Cosmological perturbations in extended massive gravity

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We study cosmological perturbations around self-accelerating solutions to two extensions of nonlinear massive gravity: the quasi-dilaton theory and the mass-varying theory. We examine stability of the cosmological solutions, and the extent to which the vanishing of the kinetic terms for scalar and vector perturbations of self-accelerating solutions in massive gravity is generic when the theory is extended. We find that these kinetic terms are in general nonvanishing in both extensions, though there are constraints on the parameters and background evolution from demanding that they have the correct sign. In particular, the self-accelerating solutions of the quasi-dilaton theory are always unstable to scalar perturbations with wavelength shorter than the Hubble length.

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I. INTRODUCTION

Recent years have seen the development of a nonlinear theory propagating the five degrees of freedom of a massive graviton (dRGT theory [1,2], see [3] for a review), without the Boulware-Deser ghost [4,5]. This theory admits self-accelerating solutions [6–12], in which the Universe is de Sitter without a cosmological constant in the action. The Hubble scale of these self-accelerating solutions is of order the mass of the graviton. Having a light graviton is technically natural [13,14], so these solutions are of great interest to account for cosmic acceleration in the late-time universe.

Given any nontrivial solution, it is natural to ask how the perturbations around it behave, and in particular whether there are interesting new effects in the propagation of the associated degrees of freedom. The perturbation theory for the self-accelerating solutions of dRGT has been studied in [15–19]. Freedom from the Boulware-Deser ghost means that around any background, at most five degrees of freedom propagate. Around a homogeneous and isotropic cosmology, these take the form of one transverse-traceless tensor, one transverse vector and one scalar. Even though the Boulware-Deser ghost is absent, the kinetic terms of these degrees of freedom can potentially have the wrong sign, in which case they are ghosts, signaling that particular background is unstable.

In fact, around the self-accelerating solutions of dRGT theory, the scalar and vector degrees of freedom have vanishing kinetic terms [15,20]. This result could have critical implications for the cosmology, and it is important to understand the extent to which it is a generic result in these types of models. There are several avenues one might consider. Quantum mechanically, kinetic terms may be generated by loops. Determining the sign of such terms, and hence whether the background propagates ghosts, is then a difficult question whose answer depends in general on the details of the matter propagating in the loops. If we wish to restore the kinetic terms at the classical level, one avenue is to move away from homogeneous and isotropic cosmologies [21,22]. Another is to keep homogeneity and isotropy, but change the theory by adding more degrees of freedom. In this paper, we take the latter approach. We study cosmological perturbations in two extensions of dRGT theory: quasi-dilaton massive gravity, and mass-varying massive gravity.

Massive gravity admits an extension to a theory with a global scale symmetry through the inclusion of a specific scalar field, dubbed the *quasi-dilaton* [23]. We examine self-accelerating background solutions to the quasi-dilaton model, and consider the behavior of cosmological perturbations (see also [24]). We find the conditions under which the backgrounds are free of ghosts. We find that there is always a ghost-like instability in the scalar sector, for fluctuations of physical wavelength shorter than the Hubble radius.

The other extension we consider is mass-varying massive gravity, the theory obtained by promoting the mass to a scalar field [10,25]. We study the behavior of cosmological perturbations in this model, and find the conditions under which the backgrounds are free of ghosts.

II. QUASI-DILATON THEORY

We start with the quasi-dilaton theory, introduced in [23]. The action governs a dynamical metric $g_{\mu\nu}$ and a scalar field σ,[1]

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[1]We thank the authors of [23] for pointing out the existence of the ξ term in the most general quasi-dilaton action.

$$S = \frac{M_p^2}{2}\int d^4x\Big\{\sqrt{-g}\Big[R[g] - 2\Lambda + 2m_g^2(\mathcal{L}_2 + \alpha_3\mathcal{L}_3 + \alpha_4\mathcal{L}_4) - \frac{\omega}{M_p^2}\partial_\mu\sigma\partial^\nu\sigma\Big] + 2m_g^2\xi\sqrt{-\bar{g}}e^{4\sigma/M_p}\Big\}. \quad (1)$$

The part of the action which provides the mass to the graviton is

$$\mathcal{L}_2 = \frac{1}{2}([\mathcal{K}]^2 - [\mathcal{K}^2]),$$
$$\mathcal{L}_3 = \frac{1}{6}([\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3]),$$
$$\mathcal{L}_4 = \frac{1}{24}([\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 3[\mathcal{K}^2]^2 + 8[\mathcal{K}][\mathcal{K}^3] - 6[\mathcal{K}^4]), \quad (2)$$

where square brackets denote a trace. While these expressions are similar in form to the dRGT theory, in the case of the quasi-dilaton theory, the building block tensor $\mathcal{K}$ is defined as

$$\mathcal{K}^\mu{}_\nu = \delta^\mu{}_\nu - e^{\sigma/M_p}\left(\sqrt{g^{-1}\bar{g}}\right)^\mu{}_\nu, \quad (3)$$

where $\bar{g}_{\mu\nu}$ is a nondynamical fiducial metric. The theory is invariant under a global dilation of the space-time coordinates, accompanied by a shift of σ. This symmetry rules out a nontrivial potential for σ.

Throughout the analysis, we choose the fiducial metric to be Minkowski,

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \delta_{ij}dx^i dx^j. \quad (4)$$

A. Background equations of motion

For the physical background metric, we adopt the flat Friedmann Robertson Walker (FRW) ansatz

$$g_{\mu\nu}dx^\mu dx^\nu = -N(t)^2dt^2 + a(t)^2\delta_{ij}dx^i dx^j. \quad (5)$$

To obtain the background equations of motion, it is convenient to introduce time reparametrization invariance, so that we may write a mini-superspace action. We replace the fiducial metric with

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu = -f'(t)^2dt^2 + \delta_{ij}dx^i dx^j, \quad (6)$$

where $f(t)$ is the Stückelberg scalar [13], and unitary gauge corresponds to the choice $f(t) = t$.

The mini-superspace action is

$$\frac{S}{V} = \int dt\Big\{M_p^2\Big[-3\frac{a\dot{a}^2}{N} - \Lambda a^3 N\Big] + \frac{\omega a^3}{2N}\dot{\sigma}^2 + M_p^2m_g^2[Na^3(X-1)(3(X-2) - (X-4)(X-1)\alpha_3 - (X-1)^2\alpha_4) + f'a^4X[(X-1)(3 - 3(X-1)\alpha_3 + (X-1)^2\alpha_4) + \xi X^3]]\Big\}, \quad (7)$$

where V is the comoving volume and we have defined

$$X \equiv \frac{e^{\sigma/M_p}}{a}. \quad (8)$$

In addition, to simplify expressions later on, we define

$$H \equiv \frac{\dot{a}}{Na}, \qquad r \equiv \frac{a}{N}. \quad (9)$$

Varying with respect to f and then choosing unitary gauge, we obtain a constraint equation:

$$\frac{\delta S}{\delta f}\Big|_{f=t} = m_g^2M_p^2\frac{d}{dt}\{a^4X[(1-X)[3 - 3(X-1)\alpha_3 + (X-1)^2\alpha_4] - \xi X^3]\} = 0. \quad (10)$$

The Friedmann equation is obtained by varying with respect to the lapse N,

$$\frac{1}{M_p^2a^3}\frac{\delta S}{\delta N}\Big|_{f=t} = 3H^2 - \Lambda - m_g^2(X-1)[-3(X-2) + (X-4)(X-1)\alpha_3 + (X-1)^2\alpha_4] - \frac{\omega}{2}\left(H + \frac{\dot{X}}{XN}\right)^2 = 0, \quad (11)$$

and varying with respect to the scale factor, a, yields the acceleration equation. This may be combined in a linear combination with (11) to yield the simpler equation

$$\frac{1}{6M_p^2a^2N}\frac{\delta S}{\delta a}\Big|_{f=t} + \frac{1}{2M_p^2a^3}\frac{\delta S}{\delta N}\Big|_{f=t} = \frac{\dot{H}}{N} + \frac{\omega}{2}\left(H + \frac{\dot{X}}{NX}\right)^2 + \frac{m_g^2}{2}(1-r)X(3 - 2X + (X-3)(X-1)\alpha_3 + (X-1)^2\alpha_4) = 0. \quad (12)$$

Finally, the equation of motion for σ is

$$-\frac{X}{M_p\omega a^3N}\frac{\delta S}{\delta\sigma}\Big|_{f=t} = \frac{1}{N}\frac{d}{dt}\left(\frac{\dot{X}}{N}\right) + 3HX\left(H + \frac{\dot{X}}{NX}\right) + X\left[\frac{\dot{H}}{N} - \left(\frac{\dot{X}}{NX}\right)^2\right] + \frac{m_g^2X^2}{\omega}[3r(1-2X) - 6X + 9 + 3(X-1)(r(3X-1) + X - 3)\alpha_3 - (X-1)^2(r(4X-1) - 3)\alpha_4 - 4rX^3\xi] = 0. \quad (13)$$

Since time reparametrization invariance was introduced with the Stückelberg field f, there is a Bianchi identity which relates the four equations,

$$\frac{\delta S}{\delta\sigma}\dot{\sigma} + \frac{\delta S}{\delta f}\dot{f} - N\frac{d}{dt}\frac{\delta S}{\delta N} + \dot{a}\frac{\delta S}{\delta a} = 0. \quad (14)$$

Therefore one equation is redundant with the others and may be dropped. In discussing solutions we will generally choose to drop the acceleration Eq. (12), although we will use it to simplify expressions for the perturbations.

B. Self-accelerating background solutions

We now discuss solutions, starting with the Stückelberg constraint (10). Integrating this equation gives

$$X\{(1-X)[3-3(X-1)\alpha_3+(X-1)^2\alpha_4]-\xi X^3\} = \frac{1}{a^4}\times \text{constant}. \tag{15}$$

In an expanding universe, the right-hand side of the above equation decays as a^{-4}. Thus after a sufficiently long time, X saturates to a constant value $X_{\rm SA}$, corresponding to a zero of the left-hand side of (15). These constant X solutions lead to an effective energy density which acts like a cosmological constant. As pointed out in [23], there are four such solutions for which X is constant. Of these, $X=0$ implies $\sigma\to-\infty$, and as in [23], we drop this solution to avoid strong coupling.[2] What remains are the three solutions to the cubic equation[3]

$$(1-X)[3-3(X-1)\alpha_3+(X-1)^2\alpha_4]-\xi X^3|_{X=X_{\rm SA}}=0. \tag{17}$$

For these solutions, we write the Friedmann Eq. (11) as

$$\left(3-\frac{\omega}{2}\right)H^2=\Lambda+\Lambda_{\rm SA}, \tag{18}$$

where the effective cosmological constant from the mass term is

$$\Lambda_{\rm SA}\equiv m_g^2(X_{\rm SA}-1)[-3X_{\rm SA}+6+(X_{\rm SA}-4)(X_{\rm SA}-1)\alpha_3 + (X_{\rm SA}-1)^2\alpha_4]. \tag{19}$$

The Friedmann Eq. (18) also provides a condition on the parameter ω; on the self-accelerating solutions, one needs to have $\omega<6$ in order to keep the left-hand side of the Friedmann Eq. (18) positive. This ensures that when ordinary matter is added to the right-hand side, we will have standard cosmology during matter domination. (Although we do not include matter fields, the sign of the matter energy density can be determined by replacing the bare Λ with ρ/M_p^2.)

