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Propagators in de Sitter space

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In a spacetime with no global timelike Killing vector, we do not have a natural choice for the vacuum state of matter fields, which leads to an ambiguity in defining the Feynman propagators. In this paper, taking the vacuum state to be the instantaneous ground state of the Hamiltonian at each moment, we develop a method for calculating wave functions associated with the vacuum and the corresponding in-in and in-out propagators. We apply this method to free scalar field theory in de Sitter space and obtain de Sitter invariant propagators in various coordinate patches. We show that the in-out propagator in the Poincaré patch has a finite massless limit in a de Sitter invariant form. We argue and numerically check that our in-out propagators agree with those obtained by a path integral with the standard iε prescription, and we identify the condition on a foliation of spacetime under which such coincidence can happen for the foliation. We also show that the in-out propagators satisfy Polyakov's composition law. Several applications of our framework are also discussed.

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I. INTRODUCTION

In a spacetime with no global timelike Killing vector, we do not have an established prescription to define the vacuum state of matter fields. The issue exists even at the level of free fields, leading to an ambiguity in defining propagators [1] (see also [2] for recent discussions).

A typical example of such spacetimes is de Sitter space, and various vacua have been studied throughout the decades. Among them, the Euclidean vacuum (or the Bunch-Davies vacuum) [3] is often used in cosmology to describe the physics in the inflationary era. This is mainly because it is invariant under the de Sitter group and further satisfies the Hadamard condition, which essentially states that a two-point function comes to behave in the same way as in flat Minkowski space as two points get closer to each other (see, e.g., [4–9] for arguments that physically natural states should satisfy the Hadamard condition). Also often studied are a series of vacua called the α vacua (or Mottola-Allen vacua) [10,11], which are parametrized by a complex number α. They are all de Sitter invariant but do not satisfy the Hadamard condition except for $\alpha = -\infty$, which corresponds to the Euclidean vacuum.[1]

An interesting feature of de Sitter space is its thermodynamic property. As is pointed out in [17], a particle detector staying in de Sitter space and interacting weakly with a scalar field in the Euclidean vacuum behaves as if it is in a thermal bath with the temperature $T = 1/2\pi\ell$, where ℓ is the de Sitter radius. This phenomenon crucially depends on the setup where the Euclidean vacuum is taken. In fact, other α vacua do not yield such thermal behavior [18]. In this sense, the choice of vacuum is also important in understanding the thermodynamic character of curved spacetimes.

In this paper, we take the vacuum of a free scalar field to be the instantaneous ground state of the Hamiltonian at each moment. We develop a general method to calculate transition amplitudes during finite time intervals for a quantum mechanical system with time-dependent Hamiltonian and define the propagators as the limit of two-point functions, when the initial and final times are sent to the past and future infinities.[2] In our method, wave functions associated with the vacuum are automatically determined with no need to consider asymptotic boundary conditions such as positive-energy conditions.

We apply the method to construct various Feynman propagators in de Sitter space. We treat the principal series (with large mass, $m > (d-1)/2$) and the complementary

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[1] It is pointed out on the basis of the in-in formalism (or the Schwinger-Keldysh formalism) [12,13] that two-point functions for α vacua have various pathological behaviors (e.g., the breaking of their analyticities) when a quantum field has an interaction [14–16]. Note that discussions in favor of the Hadamard condition are about the in-in propagators and are not applied to the in-out propagators. In this paper, we will not further touch on this fundamental issue of the Hadamard condition on two-point functions.

[2] There was a study that took the vacuum to be the instantaneous ground state, which is sometimes called the instantaneous Hamiltonian diagonalization method (see, e.g., [1,19,20] and references therein). Our framework has the same principle as that of the method in determining the vacuum, but has an advantage over the method in that it enables us to obtain an explicit form of various propagators for finite time intervals and can be applied to a wide class of nonstatic spacetimes.

series (with small mass, $m < (d-1)/2$) at the same time, and show that the obtained in-in and in-out propagators always take de Sitter invariant forms. Furthermore, we show that our de Sitter invariant in-out propagator has a finite massless limit in the Poincaré patch.[3] This is in contrast to the in-in propagators, for which the no-go theorem states that there is no de Sitter invariant Fock vacuum for massless scalar fields [11].

We argue and numerically check that our in-out propagators agree with the propagators obtained by a path integral with the standard iε prescription. This result is consistent with the de Sitter invariance of the propagators since the corresponding path integral is performed over a patch which is preserved under the infinitesimal action of de Sitter group SO(1, d). Moreover, our in-out propagators are shown to satisfy the composition law [21], which has been claimed by Polyakov recently as a principle to be satisfied in order for the propagator to be interpreted as representing a sum over paths of a particle moving in a spacetime.

This paper is organized as follows. In Sec. II, we develop a general framework for a given foliation of spacetime to calculate wave functions and propagators for quantum mechanics with a time-dependent Hamiltonian. In Sec. III we demonstrate how our prescription works in the simplest spacetime, Minkowski space. A mathematical detail is given in Appendix A. Another well-studied example of asymptotically Minkowski space is investigated in Appendix B. We analyze the de Sitter case in Sec. IV. After giving a brief review on the geometry of de Sitter space in Sec. IV A, we discuss the propagators in the Poincaré patch in Sec. IV B and those in the global patch in Sec. IV C. For both patches, we first make a mode expansion of a scalar field and calculate the propagators for each mode. We then make a sum over modes to obtain the propagators in spacetime, both of the in-in and in-out types. The obtained propagators are found to be written with the de Sitter invariant quantity. In Sec. V we introduce the concept of effective noncompactness in the time direction and show that when the foliation meets the condition of effective noncompactness, our in-out propagator coincides with that obtained by a path integral with the standard iε prescription. We further show that the Poincaré and the global patches satisfy the condition and confirm the coincidence of the two propagators by numerical calculations. In Sec. VI we prove that our in-out propagators have the heat-kernel representation, which means that the propagators satisfy Polyakov's composition law [22]. Section VII is devoted to discussions and conclusion. We give some of the mathematical details in Appendixes C, D, E, and F with useful formulas. In Appendix G we show that each of the in- and out-vacua for the two patches can be identified with a α vacuum. In Appendix H we consider two possible ways to introduce the iε prescription and confirm that the two prescriptions give the same analytic expressions after taking the limit $\varepsilon \to 0$.

[3]The in-out propagator in the global patch still diverges in the massless limit.

II. GENERAL FRAMEWORK

A. Setup

In this paper, we consider quantum theory of a free real scalar field $\phi(x)$ living in a d-dimensional curved spacetime with background metric $g_{\mu\nu}$,[4]

$$S[\phi(x)] = -\frac{1}{2}\int \mathrm{d}^d x\sqrt{-g}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2). \tag{1}$$

We assume that the spacetime is globally hyperbolic and the foliation of spacetime (i.e., the set of time slices) is already specified. We denote the temporal and spatial coordinates by t and $\boldsymbol{x}$, respectively, and the spacetime coordinates by $x = (x^\mu) = (t, \boldsymbol{x})$ $(\mu = 0, 1, \ldots, d-1)$. We further assume that the metric has the form

$$\mathrm{d}s^2 = -N^2(t)\mathrm{d}t^2 + A^2(t)h_{ij}(\boldsymbol{x})\mathrm{d}x^i\mathrm{d}x^j \quad (i, j = 1, \ldots, d-1). \tag{2}$$

The action is then written as

$$S = \frac{1}{2}\int \mathrm{d}t\int \mathrm{d}^{d-1}x\sqrt{h}NA^{d-1}(N^{-2}\partial_t\phi\partial_t\phi + A^{-2}\phi\Delta_{d-1}\phi - m^2\phi^2), \tag{3}$$

where $\sqrt{h} \equiv \sqrt{\det h_{ij}}$, and $\Delta_{d-1} \equiv (1/\sqrt{h})\partial_i(\sqrt{h}h^{ij}\partial_j)$ is the Laplacian for the spatial metric $\mathrm{d}s^2_{d-1} = h_{ij}(\boldsymbol{x})\mathrm{d}x^i\mathrm{d}x^j$. We have neglected the surface term coming from integration by parts. We introduce a complete system $\{Y_n(\boldsymbol{x})\}$ of real-valued orthonormal eigenfunctions of Δ_{d-1}, satisfying

$$\Delta_{d-1}Y_n(\boldsymbol{x}) = -\lambda_n Y_n(\boldsymbol{x}), \quad \int \mathrm{d}^{d-1}\boldsymbol{x}\sqrt{h(\boldsymbol{x})}Y_n(\boldsymbol{x})Y_{n'}(\boldsymbol{x}) = \delta_{nn'}, \tag{4}$$

and make a mode expansion of the scalar field as

$$\phi(x) = \phi(t, \boldsymbol{x}) = \sum_n \phi_n(t)Y_n(\boldsymbol{x}). \tag{5}$$

Note that $\phi_n(t) \in \mathbb{R}$ since $Y_n(\boldsymbol{x})$ are real valued. The action can then be written as a sum of the actions for mode functions $\{\phi_n(t)\}$,

$$S_\varepsilon = \sum_n \int \mathrm{d}t L_{n,\varepsilon}(\phi_n(t), \dot{\phi}_n(t), t), \tag{6}$$

with

$$L_{n,\varepsilon}(\phi_n, \dot{\phi}_n, t) = \frac{\rho(t)}{2}\dot{\phi}_n^2 - \frac{\rho(t)\omega_n^2(t)}{2}\phi_n^2, \tag{7}$$

[4]The metric has the signature $(-, +, \ldots, +)$.

$$\rho(t) \equiv e^{+\mathrm{i}\varepsilon} N^{-1}(t) A^{d-1}(t), \tag{8}$$

$$\omega_n(t) \equiv e^{-\mathrm{i}\varepsilon} N(t) \sqrt{\lambda_n A^{-2}(t) + m^2}. \tag{9}$$

This shows that the nth mode function $\phi_n(t)$ behaves as a quantum oscillator with time-dependent mass $\rho(t)$ and frequency $\omega_n(t)$. Here, we have introduced an infinitesimal imaginary part $\mathrm{i}\varepsilon$ ($\varepsilon > 0$) in order to discuss the behavior of states near the temporal boundary in a well-defined manner. Note that the combination $\rho(t)\omega_n(t)$ is always real. The quantum oscillator with time-dependent mass $\rho(t)$ and frequency $\omega_n(t)$ is described by the following time-dependent Hamiltonian in the Schrödinger picture:

$$H_{n,s}(t) = H_n(\phi_{n,s}, \pi_{n,s}, t) = \frac{1}{2\rho(t)} \pi_{n,s}^2 + \frac{\rho(t)\omega_n^2(t)}{2} \phi_{n,s}^2, \tag{10}$$

where the suffix s indicates that the operators are in the Schrödinger picture. Thus, the theory is reduced to quantum mechanics of a set of independent harmonic oscillators with time-dependent parameters,

$$H_s(t) = \sum_n H_{n,s}(t) = \sum_n \Big[\frac{1}{2\rho(t)} \pi_{n,s}^2 + \frac{\rho(t)\omega_n^2(t)}{2} \phi_{n,s}^2 \Big]. \tag{11}$$

Note that the introduction of $\mathrm{i}\varepsilon$ in (8) and (9) corresponds to the replacement $H_{n,s}(t) = e^{-\mathrm{i}\varepsilon}[H_{n,s}(t)|_{\varepsilon=0}]$, which makes the Hamiltonian a non-Hermitian operator. The quantization is accomplished by setting the commutation relations,

$$[\phi_{n,s}, \pi_{m,s}] = \mathrm{i}\delta_{n,m}, \tag{12}$$

$$[\phi_{n,s}, \phi_{m,s}] = 0 = [\pi_{n,s}, \pi_{m,s}]. \tag{13}$$

In the following subsections, we develop a general theory to describe the time evolution of states for quantum mechanics with such a time-dependent Hamiltonian.

We here make a comment on a subtlety existing in field redefinitions (for brevity we set $\varepsilon = 0$ below). By transforming the mode function $\phi_n(t)$ to $\chi_n(t) = \rho^{1/2}(t)\phi_n(t) \equiv e^{\sigma(t)}\phi_n(t)$, one can make the coefficient of the kinetic term to unity,

$$S[\chi_n(t)] = \int_{t_i}^{t_f} \mathrm{d}t \frac{1}{2} [\dot{\chi}_n^2(t) - \Omega_n^2(t)\chi_n^2(t)], \tag{14}$$

$$\Omega_n^2(t) \equiv \omega_n^2(t) - (\dot{\sigma}(t))^2 - \ddot{\sigma}(t). \tag{15}$$

However, it can often happen that $\Omega_n^2(t)$ takes negative values for some region of m^2 even though the original $\omega^2(t)$ is strictly positive.[5] Although the physics should be the same for the two descriptions using $\phi_n(t)$ and $\chi_n(t)$ (as long as $\mathrm{i}\varepsilon$ is introduced in a consistent way), the inverted harmonic potential for $\chi_n(t)$ can easily cause a catastrophe when making an analysis based on an approximation, such as the WKB approximation. In order to avoid this subtlety (and also to keep the original symmetry manifest), we will not make such transformations.[6]

B. Quantum mechanics with time-dependent Hamiltonian

To simplify expressions in the following discussions, for a while we omit the mode index n and denote the canonical variables $\{\phi_{n,s}, \pi_{n,s}\}$ in the Schrödinger picture by $\{q_s, p_s\}$. Our Hamiltonian then takes the form

$$H_s(t) = H(q_s, p_s, t) = \frac{1}{2\rho(t)} p_s^2 + \frac{\rho(t)\omega^2(t)}{2} q_s^2, \tag{16}$$

and the system is quantized by setting the commutation relation

$$[q_s, p_s] = \mathrm{i}. \tag{17}$$

Recall that $\rho(t) = e^{\mathrm{i}\varepsilon}|\rho(t)|$ and $\omega(t) = e^{-\mathrm{i}\varepsilon}|\omega(t)|$.

We denote by T_s the time at which quantization is carried out in the Schrödinger picture. The Hilbert state $\mathcal{H} = \{|\psi\rangle\}$ with a Hermitian inner product (ψ_1, ψ_2) is then constructed on the time slice at $t = T_s$, and the dual space $\mathcal{H}^* = \{\langle\psi|\}$ is defined with respect to the Hermitian inner product with the rule $\langle\psi_1| \equiv (|\psi_1\rangle)^\dagger$, i.e., $\langle\psi_1|(|\psi_2\rangle) = (\psi_1, \psi_2)$. The time evolution of a state $|\psi\rangle \in \mathcal{H}$ is governed by the Schrödinger equation

$$\partial_t |\psi, t\rangle = -\mathrm{i} H_s(t) |\psi, t\rangle, \tag{18}$$

with the initial condition $|\psi, T_s\rangle = |\psi\rangle$. The Schrödinger equation can be integrated to the form

$$|\psi, t\rangle = U(t, T_s)|\psi\rangle, \tag{19}$$

where $U(t, T_s)$ is the time-evolution operator expressed as the time-ordered exponential of $H_s(t)$,

$$\begin{aligned} U(t, T_s) &\equiv \mathrm{T}\exp\Big(-\mathrm{i}\int_{T_s}^{t} \mathrm{d}t' H_s(t')\Big) \\ &\equiv \lim_{\Delta t_k \to 0} (1 - \mathrm{i}\Delta t_N H_s(t_N)) \\ &\quad \times (1 - \mathrm{i}\Delta t_{N-1} H_s(t_{N-1})) \ldots (1 - \mathrm{i}\Delta t_1 H_s(t_1)) \\ &\quad \begin{pmatrix} t = t_N > t_{N-1} > \cdots > t_1 > t_0 = T_s \\ \Delta t_k = t_k - t_{k-1} \end{pmatrix}. \end{aligned} \tag{20}$$

The Hermitian conjugate of $|\psi, t\rangle$ is given by

[5] A typical example is a scalar field in the Poincaré patch of de Sitter space. One can easily see that $\Omega_n^2(t)$ can be negative when the mass is small and $\phi_n(t)$ represents a mode of long wavelength.

[6] The coefficient of the kinetic term can also be set to unity by making a transformation of the time coordinate. We will see that physical quantities do not change under the transformation (see the last paragraph of Sec. II E).

$$\langle\psi, t| = \langle\psi|U^\dagger(t, T_s). \tag{21}$$

In addition, we introduce a one-parameter family of states for a given state $\overline{\langle\psi|} \in \mathcal{H}^*$ as

$$\overline{\langle\psi, t|} \equiv \overline{\langle\psi|}U^{-1}(t, T_s), \tag{22}$$

which satisfy

$$\partial_t\overline{\langle\psi, t|} = +\mathrm{i}\overline{\langle\psi, t|}H_s(t), \qquad \overline{\langle\psi, T_s|} = \overline{\langle\psi|}. \tag{23}$$

Note that the pairing of $\overline{\langle\psi_1|}$ and $|\psi_2\rangle$ does not change under the time evolution,

$$\overline{\langle\psi_1, t|}\psi_2, t\rangle = \overline{\langle\psi_1|}\psi_2\rangle, \tag{24}$$

although this is not the case for $\langle\psi_1, t|\psi_2, t\rangle$ when $\varepsilon \neq 0$ because the time evolution operator is then not unitary, $U^{-1}(t, T_s) \neq U^\dagger(t, T_s)$.

The spectrum of the Hamiltonian $H_s(t)$ can be easily found by introducing, as usual, a pair of operators,[7]

$$a_s(t) \equiv \sqrt{\frac{\rho(t)\omega(t)}{2}}q_s + \mathrm{i}\sqrt{\frac{1}{2\rho(t)\omega(t)}}p_s, \tag{25}$$

$$a_s^\dagger(t) \equiv \sqrt{\frac{\rho(t)\omega(t)}{2}}q_s - \mathrm{i}\sqrt{\frac{1}{2\rho(t)\omega(t)}}p_s. \tag{26}$$

We call $a_s(t)$ and $a_s^\dagger(t)$ the annihilation and creation operators at time t. Note that $a_s(t)$ and $a_s^\dagger(t)$ are Hermitian conjugate to each other, because $\rho(t)\omega(t)$ is positive and $q_s^\dagger = q_s$ and $p_s^\dagger = p_s$. From the commutation relation (17), we have

$$[a_s(t), a_s^\dagger(t)] = 1. \tag{27}$$

The Hamiltonian (16) can then be rewritten as

$$H_s(t) = \frac{\omega(t)}{2}[a_s^\dagger(t)a_s(t) + a_s(t)a_s^\dagger(t)] = \omega(t)\left[a_s^\dagger(t)a_s(t) + \frac{1}{2}\right]. \tag{28}$$

We define the state $|0_t, t\rangle$ as that which vanishes when acted on by $a_s(t)$,

$$a_s(t)|0_t, t\rangle = 0. \tag{29}$$

Accordingly, the state $\overline{\langle 0_t, t|} \equiv \langle 0_t, t| = |0_t, t\rangle^\dagger$ satisfies

$$\overline{\langle 0_t, t|}a_s^\dagger(t) = 0. \tag{30}$$

Then the right and left eigenstates of $H_s(t)$ are given by

$$|n, t\rangle_t \equiv \frac{1}{n!}[a_s^\dagger(t)]^n|0_t, t\rangle, \tag{31}$$

[7]They are Schrödinger operators, and the time dependence comes only through the parameters $\rho(t)$ and $\omega(t)$.

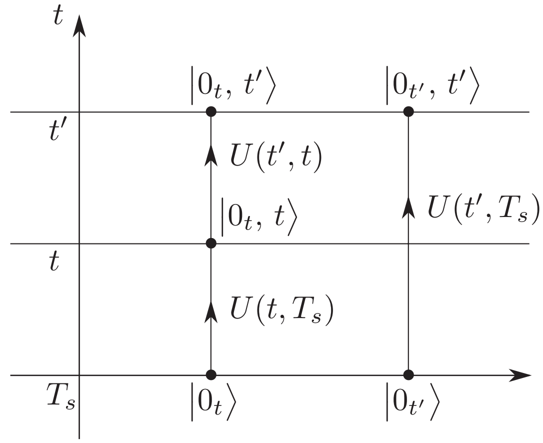

FIG. 1. Time evolution of states. The system is quantized in the Schrödinger picture on the time slice at T_s. $|0_t, t\rangle$ is the state annihilated by the Schrödinger operator $a_s(t)$, while $|0_t\rangle$ is the state annihilated by the Heisenberg operator $a(t)$.

$$\overline{{}_t\langle n, t|} \equiv \frac{1}{n!}\overline{\langle 0_t, t|}[a_s(t)]^n = |n, t\rangle_t^\dagger, \tag{32}$$

which satisfy

$$H_s(t)|n, t\rangle_t = \left(n + \frac{1}{2}\right)\omega(t)|n, t\rangle_t, \tag{33}$$

$$\overline{{}_t\langle n, t|}H_s(t) = \left(n + \frac{1}{2}\right)\omega(t)\overline{{}_t\langle n, t|}. \tag{34}$$

We call $|0_t, t\rangle = |0, t\rangle_t$ the ground state (or the vacuum) at time t, since this is the minimum energy state at the moment if $\varepsilon = 0$.

It is important to note that the ground state at time t', $|0_{t'}, t'\rangle$, is generically different from the state $|0_t, t'\rangle$; the latter is obtained as a time evolution of the ground state $|0_t, t\rangle$ at time t, $|0_t, t'\rangle = U(t', t)|0_t, t\rangle$ (see Fig. 1). Note also that since the Hamiltonian is already specified at each time, the vacuum state is uniquely determined, and there is no freedom to introduce other vacuum states through Bogoliubov transformations.

C. Heisenberg picture

Now we move from the Schrödinger picture to the Heisenberg picture. Given a Schrödinger operator $O_s(t)$ [possibly depending on t through the parameters involved when constructing the operator as in Eq. (25)], we define the corresponding Heisenberg operator as

$$O(t) \equiv U^{-1}(t, T_s)O_s(t)U(t, T_s), \tag{35}$$

which satisfies the Heisenberg equation,

$$\dot{O}(t) = \mathrm{i}[H(t), O(t)] + \frac{\partial O(t)}{\partial t} \quad \left(\frac{\partial O(t)}{\partial t} \equiv U^{-1}(t, T_s)\frac{\partial O_s(t)}{\partial t}U(t, T_s)\right). \tag{36}$$

For our harmonic oscillator, the time evolution of the canonical variables is given by

$$\dot{q}(t) = \mathrm{i}[H(t), q(t)] = \frac{p(t)}{\rho(t)}, \tag{37}$$

$$\dot{p}(t) = \mathrm{i}[H(t), p(t)] = -\rho(t)\omega^2(t)q(t), \tag{38}$$

and by eliminating $p(t)$, we obtain the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho(t)\frac{\mathrm{d}}{\mathrm{d}t}q(t)\right) + \rho(t)\omega^2(t)q(t) = 0. \tag{39}$$

This is certainly the equation of motion derived from the Lagrangian $L(q, \dot{q}, t) = \rho(t)\dot{q}^2/2 - \rho(t)\omega^2(t)q^2/2$. Note that if we had used $U^\dagger(t, T_s)$ in Eq. (35) instead of $U^{-1}(t, T_s)$, the equation of motion could not be reproduced correctly when $\varepsilon \neq 0$.

In the Heisenberg picture, the annihilation and creation operators become

$$\begin{aligned} a(t) &\equiv U^{-1}(t, T_s)a_s(t)U(t, T_s) \\ &= \sqrt{\frac{\rho(t)\omega(t)}{2}}q(t) + \mathrm{i}\sqrt{\frac{1}{2\rho(t)\omega(t)}}p(t), \end{aligned} \tag{40}$$

$$\begin{aligned} \bar{a}(t) &\equiv U^{-1}(t, T_s)a_s^\dagger(t)U(t, T_s) \\ &= \sqrt{\frac{\rho(t)\omega(t)}{2}}q(t) - \mathrm{i}\sqrt{\frac{1}{2\rho(t)\omega(t)}}p(t). \end{aligned} \tag{41}$$

They satisfy the commutation relation

$$[a(t), \bar{a}(t)] = 1, \tag{42}$$

but are not Hermitian conjugate to each other when $\varepsilon \neq 0$. The Hamiltonian is then expressed as

$$\begin{aligned} H(t) &\equiv U^{-1}(t, T_s)H_s(t)U(t, T_s) \\ &= \frac{p^2(t)}{2\rho(t)} + \frac{\rho(t)\omega^2(t)q^2(t)}{2} \\ &= \omega(t)\left[\bar{a}(t)a(t) + \frac{1}{2}\right]. \end{aligned} \tag{43}$$

Note that the states $|0_t\rangle \equiv |0_t, T_s\rangle = U^{-1}(t, T_s)|0_t, t\rangle$ and $\overline{\langle 0_t|} \equiv \overline{\langle 0_t, T_s|} = \overline{\langle 0_t, t|}U(t, T_s) = \langle 0_t, t|U(t, T_s)$ (see Fig. 1) satisfy the equations

$$a(t)|0_t\rangle = 0 = \langle 0_t|a^\dagger(t), \tag{44}$$

$$\overline{\langle 0_t|}\,\bar{a}(t) = 0 = \bar{a}^\dagger(t)\overline{|0_t\rangle}. \tag{45}$$

$\overline{\langle 0_t|}$ may differ from $\langle 0_t|$ since $U(t, T_s)$ is not unitary when $\varepsilon \neq 0$.

We denote by $\{f(t), g(t)\}$ a pair of linearly independent c-number solutions of (39). One can easily show that their Wronskian,

$$W[f, g](t) \equiv f(t)\dot{g}(t) - \dot{f}(t)g(t), \tag{46}$$

satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho W[f, g]) = f\frac{\mathrm{d}}{\mathrm{d}t}(\rho\dot{g}) - \frac{\mathrm{d}}{\mathrm{d}t}(\rho\dot{f})g = 0. \tag{47}$$

Thus the combination (to be called the weighted Wronskian),

$$W_\rho[f, g] \equiv \rho(t)W[f, g](t), \tag{48}$$

does not depend on t. Since $q(t)$ is also a solution of Eq. (39), we can expand canonical variables $q(t)$ and $p(t)$ as

$$q(t) = c_1 f(t) + c_2 g(t), \tag{49}$$

$$p(t) = \rho(t)\dot{q}(t) = \rho(t)[c_1\dot{f}(t) + c_2\dot{g}(t)], \tag{50}$$

where c_1 and c_2 are some time-independent quantum operators living in a space spanned by q_s and p_s with complex coefficients. Equations (49) and (50) can be solved with respect to c_1 and c_2 as

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{W_\rho[f, g]}\begin{pmatrix} \rho\dot{g} & -g \\ -\rho\dot{f} & f \end{pmatrix}(t)\begin{pmatrix} q(t) \\ p(t) \end{pmatrix}. \tag{51}$$

This can be further rewritten by using (40) and (41) as

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = C(t)\begin{pmatrix} a(t) \\ \bar{a}(t) \end{pmatrix}, \tag{52}$$

where

$$C(t) \equiv \frac{1}{W_\rho[f, g]\sqrt{2\rho(t)\omega(t)}}\begin{pmatrix} v(t) & \bar{v}(t) \\ -u(t) & -\bar{u}(t) \end{pmatrix} \tag{53}$$

with

$$\begin{Bmatrix} u(t) \\ \bar{u}(t) \end{Bmatrix} \equiv \rho(t)[\dot{f}(t) \pm \mathrm{i}\omega(t)f(t)], \tag{54}$$

$$\begin{Bmatrix} v(t) \\ \bar{v}(t) \end{Bmatrix} \equiv \rho(t)[\dot{g}(t) \pm \mathrm{i}\omega(t)g(t)]. \tag{55}$$

Note that

$$(u\bar{v} - v\bar{u})(t) = 2\mathrm{i}\rho(t)\omega(t)W_\rho[f, g], \tag{56}$$

$$\det C(t) = \frac{\mathrm{i}}{W_\rho[f, g]} \quad (= \mathrm{const}), \tag{57}$$

$$C^{-1}(t) = \frac{-\mathrm{i}}{\sqrt{2\rho(t)\omega(t)}}\begin{pmatrix} -\bar{u}(t) & -\bar{v}(t) \\ u(t) & v(t) \end{pmatrix}. \tag{58}$$

D. Bogoliubov coefficients for finite time intervals

The Bogoliubov coefficients from time t' to time t are defined by

$$\begin{pmatrix} a(t) \\ \bar{a}(t) \end{pmatrix} \equiv \begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ -\beta & \alpha \end{pmatrix}(t;t')\begin{pmatrix} a(t') \\ \bar{a}(t') \end{pmatrix}. \tag{59}$$

Since the operators $\{c_1, c_2\}$ in (52) do not depend on time, we have

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = C(t)\begin{pmatrix} a(t) \\ \bar{a}(t) \end{pmatrix} = C(t')\begin{pmatrix} a(t') \\ \bar{a}(t') \end{pmatrix}, \tag{60}$$

from which we find

$$\begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ -\beta & \alpha \end{pmatrix}(t;t') = C^{-1}(t)C(t') = \frac{-\mathrm{i}}{2W_\rho[f,g]\sqrt{\rho(t)\omega(t)\rho(t')\omega(t')}}\begin{pmatrix} u(t')\bar{v}(t) - v(t')\bar{u}(t) & \bar{u}(t')\bar{v}(t) - \bar{v}(t')\bar{u}(t) \\ v(t')u(t) - u(t')v(t) & \bar{v}(t')u(t) - \bar{u}(t')v(t) \end{pmatrix}. \tag{61}$$

This is the fundamental formula to express the Bogoliubov coefficients in terms of a given set of independent solutions $\{f(t), g(t)\}$.

Because of the commutation relations (42), the Bogoliubov coefficients should satisfy the relation

$$(\alpha\bar{\alpha} - \beta\bar{\beta})(t;t') = 1. \tag{62}$$

This can be directly checked by using the identity (57) as

$$(\alpha\bar{\alpha} - \beta\bar{\beta})(t;t') = \det C^{-1}(t)\det C(t') = \frac{W_\rho[f,g]}{W_\rho[f,g]} = 1. \tag{63}$$

Note that $\bar{\alpha} \neq \alpha^*$ and $\bar{\beta} \neq \beta^*$ when $\varepsilon \neq 0$.

E. Wave functions

Using the Bogoliubov coefficients, we can express the Heisenberg operators $q(t)$ with the creation and annihilation operators at a different time t_I as follows[8]:

$$\begin{aligned} q(t) &= \frac{1}{\sqrt{2\rho(t)\omega(t)}}(a(t) + \bar{a}(t)) \\ &= \frac{1}{\sqrt{2\rho(t)\omega(t)}}(\bar{\alpha}(t;t_I)a_I - \bar{\beta}(t;t_I)\bar{a}_I \\ &\quad + \alpha(t;t_I)\bar{a}_I - \beta(t;t_I)a_I) \\ &\equiv \varphi(t;t_I)a_I + \bar{\varphi}(t;t_I)\bar{a}_I, \end{aligned} \tag{64}$$

where we have defined functions $\varphi(t;t_I)$ and $\bar{\varphi}(t;t_I)$ (to be called wave functions) as

$$\begin{aligned} \varphi(t;t_I) &\equiv \frac{1}{\sqrt{2\rho(t)\omega(t)}}(\bar{\alpha}(t;t_I) - \beta(t;t_I)) \\ &= \frac{1}{W_\rho[f,g]\sqrt{2\rho_I\omega_I}}(v_I f(t) - u_I g(t)), \end{aligned} \tag{65}$$

$$\begin{aligned} \bar{\varphi}(t;t_I) &\equiv \frac{1}{\sqrt{2\rho(t)\omega(t)}}(\alpha(t;t_I) - \bar{\beta}(t;t_I)) \\ &= \frac{1}{W_\rho[f,g]\sqrt{2\rho_I\omega_I}}(\bar{v}_I f(t) - \bar{u}_I g(t)). \end{aligned} \tag{66}$$

By using the t-independence of $W_\rho[f,g]$ and Eq. (56), we can show that $\varphi(t;t_I)$ and $\bar{\varphi}(t;t_I)$ are normalized as

$$W_\rho[\varphi(t;t_I), \bar{\varphi}(t;t_I)] = \mathrm{i} \quad (\forall\ t, \forall\ t_I). \tag{67}$$

Moreover, the following relation holds for any value of t:

$$\begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ -\beta & \alpha \end{pmatrix}(t_1;t_0) = -\mathrm{i}\begin{pmatrix} -W_\rho[\bar{\varphi}(t;t_1), \varphi(t;t_0)] & -W_\rho[\bar{\varphi}(t;t_1), \bar{\varphi}(t;t_0)] \\ W_\rho[\varphi(t;t_1), \varphi(t;t_0)] & W_\rho[\varphi(t;t_1), \bar{\varphi}(t;t_0)] \end{pmatrix}. \tag{68}$$

We are now in a position to make a few comments.

