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Magnetic fields of spherical compact stars in modified theories of gravity: $f(R)$ type gravity and Hořava-Lifshitz gravity

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The stellar magnetic field configuration and solutions of Maxwell equations in the external background spacetime of a magnetized spherical stars in the Hořava-Lifshitz gravity and in modified $f(R)$ gravity are studied. The star is modeled as a sphere consisting of perfect highly magnetized fluid with infinite conductivity and frozen-in magnetic field. With respect to solutions for magnetic fields found in the Schwarzschild spacetime star in modified gravity theories, enhancing corrections are added to the exterior magnetic field. The energy losses through magnetodipolar radiation of the rotating magnetized compact star within alternative gravity theories is also considered. The question of whether these models can be considered as an alternative theory for general relativity is also discussed through astrophysical application of the obtained magnetodipolar energy loss formula. Finally we analyze the role of general relativistic effect on the decay of a neutron star's magnetic field in modified theories of gravity.

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I. INTRODUCTION

So-called $f(R)$ gravity is a type of modified gravity theory first proposed in 1970 by Buchdahl [1], where $f(R)$ is a generic function of the Ricci scalar R, which comes into the game as a straightforward extension of general relativity (GR). In $f(R)$ gravity the further geometrical degrees of freedom are considered instead of searching for new material ingredients [2]. Recently, it became an active field of research among alternative theories of gravity. It has the potential, in principle, to explain the accelerated expansion of the Universe without adding unknown forms of matter as dark energy or dark matter.

Later, the interest in spherically symmetric solutions of $f(R)$ gravity is grown up. For example, in [3], vacuum solutions of field equations have been found considering relations among functions that define the spherical metric or imposing a constant Ricci curvature scalar. The authors of papers [3,4] have reconstructed the form of some $f(R)$ models, discussing their physical relevance as well as static spherically symmetric solutions, in the presence of perfect fluid matter, adopting the metric formalism. They have shown that a given matter distribution is not capable of globally determining the functional form of $f(R)$. Other authors have discussed in detail the spherical symmetry of $f(R)$ gravity considering also the relations with the weak field limit.

Noether gauge symmetries of $f(T)$ gravity minimally coupled with a canonical scalar field, generalized first and second laws of thermodynamics, and the statefinder parameters in $f(T)$ cosmology have been studied in [5]. A flat Friedmann-Robertson-Walker universe in the context of the Palatini $f(R)$ theory of gravity has been studied in [6]. The equivalence of the Einstein-Hilbert and the Einstein-Palatini formulations of general relativity for an arbitrary connection has been explored in [7].

Recently Hořava proposed a UV complete, nonrelativistic and renormalizable theory of gravity [8,9]. Since then, many authors have paid attention to this scenario to apply it to the black hole (BH) physics [10–15], cosmology [16–22], and observational tests [23]. Energy extraction and particle acceleration [24], quantum interference effects [25], and the motion of the test particle around the BH [26] in Hořava-Lifshitz gravity have been also recently studied.

In Ref. [23] the possibility of observationally testing Hořava-Lifshitz gravity at the scale of the Solar System, by considering the classical tests of general relativity (perihelion precession of the planet Mercury, deflection of light by the Sun, and the radar echo delay) for the Kehagias-Sfetsos (KS) asymptotically flat black hole solution of Hořava-Lifshitz gravity has been considered. Recently the authors of [27] have studied the particle motion in the spacetime of a KS black hole. The stability of the Einstein static universe by considering linear homogeneous perturbations in the context of an infrared (IR) modification of Hořava-Lifshitz gravity has been studied in [28]. Potentially observable properties of black holes in the deformed Hořava-Lifshitz gravity with Minkowski vacuum—the gravitational lensing and quasinormal modes—have been studied in [11]. The authors of the paper [29] derived the full set of equations of motion and then obtained spherically symmetric solutions for the UV completed theory of gravity proposed by Hořava.

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Black hole solutions and the full spectrum of spherically symmetric solutions in the five-dimensional nonprojectable Hořava-Lifshitz-type gravity theories have been recently studied in [30]. Geodesic stability and the spectrum of entropy or area for the black hole in Hořava-Lifshitz gravity via quasinormal mode approaches are analyzed in [31]. Particle geodesics around the Kehagias-Sfetsos black hole in Hořava-Lifshitz gravity are also investigated by the authors of [32]. Recently, observational constraints on Hořava-Lifshitz gravity have been found from the cosmological data [33]. The authors of [15] have found all spherical black hole solutions for two, four, and six derivative terms in the presence of a Cotton tensor.

It is well known that magnetic fields play an important role in the life history of a majority of astrophysical objects, especially of compact relativistic stars, which possess surface magnetic fields of order of 10^{12} G. Magnetic fields of magnetars [34,35] can reach up to 10^{15}–10^{16} G, and in the deep interior of compact stars the magnetic field strength may be estimated up to 10^{18} G. The magnetic field strength of compact stars is one of the main quantities determining their observability, for example, as pulsars through magnetodipolar radiation. Therefore it is extremely important to study the effect of the different phenomena on the evolution and behavior of stellar exterior magnetic fields.

