the inequality $dM_{\mathrm{irr}}^2 \geq 0$ holds), and as $A = 16\pi M_{\mathrm{irr}}^2$, it follows that $dA \geq 0$; that is, by no continuous process can the horizon area of a black hole be decreased [38]. Therefore, in view of Eqs. (45) and (47), one is led to $TdS = dE + pdV \geq 0$.[3] This can be taken as the second law of black thermodynamics in the framework of the TEGR; although, here, it has been based on continuous processes involving only a single black hole.

Finally, we note that the entropy-area relation given by Eq. (50) can be considered "holographic," analogously to the standard relation. We tried to verify if a relation similar to Eq. (50) holds for a Kerr black hole by making use of the

[3] We recall that the equality in $TdS = dE + pdV \geq 0$ holds for a Schwarzschild black hole and that, for the latter, it is not possible to extract the energy (mass) by the Penrose process since $M_{\mathrm{irr}} = M$, where M is the mass of the black hole.

expression for TdS obtained in Ref. [19]. However, in that case, the expression for the quantity $TdS = dE + pdV$ is not simple but rather has a complicated form such that we could not make a conclusive statement. We hope to report about it in the future.

ACKNOWLEDGMENTS

The authors are grateful to J. W. Maluf for critical reading and providing comments on a earlier version of this manuscript, which has led to improvements of the presentation.

---

[1] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[2] E. Poisson, *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics* (Cambridge University Press, Cambridge, England, 2004), p. 215.
[3] J. D. Brown and J. W. York, Phys. Rev. D **47**, 1407 (1993).
[4] P.L. McGrath, R.J. Epp, and R.B. Mann, Classical Quantum Gravity **29**, 215012 (2012).
[5] F. W. Hehl, in *Proceedings of the 6th School of Cosmology and Gravitation on Spin, Torsion, Rotation and Supergravity, Erice, 1979*, edited by P.G. Bergmann and V. de Sabbata (Plenum, New York, 1980).
[6] J. M. Nester, Int. J. Mod. Phys. A **04**, 1755 (1989).
[7] J. W. Maluf, J. Math. Phys. (N.Y.) **35**, 335 (1994).
[8] F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne'eman, Phys. Rep. **258**, 1 (1995).
[9] J.W. Maluf, J.F. da Rocha-Neto, T.M.L. Toríbio, and K.H. Castello-Branco, Phys. Rev. D **65**, 124001 (2002).
[10] Y. D. Obukhov and J. G. Pereira, Phys. Rev. D **67**, 044016 (2003).
[11] Y. D. Obukhov and G. F. Rubilar, Phys. Rev. D **73**, 124017 (2006).
[12] C. Chen, J. Liu, and J. M. Nester, Mod. Phys. Lett. A **22**, 2039 (2007).
[13] J.W. Maluf, F.F. Faria, and S.C. Ulhoa, Classical Quantum Gravity **24**, 2743 (2007).
[14] J.W. Maluf and S.C. Ulhoa, Phys. Rev. D **78**, 047502 (2008); **78**, 069901 (2008).
[15] K.H.C. Castello-Branco and J.F. da Rocha-Neto, Gen. Relativ. Gravit. **44**, 3017 (2012).
[16] R. Aldrovandi and J. G. Pereira, *Teleparallel Gravity: An Introduction* (Springer, New York, 2013).
[17] J. W. Maluf, Ann. Phys. (Berlin) **525**, 339 (2013).
[18] J. W. Maluf, Ann. Phys. (Berlin) **14**, 723 (2005).
[19] J. W. Maluf, S. C. Ulhoa, and J. F. da Rocha-Neto, Phys. Rev. D **85**, 044050 (2012).
[20] D. Christodoulou and R. Ruffini, Phys. Rev. D **4**, 3552 (1971).
[21] D. Christodoulou, Phys. Rev. Lett. **25**, 1596 (1970).
[22] B.P. Dolan, Classical Quantum Gravity **28**, 235017 (2011); Phys. Rev. D **84**, 127503 (2011).
[23] M. Blagojevic, *Gravitation and Gauge Symmetries* (IOP, Bristol, 2002).
[24] T. Ortin, *Gravity and Strings* (Cambridge University Press, Cambridge, England, 2004).
[25] J.W. Maluf, F.F. Faria, and K.H. Castello-Branco, Classical Quantum Gravity **20**, 4683 (2003).
[26] V. C. de Andrade, L. C. T. Guillen, and J. G. Pereira, Phys. Rev. Lett. **84**, 4533 (2000).
[27] C. Möller, Mat. Fys. Skr. Dan. Vid. Selsk. **1**, 10 (1961).
[28] J.W. Maluf and J.F. da Rocha-Neto, Phys. Rev. D **64**, 084014 (2001).
[29] J.W. Maluf and F.F. Faria, Ann. Phys. (Berlin) **13**, 604 (2004).
[30] F. W. Hehl, J. Lemke, and E. W. Mielke, *Two Lectures on Fermions and Gravity, in Geometry and Theoretical Physics*, edited by J. Debrus and A.C. Hirshfield (Springer, Berlin, 1991).
[31] A. P. Lundgren, B. S. Schmekel, and J. W. York, Phys. Rev. D **75**, 084026 (2007).
[32] E. A. Martinez, Phys. Rev. D **50**, 4920 (1994).
[33] M. H. Dehghani and R. B. Mann, Phys. Rev. D **64**, 044003 (2001).
[34] S. Bose and T.Z. Naing, Phys. Rev. D **60**, 104027 (1999).
[35] S. Xu and J. Jing, Classical Quantum Gravity **23**, 4659 (2006).
[36] This issue is under investigation by the present authors.
[37] J. W. York, Phys. Rev. D **28**, 2929 (1983).
[38] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Clarendon, Oxford, 1983).