1. Basis independence

We can show that the Bogoliubov coefficients and the wave functions $\{\varphi(t;t_I), \bar{\varphi}(t;t_I)\}$ do no depend on the choice of a pair of independent solutions $\{f(t), g(t)\}$, as they should. In fact, suppose that we take another pair $\{f'(t), g'(t)\}$. They should be expressed as linear combinations of $\{f(t), g(t)\}$ of the form

$$(f'(t)\ g'(t)) = (f(t)\ g(t))\Xi \qquad (\Xi \in GL(2,\mathbb{C})), \tag{69}$$

from which we have

$$\begin{pmatrix} f' & g' \\ \dot{f}' & \dot{g}' \end{pmatrix}(t) = \begin{pmatrix} f & g \\ \dot{f} & \dot{g} \end{pmatrix}(t)\Xi, \quad C'(t) = \Xi^{-1}C(t). \tag{70}$$

The new Bogoliubov coefficients associated with the choice $\{f'(t), g'(t)\}$ then become

[8]In the following, we will use the shorthand notation, such as $f_I \equiv f(t_I)$ or $\dot{f}_I \equiv \dot{f}(t_I)$, when a quantity is evaluated at time t_I.

$$\begin{pmatrix} \bar{\alpha}' & -\bar{\beta}' \\ -\beta' & \alpha' \end{pmatrix}(t;t_I) = [C'(t)]^{-1}C'(t_I) = C^{-1}(t)\Xi\Xi^{-1}C(t_I) = C^{-1}(t)C(t_I) = \begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ -\beta & \alpha \end{pmatrix}(t;t_I), \quad (71)$$

which shows the basis independence of the Bogoliubov coefficients. The wave functions $\{\varphi(t;t_I), \bar{\varphi}(t;t_I)\}$ are also basis independent since they are expressed by the basis-independent Bogoliubov coefficients [see Eqs. (65) and (66)].

2. Lapse independence

We can show that the Bogoliubov coefficients and the wave functions $\varphi(t;t_I)$ behave as scalar functions under the temporal reparametrizations, preserving the foliation of spacetime. In fact, for such reparametrization $t \to \tilde{t} = \tilde{t}(t)$, the pull-back of the lapse function $N(t)$ [see Eq. (2)] is given by $N(t) \to \tilde{N}(t) = (\mathrm{d}\tilde{t}/\mathrm{d}t)N(\tilde{t}(t))$, and we can choose a new pair of solutions $\{\tilde{f}(t), \tilde{g}(t)\}$ as $\tilde{f}(t) = f(\tilde{t}(t))$ and $\tilde{g}(t) = g(\tilde{t}(t))$. Then, we can easily show that the functions $\rho(t)\omega(t)$, $u(t)$, $v(t)$, and $W_\rho[f(t), g(t)]$ transform as scalar functions under the reparametrization. Since the Bogoliubov coefficients and the wave function $\varphi(t;t_I)$ are written as combinations of these functions, they also transform as scalar functions. This means that there is no need to care about the temporal reparametrization [i.e., the choice of the lapse function $N(t)$] when we construct vacua.

F. Feynman propagators

We consider the region $t_i < t_0 \le \{t, t'\} \le t_1 < t_f$, where t_f and t_i are the future and the past boundaries of the spacetime region we consider. The in-out and in-in propagators are defined with the following two steps:

Step 1: We first introduce the following two-point functions from our wave functions $\varphi(t;t_I)$ and $\bar{\varphi}(t;t_I)$[9]:

$$G_{10}(t,t';t_1,t_0) \equiv \frac{\overline{\langle 0_{t_1}|}\mathrm{T}q(t)q(t')|0_{t_0}\rangle}{\overline{\langle 0_{t_1}|}0_{t_0}\rangle} = \frac{\mathrm{i}}{W_\rho[\varphi(s;t_1), \bar{\varphi}(s;t_0)]}\varphi(t_>;t_1)\bar{\varphi}(t_<;t_0) \quad (s\text{: arbitrary}), \quad (72)$$

$$G_{00}(t,t';t_0,t_0) \equiv \frac{\langle 0_{t_0}|q^\dagger(t_>)q(t_<)|0_{t_0}\rangle}{\langle 0_{t_0}|0_{t_0}\rangle} = \frac{\mathrm{i}}{V_\rho[\bar{\varphi}^*(s;t_0), \bar{\varphi}(s;t_0)](T_s)} \times \bar{\varphi}^*(t_>;t_0)\bar{\varphi}(t_<;t_0), \quad (73)$$

where $t_> \equiv \max(t,t')$, $t_< \equiv \min(t,t')$ and $V_\rho[f,g](s) \equiv \rho(s)f(s)\dot{g}(s) - \rho^*(s)\dot{f}(s)g(s)$.[10]

Step 2: We then define the in-out and in-in propagators by sending t_0 and t_1 to the values at the temporal boundary,

$$G^{\mathrm{out/in}}(t,t') \equiv \lim_{\substack{t_0\to t_i \\ t_1\to t_f}} G_{10}(t,t';t_1,t_0), \quad (74)$$

$$G^{\mathrm{in/in}}(t,t') \equiv \lim_{t_0\to t_i} G_{00}(t,t';t_0,t_0). \quad (75)$$

Here we make a few comments. To obtain the last expression of (72), we use the following identities which are direct consequences of Eqs. (59), (62), and (68):

$$a_1 = \bar{\alpha}(t_1;t_0)a_0 - \frac{\bar{\beta}(t_1;t_0)}{\alpha(t_1;t_0)}(\bar{a}_1 + \beta(t_1;t_0)a_0) = \frac{1}{\alpha(t_1;t_0)}(a_0 - \bar{\beta}(t_1;t_0)\bar{a}_1), \quad (76)$$

$$\frac{\overline{\langle 0_{t_1}|}a_1\bar{a}_0|0_{t_0}\rangle}{\overline{\langle 0_{t_1}|}0_{t_0}\rangle} = \frac{1}{\alpha(t_1;t_0)}\frac{\overline{\langle 0_{t_1}|}(a_0 - \bar{\beta}(t_1;t_0)\bar{a}_1)\bar{a}_0 0_{t_0}\rangle}{\overline{\langle 0_{t_1}|}0_{t_0}\rangle} = \frac{1}{\alpha(t_1;t_0)} = \frac{\mathrm{i}}{W_\rho[\varphi(s;t_1), \bar{\varphi}(s;t_0)]} \quad (s\text{: arbitrary}). \quad (77)$$

We then have

$$G_{10}(t,t';t_1,t_0) = \frac{1}{\overline{\langle 0_{t_1}|}0_{t_0}\rangle}\overline{\langle 0_{t_1}|}(\varphi(t_>;t_1)a_1 + \bar{\varphi}(t_>;t_1)\bar{a}_1) \times (\varphi(t_<;t_0)a_0 + \bar{\varphi}(t_<;t_0)\bar{a}_0)|0_{t_0}\rangle = \varphi(t_>;t_1)\bar{\varphi}(t_<;t_0)\frac{\overline{\langle 0_{t_1}|}a_1\bar{a}_0|0_{t_0}\rangle}{\overline{\langle 0_{t_1}|}0_{t_0}\rangle} = \frac{\mathrm{i}}{W_\rho[\varphi(s;t_1), \bar{\varphi}(s;t_0)]}\varphi(t_>;t_1)\bar{\varphi}(t_<;t_0) \quad (s\text{: arbitrary}). \quad (78)$$

On the other hand, to obtain the last expression of (73), we start from the identities

$$\bar{a}_0^\dagger = -\frac{V_\rho[\varphi^*(s;t_0), \bar{\varphi}(s;t_0)](T_s)}{V_\rho[\bar{\varphi}^*(s;t_0), \bar{\varphi}(s;t_0)](T_s)}a_0^\dagger + \frac{\mathrm{i}}{V_\rho[\bar{\varphi}^*(s;t_0), \bar{\varphi}(s;t_0)](T_s)}a_0, \quad (79)$$

[9] If we instead use $G'_{10}(t,t';t_1,t_0) = \langle 0_{t_1}|q^\dagger(t_>)q(t_<)|0_{t_0}\rangle / \langle 0_{t_1}|0_{t_0}\rangle$, then the corresponding in-out propagator will not coincide with the propagator obtained by the standard path integral (see Sec. V), and thus we do not consider this choice in this paper. By contrast, the in-in propagator still has options for its definition (e.g., $G'_{00}(t,t';t_0,t_0) = \langle 0_{t_0}|\mathrm{T}q(t)q(t')|0_{t_0}\rangle / \langle 0_{t_0}|0_{t_0}\rangle$), and we leave for a future work a detailed study of such options as well as an investigation of the relation to the path integral based on the Schwinger-Keldysh formalism [12,13].

[10] Note that when $\varepsilon = 0$, $V_\rho[f,g](s)$ coincides with $W_\rho[f,g](s)$ and thus is constant in s. Otherwise, it may depend on s.

$$\frac{\langle 0_{t_0}|\bar{a}_0^\dagger \bar{a}_0|0_{t_0}\rangle}{\langle 0_{t_0}0_{t_0}\rangle} = \frac{\mathrm{i}}{V_\rho[\bar{\varphi}^*(s;t_0)\bar{\varphi}(s;t_0)](T_s)}, \tag{80}$$

which can be shown by using the Hermiticity at time T_s, $q^\dagger(T_s) = q(T_s)$ and $p^\dagger(T_s) = p(T_s)$ (see Appendix A). We then have

$$\begin{aligned} G_{00}(t,t';t_0,t_0) &= \frac{1}{\langle 0_{t_0}|0_{t_0}\rangle}\langle 0_{t_0}|(\varphi(t_>;t_0)a_0 + \bar{\varphi}(t_>;t_0)\bar{a}_0)^\dagger(\varphi(t_<;t_0)a_0 + \bar{\varphi}(t_<;t_0)\bar{a}_0)|0_{t_0}\rangle \\ &= \bar{\varphi}^*(t_>;t_0)\bar{\varphi}(t_<;t_0)\frac{\langle 0_{t_0}|\bar{a}_0^\dagger\bar{a}_0|0_{t_0}\rangle}{\langle 0_{t_0}|0_{t_0}\rangle} \\ &= \frac{\mathrm{i}}{V_\rho[\bar{\varphi}^*(s;t_0),\bar{\varphi}(s;t_0)](T_s)}\bar{\varphi}^*(t_>;t_0)\bar{\varphi}(t_<;t_0). \end{aligned} \tag{81}$$

When we need to specify T_s, we will set $T_s = t_>$ as in [23] [see also discussions following Eq. (109)], which leads in the Schrödinger picture to

$$G_{00}(t,t';t_0,t_0) = \frac{(U(t_>,t_0)|0_{t_0},t_0\rangle)^\dagger q_s U(t_>,t_<) q_s U(t_<,t_0)|0_{t_0},t_0\rangle}{\|U(t_>,t_0)|0_{t_0},t_0\rangle\|^2}. \tag{82}$$

When $\rho(t)$ and $\omega(t)$ are asymptotically constant in the remote past [i.e., $\rho(t) \sim \rho_{\mathrm{in}}$ and $\omega(t) \sim \omega_{\mathrm{in}}$ as $t \to t_i$], we can choose a pair of independent solutions $\{f(t), g(t)\}$ as those which behave as

$$f(t) \sim e^{-\mathrm{i}\omega_{\mathrm{in}}t}, \qquad g(t) \sim e^{+\mathrm{i}\omega_{\mathrm{in}}t} \qquad (t \sim t_i). \tag{83}$$

If we choose such a basis, we then have

$$u_0 \sim 0, \qquad \bar{u}_0 \sim -2\mathrm{i}\rho_{\mathrm{in}}\omega_{\mathrm{in}}e^{-\mathrm{i}\omega_{\mathrm{in}}t_0}, \tag{84}$$

$$v_0 \sim 2\mathrm{i}\rho_{\mathrm{in}}\omega_{\mathrm{in}}e^{\mathrm{i}\omega_{\mathrm{in}}t_0}, \qquad \bar{v}_0 \sim 0 \quad (t_0 \sim t_i), \tag{85}$$

and from Eqs. (65) and (66) the wave functions at the remote past are found to behave as

$$\varphi(t;t_0) \sim \frac{1}{\sqrt{2\rho_{\mathrm{in}}\omega_{\mathrm{in}}}}e^{-\mathrm{i}\omega_{\mathrm{in}}(t-t_0)}, \tag{86}$$

$$\bar{\varphi}(t;t_0) \sim \frac{1}{\sqrt{2\rho_{\mathrm{in}}\omega_{\mathrm{in}}}}e^{+\mathrm{i}\omega_{\mathrm{in}}(t-t_0)} \quad (t \sim t_i; t_0 \sim t_i). \tag{87}$$

A conclusion of the same kind can be obtained for the wave functions $(\varphi(t;t_1), \bar{\varphi}(t;t_1))$ if $\rho(t)$ and $\omega(t)$ are asymptotically constant at the remote future. This behavior of the wave functions will be directly seen in concrete examples given in Sec. III and Appendix B.

III. SIMPLE EXAMPLE: SCALAR FIELD IN MINKOWSKI SPACE

In this section, to demonstrate how the prescription of the previous section works, we consider a free real scalar field $\phi(x)$ living in Minkowski space with the metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\boldsymbol{x}^2. \tag{88}$$

Another well-studied example is investigated within our framework in Appendix B.

A. Setup

In order to clarify the structure of mode functions, we first assume that the spatial part is a $(d-1)$-dimensional torus of radius $L/2\pi$, which we will take infinite afterwards. The wave vectors $\boldsymbol{k}$ then take the following values:

$$\boldsymbol{k} = \frac{2\pi}{L}\boldsymbol{n} \quad (\boldsymbol{n} \in \mathbb{Z}^{d-1}). \tag{89}$$

For $\boldsymbol{k} = (k_1, k_2, \ldots, k_{d-1})$, we write $\boldsymbol{k} > 0$ (or $\boldsymbol{k} < 0$) if the first nonvanishing element in the sequence $\{k_1, k_2, \ldots\}$ is positive (or negative). Note that $\boldsymbol{k} < 0$ is equivalent to $-\boldsymbol{k} > 0$. We write $\boldsymbol{k} = 0$ if $\boldsymbol{k}$ is the zero vector ($\boldsymbol{k} = \boldsymbol{0}$).

We introduce a complete set of (real-valued) eigenfunctions $\{Y_{\boldsymbol{k},a}(\boldsymbol{x})\}$ of the spatial Laplacian $\Delta_{d-1} = \sum_{i=1}^{d-1}\partial_i^2$ as

$$\boldsymbol{k} = 0\colon\ Y_{\boldsymbol{k}=0,a=1} \equiv \frac{1}{\sqrt{V}} \quad (V \equiv L^{d-1}), \tag{90}$$

$$\boldsymbol{k} > 0\colon\ Y_{\boldsymbol{k},a=1}(\boldsymbol{x}) \equiv \sqrt{\frac{2}{V}}\cos\boldsymbol{k}\cdot\boldsymbol{x}, \tag{91}$$

$$Y_{\boldsymbol{k},a=2}(\boldsymbol{x}) \equiv \sqrt{\frac{2}{V}}\sin\boldsymbol{k}\cdot\boldsymbol{x}. \tag{92}$$

They satisfy the orthonormal relations,

$$\int \mathrm{d}^{d-1}\boldsymbol{x}\, Y_{\boldsymbol{k},a}(\boldsymbol{x})Y_{\boldsymbol{k}',a'}(\boldsymbol{x}) = \delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{a,a'}, \tag{93}$$

and we expand the scalar field $\phi(x)$ as

$$\phi(x) = \phi(t,\boldsymbol{x}) = \sum_{\boldsymbol{k}\geq 0}\sum_a \phi_{\boldsymbol{k},a}(t)Y_{\boldsymbol{k},a}(\boldsymbol{x}). \tag{94}$$

The action then becomes

$$S[\phi(x)] = \int \mathrm{d}^d x\Big[-\frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) - \frac{m^2}{2}\phi^2\Big] = \sum_{\boldsymbol{k}\geq 0}\sum_a \int \mathrm{d}t\Big[\frac{1}{2}\dot\phi^2_{\boldsymbol{k},a}(t) - \frac{\omega_k^2}{2}\phi^2_{\boldsymbol{k},a}(t)\Big], \tag{95}$$

where

$$\omega_k \equiv \sqrt{k^2+m^2} \quad \Big(k\equiv|\boldsymbol{k}| = \Big[\sum_{i=1}^{d-1}k_i^2\Big]^{1/2}\Big). \tag{96}$$

We thus have the following correspondence with the ingredients of the previous section:

$$q(t) = \phi_{\boldsymbol{k},a}(t), \qquad \rho(t) = e^{\mathrm{i}\varepsilon}, \tag{97}$$

$$\omega(t) = \omega_{k,\varepsilon} \equiv e^{-\mathrm{i}\varepsilon}\omega_k \quad (= \text{constant in } t). \tag{98}$$

B. Propagator for each mode

The equation of motion is given by $\ddot q + \omega^2 q = 0$, and we choose a pair of independent solutions as

$$f(t) = e^{-\mathrm{i}\omega t}, \qquad g(t) = e^{\mathrm{i}\omega t}. \tag{99}$$

Their Wronskian is given by $W_\rho[f,g] = 2\mathrm{i}\rho\omega = 2\mathrm{i}\omega_k$, which is constant in t.

The functions $u(t)$ and $v(t)$ are easily found to be[11]

$$\begin{Bmatrix} u(t) \\ \bar u(t)\end{Bmatrix} = \rho[\dot f(t) \pm \mathrm{i}\omega f(t)] = \begin{cases} 0 \\ -2\mathrm{i}\omega_k e^{-\mathrm{i}\omega t}\end{cases}, \tag{100}$$

$$\begin{Bmatrix} v(t) \\ \bar v(t)\end{Bmatrix} = \rho[\dot g(t) \pm \mathrm{i}\omega g(t)] = \begin{cases} 2\mathrm{i}\omega_k e^{\mathrm{i}\omega t} \\ 0\end{cases}, \tag{101}$$

and using (65) and (66) we obtain the wave functions as

$$\varphi(t;t_0) = \frac{1}{W_\rho\sqrt{2\rho_0\omega_0}}[v_0 f(t) - u_0 g(t)] = \frac{1}{\sqrt{2\omega_k}}e^{-\mathrm{i}\omega_{k,\varepsilon}(t-t_0)}, \tag{102}$$

$$\bar\varphi(t;t_0) = \frac{1}{W_\rho\sqrt{2\rho_0\omega_0}}[\bar v_0 f(t) - \bar u_0 g(t)] = \frac{1}{\sqrt{2\omega_k}}e^{\mathrm{i}\omega_{k,\varepsilon}(t-t_0)}, \tag{103}$$

$$\varphi(t;t_1) = \frac{1}{W_\rho\sqrt{2\rho_1\omega_1}}[v_1 f(t) - u_1 g(t)] = \frac{1}{\sqrt{2\omega_k}}e^{-\mathrm{i}\omega_{k,\varepsilon}(t-t_1)}, \tag{104}$$

$$\bar\varphi(t;t_1) = \frac{1}{W_\rho\sqrt{2\rho_1\omega_1}}[\bar v_1 f(t) - \bar u_1 g(t)] = \frac{1}{\sqrt{2\omega_k}}e^{\mathrm{i}\omega_{k,\varepsilon}(t-t_1)}. \tag{105}$$

From them, we have

$$W_\rho[\varphi(s;t_1),\bar\varphi(s;t_0)] = \mathrm{i}e^{\mathrm{i}\omega_{k,\varepsilon}(t_1-t_0)}, \tag{106}$$

$$V_\rho[\bar\varphi^*(s;t_0),\bar\varphi(s;t_0)] = \mathrm{i}e^{\mathrm{i}(\omega_{k,\varepsilon}-\omega_{k,-\varepsilon})(s-t_0)}, \tag{107}$$

and the two-point functions take the forms

$$G_{\boldsymbol{k},10}(t,t';t_1,t_0) = \frac{\mathrm{i}}{W_\rho[\varphi(s;t_1),\bar\varphi(s;t_0)]}\varphi(t_>,t_1)\bar\varphi(t_<;t_0) = \frac{1}{2\omega_k}e^{-\mathrm{i}\omega_{k,\varepsilon}(t_>-t_<)}, \tag{108}$$

$$G_{\boldsymbol{k},00}(t,t';t_0,t_0) = \frac{\mathrm{i}}{V_\rho[\bar\varphi^*(s;t_0)\bar\varphi(s;t_0)](T_s)} \times \bar\varphi^*(t_>;t_0)\bar\varphi(t_<;t_0) = \frac{1}{2\omega_k}e^{-\mathrm{i}\omega_{k,-\varepsilon}(t_>-T_s)-\mathrm{i}\omega_{k,\varepsilon}(T_s-t_<)}. \tag{109}$$

Note that the dependence on t_0 and t_1 totally disappears in $G_{\boldsymbol{k},IJ}(t,t';t_I,t_J)$ for Minkowski space, and thus we need not to take the limit $t_0\to-\infty$, $t_1\to+\infty$ to obtain the in-out and in-in propagators. We see from (109) that the behavior of $G_{\boldsymbol{k},00}$ in the region $k\to\infty$ gets significantly improved if we choose T_s such that $T_s \geq t_>$. By simply setting $T_s = t_>$, we obtain

$$G_{\boldsymbol{k}}^{\mathrm{in/in}}(t,t') = G_{\boldsymbol{k}}^{\mathrm{out/in}}(t,t') = \frac{1}{2\omega_k}e^{-\mathrm{i}\omega_{k,\varepsilon}(t_>-t_<)}, \tag{110}$$

which will be denoted by $G_k(t,t')$ in the following discussions.

C. Propagator in spacetime

Once propagators are obtained for each mode $(\boldsymbol{k},a)$, the propagator in spacetime can be obtained by summing them over the modes. The manipulation is known very well for Minkowski space, but we here review it briefly for later reference.

The in-out or in-in propagator is given by the following summation ($t_i = -\infty$, $t_f = \infty$):

[11]The Bogoliubov coefficients can be calculated by using (68) as

$$\alpha(t_1;t_0) = e^{\mathrm{i}\omega_k(t_1-t_0)}, \qquad \beta(t_1;t_0) = 0,$$
$$\bar\alpha(t_1;t_0) = e^{-\mathrm{i}\omega_k(t_1-t_0)}, \qquad \bar\beta(t_1;t_0) = 0,$$

which indicates that the vacuum at a later time, $|0_{t_1}\rangle$, coincides with the vacuum at an earlier time, $|0_{t_0}\rangle$, up to a phase.

$$G^{\mathrm{out/in}}(x,x') = \sum_{\boldsymbol{k},\boldsymbol{k}'\geq 0}\sum_{a,a'} \frac{\overline{\langle 0_{t_1}|}\mathrm{T}\phi_{\boldsymbol{k},a}(t)\phi_{\boldsymbol{k}',a'}(t')|0_{t_0}\rangle}{\overline{\langle 0_{t_1}|}0_{t_0}\rangle}\Bigg|_{\substack{t_0\to t_i\\ t_1\to t_f}} \times Y_{\boldsymbol{k},a}(\boldsymbol{x})Y_{\boldsymbol{k}',a'}(\boldsymbol{x}'). \tag{111}$$

$$G^{\mathrm{in/in}}(x,x') = \sum_{\boldsymbol{k},\boldsymbol{k}'\geq 0}\sum_{a,a'} \frac{\langle 0_{t_0}|\phi^\dagger_{\boldsymbol{k},a}(t_>)\phi_{\boldsymbol{k}',a'}(t_<)|0_{t_0}\rangle}{\langle 0_{t_0}|0_{t_0}\rangle}\Bigg|_{t_0\to t_i} \times Y_{\boldsymbol{k},a}(\boldsymbol{x})Y_{\boldsymbol{k}',a'}(\boldsymbol{x}'). \tag{112}$$

Since the two-point functions are diagonalized with respect to the modes,

$$\frac{\overline{\langle 0_{t_1}|}\mathrm{T}\phi_{\boldsymbol{k},a}(t)\phi_{\boldsymbol{k}',a'}(t')|0_{t_0}\rangle}{\overline{\langle 0_{t_1}|}0_{t_0}\rangle}\Bigg|_{\substack{t_0\to t_i\\ t_1\to t_f}} = G_k(t,t')\delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{a,a'}, \tag{113}$$

$$\frac{\langle 0_{t_0}|\phi^\dagger_{\boldsymbol{k},a}(t_>)\phi_{\boldsymbol{k}',a'}(t_<)|0_{t_0}\rangle}{\langle 0_{t_0}|0_{t_0}\rangle}\Bigg|_{t_0\to t_i} = G_k(t,t')\delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{a,a'}, \tag{114}$$

we have

$$\begin{aligned} G^{\{\substack{\mathrm{out/in}\\ \mathrm{in/in}}\}}(x,x') &= \sum_{\boldsymbol{k}\geq 0} G_k(t,t')\sum_a Y_{\boldsymbol{k},a}(\boldsymbol{x})Y_{\boldsymbol{k},a}(\boldsymbol{x}') \\ &\equiv \sum_{\boldsymbol{k}\geq 0} G_k(t,t')R_{\boldsymbol{k}}(\boldsymbol{x},\boldsymbol{x}'). \end{aligned} \tag{115}$$

Here, $R_{\boldsymbol{k}}(\boldsymbol{x},\boldsymbol{x}') \equiv \sum_a Y_{\boldsymbol{k},a}(\boldsymbol{x})Y_{\boldsymbol{k},a}(\boldsymbol{x}')$ are easily calculated as

$$R_{\boldsymbol{k}=0}(\boldsymbol{x},\boldsymbol{x}') = \frac{1}{V}, \quad R_{\boldsymbol{k}>0}(\boldsymbol{x},\boldsymbol{x}') = \frac{2}{V}\cos\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}'), \tag{116}$$

and thus we have

$$\begin{aligned} G(x,x') &= \frac{1}{V}\Big[G_{k=0}(t,t') + 2\sum_{\boldsymbol{k}>0} G_k(t,t')\cos\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')\Big] \\ &= \frac{1}{V}\sum_{\boldsymbol{k}} G_k(t,t')\cos\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}') \\ &= \int\frac{\mathrm{d}^{d-1}\boldsymbol{k}}{(2\pi)^{d-1}} G_k(t,t')\cos\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}'), \end{aligned} \tag{117}$$

where we have taken the limit $L\to\infty$ in the last equality.

The integration (117) can be performed easily (see Appendix C), and we obtain

$$\begin{aligned} G(x,x') &= \frac{1}{(2\pi)^{\frac{d-1}{2}}|\boldsymbol{x}-\boldsymbol{x}'|^{\frac{d-3}{2}}} \times \int_0^\infty \mathrm{d}k k^{\frac{d-1}{2}} G_k(t,t') J_{\frac{d-3}{2}}(k|\boldsymbol{x}-\boldsymbol{x}'|) \\ &= \frac{m^{(d-2)/2}}{(2\pi)^{d/2}(\sigma+\mathrm{i}0)^{(d-2)/4}} K_{\frac{d-2}{2}}(m\sqrt{\sigma+\mathrm{i}0}) \end{aligned} \tag{118}$$

with $\sigma \equiv (x-x')^2$. Here, $J_\nu(z)$ is the Bessel function, and $K_\nu(z)$ is the modified Bessel function of the second kind.

IV. SCALAR FIELD IN DE SITTER SPACE

A. Geometry and definitions

We first recall the geometry of de Sitter space and collect the notation and definitions, where d-dimensional de Sitter space (dS_d) has the topology $\mathbb{R}\times S^{d-1}$ and is defined as a hyperboloid,

$$\eta_{MN}X^M X^N = \ell^2 \quad (M,N,\ldots = 0,\ldots,d), \qquad (\eta_{MN}) = \mathrm{diag}(-1,1,\ldots,1), \tag{119}$$

in $(d+1)$-dimensional Minkowski space with the metric

$$\mathrm{d}s^2 = \eta_{MN}\mathrm{d}X^M\mathrm{d}X^N \tag{120}$$

and ℓ is called the de Sitter radius. The constant Ricci scalar curvature is then given by $R = d(d-1)/\ell^2$.

There are several well-known coordinate patches that cover all or just a part of de Sitter space. Among them, we consider the global patch and the Poincaré (or planer) patch, which we will briefly review below.

1. Global patch

This coordinate patch covers the whole region of de Sitter space. The embedding of dS_d is given by the functions

$$\begin{aligned} X^0(\tau,\boldsymbol{\Omega}) &= \ell\sinh\tau, \\ X^I(\tau,\boldsymbol{\Omega}) &= \ell\cosh\tau\,\Omega^I \quad (I=1,\ldots,d), \end{aligned} \tag{121}$$

with $\boldsymbol{\Omega}\cdot\boldsymbol{\Omega} = 1$. Here, τ runs over the range $-\infty<\tau<\infty$, and $\boldsymbol{\Omega}$ is a unit vector in $\mathbb{R}^d$ spanning a $(d-1)$-dimensional sphere. With the coordinates $(\tau,\boldsymbol{\Omega})$ the metric has the form

$$\begin{aligned} \mathrm{d}s^2 &= \ell^2[-\mathrm{d}\tau^2 + \cosh^2\tau\,\mathrm{d}\Omega^2_{d-1}] \\ &= \ell^2[-(1-t^2)^{-2}\mathrm{d}t^2 + (1-t^2)^{-1}\mathrm{d}\Omega^2_{d-1}]. \end{aligned} \tag{122}$$

In the last equality, we have introduced another temporal coordinate t as

$$t \equiv \tanh\tau \quad (-1<t<1). \tag{123}$$

2. Poincaré patch

This coordinate patch covers only half of de Sitter space. The embedding is given by the following functions with $\eta<0$ and $\boldsymbol{x}\in\mathbb{R}^{d-1}$:

$$X^0(\eta,\boldsymbol{x}) = \frac{\ell^2-\eta^2+|\boldsymbol{x}|^2}{-2\eta}, \qquad X^i(\eta,\boldsymbol{x}) = \ell\frac{x^i}{-\eta}, \qquad X^d(\eta,\boldsymbol{x}) = \frac{\ell^2+\eta^2-|\boldsymbol{x}|^2}{-2\eta}, \tag{124}$$

where the spatial norm is defined by $|\boldsymbol{x}| \equiv \sqrt{\boldsymbol{x}\cdot\boldsymbol{x}} \equiv \sqrt{\delta_{ij}x^ix^j}$. Note that this patch only covers the region $X^0 + X^d = \ell^2/(-\eta) > 0$. In these coordinates, the metric takes the form

$$\mathrm{d}s^2 = \ell^2\frac{-\mathrm{d}\eta^2+\mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}}{\eta^2}. \tag{125}$$

The Poincaré patch is not preserved under a finite action of de Sitter group $SO(1,d)$, but is still preserved under infinitesimal actions of $SO(1,d)$. In fact, the infinitesimal actions are given by the Killing vectors $M_{MN} = X_M\partial_N - X_N\partial_M$, which take the following forms in the Poincaré patch:

$$M_{0d} = \eta\partial_\eta + x^i\partial_i, \tag{126}$$

$$M_{0i}+M_{di} = \frac{1}{\ell}[2x^i\eta\partial_\eta + 2x^ix^j\partial_j - (-\eta^2+|\boldsymbol{x}|^2)\partial_i], \tag{127}$$

$$M_{0i}-M_{di} = -\ell\partial_i, \tag{128}$$

$$M_{ij} = x^i\partial_j - x^j\partial_i. \tag{129}$$

These Killing vectors do not have the ∂_η component at the boundary of the patch $\eta = 0$ as long as $|\boldsymbol{x}| < \infty$. Similarly, if we define another time coordinate $t \equiv 1/(-\eta)$, we find that the Killing vectors do not have the ∂_t component at another boundary at $t = 0$ (i.e., $\eta = -\infty$). These results show that the infinitesimal transformations map any point inside the Poincaré patch (i.e., $|\boldsymbol{x}| < \infty$ and $-\infty < \eta < 0$) into the same region.

We define the de Sitter invariant quantity

$$Z(x,x') \equiv \ell^{-2}\eta_{MN}X^M(x)X^N(x'), \tag{130}$$

which is related to the geodesic distance $d(x,x')$ between two points x and x' via the relation

$$Z(x,x') = \cos\left(\frac{d(x,x')}{\ell}\right). \tag{131}$$

It takes the following values depending on the positional relation between x and x':

$$Z(x,x')\begin{cases} >1 & (\text{for } x \text{ and } x' \text{ timelikely separated}) \\ =1 & (\text{for } x \text{ and } x' \text{ lightlikely separated}) \\ <1 & (\text{for } x \text{ and } x' \text{ spacelikely separated}), \end{cases} \tag{132}$$

as can be seen from the identity

$$Z(x,x') = 1 - \frac{1}{2\ell^2}\eta_{MN}(X^M(x)-X^M(x'))(X^N(x)-X^N(x')). \tag{133}$$

In the global and the Poincaré coordinates, $Z(x,x')$ is written in the form

$$Z(x,x') = \begin{cases} -\sinh\tau\sinh\tau' + \cosh\tau\cosh\tau'\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}' & (\text{global coordinates}) \\ \frac{\eta^2+\eta'^2-|\boldsymbol{x}-\boldsymbol{x}'|^2}{2\eta\eta'} & (\text{Poincaré coordinates}). \end{cases} \tag{134}$$

One can easily prove that any two-point function $G(x,x')$ that is invariant under the infinitesimal actions $M_{MN} + M'_{MN} = M^\mu_{MN}(x)\partial/\partial x^\mu + M^\mu_{MN}(x')\partial/\partial x'^\mu$ must be a function of the de Sitter invariant $Z(x,x')$. We will see that all the propagators constructed in this paper turn out to be functions of $Z(x,x')$. In what follows (except in Sec. VI C), we set $\ell = 1$.