In the general relativity approach the study of the magnetic field structure outside magnetized compact gravitational objects has been initiated by the pioneering work of Ginzburg and Ozernoy [36] and further extended by a number of authors [37–45], while in some papers [46–50] the work has been completed by considering the magnetic fields' interior relativistic star for the different models of stellar matter. A general relativity treatment for the structure of external and internal stellar magnetic fields including numerical results has shown that the magnetic field is amplified by the monopolar part of the gravitational field depending on the compactness of the relativistic star. Nongeodesic corrections to orbital and epicyclic frequencies and the quasicircular motion of charged test particles in the field of magnetized slowly rotating neutron stars have been studied in [51,52].

To the best of our knowledge, the magnetic field configuration in compact stars in $f(R)$ and Hořava-Lifshitz gravity has not yet been studied. Since the magnetic field determines the reach observational phenomenology of compact stars we plan to study here static spherically symmetric highly magnetized relativistic stars in both $f(R)$ and Hořava-Lifshitz gravity theories. The magnetic field structure will be assumed to be dipolar and axisymmetric and the effect of the gravitational field of the star on the magnetic field structure is considered without feedback, amounting to the astrophysical evidence that the magnetic field energy is not strong enough to affect the spacetime geometry.

In this paper we describe our model assumptions, present the Maxwell equations for magnetic field and initial conditions for the models under consideration, and obtain analytical and numerical results for the stellar magnetic field. In Sec. II we provide a description of spherical compact objects in $f(R)$ gravity and Maxwell equations for magnetic field in the spacetime of these objects. We integrate external Maxwell equations from asymptotical infinity to the surface of the star and find numerical solutions for the magnetic field outside the star in $f(R)$ gravity. In Sec. III we reiterate the processes in Sec. II but in Hořava gravity. As astrophysical application of the obtained results, we look for the modification of the luminosity of electromagnetic magnetodipolar radiation from the rotating star in modified theories of gravity in Sec. IV. In Sec. V we discuss numerical solutions of induction equations in the curved spacetime, and an assessment of the relativistic factors influencing the field decay in both modified theories of gravity is discussed. We use a system of units in which $c = G = 1$, a spacelike signature $(-, +, +, +)$ and a spherical coordinate system (t, r, θ, φ). Greek indices are taken to run from 0 to 3. We will indicate vectors with bold symbols (e.g., $\mathbf{B}$).

II. MAGNETIC FIELD OF THE STAR WITHIN THE $f(R)$ GRAVITY

$f(R)$ or "modified" gravity consists of infrared modifications of GR that become important only at low curvatures, late in the matter era. The Einstein-Hilbert action $S_{\rm EH} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}R + S^{\rm matter}$ is modified to

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R) + S^{\rm matter}, \qquad (1)$$

where $S^{\rm matter} = \int d^4x\sqrt{-g}\mathcal{L}_m(g_{\alpha\beta,\psi})$ is a matter action with Lagranjian density $\mathcal{L}_m$ describing any matter fields ψ appearing in the theory, R is the Ricci curvature of the metric tensor $g_{\alpha\beta}$ with metric determinant g, $\kappa \equiv 8\pi G c^{-4}$, G is Newton's constant, and $f(R)$ is the nonlinear function of the argument [53,54].

Spherically symmetric solutions can be achieved starting from a pointlike $f(R)$ Lagrangian [55]. Such a Lagrangian can be obtained by imposing the spherical symmetry directly in the action (1). As a consequence, the infinite number of degrees of freedom of the original field theory will be reduced to a finite number. The technique is based on the choice of a suitable Lagrange multiplier, defined by assuming the Ricci scalar argument of the function $f(R)$ in spherical symmetry.

Starting from the above considerations, in a coordinate system (ct, r, θ, ϕ), the spacetime metric for a static spherical relativistic star in $f(R)$ gravity can be expressed as [56]

$$ds^2 = -A(r)dt^2 + D(r)dr^2 + H(r)(d\theta^2 + \sin^2\theta d\varphi^2), \qquad (2)$$

and then the pointlike $f(R)$ Lagrangian is

$$\mathcal{L}^* = \mathbf{L}^{1/2} \qquad (3)$$

with

$$\mathbf{L} = \underline{q}'^t \hat{\mathbf{L}}\, \underline{q}' = \frac{[(2+HR)f' - fH]}{H}[2H^2 f'' A' R' + 2HH'(f'A' + 2Af''R') + Af'H'^2], \quad (4)$$

where $\underline{q} = (A, D, H, R)$ and $\underline{q}' = (A', D', H', R')$ are the generalized positions and velocities associated with $\mathcal{L}$.