Finally, on the self-accelerating solutions, with constant H specified by (18), the equation of motion for σ fixes the ratio $r=a/N$. From Eq. (13), we obtain

$$r_{\rm SA}=1+\frac{\omega H^2(X_{\rm SA}-1)}{m_g^2X_{\rm SA}^2[\alpha_3(X_{\rm SA}-1)^2-2(X_{\rm SA}-1)-\xi X_{\rm SA}^2]}. \tag{20}$$

Here, to simplify the expression we have used the Stückelberg Eq. (15) to eliminate α_4.

C. Perturbations

To find the action for quadratic perturbations, we expand the physical metric in small fluctuations $\delta g_{\mu\nu}$ around a solution $g^{(0)}_{\mu\nu}$,

$$g_{\mu\nu}=g^{(0)}_{\mu\nu}+\delta g_{\mu\nu}, \tag{21}$$

and keep terms to quadratic order in $\delta g_{\mu\nu}$.

We break the perturbations into standard scalar, transverse vector and transverse-traceless tensor parts,

$$\delta g_{00}=-2N^2\Phi, \tag{22}$$

$$\delta g_{0i}=Na(B_i^T+\partial_iB), \tag{23}$$

$$\delta g_{ij}=a^2\left[h_{ij}^{TT}+\frac{1}{2}(\partial_iE_j^T+\partial_jE_i^T)+2\delta_{ij}\Psi + \left(\partial_i\partial_j-\frac{1}{3}\delta_{ij}\partial_l\partial^l\right)E\right], \tag{24}$$

where

$$\partial^ih_{ij}^{TT}=h_i^{TT\ i}=0,\qquad \partial^iB_i^T=0,\qquad \partial^iE_i^T=0. \tag{25}$$

We then introduce the perturbation of the scalar via

$$\sigma=\sigma^{(0)}+M_p\delta\sigma. \tag{26}$$

We perform the entire analysis in unitary gauge, so that there are no issues of gauge invariance to worry about, and no need to form gauge invariant combinations. We write the actions expanded in Fourier plane waves, i.e. $\vec{\nabla}^2\to -k^2$, $d^3x\to d^3k$. Raising and lowering of the spatial indices on perturbations is always carried out by δ^{ij} and δ_{ij}.

D. Tensor perturbations

We begin by considering tensor perturbations around the background (5),

$$\delta g_{ij}=a^2h_{ij}^{TT}, \tag{27}$$

where $\partial^ih_{ij}^{TT}=h_i^{TT\ i}=0$. The tensor quadratic action reads

[2] As we discuss at the end of Appendix A, the remaining solutions also lead to strong coupling in the vector and scalar sectors, when $\xi=0$ and the parameters α_3 and α_4 are such that $X\simeq 0$.

[3] For the choice $\xi=0$, the system simplifies as one of the solutions to Eq. (17) becomes $X_{\rm SA}=1$, while the remaining two are

$$X_\pm\equiv X_{\rm SA}|_{\xi=0}=\frac{3\alpha_3+2\alpha_4\pm\sqrt{9\alpha_3^2-12\alpha_4}}{2\alpha_4}. \tag{16}$$

In this special setting, the solution $X=1$ is uninteresting: the effective cosmological constant from the mass term is zero and in the present scenario, the background becomes equivalent to a de Sitter universe driven by a (bare) cosmological constant $6\Lambda/(6-\omega)$. However, in the presence of matter fields and no bare cosmological constant, this solution asymptotically approaches a Minkowski background and is unstable [23].

$$S = \frac{M_p^2}{8}\int d^3k a^3 N dt\left(\frac{1}{N^2}|\dot{h}_{ij}^{TT}|^2 - \left(\frac{k^2}{a^2} + M_{\rm GW}^2\right)|h_{ij}^{TT}|^2\right), \tag{28}$$

where the mass of the tensor modes is given by

$$M_{\rm GW}^2 \equiv \frac{m_g^2(r-1)X_{\rm SA}^3}{X_{\rm SA}-1}\left(1 + \frac{\xi X_{\rm SA}}{X_{\rm SA}-1}\right) + H^2\omega\left(\frac{r}{r-1} + \frac{2}{X_{\rm SA}-1}\right). \tag{29}$$

To obtain this, we have first used the background acceleration equation to eliminate any terms with $\ddot{a}$. Then we have used the self-accelerating branch of (15) (at late times when the right-hand side is zero), the Friedmann Eq. (11) evaluated on the self-accelerating solution (i.e. $\dot{X} = 0$), and the σ Eq. (13) evaluated on the self-accelerating solution (i.e. $\dot{X} = \dot{H} = 0$), to eliminate Λ, α_3 and α_4.

The tensor mode always has correct sign kinetic and gradient terms. However, it will be tachyonic if the mass term is negative: $M_{\rm GW}^2 < 0$. The stability of long wavelength gravitational waves is thus ensured by the condition $M_{\rm GW}^2 > 0$. Nevertheless, even if this condition is violated, the tachyonic mass is generically of order Hubble, so the instability would take the age of the Universe to develop.

E. Vector perturbations

We next turn to vector perturbations,

$$\delta g_{0i} = NaB_i^T, \qquad \delta g_{ij} = \frac{a^2}{2}(\partial_i E_j^T + \partial_j E_i^T), \tag{30}$$

where $\partial^i B_i^T = \partial^i E_i^T = 0$. The field B_i^T enters the action without time derivatives, so we may eliminate it as an auxiliary field using its own equation of motion (again we are using the equations of the background self-accelerating solution to eliminate Λ, α_3 and α_4)

$$B_i^T = \frac{k^2 a(r^2-1)}{4\omega a^2 H^2 + 2k^2(r^2-1)}\frac{\dot{E}_i^T}{N}. \tag{31}$$

Once this is inserted back into the action, what remains is a system of a single propagating vector,

$$S = \frac{M_p^2}{8}\int d^3k a^3 N dt\left(\frac{\mathcal{T}_V}{N^2}|\dot{E}_i^T|^2 - \frac{k^2 M_{\rm GW}^2}{2}|E_i^T|^2\right), \tag{32}$$

where

$$\mathcal{T}_V \equiv \frac{k^2}{2}\left(1 + \frac{k^2(r^2-1)}{2a^2H^2\omega}\right)^{-1}, \tag{33}$$

and $M_{\rm GW}$ is the mass of the tensor modes as in (29).

From Eq. (33), we see that for $(r^2-1)/\omega < 0$, there exists a critical momentum scale, $k_c = aH\sqrt{\frac{2\omega}{1-r^2}}$, above which the vector becomes a ghost. In the case $(r^2-1)/\omega \geq 0$, the kinetic terms of vector always has correct sign and thus there is no such critical momentum scale. In the first case, stability of the system requires that the physical critical momentum scale, k_c/a, be above the ultraviolet cutoff scale of the effective field theory, $\Lambda_{\rm UV}$, i.e.

$$\Lambda_{\rm UV}^2 \lesssim \frac{2H^2\omega}{1-r^2}, \quad \text{if } (r^2-1)/\omega < 0. \tag{34}$$

To determine whether the vector modes suffer from other instabilities, we define canonically normalized fields,

$$\mathcal{E}_i^T \equiv \frac{M_p}{2}\mathcal{T}_V E_i^T, \tag{35}$$

in terms of which, the action (32) reads

$$S = \frac{1}{2}\int d^3k a^3 N dt\left(\frac{1}{N^2}|\dot{\mathcal{E}}_i^T|^2 - \omega_V^2|\mathcal{E}_i^T|^2\right), \tag{36}$$

where the dispersion relation of the modes is given by

$$\omega_V^2 = (1+q^2)M_{\rm GW}^2 - \frac{H^2q^2(1+4q^2)}{(1+q^2)^2}, \tag{37}$$

and we have defined the dimensionless quantity

$$q^2 \equiv \frac{k^2}{a^2}\frac{r^2-1}{2H^2\omega}. \tag{38}$$

The second term in the dispersion relation (37), which originates from the time derivatives of $\mathcal{T}_V$, is always of order $\mathcal{O}(H^2)$, provided that $q^2 > 0$. Therefore, in this regime, this term does not introduce instabilities faster than the Hubble expansion rate. Moreover, if $q^2 < 0$, the no-ghost condition (34) imposes $|q^2| \lesssim (k^2/a^2)/\Lambda_{\rm UV}^2$. Thus, for any physical momenta sufficiently lower than the cutoff scale of the effective theory, the second term in (37) does not lead to any visible instability, i.e. the growth rate of any instability (if any exist) is at most of the cosmological scale.

The vector modes may potentially suffer from a gradient instability arising from the first term in (37), if $M_{\rm GW}^2 < 0$ and $q^2 > 0$. The growth rate of this instability can be made lower than or at most of the order of the cosmological scale for all physical momenta below the UV cutoff $\Lambda_{\rm UV}$, provided that

$$\Lambda_{\rm UV}^2 \lesssim \frac{2H^2\omega}{r^2-1}, \quad \text{if } (r^2-1)/\omega > 0 \quad \text{and} \quad M_{\rm GW}^2 < 0. \tag{39}$$

F. Scalar perturbations

Finally, we consider the action quadratic in scalar perturbations,

$$\delta g_{00} = -2N^2\Phi, \qquad \delta g_{0i} = Na\partial_i B, \qquad \delta g_{ij} = a^2\left[2\delta_{ij}\Psi + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\partial_l\partial^l\right)E\right], \tag{40}$$

$$\sigma = \sigma^{(0)} + M_p\delta\sigma. \tag{41}$$

The scalar sector should consist of two dynamical degrees of freedom: the scalar field and the longitudinal mode of the massive graviton. The perturbations Φ and B stemming from δg_{0i} and δg_{00} are free of time derivatives, and so we eliminate them as auxiliary fields using their equations of motion:

$$B = \frac{r^2-1}{3\omega a H^2}\left[3H(\omega\delta\sigma - 2\Phi) + \frac{1}{N}(k^2\dot{E} + 6\dot{\Psi})\right], \tag{42}$$

$$\begin{aligned}\Phi = {}& \frac{1}{3[\omega(6-\omega)a^2H^2 + 4k^2(r^2-1)]}\\ &\times\left[k^4\omega E + 3\omega\left(2k^2(r^2-1) - \frac{3\omega a^2H^2}{r-1}\right)\delta\sigma\right.\\ &+ 3\omega\left(2k^2 + \frac{3\omega a^2H^2}{r-1}\right)\Psi - \frac{3\omega a^2 H}{N}(\omega\delta\dot{\sigma} - 6\dot{\Psi})\\ &\left.+ \frac{2k^2}{HN}(r^2-1)(k^2\dot{E} + 6\dot{\Psi})\right].\end{aligned} \tag{43}$$