B. Scalar field in the Poincaré patch

We first consider a free real scalar field in the Poincaré patch. The action takes the form[12]

$$S[\phi(x)] = -\frac{1}{2}\int\mathrm{d}\eta\int\mathrm{d}^{d-1}\boldsymbol{x}\sqrt{-g}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2). \tag{135}$$

Using the same eigenfunctions $\{Y_{\boldsymbol{k},a}(\boldsymbol{x})\}$ as those given in Sec. III [Eqs. (90)–(92)], we expand a scalar field $\phi(x)$ as

$$\phi(x) = \phi(\eta,\boldsymbol{x}) = \sum_{\boldsymbol{k}\geq 0}\sum_a \phi_{\boldsymbol{k},a}(\eta)Y_{\boldsymbol{k},a}(\boldsymbol{x}). \tag{136}$$

The functions defined in Sec. II then take the following form [see (8) and (9)]:

$$q(\eta) = \phi_{\boldsymbol{k},a}(\eta), \qquad \rho(\eta) = e^{i\varepsilon}(-\eta)^{-(d-2)}, \tag{137}$$

[12] In this paper, we put a possible curvature-coupling term, $(\xi/2)R\phi^2 = (d(d-1)\xi/2)\phi^2$, into the mass term, $(m^2/2)\phi^2$.

$$\omega(\eta) = e^{-i\varepsilon}\sqrt{m^2(-\eta)^{-2} + k^2} \equiv e^{-i\varepsilon}\omega_k(\eta), \qquad (138)$$

from which we introduce[13]

$$m_\varepsilon \equiv e^{-i\varepsilon}m, \qquad k_\varepsilon \equiv e^{-i\varepsilon}k. \qquad (139)$$

Note that

$$\omega_{k,0} \equiv \omega_k(\eta_0) \xrightarrow{\eta_0\to-\infty} k, \qquad \omega_{k,1} \equiv \omega_k(\eta_1) \xrightarrow{\eta_1\to-0} \frac{m}{-\eta_1}. \qquad (140)$$

1. Propagators for each mode in the Poincaré patch

The equation of motion (39) takes the form

$$\eta^2\ddot{q}(\eta) - (d-2)\dot{q}(\eta) + (k_\varepsilon^2\eta^2 + m_\varepsilon^2)q(\eta) = 0, \qquad (141)$$

and we choose a set of independent solutions as

$$f(\eta) = (-\eta)^{\frac{d-1}{2}}J_{\nu_\varepsilon}(-k_\varepsilon\eta), \qquad (142)$$

$$g(\eta) = (-\eta)^{\frac{d-1}{2}}N_{\nu_\varepsilon}(-k_\varepsilon\eta), \qquad (143)$$

with

$$\nu_\varepsilon \equiv \begin{cases} \sqrt{\left(\frac{d-1}{2}\right)^2 - m_\varepsilon^2} = \nu + i\varepsilon & \left(m < \frac{d-1}{2}\right) \\ i\sqrt{m_\varepsilon^2 - \left(\frac{d-1}{2}\right)^2} = i\mu + \varepsilon & \left(m \geq \frac{d-1}{2}\right) \end{cases}$$

$$\left[\nu \equiv \sqrt{\left(\frac{d-1}{2}\right)^2 - m^2},\ \mu \equiv \sqrt{m^2 - \left(\frac{d-1}{2}\right)^2}\right]. \qquad (144)$$

Here, $N_\nu(x)$ is the Neumann function. Note that $\mathrm{Re}\,\nu_\varepsilon > 0$ for any positive value of m. The Wronskian is given by $W[f,g](\eta) = -(2/\pi)(-\eta)^{d-2}$, and thus

$$W_\rho[f,g] = \rho(t)W[f,g](t) = -e^{i\varepsilon}\frac{2}{\pi}. \qquad (145)$$

The functions $u(\eta)$ and $v(\eta)$ of (54) and (55) have the forms

$$\begin{Bmatrix} u(\eta) \\ \bar{u}(\eta) \end{Bmatrix} = -e^{i\varepsilon}(-\eta)^{-\frac{d-1}{2}} \times \left[\left(\frac{d-1}{2} + \nu_\varepsilon \pm i\omega(\eta)\eta\right)J_{\nu_\varepsilon}(-k_\varepsilon\eta) + k_\varepsilon\eta J_{1+\nu_\varepsilon}(-k_\varepsilon\eta)\right], \qquad (146)$$

$$\begin{Bmatrix} v(\eta) \\ \bar{v}(\eta) \end{Bmatrix} = -e^{i\varepsilon}(-\eta)^{-\frac{d-1}{2}} \times \left[\left(\frac{d-1}{2} + \nu_\varepsilon \pm i\omega(\eta)\eta\right)N_{\nu_\varepsilon}(-k_\varepsilon\eta) + k_\varepsilon\eta N_{1+\nu_\varepsilon}(-k_\varepsilon\eta)\right], \qquad (147)$$

where we have used the formulas

$$z\frac{\partial J_\nu(z)}{\partial z} = \nu J_\nu(z) - zJ_{\nu+1}(z), \qquad (148)$$

$$z\frac{\partial N_\nu(z)}{\partial z} = \nu N_\nu(z) - zN_{\nu+1}(z). \qquad (149)$$

The wave functions are then given by

$$\varphi(\eta;\eta_I) = -\frac{\pi e^{-i\varepsilon}}{2\sqrt{2\omega_{k,I}}}(-\eta_I)^{\frac{d-2}{2}}[v_I f(\eta) - u_I g(\eta)], \qquad (150)$$

$$\bar{\varphi}(\eta;\eta_I) = -\frac{\pi e^{-i\varepsilon}}{2\sqrt{2\omega_{k,I}}}(-\eta_I)^{\frac{d-2}{2}}[\bar{v}_I f(\eta) - \bar{u}_I g(\eta)], \qquad (151)$$

from which the two-point functions are obtained as

$$G_{10}(\eta,\eta';\eta_1,\eta_0) = \frac{i}{W_\rho[\varphi(s;\eta_1),\bar{\varphi}(s;\eta_0)]}\varphi(\eta_>;\eta_1)\bar{\varphi}(\eta_<;\eta_0) \quad (s\text{: arbitrary}), \qquad (152)$$

$$G_{00}(\eta,\eta';\eta_0,\eta_0) = \frac{i}{V_\rho[\bar{\varphi}^*(s;\eta_0),\bar{\varphi}(s;\eta_0)](\eta_s)}\bar{\varphi}^*(\eta_>;\eta_0)\bar{\varphi}(\eta_<;\eta_0). \qquad (153)$$

We now send η_0, η_1 to the boundary of the Poincaré patch: $\eta_0 \to \eta_i = -\infty$ and $\eta_1 \to \eta_f = 0$. By using the asymptotic forms of the Bessel functions,[14]

$$J_{\nu_\varepsilon}(-k_\varepsilon\eta) \overset{\eta\to-\infty}{\sim} \frac{1}{\sqrt{2\pi}}(-k_\varepsilon\eta)^{-1/2}\left[1 + i\frac{\nu_\varepsilon^2 - (1/4)}{2(-k_\varepsilon\eta)}\right] \times e^{-i(k_\varepsilon\eta + \frac{\pi(2\nu_\varepsilon+1)}{4})}, \qquad (154)$$

$$N_{\nu_\varepsilon}(-k_\varepsilon\eta) \overset{\eta\to-\infty}{\sim} \frac{-i}{\sqrt{2\pi}}(-k_\varepsilon\eta)^{-1/2}\left[1 + i\frac{\nu_\varepsilon^2 - (1/4)}{2(-k_\varepsilon\eta)}\right] \times e^{-i(k_\varepsilon\eta + \frac{\pi(2\nu_\varepsilon+1)}{4})}, \qquad (155)$$

$$J_{\nu_\varepsilon}(-k_\varepsilon\eta) \overset{\eta\to0}{\sim} \frac{1}{\Gamma(1+\nu_\varepsilon)}\left(-\frac{k_\varepsilon\eta}{2}\right)^{\nu_\varepsilon}, \qquad (156)$$

[13]Our prescription for Sec. II cannot be applied directly to the exactly massless case, where $\omega(\eta) = 0$ for $k = 0$. We actually define the massless theory as the $m \to 0$ limit of a massive theory.

[14]We have used the inequalities $\mathrm{Re}\,\nu_\varepsilon > 0$ and $\mathrm{Re}(-ik_\varepsilon\eta) > 0$.

$$N_{\nu_\varepsilon}(-k_\varepsilon\eta)\overset{\eta\to 0}{\sim} -\frac{\Gamma(\nu_\varepsilon)}{\pi}\left(-\frac{k_\varepsilon\eta}{2}\right)^{-\nu_\varepsilon}, \tag{157}$$

one can easily show that u_I, $\bar{u}_I$, v_I, and $\bar{v}_I$ ($I=0,1$) have the asymptotic forms

$$u_0 \sim \mathrm{i}v_0 \sim -\frac{e^{\mathrm{i}\varepsilon}}{\sqrt{2\pi k_\varepsilon}}\left(\frac{d-2}{2}\right)(-\eta_0)^{-\frac{d}{2}}e^{-\mathrm{i}(k_\varepsilon\eta_0+\frac{\pi(2\nu_\varepsilon+1)}{4})}, \tag{158}$$

$$\bar{u}_0 \sim \mathrm{i}\bar{v}_0 \sim -\mathrm{i}e^{\mathrm{i}\varepsilon}\sqrt{\frac{2k_\varepsilon}{\pi}}(-\eta_0)^{-\frac{d-2}{2}}e^{-\mathrm{i}(k_\varepsilon\eta_0+\frac{\pi(2\nu_\varepsilon+1)}{4})}, \tag{159}$$

$$\begin{Bmatrix} u_1 \\ \bar{u}_1 \end{Bmatrix} \sim -\frac{e^{\mathrm{i}\varepsilon}(k_\varepsilon/2)^{\nu_\varepsilon}}{\Gamma(1+\nu_\varepsilon)}(-\eta_1)^{-\frac{d-1}{2}+\nu_\varepsilon}\left(\frac{d-1}{2}+\nu_\varepsilon \mp \mathrm{i}m_\varepsilon\right), \tag{160}$$

$$\begin{Bmatrix} v_1 \\ \bar{v}_1 \end{Bmatrix} \sim \frac{e^{\mathrm{i}\varepsilon}\Gamma(\nu_\varepsilon)(k_\varepsilon/2)^{-\nu_\varepsilon}}{\pi}(-\eta_1)^{-\frac{d-1}{2}-\nu_\varepsilon} \times\left(\frac{d-1}{2}-\nu_\varepsilon \mp \mathrm{i}m_\varepsilon\right). \tag{161}$$

Since ν_ε always has a positive real part, we obtain the relation $(-\eta_1)^{-\nu_\varepsilon} \gg (-\eta_1)^{\nu_\varepsilon}$ in the limit $\eta_1 \to 0$, from which we find

$$|u_1| \ll |v_1|, \qquad |\bar{u}_1| \ll |\bar{v}_1|. \tag{162}$$

Thus, we find that the wave functions behave as

$$\varphi(\eta;\eta_0) \sim -\frac{\pi e^{-\mathrm{i}\varepsilon}}{2\sqrt{2k}}(-\eta_0)^{\frac{d-2}{2}}v_0[f(\eta)-\mathrm{i}g(\eta)], \tag{163}$$

$$\bar{\varphi}(\eta;\eta_0) \sim -\frac{\pi e^{-\mathrm{i}\varepsilon}}{2\sqrt{2k}}(-\eta_0)^{\frac{d-2}{2}}\bar{v}_0[f(\eta)-\mathrm{i}g(\eta)], \tag{164}$$

$$\varphi(\eta;\eta_1) \sim -\frac{\pi e^{-\mathrm{i}\varepsilon}}{2\sqrt{2m}}(-\eta_1)^{\frac{d-1}{2}}v_1 f(\eta), \tag{165}$$

$$\varphi(\eta;\eta_1) \sim -\frac{\pi e^{-\mathrm{i}\varepsilon}}{2\sqrt{2m}}(-\eta_1)^{\frac{d-1}{2}}\bar{v}_1 f(\eta), \tag{166}$$

with which the weighted Wronskian becomes

$$\begin{aligned} &W_\rho[\varphi(\eta;\eta_1),\bar{\varphi}(\eta;\eta_0)] \\ &\sim \frac{\pi^2 e^{-2\mathrm{i}\varepsilon}}{8\sqrt{km}}(-\eta_0)^{\frac{d-2}{2}}(-\eta_1)^{\frac{d-1}{2}}v_1\bar{v}_0 W_\rho[f(\eta),f(\eta)-\mathrm{i}g(\eta)] \\ &= \frac{\mathrm{i}\pi}{4\sqrt{km}}e^{-\mathrm{i}\varepsilon}v_1\bar{v}_0(-\eta_0)^{\frac{d-2}{2}}(-\eta_1)^{\frac{d-1}{2}}, \end{aligned} \tag{167}$$

$$\begin{aligned} &V_\rho[\bar{\varphi}^*(\eta;\eta_0),\bar{\varphi}(\eta;\eta_0)] \\ &\sim \frac{\pi^2}{8k}(-\eta_0)^{d-2}|\bar{v}_0|^2 V_\rho[(f(\eta)-\mathrm{i}g(\eta))^*, f(\eta)-\mathrm{i}g(\eta)]. \end{aligned} \tag{168}$$

The in-out propagator can be readily obtained by substituting (164), (165), and (167) into (152) as

$$\begin{aligned} &G_{\boldsymbol{k}}^{\mathrm{out/in}}(\eta,\eta') \\ &= \frac{\pi}{2}e^{-\mathrm{i}\varepsilon}f(\eta_>)[f(\eta_<)-\mathrm{i}g(\eta_<)] \\ &= \frac{\pi}{2}[(-\eta)(-\eta')]^{\frac{d-1}{2}}J_\nu(-k_\varepsilon\eta_>)H_\nu^{(2)}(-k_\varepsilon\eta_<). \end{aligned} \tag{169}$$

Here, $H_\nu^{(1,2)}(x)$ are the Hankel functions defined by $H_\nu^{(1,2)}(x) \equiv J_\nu(x) \pm \mathrm{i}N_\nu(x)$, and we have set $\varepsilon = 0$ in the last expression as far as it does not change the analytic property of the propagator. One the other hand, in order to calculate the in-in propagator,

$$\begin{aligned} &G_{\boldsymbol{k}}^{\mathrm{in/in}}(\eta,\eta') \\ &= \frac{\mathrm{i}}{V_\rho}[f(\eta_>)-\mathrm{i}g(\eta_>)]^*[f(\eta_<)-\mathrm{i}g(\eta_<)] \\ &\quad (V_\rho = V_\rho[(f(s)-\mathrm{i}g(s))^*, f(s)-\mathrm{i}g(s)](\eta_s)), \end{aligned} \tag{170}$$

we first notice that the complex conjugate of $f(\eta)-\mathrm{i}g(\eta) = (-\eta)^{\frac{d-1}{2}}H_{\nu_\varepsilon}^{(2)}(-k_\varepsilon\eta)$ is given by $[f(\eta)-\mathrm{i}g(\eta)]^* = (-\eta)^{\frac{d-1}{2}}H_{\nu_{-\varepsilon}}^{(1)}(-k_{-\varepsilon}\eta)$. Thus, for the small mass case, $m<(d-1)/2$ ($\nu\in\mathbb{R}$), we have $[H_\nu^{(2)}(z)]^* = H_\nu^{(1)}(z^*)$, so that we obtain

$$f(\eta)-\mathrm{i}g(\eta) = (-\eta)^{\frac{d-1}{2}}H_\nu^{(2)}(-k_\varepsilon\eta)+O(\varepsilon), \tag{171}$$

$$[f(\eta)-\mathrm{i}g(\eta)]^* = (-\eta)^{\frac{d-1}{2}}H_\nu^{(1)}(-k_{-\varepsilon}\eta)+O(\varepsilon), \tag{172}$$

which lead to

$$V_\rho = \frac{4\mathrm{i}}{\pi}+O(\varepsilon). \tag{173}$$

For the large mass case, $m \geq (d-1)/2$ ($\nu = \mathrm{i}\mu \in \mathrm{i}\mathbb{R}$), we have $[H_\nu^{(2)}(z)]^* = [H_{\mathrm{i}\mu}^{(2)}(z)]^* = H_{-\mathrm{i}\mu}^{(1)}(z^*) = e^{-\pi\mu}H_{\mathrm{i}\mu}^{(1)}(z^*)$, so that we obtain

$$f(\eta)-\mathrm{i}g(\eta) = (-\eta)^{\frac{d-1}{2}}H_\nu^{(2)}(-k_\varepsilon\eta)+O(\varepsilon), \tag{174}$$

$$[f(\eta)-\mathrm{i}g(\eta)]^* = e^{-\pi\mu}(-\eta)^{\frac{d-1}{2}}H_\nu^{(1)}(-k_{-\varepsilon}\eta)+O(\varepsilon), \tag{175}$$

which lead to

$$V_\rho = e^{-\pi\mu}\frac{4\mathrm{i}}{\pi}+O(\varepsilon). \tag{176}$$

Substituting Eqs. (171)–(176) for (170), we obtain

$$G^{\rm in/in}(\eta,\eta') = \frac{\pi}{4}[(-\eta)(-\eta')]^{\frac{d-1}{2}} H_\nu^{(1)}(-k_{-\varepsilon}\eta_>) H_\nu^{(2)}(-k_\varepsilon\eta_<). \tag{177}$$

We here make a few comments. With ε set to zero, the wave function $\bar\varphi(\eta;\eta_0)$ converges in the limit $\eta_0 \to -\infty$ to a finite function

$$\varphi^*_{\rm in}(\eta) \equiv \frac{\sqrt{\pi}}{2}(-\eta)^{\frac{d-1}{2}} H_\nu^{(2)}(-k\eta) \tag{178}$$

up to an oscillatory phase, while $\varphi(\eta;\eta_1)$ diverges as $(-\eta_1)^{-\nu}$ in the limit $\eta_1 \to 0$. This difference can be attributed to the fact that the timelike vector ∂_η becomes asymptotically a Killing vector in the remote past but not in the remote future.[15] The finite asymptotic function $\varphi_{\rm in}(\eta)$ coincides with the positive-mode wave function associated with the Euclidean vacuum up to a phase. One should note that, in the limit $\eta_1 \to 0$, the divergence in $\varphi(\eta;\eta_1)$ is canceled out with that in $W_\rho[\varphi(\eta,\eta_1),\bar\varphi(\eta,\eta_0)]$, and the in-out propagator is obtained with a finite value.

For completeness, we show the asymptotic forms of the Bogoliubov coefficients,

$$\alpha(\eta_1;\eta_0) \sim -\frac{\Gamma(\nu)}{2\sqrt{2\pi m}}(-k\eta_1/2)^{-\nu_\varepsilon}\left(\frac{d-1}{2}-\nu-{\rm i}m\right) \times e^{-{\rm i}(k_\varepsilon\eta_0+\frac{\pi(2\nu+1)}{4})}, \tag{179}$$

$$\beta(\eta_1;\eta_0) \sim -{\rm i}\frac{(d-2)\Gamma(\nu)}{8\sqrt{2\pi m}}(-k\eta_1/2)^{-\nu_\varepsilon}(-k\eta_0)^{-1} \times \left(\frac{d-1}{2}-\nu-{\rm i}m\right)e^{-{\rm i}(k_\varepsilon\eta_0+\frac{\pi(2\nu+1)}{4})}, \tag{180}$$

which diverge in either of the limits $\eta_0 \to -\infty$ and $\eta_1 \to 0$.

2. Propagators in the Poincaré patch

Since the eigenfunctions $\{Y_{\boldsymbol{k},a}(\boldsymbol{x})\}$ are the same as those given in Sec. III, the propagators in spacetime can be written in the form

$$\begin{aligned} G^{\{\substack{\rm out/in \\ \rm in/in}\}}(x,x') &= \sum_{\boldsymbol{k}\geq 0} G_{\boldsymbol{k}}^{\{\substack{\rm out/in \\ \rm in/in}\}}(\eta,\eta')\sum_a Y_{\boldsymbol{k},a}(\boldsymbol{x})Y_{\boldsymbol{k},a}(\boldsymbol{x}') \\ &= \frac{1}{(2\pi)^{\frac{d-1}{2}}|\boldsymbol{x}-\boldsymbol{x}'|^{\frac{d-3}{2}}}\int_0^\infty {\rm d}k k^{\frac{d-1}{2}} J_{\frac{d-3}{2}}(k|\boldsymbol{x}-\boldsymbol{x}'|) \\ &\quad \times G_{\boldsymbol{k}}^{\{\substack{\rm out/in \\ \rm in/in}\}}(\eta,\eta'), \end{aligned} \tag{181}$$

as in the case of Minkowski space (see Appendix C).

For the in-out propagator, we have

$$\begin{aligned} G^{\rm out/in}(x,x') &= \frac{e^{-{\rm i}\varepsilon}\pi}{2}\frac{[(-\eta)(-\eta')]^{\frac{d-1}{2}}}{(2\pi)^{\frac{d-1}{2}}|\boldsymbol{x}-\boldsymbol{x}'|^{\frac{d-3}{2}}} \\ &\quad \times \int_0^\infty {\rm d}k k^{\frac{d-1}{2}} J_{\frac{d-3}{2}}(k|\boldsymbol{x}-\boldsymbol{x}'|) \\ &\quad \times J_\nu(-k_\varepsilon\eta_>)H_\nu^{(2)}(-k_\varepsilon\eta_<). \end{aligned} \tag{182}$$

As is proved in Appendix D, this can be integrated to the form[16]

$$G^{\rm out/in}(x,x') = \frac{e^{-{\rm i}\pi(d-2)}}{(2\pi)^{d/2}}(u^2-1)^{-\frac{d-2}{4}}\mathcal{Q}_{\nu-1/2}^{\frac{d-2}{2}}(u) \tag{183}$$

with $u = Z(x,x') - {\rm i}0$.

The in-in propagator

$$\begin{aligned} G^{\rm in/in}(x,x') &= \frac{\pi}{4}\frac{[(-\eta)(-\eta')]^{\frac{d-1}{2}}}{(2\pi)^{\frac{d-1}{2}}|\boldsymbol{x}-\boldsymbol{x}'|^{\frac{d-3}{2}}}\int_0^\infty {\rm d}k k^{\frac{d-1}{2}} J_{\frac{d-3}{2}}(k|\boldsymbol{x}-\boldsymbol{x}'|) \\ &\quad \times H_\nu^{(1)}(-k_{-\varepsilon}\eta_>)H_\nu^{(2)}(-k_\varepsilon\eta_<) \end{aligned} \tag{184}$$

can be rewritten in a similar manner to the form

$$G^{\rm in/in}(x,x') = \frac{\Gamma(\frac{d-1}{2}+\nu)\Gamma(\frac{d-1}{2}-\nu)}{2(2\pi)^{d/2}} \times (u^2-1)^{-\frac{d-2}{4}}\mathcal{P}_{\nu-1/2}^{-\frac{d-2}{2}}(u), \tag{185}$$

with $u \equiv -Z(x,x') + {\rm i}0$. A proof is also given in Appendix D. This can be further rewritten as

$$G^{\rm in/in}(x,x') = \frac{\Gamma(\frac{d-1}{2}+\nu)\Gamma(\frac{d-1}{2}-\nu)}{(4\pi)^{d/2}\Gamma(d/2)} \times F\left(\frac{d-1}{2}+\nu,\frac{d-1}{2}-\nu;\frac{d}{2};\frac{1-u}{2}\right), \tag{186}$$

$$= \frac{\Gamma(\frac{d-1}{2})}{4\pi^{(d-1)/2}\sin[\pi(\frac{d-1}{2}-\nu)]}C_{\nu-\frac{d-1}{2}}^{\frac{d-1}{2}}(u), \tag{187}$$

where Kummer's relation (B34) has been used in the first equality and $C_\nu^\mu(x)$ is the Gegenbauer function. This propagator is the same as the well-known in-in propagator associated with the Euclidean vacuum.

If we consider the massless limit where $\nu \to (d-1)/2$, the in-in propagator diverges, as was pointed out in [11]. In contrast, we find that the in-out propagator has a finite massless limit,

[15]In fact, the Lie derivative of $g_{\mu\nu}$ with respect to the vector $\xi = \partial_\eta$ is $\mathcal{L}_\xi g_{\mu\nu} \propto (-\eta)^{-3}$.

[16]Here we have taken the limit $\varepsilon \to 0$. $\mathcal{P}_\nu^\mu(z)$ and $\mathcal{Q}_\nu^\mu(z)$ denote the associated Legendre functions of the first and second kind that are defined on the complex z plane other than the cut along the real axis to the left of the point $z = 1$. There are other types of associated Legendre functions that are defined on the interval $(-1, 1)$, which we denote by $\mathsf{P}(x)$ and $\mathsf{Q}(x)$. See Appendix E for their definitions and several useful identities.

$$G^{\rm out/in}(x,x') = \frac{e^{-i\pi(d-2)}}{(2\pi)^{d/2}}(u^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\frac{d-2}{2}}(u) \quad \text{(massless)}. \tag{188}$$

C. Scalar field in the global patch

In the global patch, as a complete set of eigenfunctions of the spatial Laplacian Δ_{d-1} on S^{d-1}, we take (real-valued) spherical harmonics $\{Y_{LM}(\mathbf{\Omega})\}$. They satisfy $\Delta_{d-1}Y_{LM} = -L(L+d-2)Y_{LM}$ $(M=1,\ldots,N_L^{(d)})$, and the degeneracy $N_L^{(d)}$ is given by

$$N_L^{(d)} = \frac{(L+d-3)!}{(d-2)!L!}(2L+d-2) \tag{189}$$

with the exceptional case $d=2$ and $L=0$, where $N_0^{(2)} \equiv 1$. We choose them such that they are orthonormal:

$$\int d\mathbf{\Omega}\, Y_{LM}(\mathbf{\Omega})Y_{L'M'}(\mathbf{\Omega}) = \delta_{LL'}\delta_{MM'}. \tag{190}$$

Then, by expanding $\phi(x)$ as

$$\phi(x) = \sum_{L,M}\phi_{LM}(t)Y_{LM}(\mathbf{\Omega}), \tag{191}$$

the mode function $q(t) \equiv \phi_{LM}(t)$ describes a harmonic oscillator with time-dependent mass and frequency of the following form [see (8) and (9)]:

$$\rho(t) = e^{i\varepsilon}(1-t^2)^{-\frac{d-3}{2}}, \tag{192}$$

$$\begin{aligned}\omega(t) &= (1-t^2)^{-1}e^{-i\varepsilon}\sqrt{L(L+d-2)(1-t^2)+m^2}\\ &\equiv (1-t^2)^{-1}\bar{m}(t).\end{aligned} \tag{193}$$

1. Propagators for each mode in the global patch

The equation of motion takes the form

$$\ddot{q}(t) + (d-3)\frac{t}{1-t^2}\dot{q}(t) + \omega^2(t)q(t) = 0. \tag{194}$$

We choose a pair of independent solutions as

$$f(t) = (1-t^2)^{\frac{d-1}{4}}\mathsf{P}^{\nu_\varepsilon}_{k_\varepsilon}(t), \qquad g(t) = (1-t^2)^{\frac{d-1}{4}}\mathsf{Q}^{\nu_\varepsilon}_{k_\varepsilon}(t), \tag{195}$$

where

$$\begin{aligned}k_\varepsilon &= -\frac{1}{2} + \sqrt{\left(\frac{d-2}{2}\right)^2 + e^{-2i\varepsilon}L(L+d-2)}\\ &= k - i\varepsilon \quad \left(k \equiv L + \frac{d-3}{2}\right),\end{aligned} \tag{196}$$

and $\mathsf{P}^\mu_\nu(t)$ and $\mathsf{Q}^\mu_\nu(t)$ are the associated Legendre functions defined on the interval $(-1,1)$ (see Appendix E). The weighted Wronskian (constant in t) then has the form

$$W_\rho[f,g] = \rho(t)W[f,g](t) = e^{i\varepsilon}\frac{\Gamma(k_\varepsilon+\nu_\varepsilon+1)}{\Gamma(k_\varepsilon-\nu_\varepsilon+1)}. \tag{197}$$

The functions $u(t)$ and $v(t)$ defined in (54) and (55) are then given by

$$\begin{aligned}\begin{Bmatrix} u(t)\\ \bar{u}(t)\end{Bmatrix} = e^{i\varepsilon}(1-t^2)^{-\frac{d-1}{4}}\bigg\{\bigg[\bigg(k_\varepsilon - \frac{d-3}{2}\bigg)t \pm i\bar{m}(t)\bigg]\mathsf{P}^{\nu_\varepsilon}_{k_\varepsilon}(t)\\ - (k_\varepsilon - \nu_\varepsilon + 1)\mathsf{P}^{\nu_\varepsilon}_{k_\varepsilon+1}(t)\bigg\},\end{aligned} \tag{198}$$

$$\begin{aligned}\begin{Bmatrix} v(t)\\ \bar{v}(t)\end{Bmatrix} = e^{i\varepsilon}(1-t^2)^{-\frac{d-1}{4}}\bigg\{\bigg[\bigg(k_\varepsilon - \frac{d-3}{2}\bigg)t \pm i\bar{m}(t)\bigg]\mathsf{Q}^{\nu_\varepsilon}_{k_\varepsilon}(t)\\ - (k_\varepsilon - \nu_\varepsilon + 1)\mathsf{Q}^{\nu_\varepsilon}_{k_\varepsilon+1}(t)\bigg\},\end{aligned} \tag{199}$$

and the wave functions $(\varphi, \bar{\varphi})$ take the form

$$\varphi(t;t_I) = \frac{e^{-i\varepsilon}\Gamma(k_\varepsilon-\nu_\varepsilon+1)}{\sqrt{2|\bar{m}_I|}\Gamma(k_\varepsilon+\nu_\varepsilon+1)}(1-t_I^2)^{\frac{d-1}{4}}(v_I f(t) - u_I g(t)), \tag{200}$$

$$\bar{\varphi}(t;t_I) = \frac{e^{-i\varepsilon}\Gamma(k_\varepsilon-\nu_\varepsilon+1)}{\sqrt{2|\bar{m}_I|}\Gamma(k_\varepsilon+\nu_\varepsilon+1)}(1-t_I^2)^{\frac{d-1}{4}}(\bar{v}_I f(t) - \bar{u}_I g(t)). \tag{201}$$

We now send t_0, t_1 to the boundary of the global patch: $t_0 \to t_i = -1$ and $t_1 \to t_f = 1$. Using (E20) and (E21), we see that $u_1 = u(t_1)$ and $v_1 = v(t_1)$ take the following asymptotic forms in the limit $t_1 \to 1$:

$$\begin{aligned}\begin{Bmatrix} u_1\\ \bar{u}_1\end{Bmatrix} &\sim \frac{2^{\nu_\varepsilon}e^{i\varepsilon}\sin(\pi\nu_\varepsilon)\Gamma(\nu_\varepsilon)}{\pi}\\ &\quad\times\left(-\frac{d-1}{2}+\nu_\varepsilon \pm i\bar{m}_1\right)(1-t_1^2)^{-\frac{d-1}{4}-\frac{\nu_\varepsilon}{2}},\end{aligned} \tag{202}$$

$$\begin{aligned}\begin{Bmatrix} v_1\\ \bar{v}_1\end{Bmatrix} &\sim 2^{\nu_\varepsilon-1}e^{i\varepsilon}\cos(\pi\nu_\varepsilon)\Gamma(\nu_\varepsilon)\\ &\quad\times\left(-\frac{d-1}{2}+\nu_\varepsilon \pm i\bar{m}_1\right)(1-t_1^2)^{-\frac{d-1}{4}-\frac{\nu_\varepsilon}{2}}\\ &\sim \frac{\pi\cos\pi\nu_\varepsilon}{2\sin\pi\nu_\varepsilon}\begin{Bmatrix} u_1\\ \bar{u}_1\end{Bmatrix},\end{aligned} \tag{203}$$

and thus we find that the wave functions behave as[17]

[17]The asymptotic forms given in Eq. (204) do not satisfy the normalization condition (67). Actually, to ensure this normalization, we need to add a subleading term proportional to $(1-t_1^2)^{\nu_\varepsilon/2}$, which is omitted from the above asymptotic forms. The asymptotic forms, however, are still sufficient for calculating various propagators.

$$\begin{Bmatrix} \varphi(t;t_1) \\ \bar{\varphi}(t;t_1) \end{Bmatrix} \sim \frac{e^{-\mathrm{i}\varepsilon}\Gamma(k_\varepsilon - \nu_\varepsilon + 1)}{\sqrt{2m}\Gamma(k_\varepsilon + \nu_\varepsilon + 1)}(1-t_1^2)^{\frac{d-1}{4}} \begin{Bmatrix} v_1 \\ \bar{v}_1 \end{Bmatrix}\left[f(t) - \frac{2\sin\pi\nu_\varepsilon}{\pi\cos\pi\nu_\varepsilon}g(t)\right] = \gamma_1 \begin{Bmatrix} v_1 \\ \bar{v}_1 \end{Bmatrix}(1-t^2)^{\frac{d-1}{4}}\mathsf{P}_{k_\varepsilon}^{-\nu_\varepsilon}(t). \quad (204)$$

Here, $\gamma_1 \equiv e^{-\mathrm{i}\varepsilon}(1-t_1^2)^{(d-1)/4}/(\sqrt{2m}\cos\pi\nu_\varepsilon)$, and we have used the formula (E10) and the fact that $\bar{m}_1 \to e^{-\mathrm{i}\varepsilon}m$ $(t \to +1)$.