According to the Noether Theorem, the existence of a symmetry for dynamics described by the Lagrangian (4) implies the existence of a conserved quantity. The Lie differentiation of Eq. (4) yields

$$L_{\mathbf{X}}\mathbf{L} = \underline{\alpha} \cdot \nabla_q \mathbf{L} + \underline{\alpha}' \cdot \nabla_{q'} \mathbf{L} = \underline{q}'^t[\underline{\alpha} \cdot \nabla_q \hat{\mathbf{L}} + 2(\nabla_q \alpha)^t \hat{\mathbf{L}}]\underline{q}', \quad (5)$$

which vanishes if the functions $\underline{\alpha}$ satisfy the following system:

$$\underline{\alpha} \cdot \nabla_q \hat{\mathbf{L}} + 2(\nabla_q \underline{\alpha})^t \hat{\mathbf{L}} = 0 \rightarrow \alpha_i \frac{\partial \hat{\mathbf{L}}_{km}}{\partial q_i} + 2\frac{\partial \alpha_i}{\partial q_k}\hat{\mathbf{L}}_{im} = 0. \quad (6)$$

Solving the system (6) means finding functions α_i, which identify the Noether vector. However, the system (6) implicitly depends on the form of $f(R)$ and then, by solving it, one obtains the forms of the function $f(R)$, which are compatible with spherical symmetry. On the other hand, by choosing the form $f(R)$, (6) can be solved explicitly. As an example, one finds that the system (6) is satisfied if one chooses

$$f(R) = f_0 R^s, \quad \underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3) = ((3-2s)kA, -kH, kR), \quad (7)$$

with s a real number, k an integration constant, and f_0 a dimensional coupling constant. This means that, for any $f(R) = R^s$, there exists, at least, a Noether symmetry and a related constant of motion Σ_0,

$$\Sigma_0 = \underline{\alpha} \cdot \nabla_{q'} \mathbf{L} = 2skHR^{2s-3}[2s + (s-1)HR][(s-2)RA' - (2s^2 - 3s + 1)AR']. \quad (8)$$

A physical interpretation of Σ_0 is possible in GR, which means for $f(R) = R$ and $s = 1$, the above procedure has to be applied to the Lagrangian of GR. We obtain the solution $\underline{\alpha}_{\mathrm{GR}} = (-kA, kH)$. The functions A and H give the Schwarzschild solution, and then the constant of motion acquires the standard form $\Sigma_0 = 2GM/c^2$, where M is the total mass of the star. General black hole solutions of the field equations regulating the function $R(r)$ give, for example, a solution corresponding to

$$s = 5/4, \qquad H = r^2, \qquad R = 5r^{-2}, \quad (9)$$

obtaining the spherically symmetric spacetime

$$ds^2 = (\alpha + \beta r)dt^2 - \frac{1}{2}\frac{\beta r}{\alpha + \beta r}dr^2 - r^2 d\Omega, \quad (10)$$

where α is a combination of Σ_0 and k, and $\beta = k_1$. The value of s for this solution is ruled out by Solar System experiments [57–59].

Now we will look for stationary solutions of the Maxwell equations in the spacetime given by (10), i.e., for solutions in which we assume that the magnetic moment of the magnetic star does not vary in time as a result of the infinite conductivity of the stellar medium. Because of discontinuities in the fields across the surface of the sphere, we will refer to as exterior solutions those valid in the range $R_* < r \le \infty$.

Assuming the magnetic field to be dipolar, we look for separable solutions of Maxwell equations in the form

$$B^{\hat{r}}(r, \theta) = F(r)\cos\theta, \quad (11)$$

$$B^{\hat{\theta}}(r, \theta) = G(r)\sin\theta, \quad (12)$$

$$B^{\hat{\phi}}(r, \theta) = 0, \quad (13)$$

where the unknown radial functions $F(r)$ and $G(r)$ will account for the relativistic corrections due to a gravitational field in the modified gravity. Since the exterior of the star is a vacuum, we can impose zero electric current density $J^{\hat{r}} = J^{\hat{\theta}} = J^{\hat{\phi}} = 0$ in Maxwell equations and obtain Maxwell equations for the radial part of the magnetic field as [60]

$$(r^2 F)_{,r} + 2rG\sqrt{D} = 0, \quad (rG\sqrt{A})_{,r} + F\sqrt{AD} = 0. \quad (14)$$

The magnetic field depends only on r and θ coordinates due to axial symmetry and stationarity. The dipolar approximation is a simple one for the interior field; however, it is consistent with the requirement that the field configuration should match at the boundary with the external dipolar field. Since the interior magnetic field has the dipolar configuration, the continuity of normal and tangential components of the magnetic field at the stellar surface has to be required.

The exterior solution for the magnetic field is simplified by the knowledge of explicit analytic expressions for the metric functions A and D. Maxwell equations for the radial part of the magnetic field (14) can be obtained as a second-order ordinary differential equation for the unknown radial function F:

$$\frac{d}{dr}\left[(\alpha + \beta r)\sqrt{\frac{2}{\beta r}}\frac{d}{dr}(r^2 F)\right] - \sqrt{2\beta r}F = 0. \quad (15)$$

The analytical solution of Eq. (15) exists for the Schwarzschild star with the total mass M [36–50], in particular after defining metric functions $N^2 = A = D^{-1} = (1 - 2M/r)$ exterior to the star. The analytical general relativity solution of a dipolar magnetic field in a vacuum expressed through the Legendre functions of the second kind [61] shows that the dipolar magnetic field is amplified by a factor

$$\frac{F_{\rm GR}(r)}{F_{\rm Newt}(r)} = -\frac{3R^3}{8M^3}\left[\ln N^2 + \frac{2M}{r}\left(1+\frac{M}{r}\right)\right] \quad (16)$$

compared to the flat spacetime solution. Here $F_{\rm Newt} = 2\mu/r^3$ is the value of magnetic field at the pole in the Newtonian limit.