Inserting these back into the action, we obtain an action with three fields, Ψ, E and $\delta\sigma$. Since the "sixth" degree of freedom (which would come from the Boulware-Deser instability in generic massive theories) is removed by construction, there is another nondynamical combination, which we determine to be

$$\tilde{\Psi} = \frac{1}{\sqrt{2}}(\Psi + \delta\sigma). \tag{44}$$

We also define an orthogonal combination,

$$\tilde{\delta\sigma} = \frac{1}{\sqrt{2}k^2}(\Psi - \delta\sigma). \tag{45}$$

With these field redefinitions, the action can be written in terms of $\tilde{\Psi}$, $\tilde{\delta\sigma}$ and E, with no time derivatives on $\tilde{\Psi}$. The latter is therefore auxiliary and can be eliminated via its equation:

$$\begin{aligned}\tilde{\Psi} = {}& \left(-k^2 - \frac{24a^2H^2}{r(r-1)}\right.\\ &\left.+ \frac{2a^2H^2k^2\{[48-(6-\omega)\omega]r - \omega^2\}}{[4k^2-(6-\omega)\omega a^2H^2](r-1)}\right)\tilde{\delta\sigma}\\ &- \frac{2\sqrt{2}k^4 E}{3[4k^2-(6-\omega)\omega a^2H^2]}\\ &+ 2a^2H\left(\frac{3}{r} + \frac{(6-\omega)[2k^2(r-1) + 3\omega a^2H^2]}{[4k^2-(6-\omega)\omega a^2H^2](r-1)}\right)\frac{\dot{\tilde{\delta\sigma}}}{N}\\ &+ \frac{\sqrt{2}(6-\omega)k^2a^2H}{3[4k^2-(6-\omega)\omega a^2H^2]}\frac{\dot{E}}{N}.\end{aligned} \tag{46}$$

Using this solution in the action, and introducing the notation $Y \equiv (\tilde{\delta\sigma}, E)$, the scalar action can then be written as

$$\begin{aligned}S = \int\frac{d^3k}{2}a^3Ndt\left[\frac{\dot{Y}^\dagger}{N}\mathcal{K}\frac{\dot{Y}}{N} + \frac{\dot{Y}^\dagger}{N}\mathcal{M}Y\right.\\ \left.+ Y^\dagger\mathcal{M}^T\frac{\dot{Y}}{N} - Y^T\Omega^2 Y\right],\end{aligned} \tag{47}$$

where $\mathcal{M}$ is a real antisymmetric 2×2 matrix, and $\mathcal{K}$ and Ω^2 are real symmetric 2×2 matrices. (Note that there is no loss of generality in taking $\mathcal{M}$ antisymmetric, since the symmetric part can be absorbed into Ω^2 by adding total derivatives.) For now, we focus on the kinetic terms. The components of the matrix $\mathcal{K}$ are

$$\begin{aligned}\mathcal{K}_{11} &= 2k^4M_p^2\omega\left[1 + \frac{9a^2H^2}{k^2(r-1)^2}\right.\\ &\qquad\left.- \frac{a^2H^2[\omega + (6-\omega)r]^2}{[4k^2-(6-\omega)\omega a^2H^2](r-1)^2}\right],\\ \mathcal{K}_{12} &= \sqrt{2}k^4M_p^2\omega\left[\frac{r}{\omega(r-1)}\right.\\ &\qquad\left.- \frac{2k^2[\omega + (6-\omega)r]}{3\omega[4k^2-(6-\omega)\omega a^2H^2](r-1)}\right],\\ \mathcal{K}_{22} &= \frac{k^4M_p^2\omega}{36}\left[1 - \frac{(6-\omega)^2a^2H^2}{4k^2-(6-\omega)\omega a^2H^2}\right].\end{aligned} \tag{48}$$

For the case at hand, it is sufficient to study the determinant of the kinetic matrix $\mathcal{K}$ to determine the sign of the eigenvalues.[4] The determinant takes the comparatively simple form,

$$\det\mathcal{K} = \frac{3M_p^4k^6\omega^2a^4H^4}{[\omega a^2H^2 - \frac{4k^2}{6-\omega}](r-1)^2}. \tag{49}$$

The sign of the determinant is determined by the sign of the quantity within the square brackets. First, note that the determinant is always negative if $\omega < 0$. Along with the condition for a realistic cosmology obtained from (18), the range of allowed ω is thus

$$0 < \omega < 6, \tag{50}$$

in agreement with [23]. In order to have no ghosts in the scalar sector, we need (see Fig. 1)

$$\frac{k}{aH} < \frac{\sqrt{\omega(6-\omega)}}{2}. \tag{51}$$

Generically, the right-hand side of the inequality (51) is of order 1. This implies that for modes with physical wavelengths that are smaller than cosmological scales, one of the two degrees of freedom is a ghost. In other words, parametrically, there is an instability in the scalar sector at physical momenta above $H \sim m_g$. As shown in Appendix A, both the physical momenta and the energies of those ghost modes near the threshold are not parametrically higher than $H \sim m_g$ and thus are below the UV cutoff scale of the effective field theory. This signals the presence of ghost instabilities in the regime of validity of the effective field theory.

[4]The absence of ghosts requires that both the determinant and the trace are positive. On the other hand, only one of these being negative is enough to deduce the existence of a ghost degree, which happens to be the case for the current system. For a detailed diagonalization treatment, we refer the reader to Appendix A.

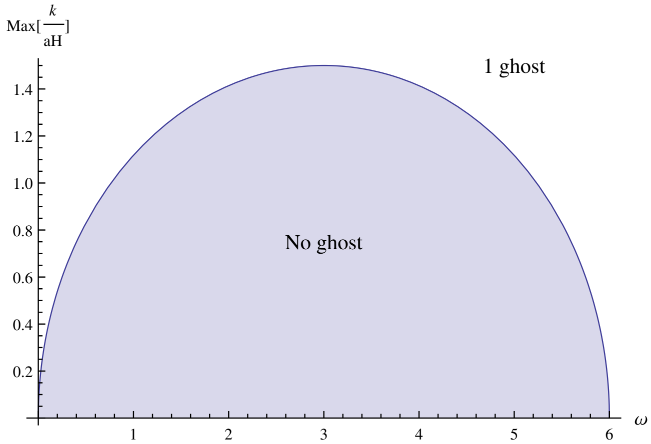

FIG. 1 (color online). The stability of the scalar sector implied by the determinant of the kinetic matrix (49). For modes with $k/(aH)$ below the solid line, the determinant is positive, so there are no ghost degrees of freedom [see Eq. (A1)] for the field basis in which this is manifest). On the other hand, above the solid line, one degree of freedom has a positive kinetic term while the other is a ghost.

We end this section by comparing our results to those found in [23,24]. Noting that at the level of the kinetic matrix (48) the only scale other than H is the momentum, the limit $H \to 0$ is equivalent to considering modes with wavelengths much shorter than the Hubble radius, i.e. $k \gg aH$, which is in contradiction with the no-ghost condition (51). In this limit, the kinetic matrix then becomes

$$\mathcal{K} = M_p^2 \omega k^4 \begin{pmatrix} 2 & \frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} & \frac{1}{36} \end{pmatrix} + \mathcal{O}\left(\frac{aH}{k}\right)^2, \tag{52}$$

which has one positive and one zero eigenvalue, as in [23] (see Appendix B for a more detailed comparison). In other words, the apparent stability of the self-accelerating solution is due to the loss of the dynamics of the ghost degree of freedom in the short wave-length limit, and so the decoupling limit is not sufficient to determine stability, in agreement with [23]. In [24], only the super-horizon limit $k \to 0$ is considered, so the instability which appears only for physical wavelengths $\lesssim$ Hubble is not visible in this limit.

G. Higher derivative terms and UV sensitivity

The quasi-dilaton theory is governed by the global scaling symmetry described in [23]. The action (1) includes all possible terms compatible with the symmetry, with up to two derivatives, and we found there was no way to render the scalar perturbations of the self-accelerating solutions stable at all momenta.

However, beyond two derivative order there are many more terms compatible with the symmetry. These higher derivative terms can be thought of as encoding UV effects from whatever physics completes the theory. Among the possible higher-derivative terms, we will focus here on two classes of distinguished interaction terms which will not add new degrees of freedom. There are the Goldstone-like terms of the form $\sim(\partial\sigma)^n$, and the three possible nontrivial covariantized Galileon terms, of the form $\sim(\partial\sigma)^2 \times (\partial^2\sigma)^n + \cdots$ [26–29]. The strong coupling scale of the quasi-dilaton on flat space is $\Lambda_3 \sim (M_p m_g^2)^{1/3}$ [23], so it is natural for the Galileon-like terms to appear suppressed by this scale. The Goldstone-like terms should carry the scale[5] $\Lambda_2 \sim (M_p m_g)^{1/2} \gg \Lambda_3$. One can repeat the calculation of the perturbations including these terms, in the hopes that the fluctuations can be stabilized at short scales $k \gg H$.

The Friedmann equation now becomes:

$$\left(3 - \frac{\omega}{2} + 3g_3 h^2\right) H^2 + \frac{1}{2} m_g^2 [f(h^2) - 2h^2 f'(h^2)] = \Lambda + \Lambda_{\rm SA}, \qquad h \equiv \frac{H}{m_g}, \tag{53}$$

where g_3 is the dimensionless coupling for the cubic covariant Galileon and we have chosen the form $\Lambda_2^4 f(x)/2$ as the Goldstone-like term, where $x = -(\nabla\sigma)^2/\Lambda_2^4$. On the other hand, the constraint equation and the value of $\Lambda_{\rm SA}$ remain the same. (We have omitted the quartic and quintic Galileon terms for simplicity.) There are still self-accelerating solutions with $H \sim m_g$ so the existence of these solutions appears insensitive to the UV effects encoded by the higher derivative operators. For a sensible cosmology, H^2 determined by the Friedmann equation should be an increasing function of the bare cosmological constant Λ (which represents the matter energy density in our setup). Demanding this, we obtain the condition

$$6 - \omega + 12 g_3 h^2 - f'(h^2) - 2h^2 f''(h^2) > 0. \tag{54}$$

The determinant of the kinetic matrix $\mathcal{K}$ for scalar fluctuations changes by order one,

$$\det \mathcal{K} = \frac{3M_p^4 k^6 [\omega - 3g_3 h^2 + f'(h^2)]^2 a^4 H^4}{\left\{[\omega - 3g_3 h^2 + f'(h^2)]a^2H^2 - \frac{(2+g_3 h^2)^2 k^2}{6-\omega+12g_3h^2 - f'(h^2) - 2h^2 f''(h^2)}\right\}(r-1)^2}, \tag{55}$$

[5]The reason the Goldstone-like terms carry a higher scale is because the Λ_3 decoupling limit of the theory has an enhanced Galilean symmetry [23], which the Goldstone-type interactions are not invariant under. This means that they will not be generated in the decoupling limit, so whatever the quantum corrections to these operators are in the full theory, they should not survive in the decoupling limit, i.e. they should be suppressed by a scale higher than Λ_3.

but the determinant is still always negative for sufficiently large momenta, provided that the condition (54) is satisfied.

The quartic and quintic covariant Galileon terms can render the determinant of the kinetic matrix for scalar perturbation positive for large momenta, depending on the values of the coupling constants. However, in this regime of parameters, the tensor and vector modes acquire negative kinetic terms. Moreover, after explicit diagonalization of the kinetic matrix one can show that there are two ghost modes in the scalar sector, provided that H^2, determined by the Friedmann equation, is an increasing function of Λ.