Similarly, we see that $u_0 = u(t_0)$ and $v_0 = v(t_0)$ have the following asymptotic forms in the limit $t_0 \to -1$:

$$\begin{Bmatrix} u_0 \\ \bar{u}_0 \end{Bmatrix} \sim \begin{cases} \frac{e^{\mathrm{i}\varepsilon}2^{-\nu_\varepsilon}\cos(\pi k_\varepsilon)\Gamma(k_\varepsilon+\nu_\varepsilon+1)}{\Gamma(\nu_\varepsilon+1)\Gamma(k_\varepsilon-\nu_\varepsilon+1)}\left(\frac{d-1}{2} + \nu_\varepsilon \pm \mathrm{i}\bar{m}_0\right)(1-t_0^2)^{-\frac{d-1}{4}+\frac{\nu_\varepsilon}{2}} & (d\text{: odd}) \\ -\frac{e^{\mathrm{i}\varepsilon}2^{\nu_\varepsilon}\sin(\pi k_\varepsilon)\Gamma(\nu_\varepsilon)}{\pi}\left(\frac{d-1}{2} - \nu_\varepsilon \pm \mathrm{i}\bar{m}_0\right)(1-t_0^2)^{-\frac{d-1}{4}-\frac{\nu_\varepsilon}{2}} & (d\text{: even}), \end{cases} \quad (205)$$

$$\begin{Bmatrix} v_0 \\ \bar{v}_0 \end{Bmatrix} \sim \begin{cases} -e^{\mathrm{i}\varepsilon}2^{\nu_\varepsilon-1}\cos(\pi k_\varepsilon)\Gamma(\nu_\varepsilon)\left(\frac{d-1}{2} - \nu_\varepsilon \pm \mathrm{i}\bar{m}_0\right)(1-t_0^2)^{-\frac{d-1}{4}-\frac{\nu_\varepsilon}{2}} & (d\text{: odd}) \\ -\frac{e^{\mathrm{i}\varepsilon}2^{-\nu_\varepsilon-1}\pi\Gamma(k_\varepsilon+\nu_\varepsilon+1)}{\sin(\pi k_\varepsilon)\Gamma(\nu_\varepsilon+1)\Gamma(k_\varepsilon-\nu_\varepsilon+1)}\left(\frac{d-1}{2} + \nu_\varepsilon \pm \mathrm{i}\bar{m}_0\right)(1-t_0^2)^{-\frac{d-1}{4}+\frac{\nu_\varepsilon}{2}} & (d\text{: even}). \end{cases} \quad (206)$$

Here, in adopting the asymptotic forms of $\mathsf{P}_{k_\varepsilon}^{\nu_\varepsilon}(t_0)$ and $\mathsf{Q}_{k_\varepsilon}^{\nu_\varepsilon}(t_0)$ for $t_0 \to -1$ [see (E22) and (E23)], we have used the fact that $\mathrm{Re}\nu_\varepsilon > 0$, which particularly means that $(1-t_0^2)^{-\frac{\nu_\varepsilon}{2}} \gg (1-t_0^2)^{\frac{\nu_\varepsilon}{2}}$. The same inequality can be used to show that

$$\begin{Bmatrix} u_0 \\ \bar{u}_0 \end{Bmatrix} \ll \begin{Bmatrix} v_0 \\ \bar{v}_0 \end{Bmatrix} \quad (d\text{: odd}), \quad (207)$$

$$\begin{Bmatrix} u_0 \\ \bar{u}_0 \end{Bmatrix} \gg \begin{Bmatrix} v_0 \\ \bar{v}_0 \end{Bmatrix} \quad (d\text{: even}), \quad (208)$$

from which we find that

$$\begin{Bmatrix} \varphi(t;t_0) \\ \bar{\varphi}(t;t_0) \end{Bmatrix} \sim \gamma_0 \begin{Bmatrix} v_0 \\ \bar{v}_0 \end{Bmatrix}(1-t^2)^{\frac{d-1}{4}}\mathsf{P}_{k_\varepsilon}^{\nu_\varepsilon}(t) \quad (d\text{: odd}), \quad (209)$$

$$\begin{Bmatrix} \varphi(t;t_0) \\ \bar{\varphi}(t;t_0) \end{Bmatrix} \sim -\gamma_0 \begin{Bmatrix} u_0 \\ \bar{u}_0 \end{Bmatrix}(1-t^2)^{\frac{d-1}{4}}\mathsf{Q}_{k_\varepsilon}^{\nu_\varepsilon}(t) \quad (d\text{: even}), \quad (210)$$

where $\gamma_0 \equiv e^{-\mathrm{i}\varepsilon}\Gamma(k_\varepsilon - \nu_\varepsilon + 1)(1-t_0^2)^{\frac{d-1}{4}}/(\sqrt{2m}\Gamma(k_\varepsilon + \nu_\varepsilon + 1))$.

With the wave functions $\varphi(t;t_I)$ at hand [see (204), (209), and (210)], the weighted Wronskian can be readily obtained as

$$W_\rho[\varphi(t;t_1), \bar{\varphi}(t;t_0)] \sim \begin{cases} +\gamma_1\gamma_0 v_1\bar{v}_0(1-t^2)W_\rho[\mathsf{P}_{k_\varepsilon}^{-\nu_\varepsilon}(t), \mathsf{P}_{k_\varepsilon}^{\nu_\varepsilon}(t)] & (d\text{: odd}) \\ -\gamma_1\gamma_0 v_1\bar{u}_0(1-t^2)W_\rho[\mathsf{P}_{k_\varepsilon}^{-\nu_\varepsilon}(t), \mathsf{Q}_{k_\varepsilon}^{\nu_\varepsilon}(t)] & (d\text{: even}) \end{cases} = \begin{cases} +e^{\mathrm{i}\varepsilon}\gamma_1\gamma_0 v_1\bar{v}_0\frac{2}{\pi}\sin\pi\nu_\varepsilon & (d\text{: odd}) \\ -e^{\mathrm{i}\varepsilon}\gamma_1\gamma_0 v_1\bar{u}_0\cos\pi\nu_\varepsilon & (d\text{: even}). \end{cases} \quad (211)$$

We then obtain the in-out propagator

$$\begin{aligned} G_L^{\mathrm{out/in}}(t,t') &= \lim_{\substack{t_0\to -1 \\ t_1 \to +1}} \frac{\mathrm{i}}{W_\rho[\varphi(t;t_1), \bar{\varphi}(t;t_0)]}\varphi(t_>;t_1)\bar{\varphi}(t_<;t_0) \\ &= \begin{cases} \frac{\mathrm{i}\pi}{2\sin\pi\nu}[(1-t_>^2)(1-t_<^2)]^{(d-1)/4}\mathsf{P}_k^{-\nu}(t_>)\mathsf{P}_k^{\nu}(t_<) & (d\text{: odd}), \\ \frac{\mathrm{i}}{\cos\pi\nu}[(1-t_>^2)(1-t_<^2)]^{(d-1)/4}\mathsf{P}_k^{-\nu}(t_>)\mathsf{Q}_k^{\nu}(t_<) & (d\text{: even}). \end{cases} \end{aligned} \quad (212)$$

In the last expression, we have set $\varepsilon = 0$.

To obtain the in-in propagator, we first notice that

$$V_\rho \equiv V_\rho[\bar{\varphi}^*(t;t_0), \bar{\varphi}(t;t_0)] = \begin{cases} |\gamma_0|^2|\bar{v}_0|^2 V_\rho[f^*, f](T_s) & (d\text{: odd}) \\ |\gamma_0|^2|\bar{u}_0|^2 V_\rho[g^*, g](T_s) & (d\text{: even}), \end{cases} \tag{213}$$

from which we have

$$G_L^{\mathrm{in/in}}(t,t') = \lim_{t_0\to -1} \frac{\mathrm{i}}{V_\rho[\bar{\varphi}^*(t;t_0), \bar{\varphi}(t;t_0)](T_s)} \bar{\varphi}^*(t_>;t_0)\bar{\varphi}(t_<;t_0) = \begin{cases} \frac{\mathrm{i}}{V_\rho[f^*,f](T_s)} f^*(t_>)f(t_<) & (d\text{: odd}), \\ \frac{\mathrm{i}}{V_\rho[g^*,g](T_s)} g^*(t_>)g(t_<) & (d\text{: even}). \end{cases} \tag{214}$$

When the mass is large [$m \geq (d-1)/2$ and thus $\nu = \mathrm{i}\mu \in \mathrm{i}\mathbb{R}$], we have

$$[f(t)]^* = (1-t^2)^{\frac{d-1}{4}}[\mathsf{P}_{k_\varepsilon}^{\nu_\varepsilon}(t)]^* = (1-t^2)^{\frac{d-1}{4}}\mathsf{P}_{k_\varepsilon}^{-\nu_\varepsilon}(t) + \mathcal{O}(\varepsilon), \tag{215}$$

$$[g(t)]^* = (1-t^2)^{\frac{d-1}{4}}[\mathsf{Q}_{k_\varepsilon}^{\nu_\varepsilon}(t)]^* = (1-t^2)^{\frac{d-1}{4}}\mathsf{Q}_{k_\varepsilon}^{-\nu_\varepsilon}(t) + \mathcal{O}(\varepsilon), \tag{216}$$

and thus

$$V_\rho[f^*, f](T_s) = \frac{2\mathrm{i}\sinh \pi\mu}{\pi} + \mathcal{O}(\varepsilon), \tag{217}$$

$$V_\rho[g^*, g](T_s) = \frac{\mathrm{i}\pi\sinh \pi\mu}{2} + \mathcal{O}(\varepsilon). \tag{218}$$

The in-in propagator is then obtained as

$$G_L^{\mathrm{in/in}}(t,t') = \begin{cases} \frac{\pi}{2\sinh(\pi\mu)}[(1-t_>^2)(1-t_<^2)]^{\frac{d-1}{4}}\mathsf{P}_k^{-\mathrm{i}\mu}(t_>)\mathsf{P}_k^{\mathrm{i}\mu}(t_<) & (d\text{: odd}), \\ \frac{2}{\pi\sinh(\pi\mu)}[(1-t_>^2)(1-t_<^2)]^{\frac{d-1}{4}}\mathsf{Q}_k^{-\mathrm{i}\mu}(t_>)\mathsf{Q}_k^{\mathrm{i}\mu}(t_<) & (d\text{: even}). \end{cases} \tag{219}$$

On the other hand, when the mass is small [$m < (d-1)/2$ and thus $\nu \in \mathbb{R}$], we have $f^*(t) = f(t) + \mathcal{O}(\varepsilon)$, $g^*(t) = g(t) + \mathcal{O}(\varepsilon)$, and thus $V_\rho[f^*, f](T_s) = \mathcal{O}(\varepsilon)$, $V_\rho[g^*, g](T_s) = \mathcal{O}(\varepsilon)$. This means that $\lim_{t_0\to -1} G_{00}(t,t';t_0,t_0)$ has the singularity of the form $\mathcal{O}(\varepsilon^{-1})$, and we cannot set $\varepsilon = 0$.

The wave functions at the remote past and future had been obtained for the heavy mass case ($m \geq (d-1)/2$) in [10,18] as[18]

$$\varphi_{\mathrm{out}}(t) \propto (1-t^2)^{\frac{d-1}{4}}\mathsf{P}_k^{-\mathrm{i}\mu}(t), \tag{220}$$

$$\varphi_{\mathrm{in}}^*(t) \propto \begin{cases} (1-t^2)^{\frac{d-1}{4}}\mathsf{P}_k^{\mathrm{i}\mu}(t) & (d\text{: odd}) \\ (1-t^2)^{\frac{d-1}{4}}\mathsf{Q}_k^{\mathrm{i}\mu}(t) & (d\text{: even}), \end{cases} \tag{221}$$

by requiring that $\varphi_{\mathrm{in}}^*(t)$ ($\varphi_{\mathrm{out}}(t)$) be regular for $t \to -1$ ($t \to +1$) and an analytic function in the lower half of complex m^2 plane (see also [24] where they are obtained by suitably choosing the Jost functions). Our propagators (212) and (219) for $m \geq (d-1)/2$ are consistent with these wave functions.

From (68), the Bogoliubov coefficient $\alpha(t_1;t_0)$ can be found to have the asymptotic form

[18]We here give the following identities which are useful in comparing our results with the literature:

$$\frac{2^{\frac{d-1}{2}+L}\cosh^L\tau e^{(-\frac{d-1}{2}-L-\nu)\tau}}{\Gamma(1+\nu)} F\left(\frac{d-1}{2}+L, \frac{d-1}{2}+L+\nu; 1+\nu; -e^{-2\tau}\right) = (1-t^2)^{\frac{d-1}{4}}\mathsf{P}_k^{-\nu}(t),$$

$$\frac{2^{\frac{d-1}{2}+L}\cosh^L\tau e^{(\frac{d-1}{2}+L-\nu)\tau}}{\Gamma(1-\nu)} F\left(\frac{d-1}{2}+L, \frac{d-1}{2}+L-\nu; 1-\nu; -e^{+2\tau}\right) = (1-t^2)^{\frac{d-1}{4}}\mathsf{P}_k^{\nu}(-t)$$

$$= \frac{\Gamma(k+1+\nu)}{\Gamma(k+1-\nu)}(1-t^2)^{\frac{d-1}{4}}\begin{cases} (-1)^k\mathsf{P}_k^{-\nu}(t) & (d\text{: odd}) \\ (-1)^{k+1/2}(2/\pi)\mathsf{Q}_k^{-\nu}(t) & (d\text{: even}) \end{cases}.$$

$$\alpha(t_1;t_0)\sim -\frac{2\mathrm{i}\sin(\pi\nu)}{\pi}\Big[\frac{2^{\nu-1}\Gamma(\nu)}{\sqrt{2\bar{m}_1}}\Big(-\frac{d-1}{2}+\nu+\mathrm{i}\bar{m}_1\Big)(1-t_1^2)^{-\frac{\nu_\varepsilon}{2}}\Big] \times\Big[-\frac{2^{\nu-1}\cos(\pi k)\Gamma(\nu)\Gamma(k-\nu+1)}{\sqrt{2\bar{m}_0}\Gamma(k+\nu+1)}\Big(\frac{d-1}{2}-\nu-\mathrm{i}\bar{m}_0\Big)(1-t_0^2)^{-\frac{\nu_\varepsilon}{2}}\Big], \tag{222}$$

$$\beta(t_1;t_0)\sim +\frac{2\mathrm{i}\sin(\pi\nu)}{\pi}\Big[\frac{2^{\nu-1}\Gamma(\nu)}{\sqrt{2\bar{m}_1}}\Big(-\frac{d-1}{2}+\nu+\mathrm{i}\bar{m}_1\Big)(1-t_1^2)^{-\frac{\nu_\varepsilon}{2}}\Big] \times\Big[-\frac{2^{\nu-1}\cos(\pi k)\Gamma(\nu)\Gamma(k-\nu+1)}{\sqrt{2\bar{m}_0}\Gamma(k+\nu+1)}\Big(\frac{d-1}{2}-\nu+\mathrm{i}\bar{m}_0\Big)(1-t_0^2)^{-\frac{\nu_\varepsilon}{2}}\Big] \tag{223}$$

in odd dimensions, and

$$\alpha(t_1;t_0)\sim -\mathrm{i}\cos(\pi\nu)\Big[\frac{2^{\nu-1}\Gamma(\nu)}{\sqrt{2\bar{m}_1}}\Big(-\frac{d-1}{2}+\nu+\mathrm{i}\bar{m}_1\Big)(1-t_1^2)^{-\frac{\nu_\varepsilon}{2}}\Big] \times\Big[\frac{2^{\nu}\sin(\pi k)\Gamma(\nu)\Gamma(k-\nu+1)}{\pi\sqrt{2\bar{m}_0}\Gamma(k+\nu+1)}\Big(\frac{d-1}{2}-\nu-\mathrm{i}\bar{m}_0\Big)(1-t_0^2)^{-\frac{\nu_\varepsilon}{2}}\Big], \tag{224}$$

$$\beta(t_1;t_0)\sim +\mathrm{i}\cos(\pi\nu)\Big[\frac{2^{\nu-1}\Gamma(\nu)}{\sqrt{2\bar{m}_1}}\Big(-\frac{d-1}{2}+\nu+\mathrm{i}\bar{m}_1\Big)(1-t_1^2)^{-\frac{\nu_\varepsilon}{2}}\Big] \times\Big[\frac{2^{\nu}\sin(\pi k)\Gamma(\nu)\Gamma(k-\nu+1)}{\pi\sqrt{2\bar{m}_0}\Gamma(k+\nu+1)}\Big(\frac{d-1}{2}-\nu+\mathrm{i}\bar{m}_0\Big)(1-t_0^2)^{-\frac{\nu_\varepsilon}{2}}\Big] \tag{225}$$

in even dimensions.

2. Propagators in the global patch

We now make a sum over all modes to obtain the propagators, $G^{\mathrm{out/in}}(x,x')$ and $G^{\mathrm{in/in}}(x,x')$. For $d\geq 3$, the summation over M can be written with the Gegenbauer polynomials as

$$\sum_{M=1}^{N_L^{(d)}} Y_{LM}(\mathbf{\Omega})Y_{LM}(\mathbf{\Omega}')=\frac{2L+d-2}{(d-2)|\Omega_{d-1}|}C_L^{\frac{d-2}{2}}(\mathbf{\Omega}\cdot\mathbf{\Omega}')\quad (|\Omega_{d-1}|=2\pi^{\frac{d}{2}}/\Gamma(d/2)). \tag{226}$$

As for $d=2$, the sum has the form

$$\sum_{M=1}^{N_L^{(2)}} Y_{LM}(\mathbf{\Omega})Y_{LM}(\mathbf{\Omega}')=\begin{cases}\frac{1}{2\pi} & (L=0),\\ \frac{\cos(L\mathbf{\Omega}\cdot\mathbf{\Omega})}{\pi} & (L\geq 1),\end{cases} \tag{227}$$

which is the same as the $d\to 2$ limit of the expression (226). Thus Eq. (226) can be understood to hold for any dimensionality $d\geq 2$. The in-out propagator in spacetime then takes the form

$$G^{\mathrm{out/in}}(x,x')=\sum_{L=0}^{\infty}G_L^{\mathrm{out/in}}(t,t')\sum_{M=1}^{N_L^{(d)}}Y_{LM}(\mathbf{\Omega})Y_{LM}(\mathbf{\Omega}')=\sum_{L=0}^{\infty}\frac{2L+d-2}{(d-2)|\Omega_{d-1}|}G_L^{\mathrm{out/in}}(t,t')C_L^{\frac{d-2}{2}}(\mathbf{\Omega}\cdot\mathbf{\Omega}'), \tag{228}$$

which becomes

$$G_{\mathrm{odd}}^{\mathrm{out/in}}(x,x')=\frac{\mathrm{i}\pi}{2\sin(\pi\nu)(d-2)|\Omega_{d-1}|}[(1-t_>^2)(1-t_<^2)]^{(d-1)/4}\sum_{L=0}^{\infty}(2L+d-2)\mathsf{P}_k^{-\nu}(t_>)\mathsf{P}_k^{\nu}(t_<)C_L^{\frac{d-2}{2}}(\cos\theta) \tag{229}$$

in odd dimensions, and

$$G_{\mathrm{even}}^{\mathrm{out/in}}(x,x')=\frac{\mathrm{i}}{\cos(\pi\nu)(d-2)|\Omega_{d-1}|}[(1-t_>^2)(1-t_<^2)]^{(d-1)/4}\sum_{L=0}^{\infty}(2L+d-2)\mathsf{P}_k^{-\nu}(t_>)\mathsf{Q}_k^{\nu}(t_<)C_L^{\frac{d-2}{2}}(\cos\theta) \tag{230}$$

in even dimensions. Here, we have defined θ via the relation $\mathbf{\Omega}\cdot\mathbf{\Omega}'\equiv\cos\theta$.

Using Eqs. (E24) and (E25) and introducing

$$u_\pm(x,x') \equiv -Z(x,x') \pm \mathrm{i}0 = -\frac{-tt' + \cos\theta}{(1-t^2)^{\frac{1}{2}}(1-t'^2)^{\frac{1}{2}}} \pm \mathrm{i}0, \tag{231}$$

we can rewrite the in-out propagator in a de Sitter invariant form,

$$G^{\mathrm{out/in}}_{\mathrm{odd}}(x,x') = \frac{-e^{-\mathrm{i}\pi\frac{d-2}{2}}}{2(2\pi)^{\frac{d}{2}}\sin(\pi\nu)}[(u_+^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(u_+) - (u_-^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(u_-)], \tag{232}$$

$$G^{\mathrm{out/in}}_{\mathrm{even}}(x,x') = \frac{\mathrm{i}e^{-\mathrm{i}\pi\frac{d-2}{2}}}{2(2\pi)^{\frac{d}{2}}\cos(\pi\nu)}[(u_+^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(u_+) + (u_-^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(u_-)]. \tag{233}$$

In the massless limit $m \to 0$ (or $\nu \to (d-1)/2$), we have

$$\sin\pi\nu \to 0 \quad (d\text{: odd}), \qquad \cos\pi\nu \to 0 \quad (d\text{: even}), \tag{234}$$

and thus the propagators (232) and (233) diverge. We thus conclude that there exists no finite massless limit in the global patch, as opposed to the case of the Poincaré patch.

On the other hand, the in-in propagator in the heavy mass case ($m > (d-1)/2$) takes the form

$$G^{\mathrm{in/in}}(x,x') = \sum_{L=0}^{\infty}\frac{2L+d-2}{(d-2)|\Omega_{d-1}|}G_L^{\mathrm{in/in}}(t,t')C_L^{\frac{d-2}{2}}(\mathbf{\Omega}\cdot\mathbf{\Omega}'), \tag{235}$$

which becomes

$$G^{\mathrm{in/in}}_{\mathrm{odd}}(x,x') = \frac{\mathrm{i}\pi}{2\sin(\pi\nu)(d-2)|\Omega_{d-1}|}[(1-t_>^2)(1-t_<^2)]^{\frac{d-1}{4}}\sum_{L=0}^{\infty}(2L+d-2)\mathsf{P}_k^{-\nu}(t_>)\mathsf{P}_k^{\nu}(t_<)C_L^{\frac{d-2}{2}}(\cos\theta) \tag{236}$$

in odd dimensions, and

$$G^{\mathrm{in/in}}_{\mathrm{even}}(x,x') = \frac{2\mathrm{i}}{\pi\sin(\pi\nu)(d-2)|\Omega_{d-1}|}[(1-t_>^2)(1-t_<^2)]^{\frac{d-1}{4}}\sum_{L=0}^{\infty}(2L+d-2)\mathsf{Q}_k^{-\nu}(t_>)\mathsf{Q}_k^{\nu}(t_<)C_L^{\frac{d-2}{2}}(\cos\theta) \tag{237}$$

in even dimensions. Here, $\nu = \mathrm{i}\mu = \mathrm{i}\sqrt{m^2-(d-1)^2/4}$ ($\mu \in \mathbb{R}_+$). Note that $G^{\mathrm{in/in}}_{\mathrm{odd}}(x,x') = G^{\mathrm{out/in}}_{\mathrm{odd}}(x,x')$, which is consistent with a well-known fact that the in-vacuum equals the out-vacuum up to a phase in odd dimensions (see [18]). The summations in (236) and (237) can be carried out analytically by using (E24) and (E27) and are again expressed in de Sitter invariant forms,

$$G^{\mathrm{in/in}}_{\mathrm{odd}}(x,x') = G^{\mathrm{out/in}}_{\mathrm{odd}}(x,x') = \frac{-e^{-\mathrm{i}\pi\frac{d-2}{2}}}{2(2\pi)^{\frac{d}{2}}\sin(\pi\nu)}\left[(u_+^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(u_+) - (u_-^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(u_-)\right], \tag{238}$$

$$\begin{aligned} G^{\mathrm{in/in}}_{\mathrm{even}}(x,x') &= \frac{e^{-\mathrm{i}\pi\frac{d-2}{2}}}{2(2\pi)^{\frac{d}{2}}\sin\pi\nu}\Bigg\{(u_+^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(u_+) - (u_-^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(u_-) \\ &\quad + \frac{\mathrm{i}\pi}{\cos\pi\nu}\left[e^{-\mathrm{i}\pi\nu}(u_+^2-1)^{-\frac{d-2}{4}}\mathcal{P}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(u_+) + e^{\mathrm{i}\pi\nu}(u_-^2-1)^{-\frac{d-2}{4}}\mathcal{P}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(u_-)\right]\Bigg\} \\ &= -\frac{e^{-\mathrm{i}\pi\frac{d-2}{2}}}{2(2\pi)^{\frac{d}{2}}\sin\pi\nu}\Bigg\{(u_+^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(u_+) - (u_-^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(u_-) \\ &\quad - \frac{\pi}{\cos(\pi\nu)}\left[(u_+^2-1)^{-\frac{d-2}{4}}\mathcal{P}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-u_+) - (u_-^2-1)^{-\frac{d-2}{4}}\mathcal{P}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-u_-)\right]\Bigg\}. \end{aligned} \tag{239}$$

V. FEYNMAN PATH INTEGRAL IN DE SITTER SPACE

In this section, we consider the Feynman propagator obtained by a path integral in curved spacetime with the background metric (2),

$$\langle\phi(x)\phi(x')\rangle \equiv \frac{\int[\mathrm{d}\phi]\phi(x)\phi(x')e^{\mathrm{i}S_\varepsilon[\phi]}}{\int[\mathrm{d}\phi]e^{\mathrm{i}S_\varepsilon[\phi]}}, \tag{240}$$

$$S_\varepsilon[\phi] \equiv \frac{1}{2}\int \mathrm{d}t\int \mathrm{d}^{d-1}x\sqrt{h}e^{\mathrm{i}\varepsilon}N^{-1}A^{d-1} \times (\partial_t\phi\partial_t\phi + e^{-2\mathrm{i}\varepsilon}N^2[A^{-2}\phi\Delta_{d-1}\phi - m^2\phi^2]). \tag{241}$$

This action gives a Hamiltonian of the form $H_s(t) = e^{-\mathrm{i}\varepsilon}[H_s(t)|_{\varepsilon=0}]$ in the Schrödinger picture. We expect that the propagator defined by the path integral agrees with the in-out propagator obtained in the preceding sections. In fact, suppose that the base spacetime where $\phi(x)$ lives has a sufficiently large noncompact region in the temporal direction near the future and past boundaries at $t = t_f$ and $t = t_i$, respectively (see Fig. 2). Then, due to the existence of iε, if we first define the path integral for a finite interval (t_0, t_1) and send the initial time t_0 and the final time t_1 to the infinite past t_i and the infinite future t_f, respectively, then the initial and final states would be well kept subject to the projection to the instantaneous ground state at each moment t_0 or t_1, and the dominant contribution to the path integral will be only from the configurations that are in the instantaneous ground states near the temporary boundaries. In this section, we first identify the (sufficient) condition under which such projection onto the instantaneous ground states can happen and show that both the Poincaré and the global patches indeed satisfy this condition.

A. Effective noncompactness in the temporal direction

We first expand $\phi(x)$ as in (5),

$$\phi(x) = \sum_n \phi_n(t)Y_n(\boldsymbol{x}). \tag{242}$$

The propagator is then written as a sum of the propagators over the mode n,

$$\langle\phi(x)\phi(x')\rangle = \sum_n \langle\phi_n(t)\phi_n(t')\rangle Y_n(\boldsymbol{x})Y_n(\boldsymbol{x}'), \tag{243}$$

where

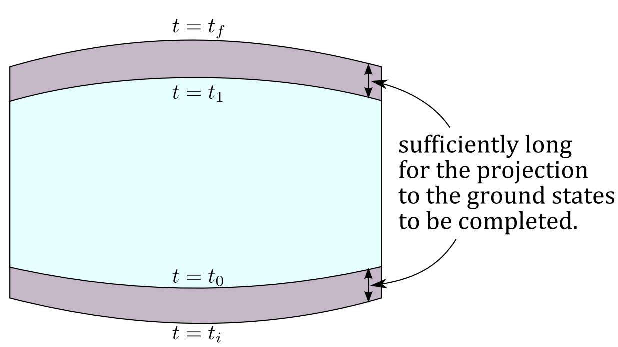

FIG. 2 (color online). The spacetime region where the path integral is performed.

$$\langle\phi_n(t)\phi_n(t')\rangle \equiv \lim_{\substack{t_1\to t_f\\ t_0\to t_i}} \frac{\int[\mathrm{d}\phi_n]_{t_0}^{t_1} e^{\mathrm{i}S_{n,\varepsilon}[\phi_n;t_1,t_0]}\phi_n(t)\phi_n(t')}{\int[\mathrm{d}\phi_n]_{t_0}^{t_1} e^{\mathrm{i}S_{n,\varepsilon}[\phi_n;t_1,t_0]}}, \tag{244}$$

$$S_{n,\varepsilon}[\phi_n; t_1, t_0] \equiv \int_{t_0}^{t_1}\mathrm{d}t\frac{1}{2}[e^{\mathrm{i}\varepsilon}|\rho(t)|\dot{\phi}_n^2(t) - e^{-\mathrm{i}\varepsilon}|\rho(t)||\omega_n(t)|^2\phi_n^2(t)]. \tag{245}$$

For a fixed mode n, the propagator $\langle\phi_n(t)\phi_n(t')\rangle$ can be given the following operator representation in the Schrödinger picture (we assume $t > t'$ in what follows):

$$\lim_{\substack{t_0\to t_i\\ t_1\to t_f}} \frac{\langle\psi_1, t_1|U(t_1,t)\phi_{n,s}U(t,t')\phi_{n,s}U(t',t_0)|\psi_0,t_0\rangle}{\langle\psi_1,t_1|U(t_1,t_0)|\psi_0,t_0\rangle}. \tag{246}$$

Here, $|\psi_1, t_1\rangle$ and $|\psi_0, t_0\rangle$ are the final and initial states to be specified as boundary conditions when performing a path integral, and are formally taken to be $\langle\phi_n|\psi_1, t_1\rangle = \langle\phi_n|\psi_0, t_0\rangle = 1$ for the path integral (244). In the following, we show that the amplitude (246) can be replaced by the in-out propagator

$$\lim_{\substack{t_0\to t_i\\ t_1\to t_f}} \frac{\langle 0_{t_1}, t_1|U(t_1,t)\phi_{n,s}U(t,t')\phi_{n,s}U(t',t_0)|0_{t_0},t_0\rangle}{\langle 0_{t_1},t_1|U(t_1,t_0)|0_{t_0},t_0\rangle} \tag{247}$$

for arbitrary $|\psi_1, t_1\rangle$ and $|\psi_0, t_0\rangle$, provided that the change of time variable $t \to \sigma(t)$ such that $|\omega_n(t)|\mathrm{d}t = \mathrm{d}\sigma$ maps the region (t_i, t_f) onto a noncompact region for both sides, (i.e., $\sigma_i \equiv \sigma(t_i) = -\infty$ and $\sigma_f \equiv \sigma(t_f) = +\infty$). When this condition is met, the foliation under consideration will be said to be effectively noncompact in the temporal direction for the mode n.

It is enough to show that the following equalities hold for an arbitrary state $|\psi\rangle$:

$$\lim_{t_1\to t_f}\langle\psi_1, t_1|U(t_1,t)|\psi\rangle = \lim_{t_1\to t_f}\langle\psi_1,t_1|0_{t_1},t_1\rangle \times\langle 0_{t_1},t_1|U(t_1,t)|\psi\rangle, \tag{248}$$

$$\lim_{t_0\to t_i}\langle\psi|U(t,t_0)|\psi_0,t_0\rangle = \lim_{t_0\to t_i}\langle\psi|U(t,t_0)|0_{t_0},t_0\rangle \times\langle 0_{t_0},t_0|\psi_0,t_0\rangle. \tag{249}$$

To show the first equality, we first introduce a new time coordinate σ such that $|\omega_n(t)|\mathrm{d}t = \mathrm{d}\sigma$, which maps the time interval (t, t_1) to a new interval (σ, σ_1). Then the Hamiltonian for the mode n becomes (we will omit the index n for brevity)[19]

$$H_s(t)\mathrm{d}t = e^{-\mathrm{i}\varepsilon}|\omega(t)|a_s^\dagger(t)a_s(t)\mathrm{d}t = e^{-\mathrm{i}\varepsilon}b_s^\dagger(\sigma)b_s(\sigma)\mathrm{d}\sigma \tag{250}$$

[19]We will discard the zero-point energy in the following discussions.