We integrate Eq. (15) using the Runge-Kutta fourth order method, with standard techniques befitting second-order ODE (e.g., Press [62]) in the program MATHEMATICA. The ODE equation is solved as an initial value problem. For initial values we have chosen $B(r) = 0$, $B'(r) = -\epsilon$ at $r = \infty$, taking into account the value of the magnetic star at the surface of the star in the Newtonian limit, where ϵ is a small positive number. In the limit of $r \to \infty$, the solution is taken to be Newtonian and does not give any contribution to the magnetic field. For the models in the present study we choose the following parameters: $R_* = 10$ km and the polar surface field strength $B(\bar{r} = r/R_* = 1) = 10^{12}$ G.

In Fig. 1 the radial dependence of the magnetic field of the neutron star in general relativity versus in the $f(R)$ gravity theory is shown. The dashed line corresponds to the dependence of the magnetic field within the framework of general relativity. The solid lines correspond to the radial dependence of the magnetic field of the neutron star in $f(R)$ gravity for the different values of parameters α and β, which are responsible for the modified terms in the theory of gravity. The mass of the star is taken as $M = 1.4M_\odot$, where $M_\odot$ is the solar mass and the radius of the star is $R_* = 10$ km. Here we consider the equatorial plane, i.e., $\theta = \pi/2$, $\dot{\theta} = 0$, and assume $\alpha = 1$ and $\beta = 2$. One can see from the dependence that the magnetic field near the compact objects is bigger in the modified gravity theory than one in GR.

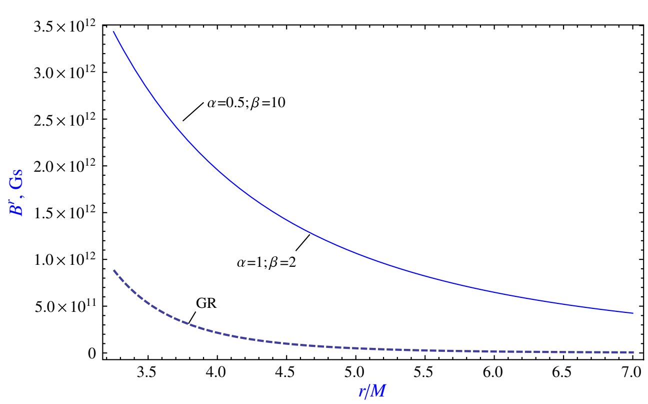

FIG. 1 (color online). The radial dependence of the magnetic field of the neutron star in general relativity versus in $f(R)$ gravity theory.

III. MAGNETIC FIELD OF THE STAR WITHIN THE HOŘAVA-LIFSHITZ GRAVITY

The four-dimensional metric of the spherical-symmetric spacetime written in the ADM formalism [11,63] has the following form:

$$ds^2 = -N^2c^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \quad (17)$$

where N, N^i are the metric functions to be defined.

The IR-modified Hořava action is given by

$$S = \int dtdx^3\sqrt{-g}N\Big[\frac{2}{\kappa^2}(K_{ij}K^{ij} - \lambda_g K^2) - \frac{\kappa^2}{2\nu_g^4}C_{ij}C^{ij} + \frac{\kappa^2\mu}{2\nu_g^2}\epsilon^{ijk}R_{il}\nabla_j R^l{}_k - \frac{\kappa^2\mu^2}{8}R_{ij}R^{ij} + \frac{\kappa^2\mu^2}{8(3\lambda_g - 1)} \times \left(\frac{4\lambda_g - 1}{4}R^2 - \Lambda_W R + 3\Lambda^2\right) + \frac{\kappa^2\mu^2\omega}{8(3\lambda_g - 1)}R\Big], \quad (18)$$

where κ, λ_g, ν_g, μ, ω, and Λ_W are constant parameters, the Cotton tensor is defined as

$$C^{ij} = \epsilon^{ikl}\nabla_k\left(R^j{}_l - \frac{1}{4}R\delta^j_l\right), \quad (19)$$

R_{ijkl} is the three-dimensional curvature tensor, and the extrinsic curvature K_{ij} is defined as

$$K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i), \quad (20)$$

where the dot denotes a derivative with respect to t.

Imposing the case $\lambda_g = 1$, which reduces to the action in the IR limit, one can obtain the Kehagias and Sfetsos (KS) asymptotically flat solution [64] for the spacetime metric outside the gravitating spherical symmetric object in Hořava gravity as

$$ds^2 = -N^2c^2dt^2 + N^{-2}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2,$$

$$N^2 = f_{\rm KS}(r) = 1 + \omega r^2 - \omega r^2\sqrt{1 + \frac{4M}{\omega r^3}}, \quad (21)$$

where ω is the KS parameter and the constant $\Lambda_W = 0$ is chosen.

Here we will also look for stationary solutions of the Maxwell equations in spacetime (21), as we did in Sec. II. Assuming the magnetic field to be dipolar, we look for separable solutions of Maxwell equations in the forms (11)–(13).