Thus, the form of the dispersion relations for the perturbations depends on and receives order one correction due to UV effects, but the presence of the ghost seems to be a robust feature.

III. VARYING MASS THEORY

We now turn to the varying mass theory, obtained by introducing a scalar into dRGT theory and allowing the graviton mass to be a function of this scalar. This theory was first considered in [10], and further studied in [25].

The action is

$$S=\int d^4x\sqrt{-g}\bigg\{\frac{M_p^2}{2}[R[g]-2\Lambda+2m_g^2(\sigma)[\mathcal{L}_2+\alpha_3(\sigma)\mathcal{L}_3 + \alpha_4(\sigma)\mathcal{L}_4]]-\frac{1}{2}\partial_\mu\sigma\partial^\nu\sigma-V(\sigma)\bigg\}. \tag{56}$$

The part of the action which provides mass to the graviton takes the same form as in Eq. (2), but here the building block tensor $\mathcal{K}$ is the same as in dRGT theory [2], i.e.

$$\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \left(\sqrt{g^{-1}\bar{g}}\right)^\mu_\nu. \tag{57}$$

One of our goals is to compare our results with the analogous analysis of perturbations in the dRGT theory. Since the original theory does not allow flat solutions for a Minkowski reference metric, it is necessary to adopt a more general form. We therefore extend the fiducial metric to be an arbitrary spatially flat homogeneous and isotropic metric,

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu = -n(t)^2dt^2 + \alpha(t)^2\delta_{ij}dx^idx^j. \tag{58}$$

A. Cosmological background equations

We first study the cosmological background equations (see [25,30–34] for more on background cosmological solutions to mass-varying massive gravity). For the physical background metric, we adopt the flat FRW ansatz

$$g_{\mu\nu}dx^\mu dx^\nu = -N(t)^2dt^2 + a(t)^2\delta_{ij}dx^idx^j. \tag{59}$$

To write the mini-superspace action, we introduce time reparametrization via a Stückelberg field $f(t)$, by replacing the fiducial metric with

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu = -n(f(t))^2f'(t)^2dt^2 + \alpha(f(t))^2\delta_{ij}dx^idx^j. \tag{60}$$

Unitary gauge corresponds to the choice $f(t) = t$.

The mini-superspace action is

$$\frac{S}{V} = \int \mathrm{d}t\bigg\{3M_\mathrm{P}^2\bigg[-\frac{\dot{a}^2a}{N} + m_g^2(NF - \dot{f}n(f)G)\bigg] + a^3\bigg[\frac{1}{2}N^{-1}\dot{\sigma}^2 - NV(\sigma) - NM_\mathrm{P}^2\Lambda\bigg]\bigg\}, \tag{61}$$

where V is the comoving volume and

$$F \equiv a(a-\alpha(f))(2a-\alpha(f)) + \frac{\alpha_3}{3}(a-\alpha(f))^2(4a-\alpha(f)) + \frac{\alpha_4}{3}(a-\alpha(f))^3, \tag{62}$$

$$G \equiv a^2(a-\alpha(f)) + \alpha_3 a(a-\alpha(f))^2 + \frac{\alpha_4}{3}(a-\alpha(f))^3. \tag{63}$$

In the following, for clarity, we will use the definitions

$$H \equiv \frac{\dot{a}}{Na}, \qquad X \equiv \frac{\alpha}{a}, \qquad \bar{H} \equiv \frac{\dot{\alpha}}{n\alpha}, \qquad r \equiv \frac{n}{NX}, \tag{64}$$

and we will omit the dependence of the functions m_g, α_3, α_4 and V on the field value σ. (We also caution the reader that the above definitions of X and r are different than the ones in the quasi-dilaton theory, which we introduced in Sec. II.)

The equation of motion for the temporal Stückelberg field f is

$$-\frac{1}{3M_p^2Nn}\frac{\delta S}{\delta f}\bigg|_{f=t} = \frac{1}{N}\frac{d}{dt}\bigg\{m_g^2a^3(X-1)\bigg[1-(X-1)\alpha_3+\frac{1}{3}(X-1)^2\alpha_4\bigg]\bigg\} + a^3\bar{H}m_g^2X[3-X(2+r)+(X-1)((1+2r)X-3)\alpha_3 - (X-1)^2(rX-1)\alpha_4] = 0. \tag{65}$$

The Friedmann equation is obtained by varying the action with respect to N,

$$\frac{1}{M_p^2a^3}\frac{\delta S}{\delta N}\bigg|_{f=t} = 3H^2 - \Lambda - \frac{1}{M_p^2}\left(\frac{\dot{\sigma}^2}{2N^2}+V\right) - m_g^2(X-1)[-3(X-2) + (X-4)(X-1)\alpha_3 + (X-1)^2\alpha_4] = 0, \tag{66}$$

and by taking a variation with respect to a, we obtain the dynamical equation which, after forming a linear combination with (66), can be expressed as

$$\frac{1}{3M_p^2Na^2}\frac{\delta S}{\delta a}\bigg|_{f=t} - \frac{1}{M_p^2a^3}\frac{\delta S}{\delta N}\bigg|_{f=t} = \frac{2\dot{H}}{N} + \frac{\dot{\sigma}^2}{M_p^2N^2} - m_g^2(r-1)X[3-2X + (X-3)(X-1)\alpha_3 + (X-1)^2\alpha_4] = 0. \quad (67)$$

Finally, the equation of motion for σ is

$$-\frac{1}{a^3N}\frac{\delta S}{\delta\sigma}\bigg|_{f=t} = \frac{1}{N}\frac{d}{dt}\left(\frac{\dot{\sigma}}{N}\right) + 3H\frac{\dot{\sigma}}{N} + V' - M_p^2m_g^2(X-1)^2 \times\bigg\{\alpha_3'(4-X(1+3r)) + \alpha_4'(X-1)(rX-1) + \frac{2m_g'}{m_g}\bigg[\frac{3(X(r+1)-2)}{X-1} - (X(1+3r)-4)\alpha_3 + (X-1)(rX-1)\alpha_4\bigg]\bigg\} = 0, \quad (68)$$

where a prime denotes differentiation with respect to σ.

It is convenient to cast these equations into a more familiar perfect fluid-like form, by defining the following quantities:

$$\begin{aligned}
\rho_m &\equiv M_p^2m_g^2(X-1)[-3(X-2) + (X-4)(X-1)\alpha_3 + (X-1)^2\alpha_4],\\
p_m &\equiv M_p^2m_g^2[6-3X(r+2) + X^2(1+2r) - (X-1)(4-X(3r+2)+rX^2)\alpha_3 - (X-1)^2(rX-1)\alpha_4],\\
Q &\equiv M_p^2m_g^2\frac{\dot{\sigma}}{N}(X-1)^2\bigg\{\alpha_3'(4-X(1+3r)) + \alpha_4'(X-1)(rX-1) + \frac{2m_g'}{m_g}\bigg[\frac{3(X(r+1)-2)}{X-1} - (X(1+3r)-4)\alpha_3 + (X-1)(rX-1)\alpha_4\bigg]\bigg\},\\
\rho_\sigma &\equiv \frac{\dot{\sigma}^2}{2N^2} + V, \qquad p_\sigma \equiv \frac{\dot{\sigma}^2}{2N^2} - V,
\end{aligned} \quad (69)$$

in terms of which Eqs. (65)–(68) can be rewritten, respectively, as

$$\begin{aligned}
&\frac{\dot{\rho}_m}{N} + 3H(\rho_m + p_m) = -Q,\\
&3H^2 = \Lambda + \frac{1}{M_p^2}(\rho_\sigma + \rho_m),\\
&\frac{\dot{H}}{N} = -\frac{1}{2M_p^2}[(\rho_\sigma + p_\sigma) + (\rho_m + p_m)],\\
&\frac{\dot{\rho}_\sigma}{N} + 3H(\rho_\sigma + p_\sigma) = Q.
\end{aligned} \quad (70)$$

From here on, we will use these forms of the cosmological background equations.

B. Tensor perturbations

We proceed with the perturbation theory in the same manner as described in Sec. II C. We start with tensor perturbations around the background (59),

$$\delta g_{ij} = a^2h_{ij}^{TT}, \quad (71)$$

where $\partial^ih_{ij}^{TT} = h_i^{TT\,i} = 0$.

The action for the tensor perturbations reads

$$S = \frac{M_p^2}{8}\int d^3ka^3Ndt\left(\frac{1}{N^2}|\dot{h}_{ij}^{TT}|^2 - \left(\frac{k^2}{a^2} + M_{\rm GW}^2\right)|h_{ij}^{TT}|^2\right), \quad (72)$$

where the mass term is

$$M_{\rm GW}^2 = \frac{(r-1)X^2}{(X-1)^2}\left[m_g^2(X-1) - \frac{\rho_m}{M_p^2}\right] - \left(\frac{1}{r-1} + \frac{2X}{X-1}\right)\frac{\rho_m + p_m}{M_p^2}. \quad (73)$$

To obtain this, we have used the background acceleration Eq. (67). In the case of self-accelerating solutions [11,15] of the standard dRGT theory, i.e. when $\dot{\sigma} = 0$ and $\rho_m = -p_m =$ const, the last term in (73) drops out of the calculation, and the mass reduces to the one found in [15]. We also stress that $M_{\rm GW}$ here is different than the one defined for the quasi-dilaton theory in Sec. II.

C. Vector perturbations

Next, we consider transverse vector perturbations to the metric

$$\delta g_{0i} = NaB_i^T, \qquad \delta g_{ij} = \frac{a^2}{2}(\partial_iE_j^T + \partial_jE_i^T), \quad (74)$$

where $\partial^iB_i^T = \partial^iE_i^T = 0$. The field B_i appears without time derivatives and may be eliminated as an auxiliary field using its own equation of motion,

$$B_i^T = \frac{a}{2[1 - \frac{2a^2}{k^2M_p^2(r^2-1)}(\rho_m + p_m)]}\frac{\dot{E}_i^T}{N}. \quad (75)$$

Once this solution is inserted back into the action, what remains is an action for one dynamical vector

$$S = \frac{M_p^2}{8}\int d^3ka^3Ndt\left(\frac{\mathcal{T}_V}{N^2}|\dot{E}_i^T|^2 - \frac{k^2M_{\rm GW}^2}{2}|E_i^T|^2\right), \quad (76)$$

where

$$\mathcal{T}_V \equiv \frac{k^2}{2}\left(1 - \frac{k^2(r^2-1)M_p^2}{2a^2(\rho_m + p_m)}\right)^{-1}, \quad (77)$$

and the tensor mode mass $M_{\rm GW}$ is as defined as in (73). For the self-accelerating solutions of the dRGT theory, where $\dot\sigma = 0$ and $\rho_m + p_m = 0$, the vector kinetic term vanishes, in agreement with the results of [15].