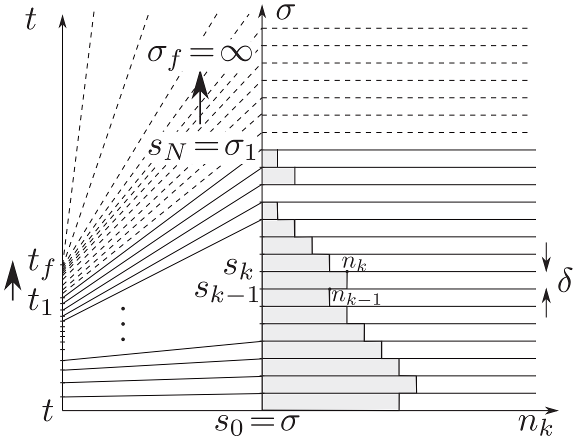

FIG. 3. A path labeled by $\{n_k, s_k\}$. The amplitude suffers from a suppression proportional to the area of the shaded region. If $\sigma_1 \to \sigma_f = \infty$ as $t_1 \to t_f$, only such paths survive that are the instantaneous ground states in the far future (i.e., $n_k = 0$ for large enough k).

with $b_s(\sigma) \equiv a_s(t(\sigma))$, and the time evolution operator becomes

$$U(t_1, t) = \mathrm{T}\exp\left[-\mathrm{i}e^{-\mathrm{i}\varepsilon}\int_\sigma^{\sigma_1} \mathrm{d}\sigma b_s^\dagger(\sigma) b_s(\sigma)\right]. \quad (251)$$

We then introduce a small interval δ and divide the new interval into N segments (see Fig. 3),

$$N = N(\delta, \sigma_1 - \sigma) \equiv \frac{\sigma_1 - \sigma}{\delta}. \quad (252)$$

By introducing $s_k \equiv \sigma + k\delta$ ($k = 0, 1, \ldots, N$) with $s_0 = \sigma$ and $s_N = \sigma_1$, the time evolution operator becomes

$$U(t_1, t) = \prod_{k=1}^{N} \exp[-\mathrm{i}\delta e^{-\mathrm{i}\varepsilon} b_s^\dagger(s_k) b_s(s_k)]. \quad (253)$$

Substituting this to the amplitude $\langle\psi_1, t_1|U(t_1, t)|\psi\rangle$ and inserting the identity $\sum_{n_k=0}^{\infty}|n_k, s_k\rangle\langle n_k, s_k| = 1$ at each time s_k, we obtain

$$\langle\psi, t_1|U(t_1, t)|\psi\rangle = \sum_{\{n_k\}} e^{-\varepsilon\delta\sum_k n_k}\langle\psi_1, t_1|n_N, s_N\rangle\langle n_0, s_0|\psi\rangle \times \prod_{k=1}^{N}\langle n_k, s_k|e^{-\mathrm{i}\delta b_s^\dagger(s_k) b_s(s_k)}|n_{k-1}, s_{k-1}\rangle. \quad (254)$$

We thus find that the amplitude $\langle\psi_1, t_1|U(t_1, t)|\psi\rangle$ is expressed as a sum over the paths, each path corresponding to an evolution of energy levels (not of "positions") and represented by a sequence $\{n_k\}$ ($k = 0, 1, \ldots, N$). We see that each path receives a suppression factor $\exp[-\varepsilon\delta\sum_{k=1}^{N} n_k] = \exp[-\varepsilon \times (\text{area})]$, where (area) is the area of the shaded region in Fig. 3.

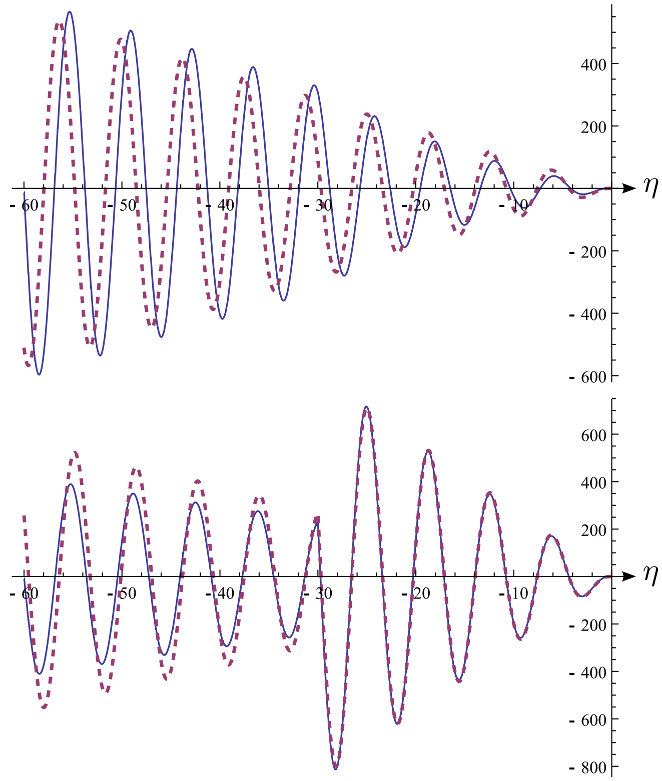

FIG. 4 (color online). Poincaré patch with light mass: the real part (upper) and the imaginary part (lower) of the in-out propagator $G_{\boldsymbol{k}}^{\mathrm{out/in}}(\eta, \eta')$ (dashed curve) and the propagator $\langle\phi_{\boldsymbol{k}}(\eta)\phi_{\boldsymbol{k}}(\eta')\rangle$ (solid curve) are shown for $d = 4$, $m = 0.5$, $k = 1$, $a = 0.02$, $\varepsilon = 0.01$, $\eta_0 = -60$, $\eta_1 = -0.01$, and $\eta' = -30.005$. Recall that $\eta_i = -\infty$ and $\eta_f = 0$.

We now take the limit $t_1 \to t_f$. If the foliation is effectively noncompact in the temporal direction (i.e., if σ_1 approaches $\sigma_f = \infty$), then $N(\delta, \sigma_1 - \sigma)$ goes to infinity as $t_1 \to t_f$ for the fixed small number δ, and thus the suppression factor removes the contribution from any path having a nonvanishing tail for large t and projects onto a set of paths satisfying the condition $n_k \to 0$ ($k \to \infty$). This proves the equality (248). Equality (249) can also be proved in the same way.

We can easily show that both the Poincaré and the global patches are effectively noncompact in the temporal direction for any mode. As for the Poincaré patch, the frequency for the mode $\boldsymbol{k}$ is given by $|\omega_{\boldsymbol{k}}(\eta)| = \sqrt{m^2(-\eta)^{-2} + k^2}$, and thus it behaves as $|\omega_{\boldsymbol{k}}(\eta)| \sim k$ ($\eta \sim \eta_i = -\infty$) and $|\omega_{\boldsymbol{k}}(\eta)| \sim m/(-\eta)$ ($\eta \sim \eta_f = 0$). We thus have

$$-\sigma_i \sim \int_{\eta_i} \mathrm{d}\eta|\omega_{\boldsymbol{k}}(\eta)| = k(-\eta_i) + \mathrm{const} = +\infty,$$
$$\sigma_f \sim \int^{\eta_f} \mathrm{d}\eta|\omega_{\boldsymbol{k}}(\eta)| = m\log\frac{1}{-\eta_f} + \mathrm{const} = +\infty. \quad (255)$$

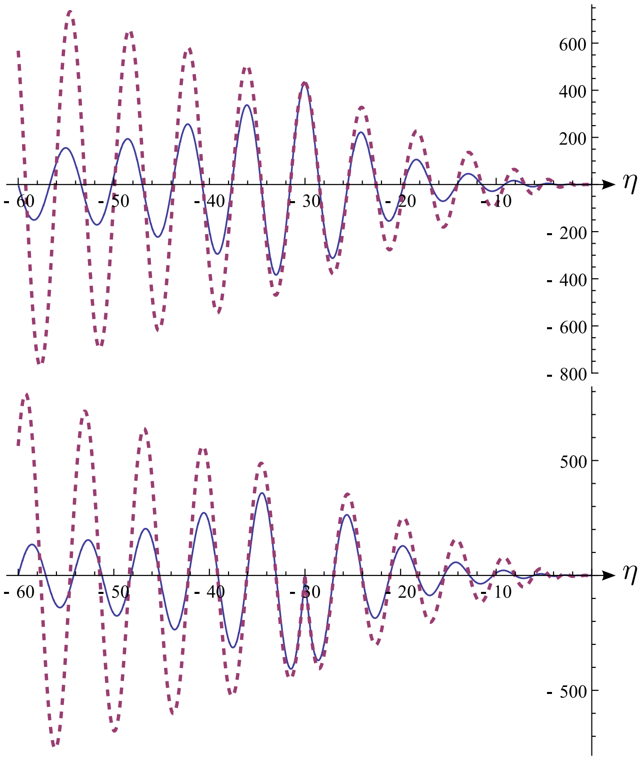

FIG. 5 (color online). Poincaré patch with heavy mass: the real part (upper) and the imaginary part (lower) of the in-out propagator $G^{\rm out/in}_{\boldsymbol{k}}(\eta, \eta')$ (dashed curve) and the propagator $\langle \phi_{\boldsymbol{k}}(\eta)\phi_{\boldsymbol{k}}(\eta')\rangle$ (solid curve) are shown for $d = 4$, $m = 9$, $k = 1$, $a = 0.02$, $\varepsilon = 0.07$, $\eta_0 = -60$, $\eta_1 = -0.01$, and $\eta' = -30.005$. Recall that $\eta_i = -\infty$ and $\eta_f = 0$.

This shows that the Poincaré patch is effectively noncompact for nonvanishing modes $\boldsymbol{k}$.[20] As for the global patch, the frequency for the mode (L, M) is given by $|\omega_L(t)| = (1 - t^2)^{-1}\sqrt{m^2 + L(L + d - 2)(1 - t^2)}$, and thus it behaves as $|\omega_L(t)| \sim (m/2)(1 + t)^{-1}$ $(t \sim t_i = -1)$ and $|\omega_L(t)| \sim (m/2)(1 - t)^{-1}$ $(t \sim t_f = +1)$. We thus have

$$-\sigma_i \sim \int_{t_i} dt|\omega_L(t)| = \frac{m}{2}\log\frac{1}{1 + t_i} + \text{const} = +\infty,$$
$$\sigma_f \sim \int^{t_f} dt|\omega_L(t)| = \frac{m}{2}\log\frac{1}{1 - t_f} + \text{const} = +\infty. \qquad (256)$$

This shows that the global patch is effectively noncompact for any mode (L, M).

Once the equivalence is established, we can easily understand why the obtained in-out propagators are written with the de Sitter invariant $Z(x, x')$. In fact, since the patches we consider are preserved under infinitesimal actions of SO$(1, d)$, and since a path integral (for a free scalar field) can be defined as respecting the symmetry under the infinitesimal actions of SO$(1, d)$ (which is indeed the case only after we take the limit $\varepsilon \to 0$), the propagator obtained by such path integral (and thus the in-out propagator) must be invariant under the infinitesimal actions of SO$(1, d)$. As was mentioned in the last paragraph of Sec. IVA, this invariance is sufficient to ensure that the propagator can be written with the de Sitter invariant $Z(x, x')$.

[20]The zero mode $\boldsymbol{k} = 0$ does not satisfy the condition. However, since the mode belongs to a continuous spectrum, this does not give rise to a problem.

B. Numerical check

In this subsection, we numerically demonstrate that the equivalence between the two propagators certainly holds, one obtained by a path integral with the iε prescription and another obtained as the in-out propagator using the instantaneous ground states.

We first rewrite the action (245) with a new variable $\chi_n(t) = |\rho(t)|^{1/2}\phi_n(t) \equiv e^{\sigma(t)}\phi_n(t)$ (we will omit the mode label n for simplicity). Then, the action for each mode has the form

$$S_\varepsilon[\chi; t_1, t_0] = \int_{t_0}^{t_1} dt\frac{1}{2}\chi(t)[-e^{i\varepsilon}\partial_t^2 - \Omega_\varepsilon^2(t)]\chi(t), \qquad (257)$$

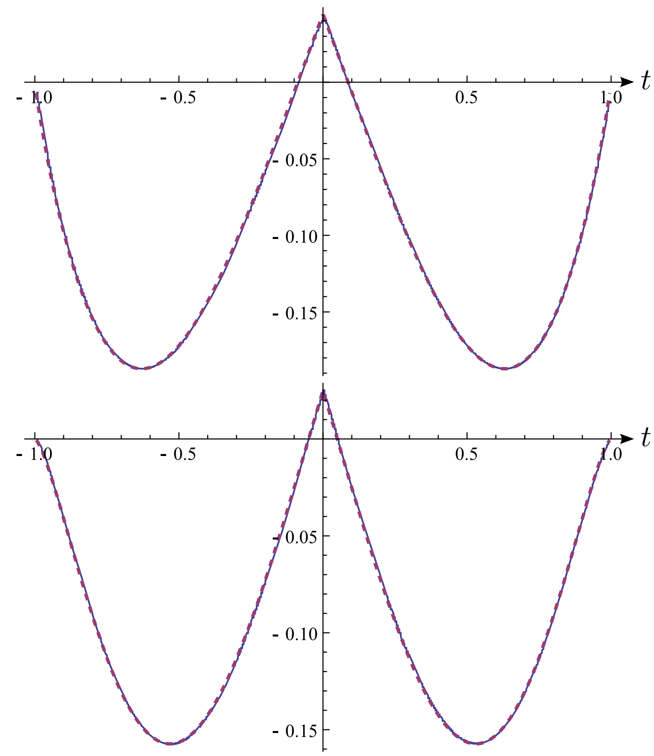

FIG. 6 (color online). Global patch with light mass (d: odd/even): the imaginary part of the in-out propagator $G^{\rm out/in}_L(t, 0)$ (dashed curve) and the propagator $\langle \phi_L(t)\phi_L(0)\rangle$ (solid curve) are shown for $d = 3$ (upper) [$d = 4$ (lower)], $m = 0.5$, $L = 2$, $a = 0.005$, $\varepsilon = 10^{-10}$, $t_0 = -0.995$, and $t_1 = 0.995$. Recall that $t_i = -1$ and $t_f = +1$. The real part is zero.

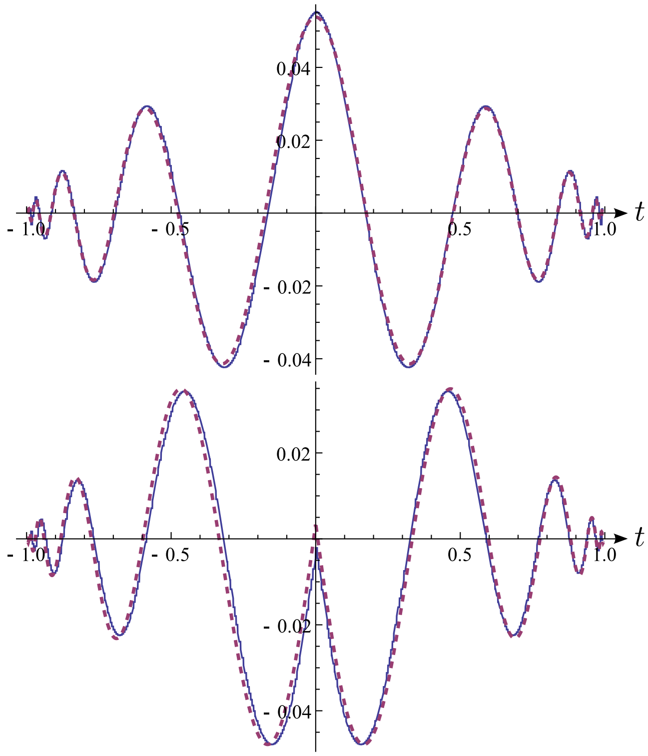

FIG. 7 (color online). Global patch with heavy mass (d: odd): the real part (upper) and the imaginary part (lower) of the in-out propagator $G_L^{\rm out/in}(t,0)$ (dashed curve) and the propagator $\langle\phi_L(t)\phi_L(0)\rangle$ (solid curve) are shown for $d=3$, $m=9$, $L=2$, $a=0.005$, $\varepsilon=0.07$, $t_0=-0.995$, and $t_1=0.995$. Recall that $t_i=-1$ and $t_f=+1$.

$$\Omega_\varepsilon^2(t)\equiv e^{-{\rm i}\varepsilon}|\omega(t)|^2-e^{{\rm i}\varepsilon}(\dot\sigma(t))^2-e^{{\rm i}\varepsilon}\ddot\sigma(t), \tag{258}$$

and the propagator for each mode is given by

$$\begin{aligned}\langle\phi(t)\phi(t')\rangle &= |\rho(t)\rho(t')|^{-1/2}\langle\chi(t)\chi(t')\rangle\\ &= |\rho(t)\rho(t')|^{-1/2}\langle t|\frac{\rm i}{-e^{{\rm i}\varepsilon}\partial_t^2-\Omega_\varepsilon^2}|t'\rangle.\end{aligned} \tag{259}$$

We numerically evaluate the propagator (259) by dividing the interval (t_i,t_f) into N parts and by calculating the inverse of the matrix corresponding to ${\rm i}^{-1}(-e^{{\rm i}\varepsilon}\partial_t^2-\Omega_\varepsilon^2)$. We take a uniform spacing $a\equiv(t_1-t_0)/N$ for brevity and write the time variable as $t=ar$ with r an integer in the region $r_0<r<r_1$ ($r_0\equiv t_0/a$ and $r_1\equiv t_1/a$). We then introduce dimensionless variables χ_r as

$$\chi(t)=a^{1/2}\chi_r, \tag{260}$$

with which the action becomes

$$S_\varepsilon[\chi]=\frac{1}{2}\sum_{r,s}S_{\varepsilon,rs}\chi_r\chi_s, \tag{261}$$

$$S_{\varepsilon,rs}\equiv[2e^{{\rm i}\varepsilon}-a^2\Omega_\varepsilon(ar)]\delta_{r,s}-e^{{\rm i}\varepsilon}\delta_{r,s+1}-e^{{\rm i}\varepsilon}\delta_{r,s-1}. \tag{262}$$

Since

$$\langle\chi_r\chi_{r'}\rangle=\frac{\int{\rm d}^N\chi\,\chi_r\chi_{r'}\exp\left(\frac{\rm i}{2}\sum S_{\varepsilon,ss'}\chi_s\chi_{s'}\right)}{\int{\rm d}^N\chi\exp\left(\frac{\rm i}{2}\sum S_{\varepsilon,ss'}\chi_s\chi_{s'}\right)}={\rm i}(S_\varepsilon^{-1})_{rr'}, \tag{263}$$

the propagator is obtained as

$$\begin{aligned}\langle\phi(t)\phi(t')\rangle &= {\rm i}a|\rho(t)\rho(t')|^{-1/2}(S_\varepsilon^{-1})_{rr'}\\ &(t=ar,\ t'=ar').\end{aligned} \tag{264}$$

We numerically calculate the inverse matrix (264) for both of the Poincaré and global patches and compare the result with our in-out propagators obtained in Sec. IV.

The results in the Poincaré case for $\{d=4,m=0.5\}$ and $\{d=4,m=9\}$ are depicted in Figs. 4 and 5, while those in the global patch for $\{d=3,4,m=0.5\}$, $\{d=3,m=9\}$, and $\{d=4,m=9\}$ are in Figs. 6–8. We find that there is a perfect agreement for the global patch, while there exists a small discrepancy for the Poincaré patch. We observe that the discrepancy gets reduced as one takes a finer mesh near $\eta=0$ and a larger value for $|\eta_0|$, and we expect that it will disappear eventually. We thus are almost

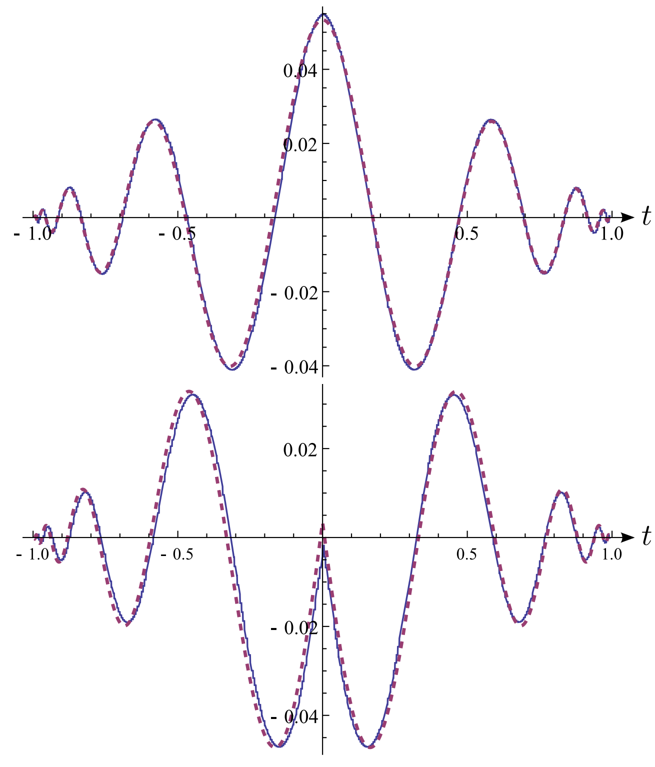

FIG. 8 (color online). Global patch with heavy mass (d: even): the real part (upper) and the imaginary part (lower) of the in-out propagator $G_L^{\rm out/in}(t,0)$ (dashed curve) and the propagator $\langle\phi_L(t)\phi_L(0)\rangle$ (solid curve) are shown for $d=4$, $m=9$, $L=1$, $a=0.005$, $\varepsilon=0.07$, $t_0=-0.995$, and $t_1=0.995$. Recall that $t_i=-1$ and $t_f=+1$.

convinced that the equivalence between the two propagators is confirmed numerically.

VI. HEAT KERNEL REPRESENTATION AND THE COMPOSITION PRINCIPLE

A. General theory

We consider the random walk of a relativistic particle moving in a Lorentzian manifold with the metric

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{265}$$

Its trajectory is uniquely specified by the functions $X^\mu(\lambda)$ $(0 \le \lambda \le 1)$, up to reparametrizations $\lambda \to f(\lambda)$ such that $\mathrm{d}f(\lambda)/\mathrm{d}\lambda > 0,\ f(0) = 0,\ f(1) = 1$. The amplitude connecting two points x and x' is then given by the Feynman path integral,

$$A(x, x') = \int_{X(0)=x'}^{X(1)=x} \frac{[\mathrm{d}X^\mu(\lambda)]}{\mathrm{Vol(Diff}_1)} e^{\mathrm{i}I_0[X(\lambda)]}, \tag{266}$$

where $\mathrm{Vol(Diff}_1)$ is the gauge volume of one-dimensional diffeomorphisms

$$X^\mu(\lambda) \to \tilde{X}^\mu(\lambda) = X^\mu(f(\lambda)), \tag{267}$$

and we propose to set the action $I_0[X]$ for the random walk in a Lorentzian manifold as

$$I_0[X(\lambda)] \equiv -(m - \mathrm{i}\varepsilon) \times \int_0^1 \mathrm{d}\lambda \sqrt{-g_{\mu\nu}(X(\lambda))\dot{X}^\mu(\lambda)\dot{X}^\nu(\lambda) - \mathrm{i}\varepsilon'}. \tag{268}$$

Note the presence of two infinitesimal imaginary parts, $\mathrm{i}\varepsilon$ and $\mathrm{i}\varepsilon'$, in $I_0[X(\lambda)]$ $(\varepsilon, \varepsilon' > 0)$. The first $(\mathrm{i}\varepsilon)$ is the standard one, which manifestly suppresses the contribution from such paths that are prolonged in the timelike direction. We further have introduced the second one $(\mathrm{i}\varepsilon')$ in order to define the path integral for any shape of path in a Lorentzian manifold. In fact, for a timelike segment $(\dot{X}^2 < 0)$, we can neglect ε' and the action becomes the standard action for a timelike path, while for a spacelike segment $(\dot{X}^2 > 0)$, we can rewrite the square root as

$$\sqrt{-\dot{X}^2 - \mathrm{i}\varepsilon'} = \sqrt{e^{-\mathrm{i}(\pi-0)}\dot{X}^2} = -\mathrm{i}\sqrt{\dot{X}^2} \tag{269}$$

and the action gives the path-integral weight which suppresses the contribution from such paths that are stretched largely in the spacelike direction.

The action (268) of the Nambu-Goto type is equivalent to the following action of the Polyakov type:

$$I[X(\lambda), e(\lambda)] = \int_0^1 \mathrm{d}\lambda \left[\frac{\dot{X}^2 + \mathrm{i}\varepsilon'}{2e} - \frac{m^2 - \mathrm{i}\varepsilon}{2} e\right], \tag{270}$$

where $e(\lambda) > 0$ is the einbein defined on the one-dimensional manifold. We can see from this expression that $\mathrm{i}\varepsilon$ and $\mathrm{i}\varepsilon'$ give imaginary parts of the same sign. The new action $I[X(\lambda), e(\lambda)]$ also has the invariance under the one-dimensional diffeomorphisms

$$X^\mu(\lambda) \to \tilde{X}^\mu(\lambda) = X^\mu(f(\lambda)), \tag{271}$$

$$e(\lambda) \to \tilde{e}(\lambda) = \frac{\mathrm{d}f(\lambda)}{\mathrm{d}\lambda} e(f(\lambda)), \tag{272}$$

and we can take a gauge fixing where $e(\lambda) = \text{constant} \equiv T$. However, as is discussed in detail in [25], such constant $T = \int_0^1 \mathrm{d}\lambda e(\lambda)$ is actually Diff_1 invariant and needs to be further integrated, so that we obtain[21]

$$\begin{aligned} A(x, x') &= \int_{X(0)=x'}^{X(1)=x} \frac{[\mathrm{d}X^\mu(\lambda)\mathrm{d}e(\lambda)]}{\mathrm{Vol(Diff}_1)} e^{\mathrm{i}I_0[X(\lambda)]} \\ &= \int_0^\infty \mathrm{d}T \int_{X(0)=x'}^{X(1)=x} [\mathrm{d}X(\lambda)] \\ &\quad \times \exp\left[\mathrm{i}\int_0^1 \mathrm{d}\lambda \left(\frac{\dot{X}^2 + \mathrm{i}\varepsilon'}{2T} - \frac{m^2 - \mathrm{i}\varepsilon}{2} T\right)\right] \\ &= \int_0^\infty \mathrm{d}T e^{-\varepsilon'/(2T)} \int_{X(0)=x'}^{X(T)=x} [\mathrm{d}X(t)] \\ &\quad \times \exp\left[\mathrm{i}\int_0^T \mathrm{d}t \left(\frac{\dot{X}^2(t) + \mathrm{i}\varepsilon'}{2} - \frac{m^2 - \mathrm{i}\varepsilon}{2}\right)\right], \end{aligned} \tag{273}$$

where in the last line we have rewritten the expression with $t \equiv T\lambda$. The path integral is nothing but that for the quantum mechanical amplitude from the state $|x'\rangle$ to the state $|x\rangle$ with the Hamiltonian

$$\begin{aligned} H &= \frac{1}{2}(-\Box + m^2 - \mathrm{i}\varepsilon) \\ &\equiv \frac{1}{2}\left[-\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) + m^2 - \mathrm{i}\varepsilon\right], \end{aligned} \tag{274}$$

and thus we obtain the expression

$$\begin{aligned} A(x, x') &= \int_0^\infty \mathrm{d}T e^{-\varepsilon'/(2T)} \langle x| e^{-\mathrm{i}(T/2)(-\Box + m^2 - \mathrm{i}\varepsilon)} |x'\rangle \\ &= \int_0^\infty \mathrm{d}T e^{-\mathrm{i}(T/2)(m^2 - \mathrm{i}\varepsilon) - \varepsilon'/(2T)} K(x, x'; T). \end{aligned} \tag{275}$$

Here, $K(x, x'; T)$ is the heat kernel of the d'Alembertian $\Box$,

$$K(x, x'; T) \equiv \langle x| e^{\mathrm{i}(T/2)\Box} |x'\rangle, \tag{276}$$

which satisfies the following equations:

$$\mathrm{i}\frac{\partial}{\partial T} K(x, x'; T) = -\frac{1}{2}\Box_x K(x, x'; T), \tag{277}$$

[21] There may arise a divergence when calculating the Jacobian to obtain the second line, but such divergence should be ultralocal in quantum mechanics (i.e., one-dimensional field theory with a coordinate λ) and can be simply dealt with by an additive renormalization of mass m, as in the Euclidean space considered in [25].

$$K(x,x';T=0)=\delta^d(x,x')\equiv\frac{1}{\sqrt{-g}}\delta^d(x-x'). \quad (278)$$

Note that we need to multiply (273) by 1/2 to obtain the propagator $G(x,x')$ of a neutral particle (i.e., particle = antiparticle),

$$\begin{aligned}G(x,x')&=\frac{1}{2}A(x,x')\\&=\frac{1}{2}\int_0^\infty \mathrm{d}Te^{-\mathrm{i}(T/2)(m^2-\mathrm{i}\varepsilon)-\varepsilon'/(2T)}K(x,x';T).\end{aligned} \quad (279)$$

Since the propagator $G(x,x')$ can formally be written as $G(x,x')=\mathrm{i}\langle x|(\Box-m^2+\mathrm{i}\varepsilon)^{-1}|x'\rangle$, one can easily show that $G(x,x')$ satisfies the following composition law [21]:

$$\frac{\partial}{\partial m^2}G(x,x')=\mathrm{i}\int\sqrt{-g(y)}\mathrm{d}^dyG(x,y)G(y,x'), \quad (280)$$

which is consistent with the asymptotic form of $G(x,x')$ for large timelike separation with large mass

$$G(x,x')\sim e^{-\mathrm{i}mL(x,x')}, \quad (281)$$

where $L(x,x')$ is the timelike geodesic distance between x and x'. The relation (280) has been proposed by Polyakov as a principle to be satisfied by quantum field theory in curved spacetime in order for the propagator to be interpreted as representing a sum over paths of a relativistic particle in the spacetime. If the spacetime has a global timelike Killing vector (as does Minkowski space), one can define a common vacuum of scalar field from the past through the future, and the relativistic particle corresponds to a one-particle state. Note that such interpretation is not always possible when spacetime has no global timelike Killing vector [1] (see also [26] for a recent discussion).

As a simple example, we consider a neutral particle propagating in a d-dimensional Minkowski space with the metric

$$\mathrm{d}s^2=-\mathrm{d}t^2+\mathrm{d}\boldsymbol{x}^2. \quad (282)$$

Then the heat kernel can be calculated with the momentum representation as

$$\begin{aligned}K(x,x';T)&=\int\frac{\mathrm{d}^dp}{(2\pi)^d}e^{\mathrm{i}p(x-x')}e^{-\mathrm{i}(T/2)p^2}\\&=\mathrm{i}\Big(\frac{1}{2\pi\mathrm{i}T}\Big)^{d/2}\exp\Big[\mathrm{i}\frac{(x-x')^2}{2T}\Big].\end{aligned} \quad (283)$$

Here, the first i reflects the fact that the Gaussian integral over p_0 has the opposite sign of the quadratic term to that for the other variables p_i $(i=1,\ldots,d-1)$. Substituting this to (279), we obtain the following expression:

$$G(x,x')=\frac{\mathrm{i}^{-(d-2)/2}}{(4\pi)^{d/2}}\int_0^\infty \mathrm{d}s s^{-\frac{d}{2}}e^{-\frac{z^2}{4s}-as}, \quad (284)$$

with $z^2=\varepsilon'-\mathrm{i}(x-x')^2$ and $a=\mathrm{i}(m^2-\mathrm{i}\varepsilon)$. This integration can be easily performed, and we obtain

$$\begin{aligned}G(x,x')&=\frac{\mathrm{i}^{-(d-2)/2}}{(4\pi)^{d/2}}a^{\frac{d-2}{2}}2^{\frac{d}{2}}(\sqrt{a}z)^{-\frac{d-2}{2}}K_{\frac{d-2}{2}}(\sqrt{a}z)\\&=\frac{m^{d-2}}{(2\pi)^{d/2}}(m\sqrt{\sigma+\mathrm{i}\varepsilon'})^{-\frac{d-2}{2}}K_{\frac{d-2}{2}}(m\sqrt{\sigma+\mathrm{i}\varepsilon'}),\end{aligned} \quad (285)$$

where $\sigma\equiv(x-x')^2$. This certainly agrees with the propagator (118) of a real scalar field.