Maxwell equations for the radial part of the magnetic field can be obtained as a second-order ordinary differential equation for the unknown radial function F:

$$\frac{d}{dr}\left[f_{\rm KS}(r)\frac{d}{dr}(r^2F)\right] - 2F = 0. \quad (22)$$

The numerical solution of Eq. (22) is presented in Fig. 2. In Fig. 2 the radial dependence of the magnetic field of the

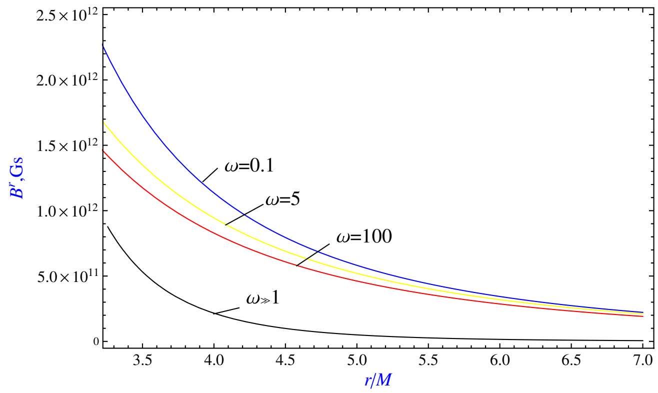

FIG. 2 (color online). The radial dependence of the magnetic field of the neutron star in general relativity versus in Hořava gravity theory.

neutron star for the cases of general relativity and Hořava gravity theory is shown. The black line corresponds to the dependence of the magnetic field within the framework of general relativity ($\omega \gg 1$). The mass of the star is taken as $M = 1.4M_{\odot}$, the radius of the star is $R_* = 10$ km. The thin colored lines correspond to the radial dependence of the magnetic field of the neutron star in Hořava gravity for the different values of ω parameter, which is responsible for the modified terms in the KS solution in Hořava gravity. Here we consider the equatorial plane, i.e., $\theta = \pi/2$, $\dot{\theta} = 0$. One can see from Figs. 1 and 2 that in the presence of modified terms in the action for the gravitational field, the surface magnetic field increases by a factor 3–4. The asymptotical values of the magnetic field tend to vanish.

IV. ASTROPHYSICAL CONSEQUENCES

Assume that the oblique magnetized star in these models of gravity is rotating, and χ is the inclination angle between the axis of rotation and magnetic momentum and observed as a pulsar through magnetic dipole radiation. Then the luminosity of the relativistic star in the case of a purely dipolar radiation, and the power radiated in the form of dipolar electromagnetic radiation, is given by [44]

$$L_{\rm em} = \frac{\Omega_{R_*}^4 R_*^6 \tilde{B}_0^2}{6c^3} \sin^2\chi, \tag{23}$$

where subscript 0 denotes the value at the star surface $r = R_*$.

When compared with the equivalent Newtonian expression for the rate of electromagnetic energy loss through dipolar radiation [65],

$$(L_{\rm em})_{\rm Newt} = \frac{\Omega^4 R_*^6 B_0^2}{6c^3} \sin^2\chi, \tag{24}$$

it is easy to realize that the general relativity $f(R)$ gravity corrections emerging in expression (23) are partly due to the magnetic field amplification $\tilde{B}_0 = F_{R_*} B_0$ at the stellar surface and partly to the increase in the effective rotational angular velocity produced by the gravitational redshift as $\Omega = \Omega_{R_*} \sqrt{A_R}$.

The modified terms in the action in $f(R)$ gravity and in Hořava gravity have the effect of enhancing the rate of energy loss through dipolar electromagnetic radiation by an amount that can easily be estimated as

$$\frac{L_{\rm em}}{(L_{\rm em})_{\rm Newt}} = \left(\frac{F_{R_*}}{A_{R_*}}\right)^2, \tag{25}$$

and whose dependence from the compactness parameter M/R_* is shown in Figs. 3 and 4 with a solid line. The dashed line corresponds to the energy loss in the case of GR. The solid line in Fig. 3 corresponds to the energy loss dependence in the case of modified gravity theory. Here we consider the equatorial plane, i.e., $\theta = \pi/2$ $\dot{\theta} = 0$, and assume $\alpha = 1$ and $\beta = 2$. The solid line in Fig. 4 corresponds to the energy loss dependence in the case of Hořava gravity. As is seen from the graph in Fig. 3, with an increase in the compactness parameter of the star in $f(R)$ gravity, the energy loss is increasing exponentially.

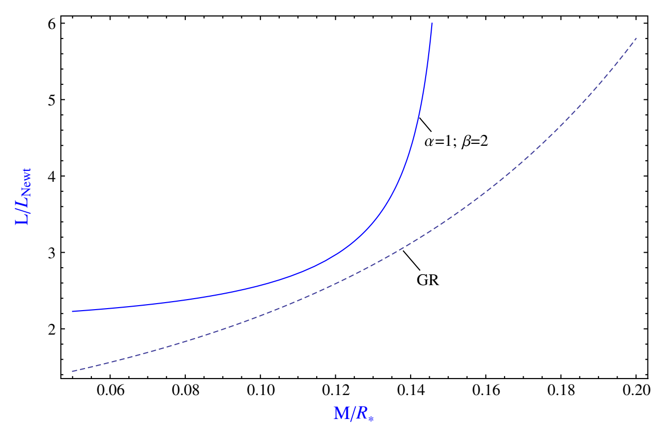

FIG. 3 (color online). The amplification of the energy loss of the NS in $f(R)$ gravity due to electromagnetic radiation.