From Eq. (77), we see that for $(r^2-1)/(\rho_m+p_m) > 0$ there is a critical momentum scale above which the vector modes become ghosts. On the other hand, in the opposite case with $(r^2-1)/(\rho_m+p_m) \le 0$, the timelike kinetic term of vector modes always has the correct sign, and thus there is no such critical momentum scale. The stability of the system thus requires that such a critical momentum scale should be either absent or above $\Lambda_{\rm UV}$, the UV cutoff scale of the effective field theory, i.e.

$$\frac{\Lambda_{\rm UV}^2(1-r^2)}{H^2R} < 2, \qquad R \equiv -\frac{\rho_m+p_m}{M_p^2H^2}. \tag{78}$$

In order to determine the stability conditions and the time scale of potential instabilities, it is useful to use canonical normalization. However, the existence of several unknown functions and the lack of a simple background prevent us from performing the stability analysis in a complete way. On the other hand, assuming that the tensor modes have positive squared-mass (73), i.e. $M_{\rm GW}^2 > 0$ and the vector sector is free of ghosts (78), we can still obtain sufficient (but not necessary) conditions to ensure the stability of the modes. These conditions cause the vector modes to damp with time, so tachyon-like instabilities can be avoided. It is convenient to perform a time reparametrization choosing the lapse function to be $N = a^3\mathcal{T}_V$,

$$S|_{N=a^3\mathcal{T}_V} = \frac{M_p^2}{8}\int d^3k dt\left(|\dot E_i^T|^2 - \frac{k^2}{2}a^6\mathcal{T}_V M_{\rm GW}^2|E_i^T|^2\right). \tag{79}$$

Hence, provided that the conditions $M_{\rm GW}^2 > 0$ and $\mathcal{T}_V > 0$ are already imposed, the amplitudes of the variables E_i^T decrease as the Universe expands if

$$\partial_t[a^6\mathcal{T}_V M_{\rm GW}^2] > 0. \tag{80}$$

Demanding that this condition holds for all momenta below $\Lambda_{\rm UV}$, we obtain

$$\mathcal{A}\,\frac{\Lambda_{\rm UV}^2(1-r^2)}{H^2R} < \frac{3}{2}\mathcal{B}, \tag{81}$$

where

$$\begin{aligned}\mathcal{A} &= 1 + \frac{1}{8NH}\frac{d}{dt}\ln\left(\frac{RM_{\rm GW}^2}{r^2-1}\right),\\ \mathcal{B} &= 1 + \frac{1}{6NH}\frac{d}{dt}\ln(M_{\rm GW}^2).\end{aligned} \tag{82}$$

D. Scalar perturbations

We now move on to the scalar perturbations. In the absence of matter, we expect the sector to contain two degrees of freedom. The scalar parts of the metric perturbations are

$$\begin{aligned}&\delta g_{00} = -2N^2\Phi, \qquad \delta g_{0i} = Na\partial_i B,\\ &\delta g_{ij} = a^2\left[2\delta_{ij}\Psi + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\partial_l\partial^l\right)E\right],\end{aligned} \tag{83}$$

while the scalar field is expanded as

$$\sigma = \sigma^{(0)} + M_p\delta\sigma. \tag{84}$$

The perturbations Φ and B coming from δg_{0i} and δg_{00} carry no time derivatives and are nondynamical, so we may determine them using their own equations of motion:

$$\begin{aligned}B = &-\frac{M_p^2(r+1)}{a[4M_p^4H^2\frac{k^2}{a^2}(r^2-1) + (\rho_m+p_m)(\rho_\sigma+p_\sigma-6M_p^2H^2)]}\left\{\frac{k^2(r-1)}{3}\left[2M_p^2H\frac{k^2}{a^2}E + (\rho_\sigma+p_\sigma-6M_p^2H^2)\frac{\dot E}{N}\right]\right.\\ &+ 2H\left[2M_p^2\frac{k^2}{a^2}(r-1) - 3(\rho_m+p_m)\right]\Psi + 2(r-1)(\rho_\sigma+p_\sigma)\frac{\dot\Psi}{N} - \frac{2M_pH(r-1)(\rho_\sigma+p_\sigma)}{\dot\sigma}\delta\dot\sigma\\ &\left.+\left[\frac{M_pHN}{\dot\sigma}\left(6(H-\bar H rX)(\rho_m+p_m) - \frac{r-1}{N}(\dot\rho_\sigma - \dot p_\sigma + 2\dot\rho_m)\right) + \frac{(r-1)(\rho_\sigma+p_\sigma-6M_p^2H^2)\dot\sigma}{M_pN}\right]\delta\sigma\right\},\\ \Phi = &-\frac{M_p^2}{4M_p^4H^2\frac{k^2}{a^2}(r^2-1) + (\rho_m+p_m)(\rho_\sigma+p_\sigma-6M_p^2H^2)}\left\{\frac{k^4}{3a^2}\left[(\rho_m+p_m)E - 2M_p^2H(r^2-1)\frac{\dot E}{N}\right]\right.\\ &+ \frac{\rho_m+p_m}{M_p^2(r-1)}\left[2M_p^2\frac{k^2}{a^2}(r-1) - 3(\rho_m+p_m)\right]\Psi - 2H\left[2M_p^2\frac{k^2}{a^2}(r^2-1) - 3(\rho_m+p_m)\right]\frac{\dot\Psi}{N}\\ &- \frac{(\rho_m+p_m)(\rho_\sigma+p_\sigma)}{M_p\dot\sigma}\delta\dot\sigma + \left[\frac{(\rho_m+p_m)N}{2M_p^2(r-1)\dot\sigma}\left(6(H-\bar H rX)(\rho_m+p_m) - \frac{r-1}{N}(\dot\rho_\sigma - \dot p_\sigma + 2\dot\rho_m)\right)\right.\\ &\left.\left.- \frac{2M_pHk^2(r^2-1)\dot\sigma}{a^2N}\right]\delta\sigma\right\}.\end{aligned} \tag{85}$$

Inserting these back into the action, we end up with a system of three degrees of freedom, Ψ, E and $\delta\sigma$. Since the would-be Boulware-Deser ghost is removed by construction, there is another nondynamical combination, which is found to be

$$\tilde{\Psi} = \frac{1}{\sqrt{2}}\left(\Psi + \frac{M_p H N}{\dot{\sigma}}\delta\sigma\right). \tag{86}$$

We also define an orthogonal combination,

$$\tilde{\delta\sigma} = \frac{1}{\sqrt{2}k^2}\left(\Psi - \frac{M_p H N}{\dot{\sigma}}\delta\sigma\right). \tag{87}$$

The action can now be written in terms of $\tilde{\Psi}$, $\tilde{\delta\sigma}$ and E, with no time derivatives on $\tilde{\Psi}$. The latter is auxiliary and can be eliminated with its own equation of motion. Thus, we obtain an action in terms of $\tilde{\delta\sigma}$ and E of the form

$$S = \int \frac{d^3k}{2} a^3 N dt \Big[\frac{\dot{Y}^\dagger}{N}\mathcal{K}\frac{\dot{Y}}{N} + \frac{\dot{Y}^\dagger}{N}\mathcal{M}Y + Y^\dagger \mathcal{M}^T \frac{\dot{Y}}{N} - Y^\dagger \Omega^2 Y\Big], \tag{88}$$

where $Y \equiv (\tilde{\delta\sigma}, E)$, $\mathcal{M}$ is a real 2×2 matrix, and $\mathcal{K}$ and Ω^2 are real symmetric 2×2 matrices. Note that by adding boundary terms, the mixing matrix $\mathcal{M}$ between fields and derivatives can be made antisymmetric.

The full kinetic matrix is rather lengthy, and so we will not display the full expression, other than to note that for the dRGT theory, where $\dot{\sigma} = 0$ and $\rho_m = -p_m$, it can be checked that $\mathcal{K} = 0$, consistent with the results of [15]. Specializing to a Minkowski reference metric ($\bar{H} = \dot{\bar{H}} = 0$) brings the kinetic matrix to a more manageable form:

$$\begin{aligned}
\mathcal{K}_{11} &= \frac{a^2k^2}{(r-1)^2}\Bigg\{18(\rho_m + p_m)\left(\frac{2M_p^2k^2r^2}{2M_p^2k^2 + 3a^2(\rho_m + p_m)} - 1\right) \\
&\quad + \frac{4M_p^2k^2[2M_p^2\frac{k^2}{a^2}(r-1) - 3(\rho_m + p_m)]^2(\rho_\sigma + p_\sigma)}{[2M_p^2k^2 + 3a^2(\rho_m + p_m)][4M_p^4H^2\frac{k^2}{a^2} - (\rho_m + p_m)(\rho_\sigma + p_\sigma - 6M_p^2H^2)]}\Bigg\}, \\
\mathcal{K}_{12} &= \frac{\sqrt{2}M_p^2k^4}{3(r-1)}\Bigg\{\frac{9r(\rho_m + p_m)}{2M_p^2\frac{k^2}{a^2} + 3(\rho_m + p_m)} + \frac{2M_p^2k^2[2M_p^2\frac{k^2}{a^2}(r-1) - 3(\rho_m + p_m)](\rho_\sigma + p_\sigma)}{[2M_p^2k^2 + 3a^2(\rho_m + p_m)][4M_p^4H^2\frac{k^2}{a^2} - (\rho_m + p_m)(\rho_\sigma + p_\sigma - 6M_p^2H^2)]}\Bigg\}, \\
\mathcal{K}_{22} &= \frac{M_p^2k^4}{18}\Bigg\{\frac{9(\rho_m + p_m)}{2M_p^2\frac{k^2}{a^2} + 3(\rho_m + p_m)} + \frac{4M_p^4k^4(\rho_\sigma + p_\sigma)}{a^2[2M_p^2k^2 + 3a^2(\rho_m + p_m)][4M_p^4H^2\frac{k^2}{a^2} - (\rho_m + p_m)(\rho_\sigma + p_\sigma - 6M_p^2H^2)]}\Bigg\},
\end{aligned} \tag{89}$$

with determinant

$$\det[\mathcal{K}] = \frac{3M_p^2a^2k^6(\rho_m + p_m)^2(\rho_\sigma + p_\sigma - 6M_p^2H^2)}{(r-1)^2[4M_p^4H^2\frac{k^2}{a^2} - (\rho_m + p_m)(\rho_\sigma + p_\sigma - 6M_p^2H^2)]}. \tag{90}$$

We stress that we have not specified any background solution up to this point; the only choice we have made is to fix the fiducial metric to be Minkowski. By requiring that the determinant is positive, we infer that in order to avoid a ghost degree of freedom, the momentum should satisfy[6]

$$\left(\frac{\rho_\sigma + p_\sigma}{4M_p^2H^2} - \frac{3}{2}\right)^{-1}\frac{k^2}{a^2} > \frac{\rho_m + p_m}{M_p^2}. \tag{91}$$

Note that this condition should be imposed for all k in the regime $0 \le k/a \le \Lambda_{\rm UV}$, where $\Lambda_{\rm UV}$ is the UV cutoff scale of the theory.