B. de Sitter case

In this subsection, we check that the in-out propagators (183), (232), and (233) indeed satisfy the composition law by giving their heat kernel representations.[22]

1. Poincaré patch

We start from the following integral representation of the associated Legendre functions (a proof is given in Appendix F):

$$\begin{aligned}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-1/2}(u)=e^{\mathrm{i}\pi\frac{d-2}{2}}\int_0^\infty \mathrm{d}\lambda\frac{\Gamma(\frac{d-1}{2}+\mathrm{i}\lambda)\Gamma(\frac{d-1}{2}-\mathrm{i}\lambda)}{\Gamma(\mathrm{i}\lambda)\Gamma(-\mathrm{i}\lambda)}\\\times\frac{\mathcal{P}^{-\frac{d-2}{2}}_{\mathrm{i}\lambda-1/2}(u)}{\nu^2+\lambda^2}\quad[d\in\mathbb{Z},\mathrm{Re}\nu>0].\end{aligned} \quad (286)$$

Since ν_ε^2 has a positive imaginary part, we have

$$\begin{aligned}&e^{-\mathrm{i}\pi\frac{d-2}{2}}(u^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu_\varepsilon-1/2}(u)\\&=\frac{1}{2\mathrm{i}}\int_0^\infty \mathrm{d}T\int_0^\infty \mathrm{d}\lambda\frac{\Gamma(\frac{d-1}{2}+\mathrm{i}\lambda)\Gamma(\frac{d-1}{2}-\mathrm{i}\lambda)}{\Gamma(\mathrm{i}\lambda)\Gamma(-\mathrm{i}\lambda)}\\&\quad\times(u^2-1)^{-\frac{d-2}{4}}\mathcal{P}^{-\frac{d-2}{2}}_{\mathrm{i}\lambda-1/2}(u)e^{\mathrm{i}\frac{T}{2}(\nu_\varepsilon^2+\lambda^2)}\\&=\frac{\mathrm{i}}{2^{2-\frac{d}{2}}}\int_0^\infty \mathrm{d}T\int_0^\infty \mathrm{d}\lambda\frac{\lambda\Gamma(\frac{d-1}{2})\sinh(\pi\lambda)}{\sqrt{\pi}\cos[\pi(\frac{d}{2}-\mathrm{i}\lambda)]}\\&\quad\times C^{\frac{d-1}{2}}_{\mathrm{i}\lambda-\frac{d-1}{2}}(u)e^{\mathrm{i}\frac{T}{2}(\nu_\varepsilon^2+\lambda^2)}.\end{aligned} \quad (287)$$

Here, $u=Z-\mathrm{i}0$, and to obtain the second line we have used the identity

$$\begin{aligned}\mathcal{P}^{-\frac{d-2}{2}}_{\mathrm{i}\lambda-\frac{1}{2}}(u)=2^{-\frac{d-2}{2}}(u^2-1)^{\frac{d-2}{4}}\\\times\frac{\Gamma(d-1)\Gamma(\mathrm{i}\lambda-\frac{d-3}{2})}{\Gamma(d/2)\Gamma(\mathrm{i}\lambda+\frac{d-1}{2})}C^{\frac{d-1}{2}}_{\mathrm{i}\lambda-\frac{d-1}{2}}(u).\end{aligned} \quad (288)$$

We thus find that the in-out propagator (183) in the Poincaré patch has the heat kernel representation of the form

[22] See [27] (also [21]) for the direct evaluation of the random walk in de Sitter space, which is based on the heat kernel for Euclidean AdS space obtained in [28].

$$G(x,x') = \frac{1}{2}\int_0^\infty \mathrm{d}T e^{-\mathrm{i}\frac{m^2-\mathrm{i}\varepsilon}{2}T} K(x,x';T), \tag{289}$$

$$\begin{aligned} K(x,x';T) &= -\frac{e^{-\mathrm{i}\pi\frac{d-3}{2}}(u^2-1)^{-\frac{d-2}{4}}}{(2\pi)^{d/2}} \int_0^\infty \mathrm{d}\lambda \frac{\Gamma(\frac{d-1}{2}+\mathrm{i}\lambda)\Gamma(\frac{d-1}{2}-\mathrm{i}\lambda)}{\Gamma(\mathrm{i}\lambda)\Gamma(-\mathrm{i}\lambda)} \mathcal{P}^{-\frac{d-2}{2}}_{\mathrm{i}\lambda-1/2}(u) e^{\mathrm{i}\frac{T}{2}(\lambda^2+(\frac{d-1}{2})^2)} \\ &= \frac{e^{-\mathrm{i}\pi\frac{d-3}{2}}\Gamma(\frac{d-1}{2})}{2\pi^{\frac{d+1}{2}}} \int_0^\infty \mathrm{d}\lambda \frac{\lambda\sinh(\pi\lambda)}{\cos[\pi(\frac{d}{2}-\mathrm{i}\lambda)]} C^{\frac{d-1}{2}}_{\mathrm{i}\lambda-\frac{d-1}{2}}(u) e^{\mathrm{i}\frac{T}{2}(\lambda^2+(\frac{d-1}{2})^2)}. \end{aligned} \tag{290}$$

The above heat kernel certainly satisfies Eqs. (277) and (278). In fact, Eq. (277) can be shown in the following way:

$$\begin{aligned} \mathrm{i}\frac{\partial}{\partial T}K(x,x';T) &= -\frac{e^{-\mathrm{i}\pi\frac{d-3}{2}}\Gamma(\frac{d-1}{2})}{4\pi^{\frac{d+1}{2}}} \int_0^\infty \mathrm{d}\lambda \frac{\lambda\sinh(\pi\lambda)(\lambda^2+(\frac{d-1}{2})^2)}{\cos[\pi(\frac{d}{2}-\mathrm{i}\lambda)]} C^{\frac{d-1}{2}}_{\mathrm{i}\lambda-\frac{d-1}{2}}(u) e^{\mathrm{i}\frac{T}{2}(\lambda^2+(\frac{d-1}{2})^2)} \\ &= -\frac{1}{2}[-(Z^2-1)\partial_Z^2 K(x,x';T) - dZ\partial_Z K(x,x';T)] = -\frac{1}{2}\Box_x K(x,x';T), \end{aligned} \tag{291}$$

where we have used the Gegenbauer differential equation,

$$(1-u^2)\partial_u^2 C^{\frac{d-1}{2}}_{\mathrm{i}\lambda-\frac{d-1}{2}}(u) - du\partial_u C^{\frac{d-1}{2}}_{\mathrm{i}\lambda-\frac{d-1}{2}}(u) - \left[\lambda^2 + \left(\frac{d-1}{2}\right)^2\right] C^{\frac{d-1}{2}}_{\mathrm{i}\lambda-\frac{d-1}{2}}(u) = 0, \tag{292}$$

and the fact that the Klein-Gordon operator for functions of the de Sitter invariant $f(Z)$ can be written as

$$\Box f(Z) = (1-Z^2)\partial_Z^2 f(Z) - dZ\partial_Z f(Z). \tag{293}$$

The initial condition (278) can be shown to hold by using the heat kernel equation and the equality

$$(\Box_x - m^2 + \mathrm{i}\varepsilon)G^{\mathrm{out/in}}(x,x') = \frac{\mathrm{i}}{\sqrt{-g}}\delta^d(x-x') \tag{294}$$

as follows:

$$\begin{aligned} &\frac{\mathrm{i}}{\sqrt{-g}}\delta^d(x-x') \\ &= \frac{1}{2}\int_0^\infty \mathrm{d}T e^{-\mathrm{i}\frac{m^2-\mathrm{i}\varepsilon}{2}T}(\Box_x - m^2 + \mathrm{i}\varepsilon)K(x,x';T) \\ &= -\mathrm{i}\int_0^\infty \mathrm{d}T \frac{\partial}{\partial T}[e^{-\mathrm{i}\frac{m^2-\mathrm{i}\varepsilon}{2}T}K(x,x';T)] \\ &= \mathrm{i}K(x,x';0). \end{aligned} \tag{295}$$

2. Global patch

In a similar way, using Eq. (287), we can show that the in-out propagators (232) and (233) have the heat kernel representation of the form

$$G^{\mathrm{out/in}}_{\{\substack{\mathrm{odd}\\ \mathrm{even}}\}}(x,x') = \frac{1}{2}\int_0^\infty \mathrm{d}T e^{-\mathrm{i}\frac{m^2-\mathrm{i}\varepsilon}{2}T} K^{\mathrm{out/in}}_{\{\substack{\mathrm{odd}\\ \mathrm{even}}\}}(x,x';T) \tag{296}$$

with

$$\begin{aligned} &K^{\mathrm{out/in}}_{\mathrm{odd}}(x,x';T) \\ &= \frac{(-1)^{\frac{d+1}{2}}\Gamma(\frac{d-1}{2})}{4\pi^{\frac{d+1}{2}}\sin(\pi\nu)} \\ &\times \int_0^\infty \mathrm{d}\lambda\lambda\Big[C^{\frac{d-1}{2}}_{\mathrm{i}\lambda-\frac{d-1}{2}}(u_+) - C^{\frac{d-1}{2}}_{\mathrm{i}\lambda-\frac{d-1}{2}}(u_-)\Big] e^{\mathrm{i}\frac{T}{2}(\lambda^2+(\frac{d-1}{2})^2)}, \end{aligned} \tag{297}$$

$$\begin{aligned} &K^{\mathrm{out/in}}_{\mathrm{even}}(x,x';T) \\ &= \frac{(-1)^{\frac{d+2}{2}}\Gamma(\frac{d-1}{2})}{4\pi^{\frac{d+1}{2}}\cos(\pi\nu)} \int_0^\infty \mathrm{d}\lambda\lambda\tanh(\pi\lambda) \\ &\times \Big[C^{\frac{d-1}{2}}_{\mathrm{i}\lambda-\frac{d-1}{2}}(u_+) + C^{\frac{d-1}{2}}_{\mathrm{i}\lambda-\frac{d-1}{2}}(u_-)\Big] e^{\mathrm{i}\frac{T}{2}(\lambda^2+(\frac{d-1}{2})^2)}. \end{aligned} \tag{298}$$

Equations (277) and (278) also hold for these heat kernels, as can be shown in the same way as above.

C. Relation to the Green function in Euclidean AdS space

As has been pointed out in [27], the in-out propagator in the Poincaré patch is directly related to the Green function in Euclidean anti–de Sitter (AdS) space through an analytic continuation. In this subsection, we demonstrate this equivalence with precise numerical constants.

We define d-dimensional Euclidean AdS space (EAdS_d) as the hypersurface in a $(d+1)$-dimensional Minkowski space with the relation

$$\eta_{MN}Y^M Y^N = -\ell'^2 \quad (M,N = 0,\ldots,d), \tag{299}$$

where ℓ' is called the AdS radius. EAdS_d has two connected components. A frequently used coordinate system which covers only a single connected component is the

Poincaré coordinates (z, y^i) $(i = 1, \ldots, d-1)$ that are defined by the following embedding:

$$Y^0 = \frac{\ell'^2 + z^2 + |\mathbf{y}|^2}{2z}, \qquad Y^i = \ell' \frac{y^i}{z}, \qquad Y^d = \frac{\ell'^2 - z^2 - |\mathbf{y}|^2}{2z}. \tag{300}$$

We here have chosen the component with $z > 0$. The metric then takes the form

$$ds^2 = \ell'^2 \frac{dz^2 + d\mathbf{y} \cdot d\mathbf{y}}{z^2}. \tag{301}$$

The Green function in Euclidean AdS space is known to have the following form (see, e.g., [29])[23]:

$$G_{\mathrm{EAdS}}(y, y') = \frac{e^{-i\pi(d-2)/2}}{(2\pi)^{d/2}\ell'^{d-2}} (Z'^2(y, y') - 1)^{-\frac{d-2}{4}} Q^{(d-2)/2}_{\nu'-1/2}(Z'(y, y')), \tag{302}$$

where $Z'(y, y')$ is the invariant of Euclidean AdS space,

$$Z'(y, y') \equiv -\ell'^{-2}\eta_{MN}Y^M(y)Y^N(y') = 1 + \frac{(z - z')^2 + |\mathbf{y} - \mathbf{y}'|^2}{2zz'}, \tag{303}$$

and

$$\nu' \equiv \sqrt{\left(\frac{d-1}{2}\right)^2 + m^2\ell'^2}. \tag{304}$$

Note that Z' is always larger than unity.

The coordinate system (z, y^i) is related to the Poincaré coordinates (η, x^i) of d-dimensional de Sitter space through the analytic continuation

$$z = e^{i\frac{\pi-0}{2}}(-\eta), \qquad y^i = x^i, \qquad \ell' = e^{i\frac{\pi-0}{2}}\ell, \tag{305}$$

or equivalently,

$$Y^0 = iX^d, \qquad Y^i = X^i, \qquad Y^d = iX^0. \tag{306}$$

In fact, one can easily show that the metrics of EAdS_d and dS_d transform to each other. One also finds the relations

$$\nu' = \sqrt{\left(\frac{d-1}{2}\right)^2 + m^2\ell'^2} = \sqrt{\left(\frac{d-1}{2}\right)^2 - m^2\ell^2 + i0} = \nu_\varepsilon, \tag{307}$$

$$Z'(y, y') = \frac{z^2 + z'^2 + |\mathbf{y} - \mathbf{y}'|^2}{2zz'} = \frac{(-\eta)^2 + (-\eta')^2 - |\mathbf{x} - \mathbf{x}'|^2}{2\eta\eta'} - i0 = Z(x, x') - i0, \tag{308}$$

with which the Green function on Euclidean AdS space can be rewritten as

$$G_{\mathrm{EAdS}}(y, y') = \frac{e^{-i\pi(d-2)}}{(2\pi)^{d/2}\ell^{d-2}} (u^2 - 1)^{-\frac{d-2}{4}} Q^{(d-2)/2}_{\nu_\varepsilon-1/2}(u) \quad (u = Z(x, x') - i0). \tag{309}$$

This agrees with the in-out propagator (183) in the Poincaré patch of de Sitter space,

$$G^{\mathrm{out/in}}(x, x') = \frac{e^{-i\pi(d-2)}}{(2\pi)^{d/2}\ell^{d-2}} (u^2 - 1)^{-\frac{d-2}{4}} Q^{(d-2)/2}_{\nu-1/2}(u) \quad (u = Z(x, x') - i0). \tag{310}$$

As pointed out in [18], the Green function of Euclidean AdS space has no direct relation with the in-in propagators associated with α vacuum of de Sitter space for any α. We see that it is the in-out propagator (in the Poincaré patch) which is actually related to the Green function of Euclidean AdS space. We thus expect that we can obtain a deep insight on the dS/CFT correspondence [30] by analytically continuing the Euclidean AdS/CFT correspondence and by interpreting the result in terms of the in-out propagators (not of the in-in propagators).

VII. DISCUSSIONS AND CONCLUSION

In this paper, we have considered quantum theory of a free scalar field in nonstatic spacetime. We first developed a framework to treat a harmonic oscillator with time-dependent parameters and then applied it to investigate a free scalar field in de Sitter space, both in the Poincaré and the global patches.

We have taken the vacuum state at each moment t_I to be the instantaneous ground state of the Hamiltonian at the moment. We developed a calculation method to obtain the wave function $\varphi(t; t_I)$ associated with the vacuum. The in-out and in-in propagators are then obtained from the wave functions by sending the initial and final times to the past and future infinities.

A major advantage of our prescription in defining the vacuum is that we do not need to introduce "positive-energy wave functions" that cannot be defined in a definite way for a spacetime with no asymptotic timelike Killing vector.

[23] In fact, solving the Klein-Gordon equation with a delta function source using the Euclidean AdS invariant $Z' = Z'(y, y')$, we see that the Green function is a linear combination of $(Z'^2 - 1)^{-(d-2)/4}\mathcal{P}^{(d-2)/2}_{\nu'-1/2}(Z')$ and $(Z'^2 - 1)^{-(d-2)/4} \times Q^{(d-2)/2}_{\nu'-1/2}(Z')$. By requiring that the Green function damps at large separation (cluster property), only the latter solution is selected, as can be seen from the asymptotic forms of the associated Legendre functions [see (E18) and (E19)]. The normalization is then determined by requiring that the Green function coincides with that in Euclidean space for infinitesimal separation of y and y' [or $Z'(y, y') \to 1$].

We have applied our method to calculate the in-out and in-in propagators in de Sitter space. The obtained propagators take de Sitter invariant forms and are consistent with the results known in the literature. What actually happens is that, when the time t_I is sent to a temporal boundary, our wave function $\varphi(t; t_I)$ may diverge, but the obtained propagator has a finite limit and coincides with the propagator in the literature [see the comments following (177)].

As a new result, we have found that a finite massless limit exists for the in-out propagator in the Poincaré patch. This is in contrast to the in-in propagator, where the no-go theorem states that no massless limit exists for the in-in propagators without breaking the de Sitter invariance [11]. The same functional form had been obtained for the in-out propagators without precise numerical coefficients in [21,27] from other approaches, and the massless case also had been considered in [21]. However, one cannot discuss the existence of a finite massless limit without knowing the precise numerical coefficients. Indeed, our in-out propagator in the global patch diverges in the massless limit just because the numerical coefficient diverges.

We have argued that our in-out propagator for a given foliation coincides with the Feynman propagator obtained by a path integral with the iε prescription, provided that the foliation is effectively noncompact in the temporal direction. We also have shown that both the Poincaré and the global patches meet the condition, and have confirmed the coincidence by numerical calculations.

We have also shown that the in-out propagators in both the Poincaré and the global patches satisfy Polyakov's composition law, demonstrating that the in-out propagators can be expressed as a sum over paths of a relativistic particle. It should be interesting to investigate whether the composition law holds universally for the in-out propagators in any spacetime. Furthermore, as a more fundamental issue, it must be important to clarify the meaning of (or to try to give an interpretation to) the relativistic *particle* in the language of quantum field theory in curved spacetime, where it is known that it is not always possible to introduce the concept of particles.

Our in-in propagator in the global patch has a finite value for $m \geq (d-1)/2$, but it diverges for $m < (d-1)/2$. It will be important to compare the in-in propagators with those obtained (numerically) by the path integral of the Schwinger-Keldysh type [12,13] (see also [23]).

As an important application of our construction, it should be interesting to investigate a thermodynamic property intrinsic to de Sitter space, especially its nonequilibrium property [31]. On the basis of our formalism, it would also be interesting to investigate some physical quantities such as the rate of vacuum decay at finite times. As another future direction, it would be interesting to apply our method to quantum field theories in spacetimes with horizon, such as a spacetime with black hole and de Sitter space in the static patch. For such a spacetime, one needs to carefully study the consistency of our formalism with boundary conditions at the horizon.

It should be important to consider interacting fields in generic nonstatic spacetimes and to establish perturbation theory on the basis of our formalism. It will be also interesting to investigate the in-out propagators for gravitons, since our method can be applied to field theory of higher spins without any essential modifications.

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APPENDIX A: PROOF OF EQ. (79)

Setting $t = T_s$ in (52) and using the Hermiticity $q^\dagger(T_s) = q(T_s)$ and $p^\dagger(T_s) = p(T_s)$, we obtain

$$\begin{pmatrix} c_1^\dagger \\ c_2^\dagger \end{pmatrix} = \frac{1}{(W_\rho[f,g])^*} \begin{pmatrix} -V_\rho[g^*, f] & -V_\rho[g^*, g] \\ V_\rho[f^*, f] & V_\rho[f^*, g] \end{pmatrix} (T_s) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\equiv M_s \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad (V_\rho[f,g] \equiv \rho f \dot{g} - \rho^* \dot{f} g). \tag{A1}$$

Then, from (53) and (58), we obtain

$$\begin{pmatrix} a^\dagger(t) \\ \bar{a}^\dagger(t) \end{pmatrix} = [C^{-1}(t)]^* M_s C(t') \begin{pmatrix} a(t') \\ \bar{a}(t') \end{pmatrix} \equiv \Lambda(t; t') \begin{pmatrix} a(t') \\ \bar{a}(t') \end{pmatrix}. \tag{A2}$$

A straightforward calculation shows that

$$\Lambda(t; t') = \mathrm{i} \begin{pmatrix} -V_\rho[\bar{\varphi}^*(T_s, t), \varphi(T_s, t')] & -V_\rho[\bar{\varphi}^*(T_s, t), \bar{\varphi}(T_s, t')] \\ V_\rho[\varphi^*(T_s, t), \varphi(T_s, t')] & V_\rho[\varphi^*(T_s, t), \bar{\varphi}(T_s, t')]) \end{pmatrix}. \tag{A3}$$

Note that $\det \Lambda(t; t') = -1$ due to the commutation relations $[a^\dagger(t), \bar{a}^\dagger(t)] = -1$ and $[a(t'), \bar{a}(t')] = 1$ (this can also be checked by a direct calculation). Then, setting $t = t' = t_0$ in (A2) and (A3), we find that

$$\bar{a}_0^\dagger = -\frac{V_\rho[\varphi^*(T_s;t_0),\bar{\varphi}(T_s;t_0)]}{V_\rho[\bar{\varphi}^*(T_s;t_0),\bar{\varphi}(T_s;t_0)]}a_0^\dagger + \frac{\mathrm{i}}{V_\rho[\bar{\varphi}^*(T_s;t_0),\bar{\varphi}(T_s;t_0)]}a_0. \tag{A4}$$

APPENDIX B: ASYMPTOTICALLY MINKOWSKI SPACE

In this Appendix, we reinvestigate within our framework a well-studied case where spacetime is asymptotically Minkowski in both the remote past and the remote future [1].

1. Setup

For brevity we consider the two-dimensional spacetime with the metric

$$\mathrm{d}s^2 = a^2(t)(-\mathrm{d}t^2 + \mathrm{d}x^2), \tag{B1}$$

where the scale factor $a^2(t)$ now depends on time and takes the form

$$a^2(t) = a_0^2\frac{1-\tanh t}{2} + a_1^2\frac{1+\tanh t}{2}. \tag{B2}$$

This spacetime is asymptotically Minkowski with scale a_0 in the remote past and with scale a_1 in the remote future. By expanding a scalar field $\phi(t,x)$ as

$$\phi(t,x) = \sum_{k\geq 0}\sum_a \phi_{k,a}(t)Y_{k,a}(x) \tag{B3}$$

as in Sec. III, the action becomes[24]

$$S[\phi(t,x)] = \int \mathrm{d}t\mathrm{d}x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{m^2}{2}\phi^2\right] = \sum_{k\geq 0}\sum_a\int \mathrm{d}t\frac{1}{2}[\dot{\phi}_{k,a}^2(t) - (k^2+m^2a^2(t))\phi_{k,a}^2(t)]. \tag{B4}$$

Thus, the correspondence with the ingredients of Sec. II is given by

$$q(t) = \phi_{k,a}(t), \qquad \rho(t) = 1, \tag{B5}$$

$$\omega(t) = \sqrt{\omega_0^2\frac{1-\tanh t}{2} + \omega_1^2\frac{1+\tanh t}{2}}, \tag{B6}$$

where

$$\omega_0 \equiv \sqrt{k^2+m^2a_0^2}, \qquad \omega_1 \equiv \sqrt{k^2+m^2a_1^2}. \tag{B7}$$

We also introduce

$$\omega_\pm \equiv \frac{1}{2}(\omega_1 \pm \omega_0). \tag{B8}$$

[24]Since iε plays no essential role in this Appendix, we have eliminated it from the action.

2. Wave functions

The equation of motion takes the form

$$0 = \ddot{q} + \frac{\dot{\rho}}{\rho}q + \omega^2 q = \ddot{q} + \left(\frac{\omega_1^2+\omega_0^2}{2} + \frac{\omega_1^2-\omega_0^2}{2}\tanh t\right)q, \tag{B9}$$

which can be solved analytically with the hypergeometric function. We set a pair of independent solutions $\{f(t), g(t)\}$ as

$$f(t) = \left(\frac{1-\zeta}{2}\right)^{\mathrm{i}\omega_1/2}\left(\frac{1+\zeta}{2}\right)^{-\mathrm{i}\omega_0/2} \times F\left(\mathrm{i}\omega_-, 1+\mathrm{i}\omega_-; 1-\mathrm{i}\omega_0; \frac{1+\zeta}{2}\right), \tag{B10}$$

$$g(t) = \left(\frac{1-\zeta}{2}\right)^{-\mathrm{i}\omega_1/2}\left(\frac{1+\zeta}{2}\right)^{\mathrm{i}\omega_0/2} \times F\left(-\mathrm{i}\omega_-, 1-\mathrm{i}\omega_-; 1+\mathrm{i}\omega_0; \frac{1+\zeta}{2}\right), \tag{B11}$$

where $\zeta \equiv \tanh t$, and $F(a,b;c;z)$ is the hypergeometric function. Their asymptotic forms for $t = t_0 \sim -\infty$ (or $\zeta = \zeta_0 \sim -1$) are easily found to be

$$f_0 = f(t_0) \sim e^{-\mathrm{i}\omega_0 t_0}, \qquad g_0 = g(t_0) \sim e^{\mathrm{i}\omega_0 t_0}, \tag{B12}$$

and the weighted Wronskian $W_\rho[f,g] = \rho(t)W[f,g](t)$ is found to be

$$W_\rho[f,g] = 2\mathrm{i}\omega_0. \tag{B13}$$

From this we find that the functions in (54) and (55) have the asymptotic forms

$$u_0 \sim 0, \qquad \bar{u}_0 \sim -2\mathrm{i}\omega_0 e^{-\mathrm{i}\omega_0 t_0}, \tag{B14}$$

$$v_0 \sim 2\mathrm{i}\omega_0 e^{\mathrm{i}\omega_0 t_0}, \qquad \bar{v}_0 \sim 0. \tag{B15}$$

The wave functions in the limit $t_0 \to -\infty$ then take the form

$$\varphi(t;t_0) \sim \frac{1}{\sqrt{2\omega_0}}e^{\mathrm{i}\omega_0 t_0}f(t) \sim \frac{1}{\sqrt{2\omega_0}}e^{\mathrm{i}\omega_0 t_0 - \mathrm{i}\omega_+ t - \mathrm{i}\omega_- \log(2\cosh t)} \times F\left(\mathrm{i}\omega_-, 1+\mathrm{i}\omega_-; 1-\mathrm{i}\omega_0; \frac{1+\tanh t}{2}\right), \tag{B16}$$

$$\bar{\varphi}(t;t_0) \sim \frac{1}{\sqrt{2\omega_0}} e^{-i\omega_0 t_0} g(t) \sim \frac{1}{\sqrt{2\omega_0}} e^{-i\omega_0 t_0 + i\omega_+ t + i\omega_- \log(2\cosh t)} \times F\left(-i\omega_-, 1 - i\omega_-; 1 + i\omega_0; \frac{1+\tanh t}{2}\right). \tag{B17}$$

In order to calculate the asymptotic forms of various functions for $t \sim +\infty$, it is convenient to rewrite $f(t)$ and $g(t)$ by using the formula

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b;a+b-c+1;1-z) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b} \times F(c-a,c-b;c-a-b+1;1-z). \tag{B18}$$

We then obtain

$$f(t) = \left(\frac{\omega_0}{\omega_1}\right)^{1/2}\left(\frac{1-\zeta}{2}\right)^{i\omega_1/2}\left(\frac{1+\zeta}{2}\right)^{-i\omega_0/2} \times\left[\tilde{\alpha}^* F\left(i\omega_-, 1+i\omega_-; 1+i\omega_1; \frac{1-\zeta}{2}\right) - \tilde{\beta}\left(\frac{1-\zeta}{2}\right)^{-i\omega_1} F\left(1-i\omega_+, -i\omega_+; 1-i\omega_1; \frac{1-\zeta}{2}\right)\right], \tag{B19}$$

$$g(t) = \left(\frac{\omega_0}{\omega_1}\right)^{1/2}\left(\frac{1-\zeta}{2}\right)^{-i\omega_1/2}\left(\frac{1+\zeta}{2}\right)^{i\omega_0/2} \times\left[\tilde{\alpha} F\left(-i\omega_-, 1-i\omega_-; 1-i\omega_1; \frac{1-\zeta}{2}\right) - \tilde{\beta}^*\left(\frac{1-\zeta}{2}\right)^{i\omega_1} F\left(1+i\omega_+, i\omega_+; 1+i\omega_1; \frac{1-\zeta}{2}\right)\right], \tag{B20}$$

where

$$\tilde{\alpha} \equiv \left(\frac{\omega_1}{\omega_0}\right)^{1/2} \frac{\Gamma(1+i\omega_0)\Gamma(i\omega_1)}{\Gamma(1+i\omega_+)\Gamma(i\omega_+)}, \tag{B21}$$

$$\tilde{\beta} \equiv -\left(\frac{\omega_1}{\omega_0}\right)^{1/2} \frac{\Gamma(1-i\omega_0)\Gamma(i\omega_1)}{\Gamma(1+i\omega_-)\Gamma(i\omega_-)}. \tag{B22}$$

With these, the asymptotic forms for $t_1 \sim +\infty$ can be obtained easily as

$$f_1 = f(t_1) \sim \left(\frac{\omega_0}{\omega_1}\right)^{1/2} (\tilde{\alpha}^* e^{-i\omega_1 t_1} - \tilde{\beta} e^{i\omega_1 t_1}), \tag{B23}$$

$$g_1 = g(t_1) \sim \left(\frac{\omega_0}{\omega_1}\right)^{1/2} (\tilde{\alpha} e^{i\omega_1 t_1} - \tilde{\beta}^* e^{-i\omega_1 t_1}), \tag{B24}$$

and

$$u_1 \sim \left(\frac{\omega_0}{\omega_1}\right)^{1/2} (-2i\tilde{\beta}\omega_1) e^{i\omega_1 t_1}, \tag{B25}$$

$$\bar{u}_1 \sim \left(\frac{\omega_0}{\omega_1}\right)^{1/2} (-2i\tilde{\alpha}^*\omega_1) e^{-i\omega_1 t_1}, \tag{B26}$$

$$v_1 \sim \left(\frac{\omega_0}{\omega_1}\right)^{1/2} (2i\tilde{\alpha}\omega_1) e^{i\omega_1 t_1}, \tag{B27}$$

$$\bar{v}_1 \sim \left(\frac{\omega_0}{\omega_1}\right)^{1/2} (2i\tilde{\beta}^*\omega_1) e^{-i\omega_1 t_1}. \tag{B28}$$

We then find that the Bogoliubov coefficients take the asymptotic forms

$$\alpha(t_1,t_0) \sim \tilde{\alpha} e^{-i(\omega_0 t_0 - \omega_1 t_1)} \equiv \alpha_1 \quad (\bar{\alpha}(t_1,t_0) \sim \alpha_1^*), \tag{B29}$$

$$\beta(t_1,t_0) \sim \tilde{\beta} e^{i(\omega_0 t_0 + \omega_1 t_1)} \equiv \beta_1 \quad (\bar{\beta}(t_1,t_0) \sim \beta_1^*). \tag{B30}$$

They coincide with the well-known values in the literature (see, e.g., in [1]) up to a phase. It is easy to see

$$|\alpha_1|^2 = \frac{\sinh^2(\pi\omega_+)}{\sinh(\pi\omega_0)\sinh(\pi\omega_1)}, \tag{B31}$$

$$|\beta_1|^2 = \frac{\sinh^2(\pi\omega_-)}{\sinh(\pi\omega_0)\sinh(\pi\omega_1)}, \tag{B32}$$

and thus the relation $|\alpha_1|^2 - |\beta_1|^2 = 1$ actually holds.

Using the asymptotic forms of u_1 and v_1, we can calculate the wave function $\varphi(t,t_1)$ for $t_1 \to +\infty$,

$$\varphi(t;t_1) = \frac{1}{\sqrt{2\omega_0}} e^{i\omega_1 t_1}[\tilde{\alpha} f(t) + \tilde{\beta} g(t)]. \tag{B33}$$

This can be further rewritten by using Kummer's relation,

$$F(a,b;c;z) = (1-z)^{c-a-b} F(c-a,c-b;c;z), \tag{B34}$$

in the form

$$\varphi(t;t_1) = \frac{1}{\sqrt{2\omega_1}} e^{i\omega_1 t_1 - i\omega_+ t - i\omega_- \log(2\cosh t)} \times F\left(i\omega_-, 1+i\omega_-; 1-i\omega_0; \frac{1-\tanh t}{2}\right). \tag{B35}$$

This certainly coincides up to a phase with the positive-energy wave function in the remote future given in [1]. One can easily see that $\varphi(t;t_1)$ actually has the form $\varphi(t;t_1) \sim (1/\sqrt{2\omega_1}) e^{-i\omega_1(t-t_1)}$ when t is also very large.

APPENDIX C: PROPAGATOR IN MINKOWSKI SPACE

In order to evaluate the integral (117), we introduce the polar coordinates for the wave vector as

$$d\boldsymbol{k}^2 = dk^2 + k^2(d\theta^2 + \sin^2\theta\, d\Omega_{d-3}^2), \tag{C1}$$

where θ is chosen such that $\theta = 0$ corresponds to the direction $\boldsymbol{x} - \boldsymbol{x}'$, i.e., $\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}') = k|\boldsymbol{x}-\boldsymbol{x}'|\cos\theta$. The volume element is then given by

$$\begin{aligned} \mathrm{d}^{d-1}\boldsymbol{k} &= \mathrm{d}k k^{d-2}\,\mathrm{d}\theta \sin^{d-3}\theta\,\mathrm{d}\Omega_{d-3} \\ &= \mathrm{d}k k^{d-2}\,\mathrm{d}\cos\theta(1-\cos^2\theta)^{(d-4)/2}\mathrm{d}\Omega_{d-3}, \end{aligned} \tag{C2}$$

and (117) becomes

$$\begin{aligned} G(x,x') = \frac{|\Omega_{d-3}|}{(2\pi)^{d-1}}\int_0^\infty \mathrm{d}k k^{d-1} G_k(t,t') \\ \times \int_{-1}^{1}\mathrm{d}s(1-s^2)^{(d-4)/2}\cos(k|\boldsymbol{x}-\boldsymbol{x}'|s), \end{aligned} \tag{C3}$$

where $|\Omega_{n-1}|$ is the area of the unit sphere in n-dimensional Euclidean space, $|\Omega_{n-1}| = \int \mathrm{d}\Omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$. The integration with respect to $s = \cos\theta$ can be carried out by using the formula (8.411-8 in [32])

$$\begin{aligned} \int_{-1}^{1}\mathrm{d}s(1-s^2)^{\nu}\cos(zs) \\ = \sqrt{\pi}\Gamma(\nu+1)\left(\frac{z}{2}\right)^{-\nu-\frac{1}{2}} J_{\nu+\frac{1}{2}}(z) \quad [\mathrm{Re}\nu > -1], \end{aligned} \tag{C4}$$

and we obtain

$$\begin{aligned} G(x,x') &= \frac{1}{(2\pi)^{\frac{d-1}{2}}|\boldsymbol{x}-\boldsymbol{x}'|^{\frac{d-3}{2}}}\int_0^\infty \mathrm{d}k k^{\frac{d-1}{2}} G_k(t,t') J_{\frac{d-3}{2}}(k|\boldsymbol{x}-\boldsymbol{x}'|) \\ &= \frac{2^{-(d+1)/2}\pi^{-(d-1)/2}}{|\boldsymbol{x}-\boldsymbol{x}'|^{(d-3)/2}}\int_0^\infty \mathrm{d}k\frac{k^{(d-1)/2}}{\omega_k} e^{-\mathrm{i}\omega_{k,\varepsilon}(t_>-t_<)} J_{\frac{d-3}{2}}(k|\boldsymbol{x}-\boldsymbol{x}'|) \\ &= \frac{2^{-(d+1)/2}\pi^{-(d-1)/2}}{|\boldsymbol{x}-\boldsymbol{x}'|^{(d-3)/2}} m^{\frac{d-1}{2}}\int_1^\infty \mathrm{d}\lambda(\lambda^2-1)^{\frac{d-3}{4}} e^{-m\lambda e^{\mathrm{i}(\pi-\varepsilon)/2}(t_>-t_<)} J_{\frac{d-3}{2}}(k|\boldsymbol{x}-\boldsymbol{x}'|), \end{aligned} \tag{C5}$$

where, assuming $m > 0$, we have set $\lambda = \omega_k/m = \sqrt{k^2+m^2}/m$ to obtain the last expression. Then applying the formula (6.645-2 in [32])

$$\int_1^\infty \mathrm{d}\lambda(\lambda^2-1)^{\frac{\nu}{2}} e^{-\alpha\lambda} J_\nu(\beta\sqrt{\lambda^2-1}) = \sqrt{\frac{2}{\pi}}\beta^\nu(\alpha^2+\beta^2)^{-\frac{\nu}{2}-\frac{1}{4}} K_{\nu+\frac{1}{2}}\left(\sqrt{\alpha^2+\beta^2}\right) \quad [\mathrm{Re}\alpha > 0, \beta\in\mathbb{R}] \tag{C6}$$

with $\nu = (d-3)/2$, $\alpha = me^{\mathrm{i}(\pi-0)/2}(t_> - t_<)$, $\beta = m|\boldsymbol{x}-\boldsymbol{x}'|$, we obtain

$$G(x,x') = \frac{m^{(d-2)/2}}{(2\pi)^{d/2}}[e^{\mathrm{i}(\pi-0)}\Delta t^2 + \Delta\boldsymbol{x}^2]^{-\frac{d-2}{4}} K_{\frac{d-2}{2}}(m\sqrt{e^{\mathrm{i}(\pi-0)}\Delta t^2+\Delta\boldsymbol{x}^2}), \tag{C7}$$

where $\Delta t \equiv t - t'$ and $\Delta\boldsymbol{x} \equiv \boldsymbol{x}-\boldsymbol{x}'$. This expression can be further rewritten by separately investigating the cases for different sign of $\sigma \equiv (x-x')^2 = -\Delta t^2 + \Delta\boldsymbol{x}^2$.