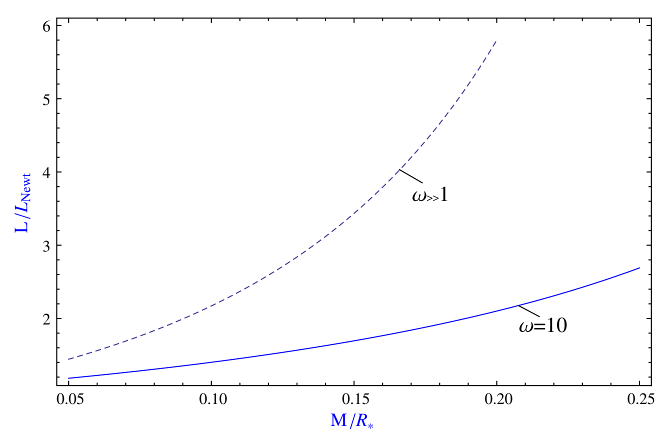

FIG. 4 (color online). The amplification of the energy loss of the NS in the Hořava gravity model due to electromagnetic radiation.

This result helps to get constraints on $f(R)$ gravity from astrophysical observations on the EM radiation from NSs. In Fig. 4 the modification of the energy loss of the NS due to electromagnetic radiation is shown. One can see from the graph that as the compactness parameter of the star increases, the energy loss is lower than in GR.

Expressions (23) and (25) could be used to study the rotational evolution of magnetized neutron stars with predominant dipolar magnetic field anchored in the crust as objects converting their rotational kinetic energy into electromagnetic radiation.

V. EVOLUTION OF MAGNETIC FIELD OF NEUTRON STARS IN MODIFIED THEORIES OF GRAVITY

From the classical electrodynamics [66], it is known that the diffusion of the magnetic field in a plasma of finite conductivity leads to a spreading of inhomogeneities, while dissipation is due to the Ohmic decay of the currents producing the field. A magnetic field $\mathbf{B}(t, \mathbf{x})$ in a plasma of uniform conductivity σ evolves, in flat spacetime, according to the following diffusion equation [66]:

$$\frac{\partial \mathbf{B}(t, \mathbf{x})}{\partial t} = \frac{c^2}{4\pi\sigma} \nabla^2 \mathbf{B}(t, \mathbf{x}). \tag{26}$$

It will decay or diffuse in a characteristic time scale $\tau_{\rm Ohm} = 4\pi\sigma L^2/c^2$, where L is a typical length scale of the field structure. Depending upon the prevailing conditions, the Ohmic decay time $\tau_{\rm Ohm}$ can range from seconds, in the case of a copper sphere of radius of a few centimeters, up to $\tau_{\rm Ohm} = 10^{10}$ years or even much longer for astrophysical settings [66].

It has been an intense effort by astrophysicists to understand the factors governing the decay of the magnetic field of neutron stars [67]. For the magnetic field of a nonrotating neutron star, it is natural to describe the field decay relative to the class of observers that find themselves at rest relative to the star, i.e., the class of Killing observers.

The spacetime geometry can be written in the form

$$ds^2 = g_{tt}(dx^t)^2 + \gamma_{ij} dx^i dx^j, \tag{27}$$

where $x^t = ct$, γ_{ij} are functions of the spatial coordinates x^i, $(i = 1, 2, 3)$ and ξ denotes the hypersurface orthogonal timelike Killing vector field obeying $\xi_\alpha \xi^\alpha = g_{tt}$. The geometry of the spacetime permits the introduction of coordinates so that the spatial three element $ds^2_{(3)}$ of (27) could be recast in the following form:

$$ds^2_{(3)} = h_r^2 (dr)^2 + h_\theta^2 (d\theta)^2 + h_\phi^2 (d\phi)^2, \tag{28}$$

where the scale factors are $h_i = h_i(r, \theta, \phi)$. For such geometries, Maxwell's equations and the current conservation law $\nabla_\alpha J^\alpha = 0$ can be rewritten in an equivalent form, involving only the components (E^i, B^i) of the electric and magnetic fields, respectively. The charge density $c\rho = -U_\mu J^\mu$ and spatial current density J^i as measured by the Killing observers can be written in the following form:

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \qquad \nabla \cdot \mathbf{B} = 0, \tag{29}$$

$$\nabla \times (\Lambda \mathbf{B}) = \frac{4\pi}{c} \Lambda \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}, \tag{30}$$

$$\nabla \times (\Lambda \mathbf{E}) = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, \tag{31}$$

$$\nabla \cdot \mathbf{J} + \mathbf{J} \cdot \nabla (\ln \Lambda) = 0, \tag{32}$$

where $\Lambda = (-\xi^\alpha \xi_\alpha)^{\frac{1}{2}} = \sqrt{g_{tt}}$ is the redshift factor, which in the language of the 3 + 1 approach to spacetime is also referred to as the lapse function [68]. Using Eqs. (29)–(32), the generalized induction equation takes the following form:

$$\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \left[\frac{c}{4\pi\sigma} \nabla \times (\Lambda \mathbf{B}) \right] = 0. \tag{33}$$