In a regime in which we have a de Sitter like expansion, i.e. $|\dot{H}| \ll H^2$, this condition becomes even simpler,

$$R + \frac{4}{R-6}\frac{k^2}{H^2a^2} > 0, \tag{92}$$

where R is defined in (78). Demanding that the condition (92) holds for all physical momenta $k/a < \Lambda_{\rm UV}$ and supposing that $\Lambda_{\rm UV}/H > 3/2$, we obtain the no-ghost condition for scalar perturbations in the regime $|\dot{H}| \ll H^2$ as

$$R > 6. \tag{93}$$

E. Consistency of stability conditions

We now discuss the regions of parameter space in which the stability requirements we obtained in Eqs. (73), (78), (81), and (93) can be satisfied. The summary of the conditions is

[6]Note that absence of ghosts requires that both the determinant and the trace are positive. However, for the scenario at hand, applying the field redefinitions given in Appendix C shows that one of the degrees of freedom always has a positive kinetic term. Thus, Eq. (90) is enough to ensure a healthy kinetic action.

(i) To avoid a tachyonic instability in the tensor sector, we need [from Eq. (73)]

$$M_{\rm GW}^2 > 0. \tag{94}$$

(ii) To avoid a ghost instability in the vector sector, from Eq. (78),

$$\frac{\Lambda_{\rm UV}^2(1-r^2)}{H^2 R} < 2. \tag{95}$$

(iii) To avoid the unchecked growth of vector perturbations, from Eq. (81),

$$\mathcal{A}\,\frac{\Lambda_{\rm UV}^2(1-r^2)}{H^2 R} < \frac{3}{2}\mathcal{B}. \tag{96}$$

(iv) To avoid a ghost instability in the scalar sector, from Eq. (93),

$$R > 6. \tag{97}$$

Here, we have defined

$$R \equiv -\frac{(\rho_m + p_m)}{H^2 M_p^2}, \qquad \mathcal{A} = 1 + \frac{1}{8NH}\frac{d}{dt}\ln\left(\frac{RM_{\rm GW}^2}{r^2-1}\right),$$
$$\mathcal{B} = 1 + \frac{1}{6NH}\frac{d}{dt}\ln(M_{\rm GW}^2), \tag{98}$$

and have assumed that the UV cutoff scale $\Lambda_{\rm UV}$ is higher than $3/2$ in units of H and that the expansion is de Sitter-like, i.e. $|\dot{H}| \ll H^2$ (relevant for the scalar sector no-ghost condition).

Note that if we satisfy the condition (iv), then the condition (ii) is trivially satisfied if $r^2 > 1$, and that the condition (iii) is also trivially satisfied in this case if both $\mathcal{A}$ and $\mathcal{B}$ are positive. In more general cases, the above set of stability conditions is less trivial, but in principle there are regimes in which all of them are simultaneously satisfied.

IV. DISCUSSION

If the cosmologies of any of the recently proposed variations of massive gravity are to be of phenomenological use, it is crucial to understand the extent to which the theories propagate well-behaved, ghost free perturbations around their cosmological backgrounds. If homogeneity and isotropy are required at the level of linear perturbations, the cosmologies in dRGT either (i) contain a nonlinear ghost [20]; (ii) contain a Higuchi ghost [19]; (iii) cannot support acceleration [35]. On the other hand, breaking the FRW symmetries may still yield stable perturbations; for instance, there exist solutions [36] where broken homogeneity and isotropy do not effect the background, but are accessible to perturbations. Alternatively, for solutions which are inhomogeneous at the background level, the observable Universe can be approximately homogeneous through a cosmological Vainshten mechanism [10]. However, both approaches have the disadvantage of rendering the perturbation analysis technically involved, due the to broken SO(3) symmetry in the spatial slices.

In this paper we explored a different possibility, where the theory is extended with a scalar field. We carried out the calculation of linear perturbations for two the cases of the quasi-dilaton theory and for the mass varying massive gravity theory. We find a host of constraints on these theories, primarily stemming from the requirement that ghost degrees of freedom do not appear in the regime of applicability of the effective field theory. In the case of the quasi-dilaton theory, it can be seen that the stability found in the decoupling limit is an artifact of that particular limit and that, in fact, a ghost degree of freedom remains in the full theory.

All the models considered in this paper are arguably the simplest extensions of dRGT, with a single scalar field coupled to the mass term in a nontrivial way. However, there are other extensions that one might consider. For instance the bimetric theory, where the fiducial metric acquires dynamics, is known to admit cosmological solutions [37,38], although perturbations around homogeneous and isotropic backgrounds suffer from gradient instability in the presence of matter [39].

Finally, if the recovery of the Fierz-Pauli form at the linear level is not required, a class of Lorentz-violating theories [40–42] become available, where the stability issues typically encountered in the FRW backgrounds in dRGT theory may potentially be avoided.

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APPENDIX A: DISPERSION RELATION FOR SCALAR PERTURBATIONS IN THE QUASI-DILATON THEORY

In this Appendix, we provide the details of the diagonalization procedure for the quadratic action for scalar perturbations in the quasi-dilaton theory, starting from (47).

1. Canonical normalization

We first introduce a new field basis

$$Z_1 \equiv \frac{k^2 M_p}{3\sqrt{\omega}} \left| \frac{4}{\omega(6-\omega)} - \frac{a^2H^2}{k^2} \right|^{-1/2} \times \left[E + 6\sqrt{2}\left(1 + \frac{3\omega a^2 H^2}{2(r-1)k^2}\right)\tilde{\delta\sigma} \right],$$
$$Z_2 \equiv \frac{k^2 M_p}{\sqrt{6}}(E + 6\sqrt{2}\tilde{\delta\sigma}), \tag{A1}$$

in terms of which the kinetic matrix (48) becomes diagonal and canonically normalized, so that the action is formally

$$S = \int \frac{d^3k}{2} a^3 N dt \left[\frac{\dot{Z}^\dagger}{N} \mathcal{K} \frac{\dot{Z}}{N} + \frac{\dot{Z}^\dagger}{N} \mathcal{M} Z - Z^\dagger \mathcal{M} \frac{\dot{Z}}{N} - Z^\dagger \Omega^2 Z \right], \tag{A2}$$

with a canonical form for the kinetic matrix:

$$\mathcal{K} = \begin{pmatrix} \mathrm{Sign}(1-\tilde{k}^2) & 0 \\ 0 & 1 \end{pmatrix}, \tag{A3}$$

where we have introduced the dimensionless (and time dependent) momentum via

$$\tilde{k} \equiv \frac{2k}{aH\sqrt{\omega(6-\omega)}}. \tag{A4}$$

By adding appropriate total derivatives, the mixing matrix $\mathcal{M}$ can be made antisymmetric, reading

$$\mathcal{M} = -\sqrt{1 - \frac{\omega}{6}} H(2r-1) \frac{\tilde{k}\sqrt{|1-\tilde{k}^2|}}{1-\tilde{k}^2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A5}$$

Finally, the components of the symmetric matrix Ω^2 are

$$(\Omega^2)_{11} = -\frac{H^2}{4} \frac{\tilde{k}^2|1-\tilde{k}^2|}{1-\tilde{k}^2} \left[(8r-\omega-2)(8r-\omega-8) + \frac{4(8r-\omega-6)}{1-\tilde{k}^2} + \frac{12}{(1-\tilde{k}^2)^2} \right],$$
$$(\Omega^2)_{12} = -\sqrt{1-\frac{\omega}{6}}\tilde{k}\sqrt{|1-\tilde{k}^2|}\left\{ M_{\rm GW}^2(r-1) + H^2\left[2r(8r-\omega-10) + \omega + 4 - \frac{(2r-1)(8r-\omega-7)}{(1-\tilde{k}^2)} - \frac{3(2r-1)}{(1-\tilde{k}^2)^2} \right] \right\},$$
$$(\Omega^2)_{22} = M_{\rm GW}^2 + \frac{(6-\omega)\tilde{k}^2}{3}\left\{ M_{\rm GW}^2(r-1) + 2H^2\left[r(4r-5) - \frac{\omega}{8} + 1 - \frac{(2r-1)^2}{1-\tilde{k}^2} \right] \right\}, \tag{A6}$$

where $M_{\rm GW}^2$ was defined in Eq. (29). For $\tilde{k} > 1$, the mode Z_1 becomes a ghost, while for momenta $\tilde{k} < 1$, both degrees of freedom are well-behaved.

At low momenta ($\tilde{k} \ll 1$), the action effectively becomes diagonal. In this long wavelength regime, both degrees of freedom have positive kinetic terms, while $\mathcal{M}_{ij} = 0$ and the two eigenfrequencies are $\omega_1^2 = 0$ and $\omega_2^2 = M_{\rm GW}^2$ [24]. On the other hand, the ghost degree of freedom appears at momenta $\tilde{k} \gtrsim 1$, and so we must still diagonalize the system to determine the amplitudes of the frequencies in this regime.

2. Diagonalization

To find the eigenfrequencies of the system, first note that since the matrices $\mathcal{M}$ and Ω^2 are time dependent, it is not possible to diagonalize the system at the level of the Lagrangian. On the other hand, the Hamiltonian can be written as a sum of decoupled oscillators, as shown in Ref. [43] and in the presence of ghosts, in Appendix D of Ref. [22]. We consider the two cases $\tilde{k} < 1$ and $\tilde{k} > 1$ separately.

a. No ghost: $\tilde{k} < 1$

For the first case (denoted by subscript $<$), the kinetic matrix is unity and there is no ghost. In this case, we can introduce a rotated basis $W \equiv R_< Z$, where the SO(2) rotation $R_<$ satisfies

$$\dot{R}_< = N R_< \mathcal{M}_<. \tag{A7}$$

This rotation allows us to remove the mixing, and the Lagrangian becomes

$$\mathcal{L}_< = \frac{\dot{W}^\dagger}{N}\frac{\dot{W}}{N} - W^\dagger R_<^T(\Omega_<^2 + \mathcal{M}_<^T\mathcal{M}_<)R_< W. \tag{A8}$$

As shown in Appendix D of Ref. [22], the eigenvalues of the matrix

$$\tilde{\Omega}_<^2 \equiv \Omega_<^2 + \mathcal{M}_<^T\mathcal{M}_<, \tag{A9}$$

correspond to the actual eigenfrequencies. The matrix $\tilde{\Omega}_<^2$ can be diagonalized by performing an SO(2) rotation $\xi_<$,

$$\xi_<^T \tilde{\Omega}_<^2 \xi_< = \omega_<^2(\text{diagonal}), \tag{A10}$$

where

$$\xi_< = \begin{pmatrix} \cos(\theta_<) & \sin(\theta_<) \\ -\sin(\theta_<) & \cos(\theta_<) \end{pmatrix}, \tag{A11}$$

and

$$\sin(2\theta_<) = \frac{(\tilde{\Omega}_<^2)_{12}}{\sqrt{[(\tilde{\Omega}_<^2)_{11} - (\tilde{\Omega}_<^2)_{22}]^2 + 4[(\tilde{\Omega}_<^2)_{12}]^2}},$$
$$\cos(2\theta_<) = \frac{(\tilde{\Omega}_<^2)_{22} - (\tilde{\Omega}_<^2)_{11}}{\sqrt{[(\tilde{\Omega}_<^2)_{11} - (\tilde{\Omega}_<^2)_{22}]^2 + 4[(\tilde{\Omega}_<^2)_{12}]^2}}. \tag{A12}$$