(1) spacelike ($\sigma > 0$):

The modified Bessel function is readily evaluated as $K_{(d-2)/2}(m\sqrt{e^{\mathrm{i}(\pi-0)}\Delta t^2+\Delta\boldsymbol{x}^2}) = K_{(d-2)/2}(m\sqrt{\sigma})$, and we have

$$G(x,x') = \frac{m^{(d-2)/2}}{(2\pi)^{d/2}}\sigma^{-\frac{d-2}{4}} K_{\frac{d-2}{2}}(m\sqrt{\sigma}). \tag{C8}$$

(2) timelike ($\sigma < 0$):

By using the relations $e^{\mathrm{i}(\pi-0)}\Delta t^2 + \Delta\boldsymbol{x}^2 = e^{\mathrm{i}\pi}(-\sigma)$ and $K_\nu(e^{\mathrm{i}\pi/2}z) = -(\mathrm{i}\pi/2)H_\nu^{(2)}(z)$, we have

$$G(x,x') = \frac{\pi}{2}\frac{m^{(d-2)/2}}{(2\pi)^{d/2}\mathrm{i}^{d-1}}(-\sigma)^{-\frac{d-2}{4}} H^{(2)}_{\frac{d-2}{2}}(m\sqrt{\sigma}). \tag{C9}$$

(3) null ($\sigma \to 0+$):

By using the expansion

$$K_\nu(z) = \frac{\Gamma(\nu)}{2}\left(\frac{z}{2}\right)^{-\nu}(1+O(z^2)), \tag{C10}$$

we obtain

$$G(x,x') \to \frac{\Gamma((d-2)/2)}{4\pi^{d/2}}(\sigma+\mathrm{i}0)^{-\frac{d-2}{2}}. \tag{C11}$$

The right-hand side actually coincides with the massless propagator.

It is easy to see that all of the expressions for the three cases can be derived from a single expression,

$$G(x,x') = \frac{m^{(d-2)/2}}{(2\pi)^{d/2}(\sigma+\mathrm{i}0)^{(d-2)/4}} K_{\frac{d-2}{2}}(m\sqrt{\sigma+\mathrm{i}0}). \tag{C12}$$

APPENDIX D: PROOFS OF EQS. (183) AND (185)

In order to show Eq. (183), we use the following equation (6.578-11 in [32]):

$$\int_0^\infty \mathrm{d}x x^{\mu+1} K_\nu(\hat{a}x) I_\nu(\hat{b}x) J_\mu(cx) = \frac{c^\mu e^{-\mathrm{i}\pi(\mu+\frac{1}{2})}}{\sqrt{2\pi}(\hat{a}\hat{b})^{\mu+1}} (u^2-1)^{-\frac{1}{2}(\mu+\frac{1}{2})} \mathcal{Q}^{\mu+\frac{1}{2}}_{\nu-1/2}(u) \left[\begin{matrix} u \equiv \frac{\hat{a}^2+\hat{b}^2+c^2}{2\hat{a}\hat{b}}, & \mathrm{Re}\hat{a} > |\mathrm{Re}\hat{b}| + |\mathrm{Im}c|, \\ \mathrm{Re}\mu > -1, & \mathrm{Re}(\mu+\nu) > -1 \end{matrix}\right]. \tag{D1}$$

By setting $\hat{a} = e^{\mathrm{i}\pi/2}a$ and $\hat{b} = e^{\mathrm{i}\pi/2}b$ for $-\pi/2 < \arg a \le \pi$ and $-\pi < \arg b \le \pi/2$, and by using the identities

$$H^{(2)}_\nu(ax) = \frac{2\mathrm{i}}{\pi} e^{\mathrm{i}\pi\nu/2} K_\nu(\hat{a}x), \tag{D2}$$

$$J_\nu(bx) = e^{-\mathrm{i}\pi\nu/2} I_\nu(\hat{b}x), \tag{D3}$$

the equation is rewritten to the form

$$\int_0^\infty \mathrm{d}x x^{\mu+1} H^{(2)}_\nu(ax) J_\nu(bx) J_\mu(cx) = \frac{\sqrt{2}c^\mu e^{-2\mathrm{i}\pi(\mu+\frac{1}{2})}}{\pi^{3/2}(ab)^{\mu+1}} (u^2-1)^{-\frac{1}{2}(\mu+\frac{1}{2})} \mathcal{Q}^{\mu+\frac{1}{2}}_{\nu-1/2}(u) \left[\begin{matrix} u \equiv \frac{a^2+b^2-c^2}{2ab}, & (-\mathrm{Im}a) > |\mathrm{Im}b| + |\mathrm{Im}c|, \\ \mathrm{Re}\mu > -1, & \mathrm{Re}(\mu+\nu) > -1 \end{matrix}\right]. \tag{D4}$$

We substitute for this $x = k$, $a = e^{-\mathrm{i}\varepsilon}(-\eta_<)$, $b = e^{-\mathrm{i}\varepsilon}(-\eta_>)$, $c = |\boldsymbol{x} - \boldsymbol{x}'|$, and $\mu = (d-3)/2$. We then obtain Eq. (183) with $u = Z(x, x') - \mathrm{i}\,0$ for infinitesimal ε.

In order to show Eq. (185), we start from the following equation (6.578-10 in [32]):

$$\int_0^\infty \mathrm{d}x x^{\mu+1} K_\nu(\hat{a}x) K_\nu(\hat{b}x) J_\mu(cx) = \frac{\sqrt{\pi}c^\mu \Gamma(\mu+\nu+1)\Gamma(\mu-\nu+1)}{2^{3/2}(\hat{a}\hat{b})^{\mu+1}} \times (u^2-1)^{-\frac{1}{2}(\mu+\frac{1}{2})} \mathcal{P}^{-(\mu+\frac{1}{2})}_{\nu-1/2}(u) \left[\begin{matrix} u \equiv \frac{\hat{a}^2+\hat{b}^2+c^2}{2\hat{a}\hat{b}}, & \mathrm{Re}(\hat{a}+\hat{b}) > |\mathrm{Im}c|, \\ \mathrm{Re}(\mu\pm\nu) > -1, & \mathrm{Re}\mu > -1 \end{matrix}\right]. \tag{D5}$$

By setting $\hat{a} = e^{-\mathrm{i}\pi/2}a$ and $\hat{b} = e^{\mathrm{i}\pi/2}b$ with $-\pi/2 < \arg a \le \pi$ and $-\pi/2 < \arg b \le \pi$, and by using the identities

$$H^{(1)}_\nu(ax) = -\frac{2\mathrm{i}}{\pi} e^{-\mathrm{i}\pi\nu/2} K_\nu(\hat{a}x), \tag{D6}$$

$$H^{(2)}_\nu(bx) = \frac{2\mathrm{i}}{\pi} e^{\mathrm{i}\pi\nu/2} K_\nu(\hat{b}x), \tag{D7}$$

the equation is rewritten to the form

$$\int_0^\infty \mathrm{d}x x^{\mu+1} H^{(1)}_\nu(ax) H^{(2)}_\nu(bx) J_\mu(cx) = \frac{\sqrt{2}c^\mu \Gamma(\mu+\nu+1)\Gamma(\mu-\nu+1)}{\pi^{3/2}(ab)^{\mu+1}} \times (u^2-1)^{-\frac{1}{2}(\mu+\frac{1}{2})} \mathcal{P}^{-\mu-\frac{1}{2}}_{\nu-1/2}(u) \left[\begin{matrix} u \equiv \frac{-a^2-b^2+c^2}{2ab}, & \mathrm{Im}(a-b) > |\mathrm{Im}c|, \\ \mathrm{Re}(\mu\pm\nu) > -1, & \mathrm{Re}\mu > -1 \end{matrix}\right]. \tag{D8}$$

We substitute for this $x = k$, $a = e^{\mathrm{i}\varepsilon}(-\eta_>)$, $b = e^{-\mathrm{i}\varepsilon}(-\eta_<)$, $c = \boldsymbol{x} - \boldsymbol{x}'$, and $\mu = (d-3)/2$. We then obtain Eq. (185) with $u = -Z(x, x') + \mathrm{i}\,0$ for infinitesimal ε.

APPENDIX E: ASSOCIATED LEGENDRE FUNCTIONS AND THE ADDITION FORMULAS

In this Appendix, we give several formulas of the associated Legendre functions that are used in Sec. IV C. For details of the associated Legendre functions, see [32,33].

The associated Legendre functions $\mathcal{P}^\mu_\nu(z)$ and $\mathcal{Q}^\mu_\nu(z)$ are defined over the complex z plane other than the cut along the real axis to the left of the point $z = 1$ (running from $-\infty$ to 1), while the associated Legendre functions $\mathsf{P}^\mu_\nu(x)$ and $\mathsf{Q}^\mu_\nu(x)$ are defined only on the interval $-1 < x < 1$,

$$\mathcal{P}^\mu_\nu(z) \equiv \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} F\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right), \tag{E1}$$

$$\mathcal{Q}^\mu_\nu(z) \equiv \frac{e^{\mathrm{i}\pi\mu}\pi^{\frac{1}{2}}\Gamma(\nu+\mu+1)}{2^{\nu+1}\Gamma(\nu+3/2)} z^{-\nu-\mu-1}(z^2-1)^{\frac{\mu}{2}} \times F\left(\frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right), \tag{E2}$$

$$\mathsf{P}^\mu_\nu(x) \equiv \frac{1}{2}\left[e^{\frac{1}{2}\mathrm{i}\pi\mu}\mathcal{P}^\mu_\nu(x+\mathrm{i}0) + e^{-\frac{1}{2}\mathrm{i}\pi\mu}\mathcal{P}^\mu_\nu(x-\mathrm{i}0)\right] = \frac{1}{\Gamma(1-\mu)}\left(\frac{1+x}{1-x}\right)^{\frac{\mu}{2}} F\left(-\nu, \nu+1; 1-\mu; \frac{1-x}{2}\right), \tag{E3}$$

$$\mathsf{Q}^\mu_\nu(x) \equiv \frac{1}{2}e^{-\mathrm{i}\pi\mu}\left[e^{-\frac{1}{2}\mathrm{i}\pi\mu}\mathcal{Q}^\mu_\nu(x+\mathrm{i}0) + e^{\frac{1}{2}\mathrm{i}\pi\mu}\mathcal{Q}^\mu_\nu(x-\mathrm{i}0)\right] = \frac{\pi}{2\sin\pi\mu}\left[\cos\pi\mu\,\mathsf{P}^\mu_\nu(x) - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\mathsf{P}^{-\mu}_\nu(x)\right]. \tag{E4}$$

1. Functional relations

The four functions $\mathcal{P}(z)$, $\mathcal{Q}(z)$, $\mathsf{P}(x)$, and $\mathsf{Q}(x)$ are related to each other as (3.4 and 3.3.1 in [33])

$$e^{\mathrm{i}\pi\frac{\mu}{2}}\mathcal{P}^\mu_\nu(x+\mathrm{i}0) = e^{-\mathrm{i}\pi\frac{\mu}{2}}\mathcal{P}^\mu_\nu(x-\mathrm{i}0) = \mathsf{P}^\mu_\nu(x), \tag{E5}$$

$$e^{-\mathrm{i}\pi\mu}[e^{-\mathrm{i}\pi\frac{\mu}{2}}\mathcal{Q}^{\mu}_{\nu}(x+\mathrm{i}0) \pm e^{\mathrm{i}\pi\frac{\mu}{2}}\mathcal{Q}^{\mu}_{\nu}(x-\mathrm{i}0)] = \begin{cases} 2\mathsf{Q}^{\mu}_{\nu}(x) \\ -\mathrm{i}\pi\mathsf{P}^{\mu}_{\nu}(x), \end{cases} \tag{E6}$$

$$\mathcal{Q}^{\mu}_{\nu}(-z) = \begin{cases} -e^{\mathrm{i}\pi\nu}\mathcal{Q}^{\mu}_{\nu}(z) & [\mathrm{Im}z > 0] \\ -e^{-\mathrm{i}\pi\nu}\mathcal{Q}^{\mu}_{\nu}(z) & [\mathrm{Im}z < 0]. \end{cases} \tag{E7}$$

We also have (8.73 in [32])

$$\mathcal{P}^{-\mu}_{\nu}(z) = \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)} \times \left[\mathcal{P}^{\mu}_{\nu}(z) - \frac{2}{\pi}e^{-\mathrm{i}\pi\mu}\sin\pi\mu\,\mathcal{Q}^{\mu}_{\nu}(z)\right], \tag{E8}$$

$$\mathcal{Q}^{-\mu}_{\nu}(z) = e^{-2\mathrm{i}\pi\mu}\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\mathcal{Q}^{\mu}_{\nu}(z), \tag{E9}$$

$$\mathsf{P}^{-\mu}_{\nu}(x) = \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)} \times \left[\cos\pi\mu\,\mathsf{P}^{\mu}_{\nu}(x) - \frac{2}{\pi}\sin\pi\mu\,\mathsf{Q}^{\mu}_{\nu}(x)\right], \tag{E10}$$

$$\mathsf{Q}^{-\mu}_{\nu}(x) = \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)} \times \left[\frac{\pi}{2}\sin\pi\mu\,\mathsf{P}^{\mu}_{\nu}(x) + \cos\pi\mu\,\mathsf{Q}^{\mu}_{\nu}(x)\right], \tag{E11}$$

$$\mathsf{P}^{\mu}_{-\nu-1}(x) = \mathsf{P}^{\mu}_{\nu}(x), \tag{E12}$$

$$\mathsf{Q}^{\mu}_{-\nu-1}(x) = \frac{1}{\sin\pi(\nu-\mu)}[-\pi\cos\pi\nu\,\mathsf{P}^{\mu}_{\nu}(x) + \sin\pi(\nu+\mu)\mathsf{Q}^{\mu}_{\nu}(x)]. \tag{E13}$$

Their Wronskians have the forms (8.741 in [32])

$$\begin{vmatrix} \mathsf{P}^{\mu}_{\nu}(x) & \mathsf{Q}^{\mu}_{\nu}(x) \\ \frac{\mathrm{d}}{\mathrm{d}x}\mathsf{P}^{\mu}_{\nu}(x) & \frac{\mathrm{d}}{\mathrm{d}x}\mathsf{Q}^{\mu}_{\nu}(x) \end{vmatrix} = \frac{1}{1-x^2}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}, \tag{E14}$$

$$\begin{vmatrix} \mathsf{P}^{-\mu}_{\nu}(x) & \mathsf{P}^{\mu}_{\nu}(x) \\ \frac{\mathrm{d}}{\mathrm{d}x}\mathsf{P}^{-\mu}_{\nu}(x) & \frac{\mathrm{d}}{\mathrm{d}x}\mathsf{P}^{\mu}_{\nu}(x) \end{vmatrix} = \frac{1}{1-x^2}\frac{2\sin\pi\mu}{\pi}. \tag{E15}$$

We also have (8.733-1 in [32])

$$(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{P}^{\mu}_{\nu}(x) = (\nu+1)x\mathsf{P}^{\mu}_{\nu}(x) - (\nu-\mu+1)\mathsf{P}^{\mu}_{\nu+1}(x), \tag{E16}$$

$$(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{Q}^{\mu}_{\nu}(x) = (\nu+1)x\mathsf{Q}^{\mu}_{\nu}(x) - (\nu-\mu+1)\mathsf{Q}^{\mu}_{\nu+1}(x). \tag{E17}$$

2. Asymptotic forms

The associated Legendre functions have the following asymptotic forms near boundaries (3.9.2 in [33]):

$$\mathcal{P}^{\mu}_{\nu}(z) \overset{z\sim\infty}{\sim} \begin{cases} 2^{\nu}\pi^{-\frac{1}{2}}\frac{\Gamma(\nu+1/2)}{\Gamma(\nu-\mu+1)}z^{\nu} & [\mathrm{Re}\nu > -1/2] \\ 2^{-\nu-1}\pi^{-\frac{1}{2}}\frac{\Gamma(-\nu-1/2)}{\Gamma(-\nu-\mu)}z^{-\nu-1} & [\mathrm{Re}\nu < -1/2], \end{cases} \tag{E18}$$

$$\mathcal{Q}^{\mu}_{\nu}(z) \overset{z\sim\infty}{\sim} e^{\mathrm{i}\pi\mu}2^{-\nu-1}\pi^{\frac{1}{2}}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+3/2)}z^{-\nu-1}, \tag{E19}$$

$$\mathsf{P}^{\nu}_{k}(x) \overset{x\sim+1}{\sim} \frac{2^{\nu/2}\sin(\pi\nu)\Gamma(\nu)}{\pi}(1-x^2)^{-\frac{\nu}{2}}, \tag{E20}$$

$$\mathsf{Q}^{\nu}_{k}(x) \overset{x\sim+1}{\sim} 2^{\nu-1}\cos(\pi\nu)\Gamma(\nu)(1-x^2)^{-\frac{\nu}{2}} \quad [\mathrm{Re}\nu > 0], \tag{E21}$$

$$\mathsf{P}^{\nu}_{k}(x) \overset{x\sim-1}{\sim} \begin{cases} 2^{-\nu}\cos(\pi k)\frac{\Gamma(k+\nu+1)}{\Gamma(\nu+1)\Gamma(k-\nu+1)}(1-x^2)^{\frac{\nu}{2}} & [k\in\mathbb{Z}] \\ -\frac{2^{\nu}\sin(\pi k)\Gamma(\nu)}{\pi}(1-x^2)^{-\frac{\nu}{2}} & [k\in\mathbb{Z}+1/2, \mathrm{Re}\nu > 0], \end{cases} \tag{E22}$$

$$\mathsf{Q}^{\nu}_{k}(x) \overset{x\sim-1}{\sim} \begin{cases} -2^{\nu-1}\cos(\pi k)\Gamma(\nu)(1-x^2)^{-\frac{\nu}{2}} & [k\in\mathbb{Z}, \mathrm{Re}\nu > 0] \\ -\frac{2^{-\nu-1}\pi\Gamma(k+\nu+1)}{\sin(\pi k)\Gamma(\nu+1)\Gamma(k-\nu+1)}(1-x^2)^{\frac{\nu}{2}} & [k\in\mathbb{Z}+1/2]. \end{cases} \tag{E23}$$

3. Addition formulas

We find the following formulas, which are useful in obtaining the propagators in the global patch:

$$\frac{\mathrm{i}\pi(\cos\varphi_1\cos\varphi_2)^{\frac{d-1}{2}}}{2(d-2)|\Omega_{d-1}|\sin(\pi\nu)}\sum_{L=0}^{\infty}(2L+d-2)\mathsf{P}_k^{-\nu}(\sin\varphi_1)\mathsf{P}_k^{\nu}(\sin\varphi_2)C_L^{\frac{d-2}{2}}(\cos\theta)$$
$$=\frac{-e^{-\mathrm{i}\pi\frac{d-2}{2}}}{2(2\pi)^{\frac{d}{2}}\sin(\pi\nu)}[(u_+^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(u_+)-(u_-^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(u_-)]\quad (d\text{: odd}), \tag{E24}$$

$$\frac{\mathrm{i}(\cos\varphi_1\cos\varphi_2)^{\frac{d-1}{2}}}{(d-2)|\Omega_{d-1}|\cos(\pi\nu)}\sum_{L=0}^{\infty}(2L+d-2)\mathsf{P}_k^{-\nu}(\sin\varphi_1)\mathsf{Q}_k^{\nu}(\sin\varphi_2)C_L^{\frac{d-2}{2}}(\cos\theta)$$
$$=\frac{\mathrm{i}e^{-\mathrm{i}\pi\frac{d-2}{2}}}{2(2\pi)^{\frac{d}{2}}\cos(\pi\nu)}\Big[(u_+^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(u_+)+(u_-^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(u_-)\Big]\quad (d\text{: even}), \tag{E25}$$

$$\frac{\mathrm{i}(\cos\varphi_1\cos\varphi_2)^{\frac{d-1}{2}}}{(d-2)|\Omega_{d-1}|}\sum_{L=0}^{\infty}(2L+d-2)\frac{\Gamma(k-\nu+1)}{\Gamma(k+\nu+1)}\mathsf{P}_k^{\nu}(\sin\varphi_1)\mathsf{Q}_k^{\nu}(\sin\varphi_2)C_L^{\frac{d-2}{2}}(\cos\theta)$$
$$=\frac{\mathrm{i}e^{-\mathrm{i}\pi\frac{d-2}{2}}}{2(2\pi)^{\frac{d}{2}}}\Big[e^{-\mathrm{i}\pi\nu}(u_+^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{-\nu-\frac{1}{2}}(u_+)+e^{\mathrm{i}\pi\nu}(u_-^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{-\nu-\frac{1}{2}}(u_-)\Big]\quad (d\text{: even}), \tag{E26}$$

$$\frac{2\mathrm{i}(\cos\varphi_1\cos\varphi_2)^{\frac{d-1}{2}}}{\pi(d-2)|\Omega_{d-1}|}\sum_{L=0}^{\infty}(2L+d-2)\mathsf{Q}_k^{-\nu}(\sin\varphi_1)\mathsf{Q}_k^{\nu}(\sin\varphi_2)C_L^{\frac{d-2}{2}}(\cos\theta)$$
$$=\frac{e^{-\mathrm{i}\pi\frac{d-2}{2}}}{2(2\pi)^{\frac{d}{2}}}\Big\{(u_+^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(u_+)-(u_-^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(u_-)$$
$$+\frac{\mathrm{i}\pi}{\cos(\pi\nu)}\Big[e^{-\mathrm{i}\pi\nu}(u_+^2-1)^{-\frac{d-2}{4}}\mathcal{P}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(u_+)+e^{\mathrm{i}\pi\nu}(u_-^2-1)^{-\frac{d-2}{4}}\mathcal{P}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(u_-)\Big]\Big\}$$
$$=-\frac{e^{-\mathrm{i}\pi\frac{d-2}{2}}}{2(2\pi)^{\frac{d}{2}}}\Big\{(u_+^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(u_+)-(u_-^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(u_-)$$
$$-\frac{\pi}{\cos(\pi\nu)}\Big[(u_+^2-1)^{-\frac{d-2}{4}}\mathcal{P}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-u_+)-(u_-^2-1)^{-\frac{d-2}{4}}\mathcal{P}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-u_-)\Big]\Big\}\quad (d\text{: even}), \tag{E27}$$

where $-\pi/2<\varphi_2<\varphi_1<\pi/2$, $0\le\theta\le\pi$, $k\equiv L+(d-3)/2$, and

$$u_\pm(\varphi_1,\varphi_2,\theta)\equiv -Z(\varphi_1,\varphi_2,\theta)\pm \mathrm{i}0\quad\text{with}\quad Z(\varphi_1,\varphi_2,\theta)\equiv\frac{-\sin\varphi_1\sin\varphi_2+\cos\theta}{\cos\varphi_1\cos\varphi_2}. \tag{E28}$$

We prove Eqs. (E24) and (E25) for the rest of this Appendix. Equation (E26) can be proved in a similar way, and (E27) is readily obtained from (E25) and (E26).

We start from Eq. (12) of [34],[25]

$$(\sinh\gamma)^{-\frac{d-2}{2}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(\cosh\gamma)=2^{\frac{d}{2}-2}\Gamma\Big(\frac{d-2}{2}\Big)e^{\mathrm{i}\pi(-\nu+\frac{d-2}{2})}(\sinh\beta_1\sinh\beta_2)^{\frac{d-1}{2}}$$
$$\times\sum_{L=0}^{\infty}(2L+d-2)\mathcal{P}_k^{-\nu}(\cosh\beta_1)\mathcal{Q}_k^{\nu}(\cosh\beta_2)C_L^{\frac{d-2}{2}}(\cos\theta), \tag{E29}$$

where

$$\cosh\gamma(\beta_1,\beta_2,\theta)\equiv\frac{\cosh\beta_1\cosh\beta_2-\cos\alpha}{\sinh\beta_1\sinh\beta_2},\qquad \mathrm{Re}\beta_2>|\mathrm{Re}\beta_1|+|\mathrm{Im}\theta|. \tag{E30}$$

[25] Equation (E26) can be proved by replacing $\mathcal{Q}_k^{\nu}(\cosh\beta_2)$ in (E29) by $\mathcal{Q}_k^{-\nu}(\cosh\beta_2)$ with the help of (E9).

Both sides of Eq. (E29) should be understood as the quantities that are continued analytically from the region where β_1, β_2, and θ take all real values (for which $\cosh\gamma > 1$). We reparametrize the variables in Eq. (E29) as

$$\beta_\alpha^\pm \equiv \pm \mathrm{i}\left(\frac{\pi}{2} - \varphi_\alpha\right) + \varepsilon_\alpha \quad (\alpha = 1, 2)$$
$$\left[-\frac{\pi}{2} < \varphi_2 < \varphi_1 < \frac{\pi}{2}, 0 < \varepsilon_1 < \varepsilon_2 \ll 1\right] \quad \text{(E31)}$$

and only keep the contributions from ε_α to the linear order. We then have

$$\cosh\beta_\alpha^\pm = \sin\varphi_\alpha \pm \mathrm{i}\varepsilon_\alpha \cos\phi_\alpha, \quad \text{(E32)}$$

$$\sinh\beta_\alpha^\pm = \pm \mathrm{i}\cos\varphi_\alpha + \varepsilon_\alpha \sin\phi_\alpha, \quad \text{(E33)}$$

and

$$\cosh\gamma_\pm \equiv \cosh\gamma(\beta_1^\mp, \beta_2^\pm, \theta)$$
$$= -Z(\varphi_1, \varphi_2, \theta) + \mathrm{i}\mathcal{O}(\varepsilon_1, \varepsilon_2). \quad \text{(E34)}$$

If we fix the parameters ε_1 and ε_2, and vary φ_1, φ_2, and θ within the regions $-\pi/2 < \varphi_2 < \varphi_1 < \pi/2$ and $0 \le \theta \le \pi$, then $\cosh\gamma_\pm$ ranges in the region depicted in Fig. 9.

In the following, we divide the parameter region of φ_1, φ_2, and θ into three parts, where Z takes values in (1) $Z > 1$, (2) $1 > Z > -1$, and (3) $Z < -1$, respectively. We then derive a simpler expression of Eq. (E29) for each case and show that the obtained expressions for the three cases can be summarized in the form (E24) and (E25).

(1) $Z > 1$

In this case, $\cosh\gamma_\pm = -Z \pm \mathrm{i}0$ as can be seen from Fig. 9.[26] Thus, we have

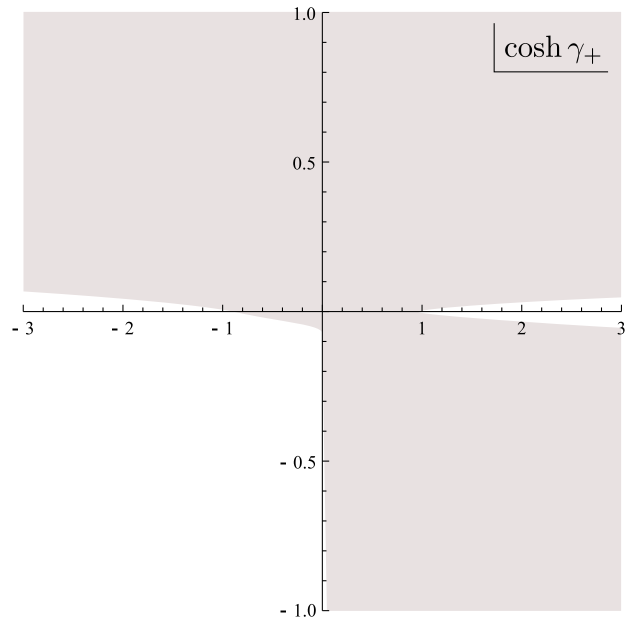

FIG. 9 (color online). Schematic view illustrating the range of $\cosh\gamma_+$ for $\varepsilon_1 = 0.001$ and $\varepsilon_2 = 0.02$. Here, φ_1 and φ_2 run over the range $-\pi/2 < \varphi_2 < \varphi_1 < \pi/2$, and θ runs over its full range $0 \le \theta \le \pi$. The range of $\cosh\gamma_-$ can be obtained by turning the figure by 180° over the horizontal axis.

$$\sinh\gamma_\pm = e^{\pm \mathrm{i}\pi}(Z^2 - 1)^{\frac{1}{2}}, \quad \text{(E35)}$$

$$\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(\cosh\gamma_\pm) = -e^{\mp \mathrm{i}\pi(\nu-\frac{1}{2})}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(Z \mp \mathrm{i}0)$$
$$= -e^{\mp \mathrm{i}\pi(\nu-\frac{1}{2})}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(Z), \quad \text{(E36)}$$

where we have used Eq. (E7) and the fact that $\mathcal{Q}^\mu_\nu(z)$ does not have a cut in the region $\mathrm{Re}z > 1$. Then, Eq. (E29) becomes

$$-e^{\mp \mathrm{i}\pi(\frac{d-3}{2}+\nu)}(Z^2 - 1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(Z)$$
$$= 2^{\frac{d}{2}-2}\Gamma\left(\frac{d-2}{2}\right)e^{\mathrm{i}\pi(-\nu\mp\frac{\nu}{2}+\frac{d-2}{2})}(\cos\varphi_1\cos\varphi_2)^{\frac{d-1}{2}}\sum_{L=0}^{\infty}(2L + d - 2)\mathsf{P}_k^{-\nu}(\sin\varphi_1)\mathcal{Q}_k^\nu(\sin\varphi_2 \pm \mathrm{i}0)C_L^{\frac{d-2}{2}}(\cos\theta). \quad \text{(E37)}$$

By taking the difference between the above equations with the upper and the lower signs, we obtain

$$2\mathrm{i}\sin\left[\pi\left(\frac{d-3}{2} + \nu\right)\right](Z^2 - 1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(Z)$$
$$= -\mathrm{i}\pi 2^{\frac{d}{2}-2}\Gamma\left(\frac{d-2}{2}\right)e^{\mathrm{i}\pi\frac{d-2}{2}}(\cos\varphi_1\cos\varphi_2)^{\frac{d-1}{2}}\sum_{L=0}^{\infty}(2L + d - 2)\mathsf{P}_k^{-\nu}(\sin\varphi_1)\mathsf{P}_k^\nu(\sin\varphi_2)C_L^{\frac{d-2}{2}}(\cos\theta), \quad \text{(E38)}$$

where Eq. (E6) has been used. Similarly, by taking their sum, we obtain

[26] ε_1 and ε_2 are to be taken to zero, keeping $\varepsilon_2 > \varepsilon_1$.

$$-2\cos\left[\pi\left(\frac{d-3}{2}+\nu\right)\right](Z^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(Z)$$
$$=2^{\frac{d}{2}-1}\Gamma\left(\frac{d-2}{2}\right)e^{\mathrm{i}\pi\frac{d-2}{2}}(\cos\varphi_1\cos\varphi_2)^{\frac{d-1}{2}}\sum_{L=0}^{\infty}(2L+d-2)\mathsf{P}_k^{-\nu}(\sin\varphi_1)\mathsf{Q}_k^{\nu}(\sin\varphi_2)C_L^{\frac{d-2}{2}}(\cos\theta). \tag{E39}$$

The right-hand side of Eqs. (E24) and (E25) can then be written as

$$\begin{cases} -\frac{\mathrm{i}\sin\pi(\frac{d-2}{2})}{(2\pi)^{\frac{d}{2}}}e^{-\mathrm{i}\pi\frac{d-2}{2}}(Z^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(Z) & (d\text{: odd}),\\ -\frac{\mathrm{i}\cos\pi(\frac{d-2}{2})\tan\pi\nu}{(2\pi)^{\frac{d}{2}}}e^{-\mathrm{i}\pi\frac{d-2}{2}}(Z^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(Z) & (d\text{: even}).\end{cases} \tag{E40}$$

(2) $1 > Z > -1$

In this case, as can be seen from Fig. 9, $\cosh\gamma_+$ crosses the branch cut between $-1 < z < 1$ from above and moves to another Riemann sheet. On the other hand, $\cosh\gamma_-$ crosses the branch cut between $-1 < z < 1$ from below. Thus, in this region, we have

$$\sinh\gamma_\pm = e^{\pm\mathrm{i}\frac{\pi}{2}}(1-Z^2)^{\frac{1}{2}}, \tag{E41}$$

$$\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(\cosh\gamma_\pm)=\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-Z\pm\mathrm{i}0)=e^{\mathrm{i}\pi\frac{d-2}{2}(1\pm\frac{1}{2})}\left[\mathsf{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-Z)\mp\frac{\mathrm{i}\pi}{2}\mathsf{P}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-Z)\right], \tag{E42}$$

where we have used Eq. (E6). Then, Eq. (E29) becomes

$$e^{\mathrm{i}\pi\frac{d-2}{2}}(1-Z^2)^{-\frac{d-2}{4}}\left[\mathsf{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-Z)\mp\frac{\mathrm{i}\pi}{2}\mathsf{P}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-Z)\right]$$
$$=2^{\frac{d}{2}-2}\Gamma\left(\frac{d-2}{2}\right)e^{\mathrm{i}\pi(-\nu\mp\frac{\nu}{2}+\frac{d-2}{2})}(\cos\varphi_1\cos\varphi_2)^{\frac{d-1}{2}}\sum_{L=0}^{\infty}(2L+d-2)\mathsf{P}_k^{-\nu}(\sin\varphi_1)\mathcal{Q}_k^{\nu}(\sin\varphi_2\pm\mathrm{i}0)C_L^{\frac{d-2}{2}}(\cos\theta). \tag{E43}$$

By taking the difference of the equations with the upper and the lower signs, we obtain

$$(1-Z^2)^{-\frac{d-2}{4}}\mathsf{P}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-Z)=2^{\frac{d}{2}-2}\Gamma\left(\frac{d-2}{2}\right)(\cos\varphi_1\cos\varphi_2)^{\frac{d-1}{2}}\sum_{L=0}^{\infty}(2L+d-2)\mathsf{P}_k^{-\nu}(\sin\varphi_1)\mathsf{P}_k^{\nu}(\sin\varphi_2)C_L^{\frac{d-2}{2}}(\cos\theta). \tag{E44}$$

Similarly, by taking their sum, we obtain

$$(1-Z^2)^{-\frac{d-2}{4}}\mathsf{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-Z)=2^{\frac{d}{2}-2}\Gamma\left(\frac{d-2}{2}\right)(\cos\varphi_1\cos\varphi_2)^{\frac{d-1}{2}}\sum_{L=0}^{\infty}(2L+d-2)\mathsf{P}_k^{-\nu}(\sin\varphi_1)\mathsf{Q}_k^{\nu}(\sin\varphi_2)C_L^{\frac{d-2}{2}}(\cos\theta). \tag{E45}$$

The right-hand side of Eqs. (E24) and (E25) can then be written as follows:

$$\begin{cases} \frac{\mathrm{i}\pi}{2(2\pi)^{\frac{d}{2}}\sin(\pi\nu)}(1-Z^2)^{-\frac{d-2}{4}}\mathsf{P}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-Z) & (d\text{: odd}),\\ \frac{\mathrm{i}}{(2\pi)^{\frac{d}{2}}\cos(\pi\nu)}(1-Z^2)^{-\frac{d-2}{4}}\mathsf{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-Z) & (d\text{: even}).\end{cases} \tag{E46}$$

(3) $Z < -1$

In this case, from Fig. 9, we know that, in the region $\mathrm{Re}z > 1$, $\cosh\gamma_+$ runs above the real axis, or below the real axis in the next Riemann sheet after passing through the cut on $-1 < z < 1$ from above. On the other hand, in the region $\mathrm{Re}z > 1$, $\cosh\gamma_-$ runs below the real axis, or above the real axis in another sheet after passing through the cut on $-1 < z < 1$ from below.