We shall explore the content of the relativistic induction equation (33) by applying it to study the evolution of magnetic fields associated with neutron stars in modified theories of gravity. To investigate the impact of the spacetime curvature upon the magnetic field decay in the modified gravity, we shall take over a simple neutron star model. Accordingly, and to avoid hard numerical computations, we shall ignore the rotation of the neutron star and thus adopt as the background geometry a nonsingular, static, and spherically symmetric one. It has been shown in [42,50] that the neutron star rotation in general relativity affects the decay of the magnetic field of neutron stars through small dimensionless parameter ω/σ (where ω is the frequency of dragging of inertial frames and negligible). The evolution of the magnetic field decay is considerably affected by the electrical conductivity σ, and in order to emphasize the effects of spacetime curvature, we shall take σ to be spherically symmetric and shall ignore any cooling effects that may influence its temporal evolution. For an axially symmetric field $\mathbf{B}$, it is convenient to decompose it into the so-called poloidal $\mathbf{B}_{(p)} = B^r \mathbf{e}_r + B^\theta \mathbf{e}_\theta$ and toroidal parts $\mathbf{B}_{(t)} = B^\phi \mathbf{e}_\phi$. For simplicity, in the present paper we shall examine the effects of the spacetime curvature only on the evolution of a purely poloidal field. Taking into account the poloidal and axisymmetric nature of magnetic field $\mathbf{B}$, one easily finds that $\nabla \cdot \mathbf{B} = 0$ implies

$$\frac{h_r^{-1}}{r} \frac{\partial (r^2 B^r)}{\partial r} + \frac{1}{\sin\theta} \frac{\partial (B^\theta \sin\theta)}{\partial \theta} = 0. \tag{34}$$

We shall look for separable solutions of the above equations in the form (11) and (12), where

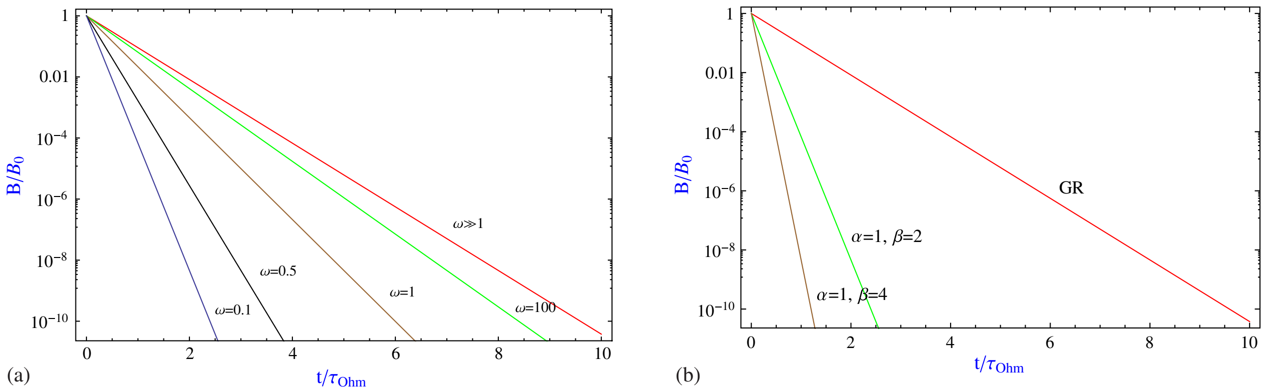

FIG. 5 (color online). Dipolar magnetic field decay for a uniform density star in (a) Hořava-Lifshitz gravity and (b) f(R) gravity model for the different values of the parameter ω and β. The lines in the case when $\omega \gg 1$ and labeled as GR correspond to the general relativistic case.

$$G(r) = \frac{h_r^{-1}}{2r}\frac{\partial(r^2 F)}{\partial r}. \tag{35}$$

Using Eq. (35) for any poloidal axisymmetric field with components (B^r, B^θ), the induction equation (33) in geometry [27] gets the following evolution equation:

$$\frac{4\pi\sigma}{c}\frac{\partial F}{\partial x^o} = \frac{h_r^{-1}}{r^2}\frac{\partial}{\partial r}\left[\frac{\Lambda}{h_r}\frac{\partial(r^2 F)}{\partial r}\right] - \frac{2\Lambda F}{r^2}. \tag{36}$$

Here we consider the decay of a magnetic field in a neutron star of constant density in modified gravity. The time evolution of a chosen initial distribution on general grounds $F(t = 0, r)$ can be expanded in a series of the following form [48]:

$$F(t, x) = \Sigma a_n e^{-c^2\lambda_n t/4\pi\sigma R^2} g_n(x), \tag{37}$$

where the summation is extended over all eigenmodes $g_n(x)$ of the corresponding (singular) Sturm-Liouville eigenvalue problem arising from Eq. (36).