The eigenvalues are then

$$(\omega^2_<)_1 = \frac{1}{2}\Big[(\tilde\Omega^2_<)_{11} + (\tilde\Omega^2_<)_{22} - \sqrt{[(\tilde\Omega^2_<)_{11} - (\tilde\Omega^2_<)_{22}]^2 + 4[(\tilde\Omega^2_<)_{12}]^2}\Big],$$
$$(\omega^2_<)_2 = \frac{1}{2}\Big[(\tilde\Omega^2_<)_{11} + (\tilde\Omega^2_<)_{22} + \sqrt{[(\tilde\Omega^2_<)_{11} - (\tilde\Omega^2_<)_{22}]^2 + 4[(\tilde\Omega^2_<)_{12}]^2}\Big]. \quad \text{(A13)}$$

b. One ghost: $\tilde k > 1$

In this regime (denoted by subscript $>$), the kinetic matrix has Lorentzian signature and the first mode is a ghost. Again, we introduce a rotated basis $W \equiv R_> Z$, where the SO(1,1) rotation $R_>$ satisfies,

$$\dot R_> = N R_> \mathcal{M}_> \eta, \quad \text{(A14)}$$

with $\eta \equiv \mathrm{diag}(-1, 1)$. This rotation removes the mixing, and the Lagrangian becomes

$$\mathcal{L}_> = \frac{\dot W^\dagger}{N}\eta\frac{\dot W}{N} - W^\dagger R^T_>(\Omega^2_> + \mathcal{M}^T_>\eta\mathcal{M}_>)R_> W. \quad \text{(A15)}$$

In this case, the matrix we need to diagonalize is [22],

$$\tilde\Omega^2_> \equiv \Omega^2_> + \mathcal{M}^T_>\eta\mathcal{M}_>, \quad \text{(A16)}$$

which gives the actual eigenfrequencies

$$\xi^T_>\tilde\Omega^2_>\xi_> = \eta\omega^2_>(\text{diagonal}). \quad \text{(A17)}$$

For an SO(1,1) rotation given by

$$\xi_> = \begin{pmatrix} \cosh(\theta_>) & \sinh(\theta_>) \\ \sinh(\theta_>) & \cosh(\theta_>) \end{pmatrix}, \quad \text{(A18)}$$

where

$$\sinh(2\theta_>) = -\frac{2(\tilde\Omega^2_>)_{12}}{\sqrt{[(\tilde\Omega^2_>)_{11} + (\tilde\Omega^2_>)_{22}]^2 - 4[(\tilde\Omega^2_>)_{12}]^2}},$$
$$\cosh(2\theta_>) = \frac{(\tilde\Omega^2_>)_{11} + (\tilde\Omega^2_>)_2}{\sqrt{[(\tilde\Omega^2_>)_{11} + (\tilde\Omega^2_>)_{22}]^2 - 4[(\tilde\Omega^2_>)_{12}]^2}}, \quad \text{(A19)}$$

the eigenvalues are

$$(\omega^2_>)_1 = \frac{1}{2}\Big[(\tilde\Omega^2_>)_{22} - (\tilde\Omega^2_>)_{11} - \sqrt{[(\tilde\Omega^2_>)_{11} + (\tilde\Omega^2_>)_{22}]^2 - 4[(\tilde\Omega^2_>)_{12}]^2}\Big],$$
$$(\omega^2_>)_2 = \frac{1}{2}\Big[(\tilde\Omega^2_>)_{22} - (\tilde\Omega^2_>)_{11} + \sqrt{[(\tilde\Omega^2_>)_{11} + (\tilde\Omega^2_>)_{22}]^2 - 4[(\tilde\Omega^2_>)_{12}]^2}\Big]. \quad \text{(A20)}$$

c. Combining the two regimes

Now that we have the necessary tools to diagonalize the system for the two regimes of momenta, we unify the two results. We first note that from Eqs. (A5) and (A6), we have

$$(\Omega^2_<)_{11} = -(\Omega^2_>)_{11}, \quad (\Omega^2_<)_{22} = -(\Omega^2_>)_{22},$$
$$[(\Omega^2_<)_{12}]^2 = -[(\Omega^2_>)_{12}]^2, \quad [(\mathcal{M}_<)_{12}]^2 = -[(\mathcal{M}_>)_{12}]^2, \quad \text{(A21)}$$

which imply

$$(\tilde\Omega^2_<)_{11} = -(\tilde\Omega^2_>)_{11}, \quad (\tilde\Omega^2_<)_{22} = -(\tilde\Omega^2_>)_{22},$$
$$[(\tilde\Omega^2_<)_{12}]^2 = -[(\tilde\Omega^2_>)_{12}]^2, \quad \text{(A22)}$$

or

$$(\omega_<)^2_1 = (\omega_>)^2_1, \quad (\omega_<)^2_2 = (\omega_>)^2_2. \quad \text{(A23)}$$

Thus, it is straightforward to write down a unified expression for the dispersion relation, independent of the momentum regime of the modes. We obtain

$$\omega^2_{1,2} = \frac{\tilde k^2}{6}\Big\{-H^2\Big[\omega\Big(8r^2 + \frac{\omega}{2} - 11(2r-1)\Big) + 4\frac{(6-\omega)r^2 + \omega(r-1) + 3}{1-\tilde k^2} + \frac{9}{(1-\tilde k^2)^2}\Big] + M^2_{\rm GW}(6-\omega)(r-1)\Big\} + \frac{M^2_{\rm GW}}{2} \mp \frac{1}{2}\sqrt{\mathcal{A}^2 + \frac{2(6-\omega)\tilde k^2(1-\tilde k^2)}{3}\mathcal{B}^2}, \quad \text{(A24)}$$

where a $-$ $(+)$ sign corresponds to the first (second) eigenmode, and we have defined

$$\mathcal{A} \equiv \frac{\tilde k^2}{3}\Big\{-H^2\Big[\omega(6-\omega) - 2(r-1)[4(12-\omega)r - 5\omega - 12] + \frac{6+\omega+8(r-1)[r(6-\omega)-3]}{1-\tilde k^2} - \frac{9}{(1-\tilde k^2)^2}\Big] + M^2_{\rm GW}(6-\omega)(r-1)\Big\} + M^2_{\rm GW},$$
$$\mathcal{B} \equiv H^2\Big[2(r-1)[6-\omega+8(r-1)] - \omega - \frac{(8r-\omega-7)(2r-1)}{1-\tilde k^2} - \frac{3(2r-1)}{(1-\tilde k^2)^2}\Big] + M^2_{\rm GW}(r-1). \quad \text{(A25)}$$

3. Stability

Since the modes with momenta $\tilde k > 1$ are ghosts, we need to determine how serious this problem is. The ghost mode appears at (physical) momenta parametrically of the order of the Hubble rate. If the frequencies of these modes are larger than the UV cutoff of the theory, they are not within the regime of validity of the low energy effective theory and may be ignored.

In the transition region where $\tilde{k} \to 1$, the frequencies are

$$\omega_1^2 = -\frac{3H^2}{4(\tilde{k}-1)^2} + \mathcal{O}\left(\frac{1}{\tilde{k}-1}\right),$$
$$\omega_2^2 = M_{\rm GW}^2 + \frac{6-\omega}{3}\Big\{M_{\rm GW}^2(r-1) + \frac{H^2}{6}[\omega(16r-5) + 4r^2[2(6-\omega)r^2 + 4\omega r - 6\omega - 9]]\Big\} + \mathcal{O}(\tilde{k}-1). \quad \text{(A26)}$$

We note that the problematic mode, right after $\tilde{k} \sim 1$, has a very large frequency. As an example, we consider the set of parameters

$$\Lambda = 0, \quad \omega = 1, \quad \alpha_3 = -10, \quad \alpha_4 = 6, \quad \xi = 0, \quad m_g^2 < 0, \quad + \text{branch}, \quad \text{(A27)}$$

where "+ branch" corresponds to the positive sign solution in Eq. (16). These parameters lead to $r \simeq 1.01$ and $M_{\rm GW}^2 \simeq 0.71|m_g^2|$, which satisfy the stability conditions for the tensor and vector modes. For this example, we show the momentum dependence of the scalar dispersion relations in Fig. 2. As discussed in the paragraph after Eq. (A6), at low momenta, $\omega_1^2 \to 0$, while $\omega_2^2 \to M_{\rm GW}^2$. After the transition region, where ω_1^2 exhibits divergent behavior, both modes increase with $\omega^2 \propto \tilde{k}^2$. The "light" mode, which becomes a ghost in the $\tilde{k} > 1$ region, has (for this specific example) frequency $\omega_1/H \propto \mathcal{O}(1)$, so that apart from in the immediate neighborhood of $\tilde{k} \sim 1$, it cannot be integrated out from the low energy effective theory.

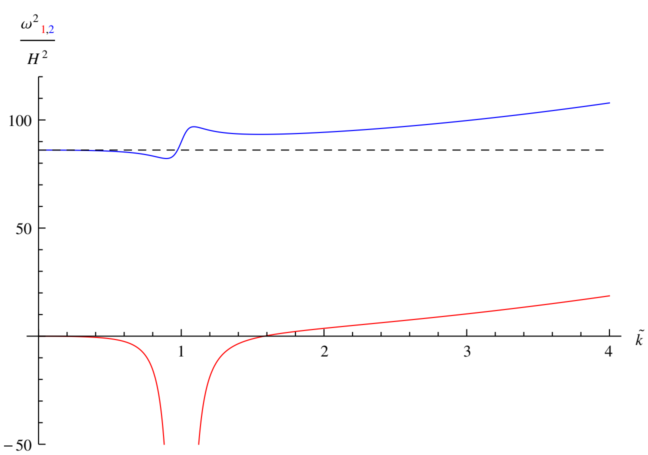

FIG. 2 (color online). Plot of the dispersion relation of scalar modes versus the rescaled momenta $\tilde{k}$, for the example (A27), with $\Lambda = 0$, $\omega = 1$, $\alpha_3 = -10$, $\alpha_4 = 6$, $\xi = 0$, $m_g^2 < 0$, in the positive branch defined in Eq. (16). The dashed line corresponds to $M_{\rm GW}^2/H^2$ which is the mass term for mode 2 (blue) in the low momentum regime. At the critical point $\tilde{k} = 1$, ω_1^2 (red) diverges to $-\infty$, and then becomes positive and finite after the transition.

Next, we consider a more general example, and extend our analysis to the α_3, α_4 parameter space. Instead of analyzing the immediate neighborhood of the critical point, we chose $\tilde{k} = 2$, such that the frequency of the ghost mode becomes finite, while differing from the critical point value by an order one factor. In Fig. 3, we show the order of magnitude of ω^2/H^2 for an example with $\Lambda = 0$, $\omega = 1$ and $\xi = 0$, in both branches defined in (16). In the regimes where $\omega_1^2 > 0$, the frequency is always of $\mathcal{O}(H)$, and the ghost mode cannot be removed. On the other hand, we see that in a special region, the ratio $|\omega_1|/H$ may exceed 10^2. The source of large $|\omega_1|$ is related to a specific relation between parameters,

$$\alpha_4 = -3(1+\alpha_3), \qquad \begin{cases} \alpha_3 > -2, & + \text{branch} \\ \alpha_3 < -2, & - \text{branch} \end{cases}, \quad \text{(A28)}$$

which leads to $X = 0$ and $r \propto X^{-2} \to \infty$. Although we have excluded the line (A28) from our analysis, parameters close to this line lead to the large (negative) values we observe in the squared-frequency of the ghost mode. In principle, the value of $|\omega_1|/H$ can be made arbitrarily large by tuning α_3 and α_4 to be close enough to this line.