When $\cosh\gamma_\pm = -Z\pm\mathrm{i}0$, we have

$$\sinh\gamma_\pm=(Z^2-1)^{\frac{1}{2}},\qquad \mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(\cosh\gamma_\pm)=\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-Z), \tag{E47}$$

and then the following equation is obtained:

$$(Z^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-Z)$$
$$=2^{\frac{d}{2}-2}\Gamma\left(\frac{d-2}{2}\right)e^{\mathrm{i}\pi(-\nu\mp\frac{\nu}{2}+\frac{d-2}{2})}(\cos\varphi_1\cos\varphi_2)^{\frac{d-1}{2}}\sum_{L=0}^{\infty}(2L+d-2)\mathsf{P}_k^{-\nu}(\sin\varphi_1)\mathcal{Q}_k^{\nu}(\sin\varphi_2\pm\mathrm{i}0)C_L^{\frac{d-2}{2}}(\cos\theta). \quad \text{(E48)}$$

By taking the difference and the sum of the above equations with the upper and the lower signs, we obtain

$$0=-\mathrm{i}\pi 2^{\frac{d}{2}-2}\Gamma\left(\frac{d-2}{2}\right)e^{\mathrm{i}\pi\frac{d-2}{2}}(\cos\varphi_1\cos\varphi_2)^{\frac{d-1}{2}}\sum_{L=0}^{\infty}(2L+d-2)\mathsf{P}_k^{-\nu}(\sin\varphi_1)\mathsf{P}_k^{\nu}(\sin\varphi_2)C_L^{\frac{d-2}{2}}(\cos\theta), \quad \text{(E49)}$$

$$2(Z^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-Z)=2^{\frac{d}{2}-1}\Gamma\left(\frac{d-2}{2}\right)e^{\mathrm{i}\pi\frac{d-2}{2}}(\cos\varphi_1\cos\varphi_2)^{\frac{d-1}{2}}\sum_{L=0}^{\infty}(2L+d-2)\mathsf{P}_k^{-\nu}(\sin\varphi_1)\mathsf{Q}_k^{\nu}(\sin\varphi_2)C_L^{\frac{d-2}{2}}(\cos\theta). \quad \text{(E50)}$$

On the other hand, when $\cosh\gamma_\pm=-Z\mp\mathrm{i}0$, we need to evaluate the functions $\sinh\gamma_\pm$ and $\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(\cosh\gamma_\pm)$ on a new Riemann sheet, since $\cosh\gamma_\pm$ has already crossed the branch cut. We then have

$$\sinh\gamma_\pm=e^{\pm\mathrm{i}\pi}(Z^2-1)^{\frac{1}{2}}, \quad \text{(E51)}$$

$$\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(\cosh\gamma_\pm)=e^{\pm\mathrm{i}\pi\frac{d-2}{2}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-Z)\mp\mathrm{i}\pi e^{\mathrm{i}\pi\frac{d-2}{2}}\mathcal{P}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-Z), \quad \text{(E52)}$$

and Eq. (E29) takes the form

$$(Z^2-1)^{-\frac{d-2}{4}}\left[\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-Z)\mp\mathrm{i}\pi e^{\pm\mathrm{i}\pi\frac{d-2}{2}(1\mp1)}\mathcal{P}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-Z)\right]$$
$$=2^{\frac{d}{2}-2}\Gamma\left(\frac{d-2}{2}\right)e^{\mathrm{i}\pi(-\nu\mp\frac{\nu}{2}+\frac{d-2}{2})}(\cos\varphi_1\cos\varphi_2)^{\frac{d-1}{2}}\sum_{L=0}^{\infty}(2L+d-2)\mathsf{P}_k^{-\nu}(\sin\varphi_1)\mathcal{Q}_k^{\nu}(\sin\varphi_2\pm\mathrm{i}0)C_L^{\frac{d-2}{2}}(\cos\theta). \quad \text{(E53)}$$

By taking the difference of the above equations with the upper and the lower signs, we obtain

$$\cos\left(\frac{d-2}{2}\pi\right)(Z^2-1)^{-\frac{d-2}{4}}\mathcal{P}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-Z)$$
$$=2^{\frac{d}{2}-2}\Gamma\left(\frac{d-2}{2}\right)e^{\mathrm{i}\pi\frac{d-2}{2}}(\cos\varphi_1\cos\varphi_2)^{\frac{d-1}{2}}\sum_{L=0}^{\infty}(2L+d-2)\mathsf{P}_k^{-\nu}(\sin\varphi_1)\mathsf{P}_k^{\nu}(\sin\varphi_2)C_L^{\frac{d-2}{2}}(\cos\theta). \quad \text{(E54)}$$

In particular, in odd dimensions, we have

$$0=2^{\frac{d}{2}-2}\Gamma\left(\frac{d-2}{2}\right)e^{\mathrm{i}\pi\frac{d-2}{2}}(\cos\varphi_1\cos\varphi_2)^{\frac{d-1}{2}}\sum_{L=0}^{\infty}(2L+d-2)\mathsf{P}_k^{-\nu}(\sin\varphi_1)\mathsf{P}_k^{\nu}(\sin\varphi_2)C_L^{\frac{d-2}{2}}(\cos\theta), \quad \text{(E55)}$$

which is equivalent to Eq. (49). Similarly, by taking the sum of Eq. (53) with the upper and lower signs, we have

$$(Z^2-1)^{-\frac{d-2}{4}}\left[\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-Z)-\pi e^{\mathrm{i}\pi\frac{d-2}{2}}\sin\left(\frac{d-2}{2}\pi\right)\mathcal{P}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-Z)\right]$$
$$=2^{\frac{d}{2}-2}\Gamma\left(\frac{d-2}{2}\right)e^{\mathrm{i}\pi\frac{d-2}{2}}(\cos\varphi_1\cos\varphi_2)^{\frac{d-1}{2}}\sum_{L=0}^{\infty}(2L+d-2)\mathsf{P}_k^{-\nu}(\sin\varphi_1)\mathsf{Q}_k^{\nu}(\sin\varphi_2)C_L^{\frac{d-2}{2}}(\cos\theta). \quad \text{(E56)}$$

In particular, in even dimensions, we have

$$(Z^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-Z)=2^{\frac{d}{2}-2}\Gamma\left(\frac{d-2}{2}\right)e^{\mathrm{i}\pi\frac{d-2}{2}}(\cos\varphi_1\cos\varphi_2)^{\frac{d-1}{2}}\sum_{L=0}^{\infty}(2L+d-2)\mathsf{P}_k^{-\nu}(\sin\varphi_1)\mathsf{Q}_k^{\nu}(\sin\varphi_2)C_L^{\frac{d-2}{2}}(\cos\theta), \quad \text{(E57)}$$

which is equivalent to Eq. (E50). Thus, when $Z<-1$, the right-hand sides of Eqs. (E24) and (E25) always take the forms

$$\begin{cases} 0 & (d\text{: odd}), \\ \frac{\mathrm{i}}{(2\pi)^{\frac{d}{2}}\cos(\pi\nu)} e^{-\mathrm{i}\pi\frac{d-2}{2}}(Z^2-1)^{-\frac{d-2}{4}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-\frac{1}{2}}(-Z) & (d\text{: even}). \end{cases} \tag{E58}$$

We thus have obtained simplified expressions for Eq. (E29) for three different regions of Z in the form Eqs. (E40), (E46), and (E58). One can readily see that three equations can be obtained from Eqs. (E24) and (E25). This completes the proof of our assertion.

APPENDIX F: INTEGRAL REPRESENTATION OF THE ASSOCIATED LEGENDRE FUNCTIONS

In this Appendix, we give a proof of Eq. (286) (we write it again here for convenience),

$$\mathcal{Q}^{\frac{d-2}{2}}_{\nu-1/2}(u) = e^{\mathrm{i}\pi\frac{d-2}{2}}\int_0^\infty \mathrm{d}\lambda \frac{\Gamma(\frac{d-1}{2}+\mathrm{i}\lambda)\Gamma(\frac{d-1}{2}-\mathrm{i}\lambda)}{\Gamma(\mathrm{i}\lambda)\Gamma(-\mathrm{i}\lambda)}\frac{\mathcal{P}^{-\frac{d-2}{2}}_{\mathrm{i}\lambda-1/2}(u)}{\nu^2+\lambda^2} \quad [d\in\mathbb{Z}, \mathrm{Re}\,\nu>0], \tag{F1}$$

treating the odd- and even-dimensional cases separately. Our discussion is heavily based on the derivation of the heat kernel in Euclidean AdS space performed in [28] (see also [27]).

1. Odd dimensions

When d is odd, using the direct relation between the associated Legendre functions and the Gegenbauer functions,

$$\mathcal{P}^{-\frac{d-2}{2}}_{\mathrm{i}\lambda-\frac{1}{2}}(u) = \frac{2^{\frac{d-2}{2}}\pi^{1/2}\Gamma(\frac{d-1}{2})}{\sin[\pi(\frac{d-1}{2}-\mathrm{i}\lambda)]\Gamma(\frac{d-1}{2}+\mathrm{i}\lambda)\Gamma(\frac{d-1}{2}-\mathrm{i}\lambda)}(u^2-1)^{\frac{d-2}{4}}C^{\frac{d-1}{2}}_{\mathrm{i}\lambda-\frac{d-1}{2}}(u), \tag{F2}$$

we can rewrite the integral on the right-hand side of (F1) as

$$\int_0^\infty \mathrm{d}\lambda \frac{\Gamma(\frac{d-1}{2}+\mathrm{i}\lambda)\Gamma(\frac{d-1}{2}-\mathrm{i}\lambda)}{\Gamma(\mathrm{i}\lambda)\Gamma(-\mathrm{i}\lambda)}\frac{\mathcal{P}^{-\frac{d-2}{2}}_{\mathrm{i}\lambda-1/2}(u)}{\nu^2+\lambda^2} = \frac{2^{\frac{d-2}{2}}e^{\mathrm{i}\pi d/2}\Gamma(\frac{d-1}{2})}{\pi^{1/2}}(u^2-1)^{\frac{d-2}{4}}\int_0^\infty \mathrm{d}\lambda\frac{\lambda}{\nu^2+\lambda^2}C^{\frac{d-1}{2}}_{\mathrm{i}\lambda-\frac{d-1}{2}}(u). \tag{F3}$$

By further using the identities for the Gegenbauer function

$$\mathrm{i}\lambda C^n_{\mathrm{i}\lambda-n}(\cosh\gamma) = \frac{2^{1-n}}{\Gamma(n)}\left[\frac{\mathrm{d}}{\mathrm{d}(\cosh\gamma)}\right]^n\cos(\lambda\gamma) \tag{F4}$$

with nonnegative integers n, Eq. (F3) can be rewritten as follows:

$$\begin{aligned} &= \frac{2^{1/2}e^{\mathrm{i}\pi\frac{d-1}{2}}}{\pi^{1/2}}(\cosh^2\gamma-1)^{\frac{d-2}{4}}\left[\frac{\mathrm{d}}{\mathrm{d}(\cosh\gamma)}\right]^{\frac{d-1}{2}}\int_0^\infty \mathrm{d}\lambda\frac{\cos(\lambda\gamma)}{\nu^2+\lambda^2} \\ &= \frac{\pi^{1/2}e^{\mathrm{i}\pi\frac{d-1}{2}}}{2^{1/2}\nu}(\cosh^2\gamma-1)^{\frac{d-2}{4}}\left[\frac{\mathrm{d}}{\mathrm{d}(\cosh\gamma)}\right]^{\frac{d-1}{2}}e^{\mp\gamma\nu} \quad [\mathrm{Re}\,\gamma\gtrless 0, \mathrm{Re}\,\nu>0] \\ &= e^{-\mathrm{i}\pi\frac{d-1}{2}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-1/2}(\cosh\gamma) \quad [\mathrm{Re}\nu>0]. \end{aligned} \tag{F5}$$

Here, in order to show the second equality, we have used the formula [3.723-2] of [32],

$$\int_0^\infty \mathrm{d}\lambda\frac{\cos(\lambda\gamma)}{\nu^2+\lambda^2} = \begin{cases} \frac{\pi}{2\nu}e^{-\gamma\nu} & [\mathrm{Re}\gamma>0, \mathrm{Re}\nu>0] \\ \frac{\pi}{2\nu}e^{\gamma\nu} & [\mathrm{Re}\gamma<0, \mathrm{Re}\nu>0], \end{cases} \tag{F6}$$

and to show the third equality we have used the following formula [see (A.20) of [28]]:

$$e^{-\mathrm{i}\pi\frac{d-2}{2}}\mathcal{Q}^{\frac{d-2}{2}}_{\nu-1/2}(\cosh\gamma) = \begin{cases} \frac{\pi^{1/2}e^{\mathrm{i}\pi\frac{d-1}{2}}}{2^{1/2}\nu}(\cosh^2\gamma-1)^{\frac{d-2}{4}}\left[\frac{\mathrm{d}}{\mathrm{d}(\cosh\gamma)}\right]^{\frac{d-1}{2}}e^{-\gamma\nu} & [\mathrm{Re}\gamma>0] \\ \frac{\pi^{1/2}e^{\mathrm{i}\pi\frac{d-1}{2}}}{2^{1/2}\nu}(\cosh^2\gamma-1)^{\frac{d-2}{4}}\left[\frac{\mathrm{d}}{\mathrm{d}(\cosh\gamma)}\right]^{\frac{d-1}{2}}e^{\gamma\nu} & [\mathrm{Re}\gamma<0]. \end{cases} \tag{F7}$$

2. Even dimensions

When d is even, the equality

$$\mathcal{P}^{-\frac{d-2}{2}}_{\mathrm{i}\lambda-\frac{1}{2}}(u) = \frac{\Gamma(\mathrm{i}\lambda - \frac{d-1}{2} + 1)}{\Gamma(\mathrm{i}\lambda + \frac{d-1}{2})} \mathcal{P}^{\frac{d-2}{2}}_{\mathrm{i}\lambda-\frac{1}{2}}(u) \tag{F8}$$

holds, and we can show (F1) as follows:

$$\begin{aligned}\int_0^\infty \mathrm{d}\lambda \frac{\Gamma(\frac{d-1}{2}+\mathrm{i}\lambda)\Gamma(\frac{d-1}{2}-\mathrm{i}\lambda)}{\Gamma(\mathrm{i}\lambda)\Gamma(-\mathrm{i}\lambda)} \frac{\mathcal{P}^{-\frac{d-2}{2}}_{\mathrm{i}\lambda-1/2}(u)}{\nu^2+\lambda^2} &= (-1)^{\frac{d-2}{2}} \int_0^\infty \mathrm{d}\lambda \frac{\lambda \tanh(\pi\lambda)}{\nu^2+\lambda^2} \mathcal{P}^{\frac{d-2}{2}}_{\mathrm{i}\lambda-1/2}(u) \\ &= (-1)^{\frac{d-2}{2}} (u^2-1)^{\frac{d-2}{4}} \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{\frac{d-2}{2}} \int_0^\infty \mathrm{d}\lambda \frac{\lambda \tanh(\pi\lambda)}{\nu^2+\lambda^2} \mathcal{P}_{\mathrm{i}\lambda-1/2}(u) \\ &= (-1)^{\frac{d-2}{2}} (u^2-1)^{\frac{d-2}{4}} (\frac{\mathrm{d}}{\mathrm{d}u})^{\frac{d-2}{2}} \mathcal{Q}_{\nu-1/2}(u) \\ &= e^{-\mathrm{i}\pi\frac{d-2}{2}} \mathcal{Q}^{\frac{d-2}{2}}_{\nu-1/2}(u),\end{aligned} \tag{F9}$$

where we have used the identities for the associated Legendre functions with integer order,

$$(u^2-1)^{n/2} \frac{\mathrm{d}^n}{\mathrm{d}u^n} \mathcal{P}_{\mathrm{i}\lambda-\frac{1}{2}}(u) = \mathcal{P}^n_{\mathrm{i}\lambda-\frac{1}{2}}(u), \tag{F10}$$

$$(u^2-1)^{n/2} \frac{\mathrm{d}^n}{\mathrm{d}u^n} \mathcal{Q}_{\nu-\frac{1}{2}}(u) = \mathcal{Q}^n_{\nu-\frac{1}{2}}(u), \tag{F11}$$

and the formula [7.213] of [32],

$$\int_0^\infty \mathrm{d}\lambda \frac{\lambda \tanh(\pi\lambda)}{\nu^2+\lambda^2} \mathcal{P}_{\mathrm{i}\lambda-1/2}(u) = \mathcal{Q}_{\nu-1/2}(u) \quad [\mathrm{Re}\nu > 0]. \tag{F12}$$

APPENDIX G: α VACUA

Since the in-in and in-out propagators in de Sitter space always have de Sitter invariant forms, it is natural to expect that the in- and out-vacua both in the Poincaré and the global patch belong to a family of de Sitter invariant vacua, i.e., the α vacua (or the Mottola-Allen vacua) [10,11]. In this Appendix, we calculate the values of $\alpha \in \mathbb{C}$ associated to the in- and out-vacua (both in the Poincaré and the global patch) explicitly. We will set $\varepsilon = 0$ in the following discussions.

Given a mode expansion, a α vacuum is defined for a complex number α with $\mathrm{Re}\alpha < 0$ such that the corresponding wave function for each mode n is given by

$$\varphi_n^{(\alpha)}(t) = \frac{1}{\sqrt{1-|e^\alpha|^2}} [\varphi_n^{(\mathrm{E})}(t) + e^\alpha \varphi_n^{(\mathrm{E})*}(t)], \tag{G1}$$

where $\varphi_n^{(\mathrm{E})}(t)$ is the wave function of the Euclidean vacuum. The Feynman propagator associated with the α vacuum is then given (for each mode) as

$$\begin{aligned}G_n^{(\alpha)}(t,t') &= \frac{\mathrm{i}}{W_\rho[\varphi_n^{(\alpha)}, \varphi_n^{(\alpha)*}]} \varphi_n^{(\alpha)}(t_>) \varphi_n^{(\alpha)*}(t_<) \\ &= \frac{1}{1-|e^\alpha|^2} \frac{\mathrm{i}}{W_\rho[\varphi_n^{(\mathrm{E})}, \varphi_n^{(\mathrm{E})*}]} [\varphi_n^{(\mathrm{E})}(t_>) \varphi_n^{(\mathrm{E})*}(t_<) \\ &\quad + |e^\alpha|^2 \varphi_n^{(\mathrm{E})*}(t_>) \varphi_n^{(\mathrm{E})}(t_<) \\ &\quad + e^{\alpha^*} \varphi_n^{(\mathrm{E})}(t_>) \varphi_n^{(\mathrm{E})}(t_<) \\ &\quad + e^\alpha \varphi_n^{(\mathrm{E})*}(t_>) \varphi_n^{(\mathrm{E})*}(t_<)].\end{aligned} \tag{G2}$$

We also have multiplied the factor $\mathrm{i}/W_\rho[\varphi_n^{(\alpha)}, \varphi_n^{(\alpha)*}]$ for which we need not care about the normalization of wave functions. Thus, if we expand the in-in or out-out propagator (for each mode) as a quadratic form of $\varphi_n^{(\mathrm{E})}$ and $\varphi_n^{(\mathrm{E})*}$, we can find the values of α associated to the in- or out-vacua. Here, the in-in propagator is defined as in (73) and (75), and the out-out propagator is defined by

$$\begin{aligned}G^{\mathrm{out/out}}(t,t') &\equiv \lim_{t_1 \to t_f} G_{11}(t,t';t_1,t_1), \\ G_{11}(t,t';t_1,t_1) &\equiv \frac{\overline{\langle 0_{t_1}|}\mathrm{T}q(t)q^\dagger(t')\overline{|0_{t_1}\rangle}}{\overline{\langle 0_{t_1}|0_{t_1}\rangle}},\end{aligned} \tag{G3}$$

which can be shown to take the form

$$G^{\mathrm{out/out}}(t,t') = \lim_{t_1 \to t_f} \frac{\mathrm{i}}{V_\rho[\varphi(t;t_1), \varphi^*(t;t_1)](T_s)} \varphi(t_>;t_1)\varphi^*(t_<;t_1). \tag{G4}$$

Since the in-in propagator in the Poincaré patch (185) coincides with the Feynman propagator in the Euclidean vacuum, the in-vacuum in the Poincaré patch is identified with the Euclidean vacuum (i.e., the α vacuum with $\alpha = -\infty$). On the other hand, by using (166), we can show that the out-out propagator for each mode in the Poincaré patch [which is finite only if $m > (d-1)/2$] takes the following form:

$$G_{\boldsymbol{k}}^{\text{out/out}}(\eta,\eta') = \frac{\pi[(-\eta)(-\eta')]^{(d-1)/2}}{2\sinh(\pi\mu)} J_{\mathrm{i}\mu}(-k_\varepsilon \eta_>) J_{-\mathrm{i}\mu}(-k_{-\varepsilon}\eta_<). \tag{G5}$$

If we use the wave function associated with the Euclidean vacuum,

$$\varphi_k^{(\mathrm{E})}(t) = \beta \frac{\sqrt{\pi}}{2} e^{-\frac{\pi\mu}{2}} (-\eta)^{\frac{d-1}{2}} H_{\mathrm{i}\mu}^{(1)}(-k\eta), \tag{G6}$$

with β an arbitrary complex constant, we can expand the out-out propagator for each mode as follows:

$$\begin{aligned} G_{\boldsymbol{k}}^{\text{out/out}}(t,t') = \frac{\pi}{1-e^{-2\pi\mu}} \frac{\mathrm{i}}{W_\rho[\varphi_k^{(\mathrm{E})}(t), \varphi_k^{(\mathrm{E})*}(t)]} \Big[&\varphi_n^{(\mathrm{E})}(t_>)\varphi_n^{(\mathrm{E})*}(t_<) + e^{-2\pi\mu}\varphi_n^{(\mathrm{E})*}(t_>)\varphi_n^{(\mathrm{E})}(t_<) \\ &+ \frac{\beta^*}{\beta} e^{-\pi\mu}\varphi_n^{(\mathrm{E})}(t_>)\varphi_n^{(\mathrm{E})}(t_<) + \frac{\beta}{\beta^*} e^{-\pi\mu}\varphi_n^{(\mathrm{E})*}(t_>)\varphi_n^{(\mathrm{E})*}(t_<)\Big]. \end{aligned} \tag{G7}$$

If we choose the constant β real, the out-vacuum is shown to correspond to the α vacuum with $\alpha = -\pi\mu$.

In the global patch, the in-in and out-out propagators for each mode in the heavy mass case ($m > (d-1)/2$) take the following forms:

$$\begin{aligned} G_L^{\text{in/in}}(t,t') &= \begin{cases} \frac{\pi}{2\sinh(\pi\mu)}[(1-t_>^2)(1-t_<^2)]^{\frac{d-1}{4}} \mathsf{P}_k^{-\mathrm{i}\mu}(t_>)\mathsf{P}_k^{\mathrm{i}\mu}(t_<) & (d\text{: odd}), \\ \frac{2}{\pi\sinh(\pi\mu)}[(1-t_>^2)(1-t_<^2)]^{\frac{d-1}{4}} \mathsf{Q}_k^{-\mathrm{i}\mu}(t_>)\mathsf{Q}_k^{\mathrm{i}\mu}(t_<) & (d\text{: even}), \end{cases} \\ G_L^{\text{out/out}}(t,t') &= \frac{\pi}{2\sinh(\pi\mu)}[(1-t_>^2)(1-t_<^2)]^{\frac{d-1}{4}} \mathsf{P}_k^{-\mathrm{i}\mu}(t_>)\mathsf{P}_k^{\mathrm{i}\mu}(t_<). \end{aligned} \tag{G8}$$

On the other hand, the wave function associated with the Euclidean vacuum is known to have the following form (see, e.g., [18]):

$$\begin{aligned} \varphi_L^{(\mathrm{E})}(t) &= \beta \frac{\sqrt{\pi}\Gamma(L+\frac{d-1}{2}+\mathrm{i}\mu)\cosh^L\tau\, e^{(L+\frac{d-1}{2}+\mathrm{i}\mu)\tau}}{2^{L+\frac{d-1}{2}} e^{-\mathrm{i}\pi(L+\frac{d-1}{2})} e^{\pi\mu}\Gamma(L+\frac{d}{2})} F\Big(L+\frac{d-1}{2}, L+\frac{d-1}{2}+\mathrm{i}\mu, 2L+d-1; 1+e^{2\tau}-\mathrm{i}0\Big) \\ &= \beta(1-t^2)^{\frac{d-1}{4}}\Big[\mathsf{Q}_k^{\mathrm{i}\mu}(t) + \frac{\mathrm{i}\pi}{2}\mathsf{P}_k^{\mathrm{i}\mu}(t)\Big] \quad (t \equiv \tanh\tau). \end{aligned} \tag{G9}$$

Then, by using the relations

$$\mathsf{P}_k^{\mathrm{i}\mu}(t) = \frac{-\mathrm{i}}{\pi\beta}\Big(\varphi_L^{(\mathrm{E})}(t) - e^{\pi\mu}\frac{\beta\Gamma(k+1+\mathrm{i}\mu)}{\beta^*\Gamma(k+1-\mathrm{i}\mu)}\varphi_L^{(\mathrm{E})*}(t)\Big), \qquad \mathsf{Q}_k^{\mathrm{i}\mu}(t) = \frac{1}{2\beta}\Big(\varphi_L^{(\mathrm{E})}(t) + e^{\pi\mu}\frac{\beta\Gamma(k+1+\mathrm{i}\mu)}{\beta^*\Gamma(k+1-\mathrm{i}\mu)}\varphi_L^{(\mathrm{E})*}(t)\Big), \tag{G10}$$

if we choose the constant β such that $\arg\beta = \arg\Gamma(k+1-\mathrm{i}\mu)$, we find that, in odd dimensions, the in- and out-vacua are the α vacua with $\alpha = -\pi\mu + \mathrm{i}\pi$, while in even dimensions, the in-vacuum is the α vacuum with $\alpha = -\pi\mu$ and the out-vacuum is that with $\alpha = -\pi\mu + \mathrm{i}\pi$.

APPENDIX H: ANOTHER iε PRESCRIPTION

In this paper, the iε prescription is defined by the replacement

$$\rho(t) \to e^{+\mathrm{i}\varepsilon}\rho(t), \qquad \omega_n(t) \to e^{-\mathrm{i}\varepsilon}\omega_n(t), \tag{H1}$$

which corresponds to the replacement $H_{n,s}(t) = e^{-\mathrm{i}\varepsilon}[H_{n,s}(t)|_{\varepsilon=0}]$. Another standard definition of the iε prescription (which does not break the symmetry existing in the background spacetime) is given by

$$m^2 \to m^2 - \mathrm{i}\varepsilon. \tag{H2}$$

In this Appendix, we comment on the difference between the two iε prescription.

In fact, for the global patch, there is no difference in the analytical results between the two iε prescription. On the other hand, for the Poincaré patch, if we use the iε prescription given by $m^2 \to m^2 - \mathrm{i}\varepsilon$, the wave functions have the form

$$\varphi(\eta;\eta_0) \sim \frac{\sqrt{\pi}}{2} e^{\mathrm{i}(k\eta_0+\frac{\pi(2\nu_\varepsilon+1)}{4})}(-\eta)^{\frac{d-1}{2}} H^{(1)}_{\nu_\varepsilon}(-k\eta), \tag{H3}$$

$$\bar{\varphi}(\eta;\eta_0) \sim \frac{\sqrt{\pi}}{2} e^{-\mathrm{i}(k\eta_0+\frac{\pi(2\nu_\varepsilon+1)}{4})}(-\eta)^{\frac{d-1}{2}} H^{(2)}_{\nu_\varepsilon}(-k\eta), \tag{H4}$$

$$\varphi(\eta;\eta_1) \sim -\frac{\Gamma(\nu_\varepsilon)(k/2)^{-\nu_\varepsilon}}{2\sqrt{2\bar{m}_1}}(-\eta_1)^{-\nu_\varepsilon}\left(\frac{d-1}{2} - \nu_\varepsilon - \mathrm{i}\bar{m}_1\right)(-\eta)^{\frac{d-1}{2}} J_{\nu_\varepsilon}(-k\eta), \tag{H5}$$

$$\bar{\varphi}(\eta;\eta_1) \sim -\frac{\Gamma(\nu_\varepsilon)(k/2)^{-\nu_\varepsilon}}{2\sqrt{2\bar{m}_1}}(-\eta_1)^{-\nu_\varepsilon}\left(\frac{d-1}{2} - \nu_\varepsilon + \mathrm{i}\bar{m}_1\right)(-\eta)^{\frac{d-1}{2}} J_{\nu_\varepsilon}(-k\eta). \tag{H6}$$

These wave functions agree with (163)–(166) after we take the limit $\varepsilon \to 0$, except for the wave function $\varphi(\eta;\eta_0)$. Since the in-in or in-out propagator does not use $\varphi(\eta;\eta_0)$, the propagators in the Poincaré patch do not depend on the manner of the iε prescription. Thus, in both patches, there is no difference in the analytical results between the two iε prescriptions. However, for the numerical calculations given in Sec. V, these two prescriptions give slightly different results, and the iε prescription used in this paper seems to be better in comparison with the analytical results.

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