It follows now from Eq. (37) that if the eigenvalues are positive and well spaced, then after $t \gg t_{\rm Ohm}/\lambda_1$ where λ_1 is the lowest eigenvalue of the above system, the evolution of the distribution will channel into an exponentially decreasing phase with the dominant contribution in the sum (37) coming from the "first" term. The evolution of $F(t, x)$ channels into an exponentially decaying mode, which means, according to (37), that the evolution of the initial distribution is eventually described by the first nonvanishing term in the series expansion (37) [48]. This behavior of $F(t, x)$ allows us to determine only the lowest eigenvalue λ_1 of (37) from numerical outputs. Besides the explicit determination of λ_1, our numerical treatment allows us to construct the magnetic field as well. In Fig. 5, we plot as a function of coordinate time t, the magnetic field as perceived by a Killing observer located at the star's pole for the various values of the parameters of Hořava-Lifshitz theory and $f(R)$ gravity models considered in the previous sections. In Fig. 5 the horizontal axis represents (coordinate) time in units of the Ohmic decay time $\tau_{\rm ohm} \equiv 4\pi\sigma R^2/c^2$ in flat spacetime and the vertical axis shows the value of $B/B_0 = B(t, r = R, \theta = 0)/B(t = 0, r = R, \theta = 0)$. All models have the same areal radius ($R = 10$ km) and a constant uniform conductivity ($\sigma = 10^{25}$ s^{-1}), which are typical for neutron star models. The initial field profile is taken as the $n = 1$ eigenmode for all cases. The graphs show quite clearly the exponential decay of magnetic field in the modified theories of gravity as in general relativity.

In Fig. 5(a) the magnetic field decay of a neutron star in the Hořava-Lifshitz gravity model is shown for the different values of the KS parameter ω in spacetime metric (21). One can observe that in the Hořava-Lifshitz gravity model, the magnetic field decay rate is faster than that in general relativity. The contribution of IR modified Hořava-Lifshitz gravity into the evolution of the stellar magnetic field is becoming dominant in the interior part of the star i.e., with the decrease of the radial coordinate. Since magnetic field decay corresponds to physical processes in inner parts of the star, the effects of Hořava-Lifshitz gravity become more dominant in comparison to the general relativity effects.

In Fig. 5(b) the magnetic field decay of the neutron star in the $f(R)$ gravity model is shown for the different values of the parameter β in metric (10). One can observe that in the $f(R)$ gravity model, the magnetic field decay rate is much faster than that in general relativity. $f(R)$ gravity model modification to the spacetime metric is becoming very sufficient inside of the neutron star. This implies an increase in the magnetic field decay rate in comparison to general relativity.

VI. CONCLUSION

We have studied the magnetic field of isolated relativistic compact star in both $f(R)$ and HL theories of gravity, assuming that their magnetic fields are confined to the stellar crust. We have been working with modified theories of gravity effects on the stellar magnetic field, accompanied by proper boundary conditions. In other words we generalized the general relativity approach in the sense that we took into account the effect of additional tension from the modified gravity on electromagnetic fields.

First we have found the numerical solutions that take into account the effect of $f(R)$ gravity tension and also KS parameter ω in HL gravity tension on the structure of the magnetic field outside the star. In Fig. 1 one can find out that in the presence of modified terms in the action for the gravitational field, the surface magnetic field increases by a factor of 3 to 4, depending on the compactness of the star. The asymptotic values of the magnetic field tend to 0. Comparing the behavior of the magnetic field when HL gravity effects are included with the one in GR, one can see enhancement of the magnetic field especially near the surface of the relativistic star for the external field. This effect grows stronger as the compactness parameter increases.

We have found that the effect of $f(R)$ gravity and the effect of KS parameter ω on magnetic fields of compact stars can be very important, and the expression for the magnetodipolar luminosity of a rotating magnetized star in modified gravity gives enhancement up to two orders.

From the obtained results in Sec. IV, we are able to conclude that the $f(R)$ gravity model, at least the configuration that we used in this context, is valid only for relativistic stars with small compactness parameters. Since the energy loss increases exponentially with the increased compactness parameter, and there is no observation data on such big electromagnetic energy loss of the neutron stars, one may conclude that $f(R)$ gravity of the Noether approach does not support the existence of relativistic stars with high compactness parameters in $f(R)$ gravity. Actually, the monopolar part of the gravitational field is responsible for the amplification of the stellar magnetic field. In $f(R)$ gravity the monopolar part of the gravitational field is essentially changed, which is the effective reason for the high amplification of the stellar magnetic field. In the Hořava model the main modification effects on the higher-order terms of the gravitational field and the effect to increase the magnetic field are not so strong compared to those in $f(R)$ gravity.

In [69] the authors have shown that the other model of $f(R)$ gravity, called the Palatini approach, cannot be considered as an alternative theory for describing physics of compact stars. Here we have obtained a similar result for the existence of magnetized compact stars with the high compactness parameter in the Noether $f(R)$ gravity model based on the astrophysical observations of no exponential spindown rate of the neutron stars [70]. In the last section we have presented a limited framework, taking into account the effects of spacetime curvature on the magnetic field decay in modified theories of gravity like Hořava gravity and $f(R)$ gravity of the Noether approach. One can see that magnetic field decay of a neutron star in both models of modified theories of gravity is faster than it is in Einstein's theory of gravity.

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