Finally, we note that in general, close to line (A28), the parameter $M_{\rm GW}^2$ defined in Eq. (29) becomes negative, with $M_{\rm GW}^2 = -\omega H^2 + \mathcal{O}(X)$. In other words, even if the parameters are fine-tuned to remove the ghost mode in the scalar sector, the tensor and vector modes have a tachyonic instability (although its rate is at most of the Hubble scale). Additionally, the fact that $X \to 0$ and $r \to \infty$ is also an indication of strong coupling, since the determinant of the kinetic matrix (49) is of the same small order of magnitude as the fine-tuning between α_4 and α_3.

APPENDIX B: COMPARISON OF THE SCALAR SECTOR OF THE QUASI-DILATON THEORY WITH RESULTS IN THE DECOUPLING LIMIT

In this Appendix, we compare our results for the scalar sector of the quasi-dilaton theory with those obtained in [23] in a decoupling limit. We first identify our degrees of freedom with the ones used by Ref. [23]. By comparing perturbations of δg_{ij}, we find that

$$\Psi = \frac{\partial^2 b}{3m_g a^2} - \frac{H}{m_g}A_0, \qquad E = \frac{2}{m_g a^2}b, \quad \text{(B1)}$$

where Ψ and E are the perturbations introduced by the decomposition (40) and A_0, b are the perturbations used in [23]. For the quasi-dilaton perturbations, $\delta\sigma$ in (41) coincides with ζ in [23].

We now turn to the action (47), which has the following kinetic term:

$$S \ni \frac{M_p^2}{2}\int d^3k a^3 N dt \frac{\dot{Y}^\dagger}{N}\mathcal{K}\frac{\dot{Y}}{N}, \quad \text{(B2)}$$

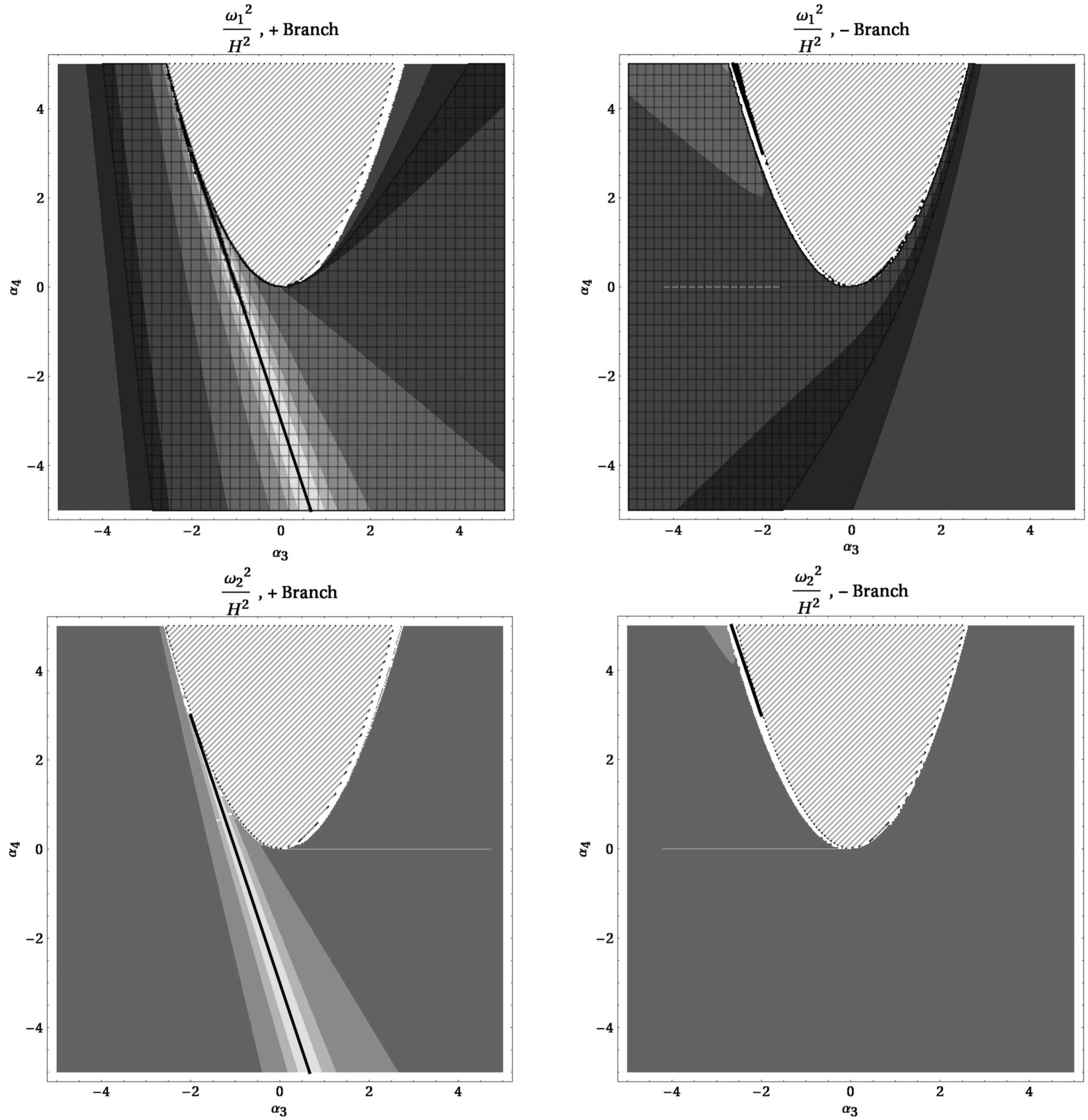

FIG. 3. Order of magnitude plots of the squared-frequency in Hubble units for the case $\Lambda = 0$, $\omega = 1$ and $\xi = 0$, and for a benchmark momentum value $\tilde{k} = 2$. The left (right) column shows the positive (negative) branch of solutions given in Eq. (16), while the upper (lower) panel shows ω_1^2 (ω_2^2). From dark to light, the colors represent the following values of the squared-frequency: (0,1]; (1,10]; (10, 10^2]; (10^2, 10^3]; (10^3, 10^4]; (10^4, ∞). In the shaded parameter region, there are no background solutions. The gridded region corresponds to $\omega^2 < 0$, while the thick black line corresponds to $\alpha_4 = -3(1 + \alpha_3)$ [for the positive (negative) branch, $\alpha_3 > -2$ ($\alpha_3 < -2$) only], where $X \sim 0$.

where the components of the kinetic matrix are given in Eq. (48) and the field basis is given by

$$Y \equiv \begin{pmatrix} \frac{1}{\sqrt{2}k^2}(\Psi - \delta\sigma) \\ E \end{pmatrix}, \tag{B3}$$

or, using Eq. (B1),

$$Y \equiv \begin{pmatrix} \frac{1}{\sqrt{2}k^2}\left(\frac{\partial^2 b}{3m_g a^2} - \rho\right) \\ \frac{2b}{m_g a^2} \end{pmatrix}, \tag{B4}$$

with $\rho = \zeta + (H/m)A_0$.

Since the decoupling limit action in Ref. [23] is given in the $Z \equiv (\rho, b)$ basis (up to the nondynamical degree of freedom A_0), we transform our action via

$$Y = RZ = \begin{pmatrix} -\frac{1}{\sqrt{2}k^2} & -\frac{1}{3\sqrt{2}m_g a^2} \\ 0 & \frac{2}{m_g a^2} \end{pmatrix} \begin{pmatrix} \rho \\ b \end{pmatrix}, \tag{B5}$$

so that the kinetic term in the Z basis becomes

$$\mathcal{K}_Z = R^T \mathcal{K} R. \tag{B6}$$

Before taking the decoupling limit, we consider the determinant of the kinetic matrix, given by

$$\det \mathcal{K}_Z = \left(-\frac{\sqrt{2}}{k^2 m_g a^2}\right)^2 \det \mathcal{K}. \tag{B7}$$

In other words, the momentum dependent ghost-free condition (51) is still valid. On the other hand, if we go to the decoupling limit, given by

$$m_g \to 0, \qquad H \to 0, \qquad \frac{H}{m_g} = \text{finite}, \tag{B8}$$

the kinetic matrix in the Z basis becomes

$$\mathcal{K}_Z = \begin{pmatrix} \omega + \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon) \\ \mathcal{O}(\epsilon) & \mathcal{O}(\epsilon^2) \end{pmatrix}, \tag{B9}$$

where ϵ denotes the order of m_g and H. Thus, we see that at momenta comparable to and smaller than the expansion rate, one degree of freedom becomes a ghost, as we found in the main text. Therefore, this decoupling limit is not sufficient for determining the stability of one of the degrees of freedom. (Also see the discussion at the end of Sec. II F.) This result coincides with the conclusion in [23], from the determinant in Eq. (B7).

APPENDIX C: DIAGONAL BASIS FOR THE SCALAR SECTOR OF THE VARYING MASS THEORY

In this Appendix, we diagonalize the kinetic matrix of the scalar sector in the varying mass gravity theory, studied in Sec. III D. Specifically, we want to show that the condition

$$\det \mathcal{K} > 0, \tag{C1}$$

for the kinetic matrix is enough to ensure the absence of ghost degrees of freedom.

The kinetic part of the action is given by

$$S \ni \int \frac{d^3k}{2} a^3 N dt \frac{\dot{Y}^\dagger}{N} \mathcal{K} \frac{\dot{Y}}{N}, \tag{C2}$$

where the components of the kinetic matrix are given in Eq. (89) and $Y = (\tilde{\delta\sigma}, E)$. We now define a new basis,

$$Z_1 \equiv \frac{k^3 M_p}{3aH}\left[E + 6\sqrt{2}\left(1 - \frac{3a^2(\rho_m + p_m)}{2k^2 M_p^2 (r-1)}\right)\tilde{\delta\sigma}\right],$$
$$Z_2 \equiv \frac{k^2 M_p}{\sqrt{6}}(E + 6\sqrt{2}\tilde{\delta\sigma}), \tag{C3}$$

after which, the kinetic terms become diagonal

$$S \ni \int \frac{d^3k}{2} a^3 N dt \left(\kappa_1 \frac{\dot{Z}_1^\dagger}{N}\frac{\dot{Z}_1}{N} + \kappa_2 \frac{\dot{Z}_2^\dagger}{N}\frac{\dot{Z}_2}{N}\right), \tag{C4}$$

with

$$\kappa_1 = \left[\frac{k^2}{a^2 H^2\left(\frac{\rho_\sigma + p_\sigma}{4M_p^2 H^2} - \frac{3}{2}\right)} - \frac{\rho_m + p_m}{M_p^2 H^2}\right]^{-1}, \qquad \kappa_2 = 1. \tag{C5}$$

Thus, the condition (91), obtained from the positivity of $\det \mathcal{K}$, actually corresponds to the sign of the kinetic term of Z_1, while Z_2 always has positive kinetic term.

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